BRAVE NEW MOTIVIC HOMOTOPY THEORY I: THE LOCALIZATION THEOREM

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Abstract. This series of papers is dedicated to the study of motivic homotopy theory in the context of brave new or spectral algebraic geometry. In Part I we set up the theory and prove an analogue of the localization theorem of Morel–Voevodsky in classical motivic homotopy theory.

Contents

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1. Introduction

1.1. **Main result.**

1.1.1. Given a scheme S, let $\mathbf{H}(\mathrm{S})$ denote the ∞-category of motivic spaces over S.

One of the most fundamental results in motivic homotopy theory is the localization theorem of F. Morel and V. Voevodsky, one formulation of which is as follows. Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$, and consider the direct image functor $i_* : \mathbf{H}(\mathrm{Z}) \to \mathbf{H}(\mathrm{S})$. The localization theorem states that this functor is fully faithful, with essential image spanned by motivic spaces $\mathcal{F}$ whose restriction $j^*(\mathcal{F})$ is contractible.

The main goal of this paper is to prove the analogue of this theorem in *spectral algebraic geometry.*

1.1.2. Before proceeding, let us explain why we chose this theorem as the starting point of our study of motivic homotopy theory in spectral algebraic geometry.

Every motivic spectrum $\mathbb{E}$ over S represents a generalized motivic cohomology theory $\mathrm{H}^q(-, \mathbb{E}(p))$ on S-schemes ($p, q \in \mathbf{Z}$), like motivic cohomology, homotopy invariant algebraic K-theory, or algebraic cobordism.

The localization theorem gives rise to the expected long exact localization sequences in $\mathbb{E}$-cohomology, for any motivic spectrum $\mathbb{E}$:

$$\cdots \to \mathrm{H}^q_{\mathrm{Z}}(\mathrm{S}, \mathbb{E}(p)) \xrightarrow{i_!} \mathrm{H}^q(\mathrm{S}, \mathbb{E}(p)) \xrightarrow{j^*} \mathrm{H}^q(\mathrm{U}, \mathbb{E}(p)) \xrightarrow{\partial} \mathrm{H}^{q+1}_{\mathrm{Z}}(\mathrm{S}, \mathbb{E}(p)) \to \cdots ,$$

for any closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ with open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. Here $\mathrm{H}^q_{\mathrm{Z}}(\mathrm{S}, \mathbb{E}(p))$ denotes[1] $\mathbb{E}$-cohomology of S with support in Z.

The localization theorem is also a crucial input into J. Ayoub's construction of the formalism of six operations in motivic homotopy theory. The construction of the six operations in the setting of spectral schemes will be the subject of one of the sequels of this paper, and the localization theorem will play an equally important role.

1.1.3. Let us also say a few words about why we are interested in developing motivic homotopy theory in the setting of spectral algebraic geometry.

In this setting, the localization theorem has another very interesting consequence which we did not mention yet. Recall that a spectral scheme S is, roughly speaking[2], a pair $(\mathrm{S}_{\mathrm{cl}}, \mathcal{O}_{\mathrm{S}})$, where S_{cl} is a classical scheme and $\mathcal{O}_{\mathrm{S}}$ is a sheaf of connective $\mathcal{E}_\infty$-ring spectra whose underlying sheaf of commutative rings $\pi_0(\mathcal{O}_{\mathrm{S}})$ coincides with the structure sheaf $\mathcal{O}_{\mathrm{S}_{\mathrm{cl}}}$. The underlying classical scheme S_{cl} may be viewed as a spectral scheme with discrete structure sheaf, and there is a canonical map $\mathrm{S}_{\mathrm{cl}} \to \mathrm{S}$ which is a closed immersion in the sense of spectral algebraic geometry. Since its open complement is empty, the localization theorem will have the consequence that the motivic homotopy category over S is equivalent to the motivic homotopy category over S_{cl}. We refer to this phenomenon as "derived nilpotent invariance" (we think of sections of $\pi_i(\mathcal{O}_{\mathrm{S}})$, for $i > 0$, as derived nilpotents).

We should note that it will not be obvious from our construction that, over the classical scheme S_{cl}, the motivic homotopy category we construct is equivalent to the usual motivic homotopy category. This turns out to be nevertheless true, and it will also be a consequence of the localization theorem, but its proof will be postponed to Part II [CK17] of this series.

In other words, any generalized motivic cohomology theory can be extended to spectral algebraic geometry in a trivial way:

$$\mathrm{H}^q(\mathrm{X}, \mathbb{E}(p)) := \mathrm{H}^q(\mathrm{X}_{\mathrm{cl}}, \mathbb{E}(p)),$$

and the claim is that *every generalized motivic cohomology theory in spectral algebraic geometry arises in this way.*

Let us mention one interesting application. Let X be an arbitrary scheme. Given any regular closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{X}$, where we allow Z to be an arbitrary *spectral* scheme[3], one can

[1]In the case where Z and S are both smooth over some base, i is pure of codimension d, and the spectrum $\mathbb{E}$ is oriented, Morel–Voevodsky's relative purity theorem will further identify this term with $\mathrm{H}^{q-2d}(\mathrm{Z}, \mathbb{E}(p-d))$.

[2]See [Lur16b, Def. 1.1.2.8] for a precise version of this definition; we will give an alternative definition in Sect. 2.

[3]See Definition 2.13.2 for our definition of regular closed immersion in this context.

construct an associated (virtual) *fundamