VIRTUAL CARTIER DIVISORS AND BLOW-UPS

ADEEL A. KHAN

1. Introduction

1.1. Let X be a scheme, Z a regularly immersed closed subscheme, and $\tilde{X}$ the blow-up of X in Z. Recall that $\tilde{X}$ admits the following universal property: for any morphism $f : S \to X$ such that the schematic fibre $f^{-1}(Z)$ is an effective Cartier divisor on S, there exists a unique morphism $S \to \tilde{X}$ over X.

The purpose of this note is to prove a stronger universal property, where the morphism f is allowed to be arbitrary. In particular, we provide a complete description of the functor represented by the blow-up $\tilde{X}$. Namely, we show that there is a canonical bijection between the set of X-morphisms $S \to \tilde{X}$ and the set of *virtual effective Cartier divisors on S lying over* (X, Z). In a word, a virtual effective Cartier divisor is a closed subscheme that is equipped with some additional structure that remembers that it is cut out locally by a single equation (or "of virtual codimension 1").

1.2. We also use the notion of virtual Cartier divisors to construct blow-ups of quasi-smooth closed immersions of *derived schemes*. One application is a derived version of Fulton–MacPherson's technique of deformation to the normal cone. This gives a new construction of virtual fundamental classes in Chow groups that we intend to explain in a different paper.

1.3. The organization of this paper is as follows. In Sect. 2 we study regular closed immersions from the perspective of derived algebraic geometry; this material is well-known to the experts. The definition of virtual effective Cartier divisor is 2.3.6. Sect. 3 contains our results on blow-ups. The derived blow-up is constructed in 3.1.1 and its properties are summarized in 3.1.2. The universal property mentioned above is in 3.1.3. The remaining pages are concerned with the proofs.

1.4. I would like to thank Denis-Charles Cisinski and Akhil Mathew for comments on previous revisions. I am especially grateful to David Rydh for help, corrections, and the extensive discussion that eventually led to the definition in 3.1.1.

2. Quasi-smooth immersions

2.1. We begin by reviewing the notion of regular closed immersion in classical algebraic geometry.

2.1.1. Let A be a commutative ring. For an element $f \in A$, the Koszul complex $\mathrm{Kosz}_A(f)$ is the chain complex

$$\mathrm{Kosz}_A(f) := \left(A \xrightarrow{f} A\right),$$

concentrated in degrees 0 and 1. Thus $H_0(\mathrm{Kosz}_A(f)) = A/f$ and $H_1(\mathrm{Kosz}_A(f)) = \mathrm{Ann}_A(f)$ is the annihilator. In particular $\mathrm{Kosz}_A(f)$ is acyclic in positive degrees, and hence quasi-isomorphic to A/f, if and only if f is regular (a non-zero divisor). More generally, given a sequence of elements $(f_1, \ldots, f_n)$, the Koszul complex $\mathrm{Kosz}_A(f_1, \ldots, f_n)$ is defined as the tensor product

$$\mathrm{Kosz}_A(f_1, \ldots, f_n) = \bigotimes_i \left(A \xrightarrow{f_i} A\right).$$

We say that the sequence $(f_1, \ldots, f_n)$ is *regular* if the Koszul complex is acyclic in positive degrees. When A is a noetherian ring and f_i belong to the radical, this is equivalent to the usual inductive definition: f_1 is regular, f_2 is regular in $A/(f_1)$, etc. (see [Gro67, Cor. 19.5.2] and [BGI71, Prop. 1.3]).

2.1.2. Any sequence $(f_1, \ldots, f_n)$ determines a homomorphism $\mathbf{Z}[T_1, \ldots, T_n] \to A$, $T_i \mapsto f_i$, and the Koszul complex $\mathrm{Kosz}_A(f_1, \ldots, f_n)$ is quasi-isomorphic to the derived tensor product

$$A \otimes^{\mathbf{L}}_{\mathbf{Z}[T_1,\ldots,T_n]} \mathbf{Z}[T_1, \ldots, T_n]/(T_1, \ldots, T_n);$$

indeed the sequence $(T_1, \ldots, T_n)$ is regular, so that $\mathrm{Kosz}_{\mathbf{Z}[T_1,\ldots,T_n]}(T_1, \ldots, T_n)$ provides a free resolution of $\mathbf{Z}[T_1, \ldots, T_n]/(T_1, \ldots, T_n)$, and the Koszul complex is stable under arbitrary extension of scalars.

2.1.3. Let $i : Z \hookrightarrow X$ be a closed immersion of schemes. We say that i is *regular* if its ideal of definition $\mathcal{I} \subset \mathcal{O}_X$ is Zariski-locally generated by a regular sequence. This is equivalent to the definition in [BGI71, Exp. VII, Def. 1.4], and to the definition in [Gro67, Def. 16.9.2] when X is locally noetherian. When i is regular, the conormal sheaf $\mathcal{N}_{Z/X} = \mathcal{I}/\mathcal{I}^2$ is locally free of finite rank. Moreover, the relative cotangent complex $\mathcal{L}_{Z/X}$ is canonically identified with $\mathcal{N}_{Z/X}[1]$. For us an *effective Cartier divisor* on a scheme X will be a scheme D equipped with a regular closed immersion $i_D : D \hookrightarrow X$ of codimension 1.

2.2. We now re-interpret the above discussion in the language of derived algebraic geometry. The basic idea is that the Koszul complex $\mathrm{Kosz}_A(f_1, \ldots, f_n)$ can be viewed as the "ring of functions" on a certain *derived* subscheme of $\mathrm{Spec}(A)$. In order to make sense of this, one should work with simplicial commutative rings, which following Quillen is the natural setting for derived tensor products of commutative algebras.

2.2.1. We use the language of ∞-categories; our reference is [Lur09]. Let Spc be the ∞-category of spaces[1]. For any ∞-category $\mathbf{C}$, there are mapping spaces $\mathrm{Maps}_{\mathbf{C}}(x,y) \in \mathrm{Spc}$ for any pair of objects $x, y \in \mathbf{C}$.

2.2.2. Let SCRing be the ∞-category of simplicial commutative rings[2]; we refer to [Lur, § 25.1] for a detailed account. If we forget the multiplication on a simplicial commutative ring A, we get by the Dold–Kan correspondence an "underlying chain complex". If we forget both the addition and multiplication we get an "underlying space". We say that a simplicial commutative ring A is 0-truncated or "discrete" if the underlying space (pointed at 0) has no higher homotopy groups (i.e., can be identified with a set); the full subcategory of discrete simplicial commutative rings is canonically equivalent to the category CRing of ordinary commutative rings, and under this identification the 0-truncation functor is given by $A \mapsto \pi_0(A)$.

Using Quillen's machinery of non-abelian derived functors one defines derived tensor products in SCRing (they are computed using simplicial resolutions by polynomial algebras $\mathbf{Z}[T_1, \dots, T_n]$). For example, the derived tensor product

$$A \otimes^{\mathbf{L}}_{\mathbf{Z}[T_1,\dots,T_n]} \mathbf{Z}[T_1, \dots, T_n]/(T_1, \dots, T_n)$$

can be viewed as a simplicial commutative ring, for any commutative ring A and any homomorphism $\mathbf{Z}[T_1, \dots, T_n] \to A$. The underlying chain complex recovers the Koszul complex of the corresponding elements (as an object of $\mathrm{Mod}_{\mathbf{Z}}$, the stable ∞-category of chain complexes of $\mathbf{Z}$-modules).

2.2.3. Derived algebraic geometry is an extension of classical algebraic geometry that allows one to make sense of "Spec(A)" where A is a simplicial commutative ring. We refer to [Lur] or [TV08] for details.

The object $\mathrm{Spec}(A)$ is an affine derived scheme which has an "underlying classical scheme" $\mathrm{Spec}(A)_{\mathrm{cl}} = \mathrm{Spec}(\pi_0(A))$, but is further equipped with a quasi-coherent sheaf of simplicial commutative rings $\mathcal{O}_{\mathrm{Spec}(A)}$. Thus it can be viewed as an infinitesimal thickening of $\mathrm{Spec}(A)_{\mathrm{cl}}$ where the nilpotents live in the higher homotopy groups. For example, given affine schemes $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ over $S = \mathrm{Spec}(R)$, there is a derived version of the fibred product:

$$X \overset{\mathbf{R}}{\underset{S}{\times}} Y = \mathrm{Spec}(A \otimes^{\mathbf{L}}_R B),$$

which is a derived scheme with $(X \times^{\mathbf{R}}_S Y)_{\mathrm{cl}} = X \times_S Y$.

A general derived scheme is Zariski-locally of the form $\mathrm{Spec}(A)$ for $A \in \mathrm{SCRing}$. Derived fibred products are homotopy limits in the ∞-category of derived schemes. The category of classical schemes embeds fully faithfully into the ∞-category of derived schemes, and its essential image is spanned by derived schemes X whose

[1] We will not rely on any particular model, say by topological spaces or simplicial sets; instead we will use the language of ∞-groupoids. Thus a "space" has "points" (objects), "paths" (invertible morphisms) between any two points, etc. In particular, the term "isomorphism of spaces" will never refer to an isomorphism of set-theoretic models, but rather to an isomorphism in the ∞-category Spc.

[2] This ∞-category is the free completion by sifted homotopy colimits of the category of polynomial algebras $\mathbf{Z}[T_1, \dots, T_n]$, $n \geqslant 0$, and ring homomorphisms. As for spaces, we will not use any set-theoretic models; the term "simplicial commutative ring" just means "object of SCRing".

structure sheaf $\mathcal{O}_X$ is discrete (i.e., takes values in discrete simplicial commutative rings). We will say that a derived scheme is "classical" if it is in the essential image.

The discussion of Subsect. 2.1 can now be rephrased as follows.

Proposition. *Let $i : Z \hookrightarrow X$ be a closed immersion of schemes. Then i is regular if and only if Zariski-locally on X, there exists a morphism $f : X \to \mathbf{A}^n$ and a commutative square*

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \mathbf{A}^n, \end{array}$$

which is homotopy cartesian in the ∞-category of derived schemes.

Here $\mathbf{A}^n = \mathrm{Spec}(\mathbf{Z}[T_1, \ldots, T_n])$ denotes n-dimensional affine space over $\mathrm{Spec}(\mathbf{Z})$, and $\{0\} = \mathrm{Spec}(\mathbf{Z}[T_1, \ldots, T_n]/(T_1, \ldots, T_n))$ denotes the inclusion of the origin.

2.3. We will now extend the notion of regularity to the derived setting.

2.3.1. Let A be a simplicial commutative ring. Let $f_1, \ldots, f_n \in A$ be a sequence of elements (i.e., points in the underlying space of A). Let $A/\!\!/(f_1, \ldots, f_n)$ denote the simplicial commutative ring defined by the homotopy cocartesian square

$$\begin{array}{ccc} \mathbf{Z}[T_1, \ldots, T_n] & \longrightarrow & \mathbf{Z}[T_1, \ldots, T_n]/(T_1, \ldots, T_n) \\ \downarrow{\scriptstyle T_i \mapsto f_i} & & \downarrow \\ A & \longrightarrow & A/\!\!/(f_1, \ldots, f_n). \end{array}$$

That is, $A/\!\!/(f_1, \ldots, f_n)$ is given by the derived tensor product $A \otimes^{\mathbf{L}}_{\mathbf{Z}[T_1,\ldots,T_n]} \mathbf{Z}[T_1, \ldots, T_n]/(T_1, \ldots, T_n)$. We have $\pi_0(A/\!\!/(f_1, \ldots, f_n)) = \pi_0(A)/(f_1, \ldots, f_n)$ (where we also write f_i for its connected component in $\pi_0(A)$). The underlying A-module of $A/\!\!/(f_1, \ldots, f_n)$ is given by

$$\bigotimes_i^{\mathbf{L}} \mathrm{Cofib}(A \xrightarrow{f_i} A),$$

where Cofib denotes the homotopy cofibre (in the stable ∞-category of A-modules). This gives in particular the following description of the space of paths $g \approx 0$ in (the underlying space of) $A/\!\!/(f)$, which will be important later:

Lemma. *Let A be a simplicial commutative ring and $f \in A$. Then for any point $g \in A$, there is a canonical isomorphism of spaces*

$$\mathrm{Maps}_{A/\!\!/(f)}(g, 0) \xrightarrow{\sim} \mathrm{Fib}_g(A \xrightarrow{f} A),$$

between the space of paths $g \approx 0$ (in the underlying space of $A/\!\!/(f)$), and the space of pairs (a, α), where a is a point and $\alpha : af \approx g$ is a path in A.

Proof. There is a fibre sequence

$$A \xrightarrow{f} A \to A/\!\!/(f)$$

of underlying spaces. $\square$

Examples. (i) If A is discrete and the sequence $(f_1, \ldots, f_n)$ is regular, then the canonical homomorphism $A/\!\!/(f_1, \ldots, f_n) \to A/(f_1, \ldots, f_n)$ is an isomorphism.

(ii) Let A be discrete and consider $A/\!\!/(0) \in \mathrm{SCRing}$. Its underlying chain complex is given by $A \oplus A[1]$ (with zero differential). In particular we have $\pi_0(A/\!\!/(0)) = \pi_1(A/\!\!/(0)) = A$.

2.3.2. Let $i : Z \hookrightarrow X$ be a closed immersion of derived schemes (i.e., the underlying morphism of classical schemes $i_{\mathrm{cl}} : Z_{\mathrm{cl}} \to X_{\mathrm{cl}}$ is a closed immersion). We say that i is *quasi-smooth* if Zariski-locally on X, there exists a morphism $f : X \to \mathbf{A}^n$ and a homotopy cartesian square

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & X \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbf{A}^n, \end{tikzcd}$$

in the ∞-category of derived schemes. In other words, $Z \hookrightarrow X$ is locally of the form $\mathrm{Spec}(A/\!\!/(f_1, \ldots, f_n)) \hookrightarrow \mathrm{Spec}(A)$, for some $f_1, \ldots, f_n \in A$. If X and Z are classical, then a closed immersion $i : Z \hookrightarrow X$ is quasi-smooth if and only if it is a regular immersion. By Proposition 2.2.3 this agrees with our previous definition when X and Z are classical. However, even if X is classical, there exist quasi-smooth immersions $Z \hookrightarrow X$ with Z non-classical (unless X is empty).

2.3.3. We now give a differential characterization of quasi-smooth immersions.

Proposition. *Let $i : Z \hookrightarrow X$ be a closed immersion of derived schemes. Then i is quasi-smooth if and only if the shifted cotangent complex $\mathcal{L}_{Z/X}[-1]$ is a locally free $\mathcal{O}_Z$-module of finite rank.*

Proof. The condition is clearly necessary: since it is Zariski-local and stable under arbitrary derived base change, we may assume that i is the zero section $\{0\} \hookrightarrow \mathbf{A}^n$, $n \geqslant 0$, in which case $\mathcal{L}_{\{0\}/\mathbf{A}^n}[-1] = \mathcal{N}_{\{0\}/\mathbf{A}^n}$ is free of rank n.

Conversely, suppose that $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ are affine, and the shifted cotangent complex $\mathrm{L}_{B/A}[-1] \in \mathrm{Mod}_B$ is free of rank n. Let F denote the homotopy fibre of the morphism $\varphi : A \to B$ in the stable ∞-category Mod_A, so that there is a canonical isomorphism of $\pi_0(B)$-modules $\pi_1(\mathrm{L}_{B/A}) \approx \pi_0(F \otimes^{\mathbf{L}}_A B)$ [Lur, Cor. 25.3.6.1]. Choose a basis $df_1, \ldots, df_n$ for $\pi_1(\mathrm{L}_{B/A})$ and note that the corresponding elements of $\pi_0(F \otimes^{\mathbf{L}}_A B)$ lift to elements $\tilde{f}_1, \ldots, \tilde{f}_n \in \pi_0(F)$, since $\varphi : A \to B$ is surjective on π_0; moreover, we can assume by Nakayama's lemma that the $\tilde{f}_i$ generate $\pi_0(F)$ as a $\pi_0(A)$-module. Lifting them to points in the underlying space of F, we get points $f_i \in A$ equipped with paths $\varphi(f_i) \approx 0$ in B, and hence a canonical homomorphism of simplicial commutative rings $A/\!\!/(f_1, \ldots, f_n) \to B$. By construction it induces an isomorphism $\pi_0(A)/(f_1, \ldots, f_n) \approx \pi_0(B)$ on connected components, so by [Lur, Cor. 25.3.6.6] it suffices to show that its relative cotangent complex vanishes, which follows by examining the exact triangle

$$\mathrm{L}_{(A/\!\!/(f_i)_i)/A} \otimes^{\mathbf{L}}_{A/\!\!/(f_i)_i} B \to \mathrm{L}_{B/A} \to \mathrm{L}_{B/(A/\!\!/(f_i)_i)}$$

in Mod_B. $\square$

Given a quasi-smooth closed immersion $i : Z \hookrightarrow X$, we write $\mathcal{N}_{Z/X} = \mathcal{L}_{Z/X}[-1]$ and take this as the definition of the *conormal sheaf*. By Proposition 2.3.3 this is a locally free $\mathcal{O}_Z$-module of finite rank.

Examples. (i) If X and Z are *smooth* over some base S, then any closed immersion $i : Z \hookrightarrow X$ is quasi-smooth. This follows from the exact triangle

$$i^*\mathcal{L}_{X/S} \to \mathcal{L}_{Z/S} \to \mathcal{L}_{Z/X}.$$

(ii) If i admits a smooth retraction, then it is quasi-smooth. This is a special case of (i).

2.3.4. Let $i : Z \hookrightarrow X$ be a quasi-smooth closed immersion of derived schemes. The *virtual codimension* of i, defined Zariski-locally on Z, is the rank of the locally free $\mathcal{O}_Z$-module $\mathcal{N}_{Z/X}$. We will sometimes denote it by $\mathrm{codim.vir}(Z, X)$. Locally this number is determined by the formula $\mathrm{codim.vir}(\mathrm{Spec}(A/\!\!/(f_1, \ldots, f_n)), \mathrm{Spec}(A)) = n$ for any $A \in \mathrm{SCRing}$ and points $f_1, \ldots, f_n \in A$. Note that virtual codimension is stable under arbitrary derived base change: $\mathrm{codim.vir}(Z, X) = \mathrm{codim.vir}(Z \times^{\mathbf{R}}_X X', X')$ for any morphism $f : X' \to X$ of derived schemes.

2.3.5. Define the *topological (Krull) codimension* $\mathrm{codim.top}(Z, X)$ as the topological codimension of Z_{cl} in X_{cl}. If X_{cl} is locally noetherian then we have an inequality $\mathrm{codim.vir}(Z, X) \geqslant \mathrm{codim.top}(Z, X)$. This is an equality (at a point $x \in X$) if the derived scheme $Z \times^{\mathbf{R}}_X X_{\mathrm{cl}}$ is classical (in a Zariski neighbourhood of x). If X_{cl} is Cohen–Macaulay (e.g. regular) at the point x, then the converse also holds.

2.3.6. Let X be a derived scheme. A *virtual effective Cartier divisor* on X is a derived scheme D together with a quasi-smooth closed immersion $i_D : D \hookrightarrow X$ of virtual codimension 1. Thus locally, a virtual effective Cartier divisor on $\mathrm{Spec}(A)$ is of the form $\mathrm{Spec}(A/\!\!/(f))$ for some $f \in A$. We will omit the adjective "effective" in this note (and never talk about non-effective divisors).

Example. Suppose that X is classical. Then any classical Cartier divisor on X is a virtual Cartier divisor.

3. Blow-ups

3.1. *Construction and main properties.*

3.1.1. Let $i : Z \hookrightarrow X$ be a quasi-smooth closed immersion of derived schemes. For any derived scheme S and a morphism $f : S \to X$, a *virtual Cartier divisor on S lying over* (X, Z) is the datum of a commutative square

$$\begin{array}{ccc} D & \xrightarrow{i_D} & S \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array} \tag{3.1.a}$$

satisfying the following conditions:

(a) The morphism $i_D : D \to S$ exhibits D as a virtual Cartier divisor on S.

(b) The underlying square of classical schemes is cartesian.

(c) The canonical morphism

$$g^*\mathcal{N}_{Z/X} \to \mathcal{N}_{D/S} \tag{3.1.b}$$

is surjective (on π_0).

Example. Suppose that X, Z, and S are classical schemes. If the classical schematic fibre $f^{-1}(Z)$ is a classical Cartier divisor on S, then $f^{-1}(Z)$ also defines a virtual Cartier divisor lying over (X, Z). Condition (c) follows from [Lur, Cor. 25.3.6.3].

Remarks. (i) If the square (3.1.a) is homotopy cartesian, then the morphism (3.1.b) is an isomorphism. If i is of virtual codimension $n > 1$ then this is never the case.

(ii) Let S_Z denote the derived fibred product $S \times^{\mathbf{R}}_X Z$. We can think of a virtual Cartier divisor on S lying over (X, Z) equivalently as a derived scheme D over S_Z such that (a) the induced morphism $i_D : D \to S_Z \hookrightarrow S$ is a virtual Cartier divisor; (b) the morphism $D \to S_Z$ induces an isomorphism $D_{\mathrm{cl}} \approx (S_Z)_{\mathrm{cl}}$ on underlying classical schemes; and (c) the canonical morphism $h^*\mathcal{N}_{S_Z/S} \to \mathcal{N}_{D/S}$ is surjective, where $h : D \to S_Z$. This latter condition is equivalent to the relative cotangent complex $\mathcal{L}_{D/S_Z}$ being 1-connected ($\pi_{i\leqslant 1} = 0$).

3.1.2. The collection of virtual Cartier divisors on S lying over (X, Z) forms a space which we denote $\mathrm{Bl}_{Z/X}(S \to X)$. Moreover, the construction is functorial and defines a presheaf of spaces

$$\mathrm{Bl}_{Z/X} : (\mathrm{DSch}_{/X})^{\mathrm{op}} \to \mathrm{Spc}$$

on the site of derived schemes over X. Indeed consider the presheaf $\mathcal{F} : (S \to X) \mapsto (\mathrm{DSch}_{/S_Z})^{\approx}$, which sends $S \to X$ to the space obtained by discarding non-invertible morphisms in the ∞-category $\mathrm{DSch}_{/S_Z}$, and note that $\mathrm{Bl}_{Z/X}$ defines a sub-presheaf since the conditions (a), (b) and (c) are stable under derived base change. Since these conditions are also Zariski-local, and $\mathcal{F}$ satisfies (hyper)descent for the Zariski topology, the presheaf $\mathrm{Bl}_{Z/X}$ is also a Zariski (hyper)sheaf. In particular, it defines a derived stack $\mathrm{Bl}_{Z/X}$ over X, which we call the *blow-up*; we denote by $\pi_{Z/X}$ the structural morphism $\mathrm{Bl}_{Z/X} \to X$. The main properties of the construction $\mathrm{Bl}_{Z/X}$ are summed up below:

Theorem.

(i) *The derived stack* $\mathrm{Bl}_{Z/X}$ *is (representable by) a derived scheme.*

(ii) *The derived scheme* $\mathrm{Bl}_{Z/X} \to X$ *is stable under arbitrary derived base change.*

(iii) *There is a canonical closed immersion* $\mathbf{P}_Z(\mathcal{N}_{Z/X}) \hookrightarrow \mathrm{Bl}_{Z/X}$ *which exhibits the projectivized normal bundle as the universal virtual Cartier divisor lying over* (X, Z)*.*

(iv) *The structural morphism* $\pi_{Z/X} : \mathrm{Bl}_{Z/X} \to X$ *is proper, and induces an isomorphism* $\mathrm{Bl}_{Z/X} - \mathbf{P}_Z(\mathcal{N}_{Z/X}) \xrightarrow{\sim} X - Z$*.*

(v) *Suppose that* X *and* Z *are classical schemes. Then the derived scheme* $\mathrm{Bl}_{Z/X}$ *is classical, and coincides with the classical blow-up* $\mathrm{Bl}^{\mathrm{cl}}_{Z/X}$*.*

The proof will be delayed a few pages (see Subsect. 3.3).

3.1.3. The following universal property for the classical blow-up $\mathrm{Bl}^{\mathrm{cl}}_{Z/X}$ follows from point (v) and the definition of $\mathrm{Bl}_{Z/X}$:

Corollary. *Let $i : Z \hookrightarrow X$ be a regular closed immersion between classical schemes. For any classical scheme S over X, the set of X-morphisms $S \to \mathrm{Bl}^{\mathrm{cl}}_{Z/X}$ is in bijection with the set of virtual Cartier divisors on S lying over (X, Z).*

Remark. In the situation of the Corollary, assertion (v) in the Theorem implies that for any classical scheme $S \to X$, the space $\mathrm{Bl}_{Z/X}(S \to X)$ of virtual Cartier divisors on S lying over (X, Z) is *discrete* (i.e., can indeed be identified with a set).

3.1.4. Suppose that $X = \mathrm{Spec}(A)$ is a classical noetherian affine scheme, $Z = \mathrm{Spec}(A/I)$ is a closed subscheme, and $f_1, \ldots, f_n$ are generators of the ideal I. In this situation the authors of [KST17] consider the derived scheme

$$X \overset{\mathbf{R}}{\underset{\mathbf{A}^n}{\times}} \mathrm{Bl}^{\mathrm{cl}}_{\{0\}/\mathbf{A}^n},$$

where $f : X \to \mathbf{A}^n$ corresponds to the elements f_i. Points (ii) and (v) of the Theorem above show that this derived scheme is nothing else than the blow-up of the quasi-smooth immersion $\mathrm{Spec}(A/\!\!/(f_1, \ldots, f_n)) \hookrightarrow \mathrm{Spec}(A)$. In particular it follows that the construction of *loc. cit.* can be defined intrinsically in terms of the derived subscheme $\mathrm{Spec}(A/\!\!/(f_1, \ldots, f_n))$.

3.2. *A special case.* In this subsection we will study the special case of the quasi-smooth closed immersion $\{0\} \hookrightarrow \mathbf{A}^n$, $n \geqslant 1$. We will identify $\mathbf{A}^n$ with $\mathrm{Spec}(\mathbf{Z}[T_1, \ldots, T_n])$ and write $Y := \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$ for convenience.

3.2.1. For each $1 \leqslant k \leqslant n$ let $A_k = \mathbf{Z}[T_1/T_k, \ldots, T_n/T_k, T_k]$. The commutative squares

$$\begin{tikzcd}
\mathrm{Spec}(A_k/(T_k)) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(A_k) \arrow[d] \\
\mathrm{Spec}(\mathbf{Z}[T_1, \ldots, T_n]/(T_1, \ldots, T_n)) \arrow[r, hook] & \mathrm{Spec}(\mathbf{Z}[T_1, \ldots, T_n])
\end{tikzcd}$$

define virtual Cartier divisors lying over $(\mathbf{A}^n, \{0\})$, which are classified by canonical morphisms $\mathrm{Spec}(A_k) \to Y$. Let Y_k denote the respective images of these morphisms (in the ∞-topos of derived stacks); that is, Y_k is the derived substack of Y classifying virtual Cartier divisors lying over $(\mathbf{A}^n, \{0\})$ of the form

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[d] & S \arrow[d] \\
\mathrm{Spec}(A_k/(T_k)) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(A_k) \arrow[d] \\
\{0\} \arrow[r, hook] & \mathbf{A}^n
\end{tikzcd} \tag{3.2.a}$$

where the upper square is homotopy cartesian, for any derived scheme S and any morphism $f : S \to \mathbf{A}^n$. In particular, if $S = \mathrm{Spec}(R)$ for a simplicial commutative ring R and f corresponds to points $f_1, \ldots, f_n \in R$, then there is a canonical isomorphism $D = \mathrm{Spec}(R/\!\!/f_k)$. We claim that the induced maps $\mathrm{Spec}(A_k) \to Y_k$, which are effective epimorphisms by construction, are in fact invertible.

Lemma. *For each $1 \leqslant k \leqslant n$, the canonical morphism of derived stacks* $\mathrm{Spec}(A_k) \to Y_k$ *is invertible.*

Proof. Let $S = \mathrm{Spec}(R)$ be an affine derived scheme and $f : S \to \mathbf{A}^n$ a morphism corresponding to points $f_1, \ldots, f_n \in R$. It suffices to show that the induced map of spaces

$$\theta : \mathrm{Maps}_{/\mathbf{A}^n}(S, \mathrm{Spec}(A_k)) \to Y_k(S \to \mathbf{A}^n)$$

is a monomorphism, i.e., an inclusion of connected components. We will write $A := \mathrm{Maps}_{/\mathbf{A}^n}(S, \mathrm{Spec}(A_k))$ and $B := Y_k(S \to \mathbf{A}^n)$. The target B is by definition a subspace of the space $(\mathrm{DSch}^{\mathrm{aff}}_{/S \times^{\mathbf{R}}_{\mathbf{A}^n} \{0\}})^{\approx} = (\mathrm{SCRing}_{R/\!\!/(f_1,\ldots,f_n)})^{\approx}$, where the notation $(-)^{\approx}$ means that we take the subspace of the ∞-category consisting of only invertible morphisms. Thus we can identify its points with morphisms $R/\!\!/(f_1, \ldots, f_n) \to R/\!\!/(f_k)$, or equivalently with paths $f_j \approx 0$ in $R/\!\!/(f_k)$, $1 \leqslant j \leqslant n$. The source A can be described as

$$A = \mathrm{Maps}_{\mathrm{SCRing}_{\mathbf{Z}[T_1,\ldots,T_n]}}(A_k, R) \approx \prod_{r \neq k} \mathrm{Fib}_{f_r}(R \xrightarrow{f_k} R),$$

since $A_k = \mathbf{Z}[T_1/T_k, \ldots, T_n/T_k, T_k]$ is free, as a simplicial commutative $\mathbf{Z}[T_1, \ldots, T_n]$-algebra, on generators $X_1, \ldots, \hat{X}_k, \ldots, X_n$ with relations $X_r T_k - T_r$, $r \neq k$ (because the sequence $(X_r T_k - T_r)_{r \neq k}$ is regular, see the proof of [BGI71, Exp. VII, Prop. 1.8, (ii)]). Thus its points can be identified with tuples $(a_r, \alpha_r)_{r \neq k}$, where $a_r \in R$ are points and $\alpha_r : a_r f_k \approx f_r$ are paths in (the underlying space of) R.

On points, the map $\theta : A \to B$ sends a point $a = (a_r, \alpha_r)_{r \neq k} \in A$ to the point $\theta(a) \in B$ corresponding to paths $\theta(a)_j : f_j \approx 0$ in $R/\!\!/(f_k)$, $1 \leqslant j \leqslant n$, where $\theta(a)_k$ is the "tautological" path and $\theta(a)_r$, $r \neq k$, are induced by composing α_r with $\theta(a)_k$. We will show that θ is fully faithful as a functor of ∞-groupoids. Let a' be another point in A and $\theta(a')$ its image in B. Observe that any morphism of $R/\!\!/(f_1, \ldots, f_n)$-algebras $R/\!\!/(f_k) \to R/\!\!/(f_k)$ is invertible; in fact, as a morphism of R-algebras, it is the identity. Therefore the space of paths $\theta(a) \approx \theta(a')$ can be described as follows:

$$\begin{aligned}
\mathrm{Maps}_B(\theta(a), \theta(a')) &= \mathrm{Maps}_{\mathrm{SCRing}_{R/\!\!/(f_1,\ldots,f_n)}}(R/\!\!/(f_k), R/\!\!/(f_k)) \\
&= \mathrm{Fib}_{\theta(a')}(\mathrm{Maps}_{\mathrm{SCRing}_R}(R/\!\!/(f_k), R/\!\!/(f_k)) \\
&\qquad \xrightarrow{\theta(a)^*} \mathrm{Maps}_{\mathrm{SCRing}_R}(R/\!\!/(f_1, \ldots, f_n), R/\!\!/(f_k))) \\
&= \mathrm{Fib}_{(\theta(a')_j)_j}(\mathrm{Maps}_{R/\!\!/(f_k)}(f_k, 0) \to \prod_{j=1}^{n} \mathrm{Maps}_{R/\!\!/(f_k)}(f_j, 0)) \\
&= \prod_{r \neq k} \mathrm{Maps}_{R/\!\!/(f_k)}(\theta(a)_r, \theta(a')_r).
\end{aligned}$$

Under the above identifications the map $\mathrm{Maps}_A(a, a') \to \mathrm{Maps}_B(\theta(a), \theta(a'))$ is identified with the canonical map

$$\prod_{r \neq k} \mathrm{Maps}_{\mathrm{Fib}_{f_r}(R \xrightarrow{f_k} R)}((a_r, \alpha_r), (a'_r, \alpha'_r)) \to \prod_{r \neq k} \mathrm{Maps}_{R/\!\!/(f_k)}(\theta(a)_r, \theta(a')_r)$$

which is invertible by Lemma 2.3.1. $\square$

3.2.2. We next observe that the classical affine schemes Y_k cover Y.

Lemma. *The family $(Y_k \hookrightarrow Y)_k$ defines a Zariski atlas for the derived stack Y. That is, the induced morphism $\coprod_k Y_k \to Y$ is an effective epimorphism of Zariski sheaves.*

Proof. Let S be a derived scheme, $f : S \to \mathbf{A}^n$ a morphism, and D a virtual Cartier divisor on S lying over $(\mathbf{A}^n, \{0\})$. The claim is that Zariski-locally on S, D is of the form (3.2.a) for some k. We can assume that S is affine, say $S = \operatorname{Spec}(R)$ for some $R \in \mathrm{SCRing}$, so that f corresponds to points $f_1, \ldots, f_n \in R$. The fact that D lies over $(\mathbf{A}^n, \{0\})$ then implies that locally, the conormal sheaf $\mathcal{N}_{D/S}$ has a basis given by df_k for some k. It follows that the induced morphism $D \to \operatorname{Spec}(R/\!\!/(f_1, \ldots, f_n)) \to \operatorname{Spec}(R/\!\!/(f_k))$ is invertible, arguing as in the end of the proof of Proposition 2.3.3. □

3.2.3. The lemmas in 3.2.1 and 3.2.2 provide a Zariski cover for Y by classical affine schemes, so we deduce:

Corollary. *The derived stack $Y = \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$ is a classical scheme. Moreover, it is isomorphic to the classical blow-up $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}/\mathbf{A}^n}$.*

Proof. For the second statement, note that the affine cover we have constructed for Y coincides with the standard affine cover of $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}/\mathbf{A}^n}$. □

Remark. Alternatively, it is easy to check directly that Y satisfies the classical universal property of $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}/\mathbf{A}^n}$. That is, suppose S is a classical scheme and $f : S \to \mathbf{A}^n$ is a morphism. If the classical schematic fibre $f^{-1}(\{0\})$ is a classical Cartier divisor on S, then it lies over $(\mathbf{A}^n, \{0\})$ as a virtual Cartier divisor, and is moreover the *unique* such; in particular, there exists a unique morphism $S \to Y$ over $\mathbf{A}^n$.

3.3. *Proof of main theorem (3.1.2).*

3.3.1. *Proof of (ii).* Let $i : Z \hookrightarrow X$ be a quasi-smooth closed immersion of derived schemes, and $i' : Z' \hookrightarrow X'$ its derived base change along a morphism $p : X' \to X$. Given a derived scheme S' over X', any virtual Cartier divisor D' on S' lying over (X', Z') also lies over (X, Z). In particular there is a canonical morphism $\mathrm{Bl}_{Z'/X'} \to \mathrm{Bl}_{Z/X} \times^{\mathbf{R}}_X X'$. We prove the following more precise formulation of the statement:

Claim. *The canonical morphism of derived stacks $\mathrm{Bl}_{Z'/X'} \to \mathrm{Bl}_{Z/X} \times^{\mathbf{R}}_X X'$ is invertible.*

Proof. Use the description mentioned in Remark (ii) in 3.1.1: for any $S' \to X'$, the spaces $\mathrm{Bl}_{Z'/X'}(S' \to X')$ and $(\mathrm{Bl}_{Z/X} \times^{\mathbf{R}}_X X')(S' \to X')$ both define the same subspace of $(\mathrm{DSch}_{/S' \times^{\mathbf{R}}_{X'} Z'})^{\approx} = (\mathrm{DSch}_{/S' \times^{\mathbf{R}}_X Z})^{\approx}$. □

3.3.2. *Proof of (i).* It suffices to show this Zariski-locally on the base X, so we can assume that $i : Z \hookrightarrow X$ is a derived base change of $\{0\} \hookrightarrow \mathbf{A}^n$. Derived fibred products of derived schemes are representable, so by assertion (ii) (proven in 3.3.1 above) we can reduce to the special case considered in 3.2.3.

3.3.3. *Proof of (iii).* Let $D^{\mathrm{univ}}_{Z/X}$ denote the "universal virtual Cartier divisor" lying over (X, Z), classified by the identity morphism $\mathrm{Bl}_{Z/X} \to \mathrm{Bl}_{Z/X}$. This is a derived scheme $D^{\mathrm{univ}}_{Z/X}$ equipped with a canonical morphism $\pi_{\mathrm{univ}} : D^{\mathrm{univ}}_{Z/X} \to Z$, a canonical locally free sheaf $\mathcal{L}^{\mathrm{univ}}_{Z/X} := \mathcal{N}_{D^{\mathrm{univ}}_{Z/X}/\,\mathrm{Bl}_{Z/X}}$ of rank 1, and a canonical surjection $(\pi_{\mathrm{univ}})^*\mathcal{N}_{Z/X} \to \mathcal{L}^{\mathrm{univ}}_{Z/X}$. This data is classified by a canonical morphism

$$D^{\mathrm{univ}}_{Z/X} \to \mathbf{P}_Z(\mathcal{N}_{Z/X})$$

of derived schemes over Z.

Claim. *The morphism $D^{\mathrm{univ}}_{Z/X} \to \mathbf{P}_Z(\mathcal{N}_{Z/X})$ is invertible. In particular, there is a canonical closed immersion $\mathbf{P}_Z(\mathcal{N}_{Z/X}) \hookrightarrow \mathrm{Bl}_{Z/X}$ which exhibits the projectivized normal bundle as the universal virtual Cartier divisor lying over (X, Z).*

Proof. The assertion is local and stable under derived base change, so we reduce to the case of $\{0\} \hookrightarrow \mathbf{A}^n$. Then we can apply the well-known universal property of the classical blow-up (Remark 3.2.3): since the classical fibred product $\mathrm{Bl}_{\{0\}/\mathbf{A}^n} \times_{\mathbf{A}^n} \{0\}$ is the classical effective Cartier divisor $\mathbf{P}_{\{0\}}(\mathcal{N}_{\{0\}/\mathbf{A}^n})$, the conclusion is that there is a unique virtual Cartier divisor $\mathbf{P}_{\{0\}}(\mathcal{N}_{\{0\}/\mathbf{A}^n}) \hookrightarrow \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$ lying over $(\mathbf{A}^n, \{0\})$, classified by the identity of $\mathrm{Bl}_{\{0\}/\mathbf{A}^n}$. □

3.3.4. *Proof of (iv).* The properties in question are Zariski-local on the target and stable under arbitrary derived base change, so we again reduce to the case of $\{0\} \hookrightarrow \mathbf{A}^n$. Then these are well-known properties of the classical blow-up.

3.3.5. *Proof of (v).* Suppose that X and Z are classical schemes. To show that $\mathrm{Bl}_{Z/X}$ is classical, we can assume that $X = \mathrm{Spec}(R)$ and $Z = \mathrm{Spec}(R/\!\!/(f_1, \ldots, f_n)) = \mathrm{Spec}(R/(f_1, \ldots, f_n))$, where R is a commutative ring and $(f_1, \ldots, f_n)$ is a regular sequence. Then by (ii) and 3.2.2, the derived scheme $\mathrm{Bl}_{Z/X}$ admits a Zariski cover by the schemes

$$\mathrm{Spec}(R) \overset{\mathbf{R}}{\underset{\mathbf{A}^n}{\times}} \mathrm{Spec}(A_k) = \mathrm{Spec}(R[X_1, \ldots, \hat{X}_k, \ldots, X_n]/\!\!/(X_r f_i - f_r)_{r \neq j}),$$

where $A_k = \mathbf{Z}[T_1/T_k, \ldots, T_n/T_k, T_k]$ as in Subsect. 3.2. Thus the claim follows from the fact that the sequence $(X_r f_i - f_r)_{r \neq j}$ is regular (see the proof of [BGI71, Exp. VII, Prop. 1.8(ii)]).

To show that $\mathrm{Bl}_{Z/X}$ moreover coincides with $\mathrm{Bl}^{\mathrm{cl}}_{Z/X}$, we can assume $Z \hookrightarrow X$ is a derived base change of $\{0\} \hookrightarrow \mathbf{A}^n$ along some morphism $f : X \to \mathbf{A}^n$. Then we have canonical isomorphisms $\mathrm{Bl}_{Z/X} \approx \mathrm{Bl}_{\{0\}/\mathbf{A}^n} \times^{\mathbf{R}}_{\mathbf{A}^n} X \approx \mathrm{Bl}_{\{0\}/\mathbf{A}^n} \times_{\mathbf{A}^n} X$ by (ii) and the first part of (v). On the other hand the classical blow-up $\mathrm{Bl}^{\mathrm{cl}}_{Z/X}$ is the classical base change $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}/\{\mathbf{A}^n\}} \times_{\mathbf{A}^n} X$ by [BGI71, Exp. VII, Prop. 1.8(i)]. Therefore the claim follows from the special case of $\{0\} \hookrightarrow \mathbf{A}^n$ (3.2.3).

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Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

E-mail address: adeel.khan@mathematik.uni-regensburg.de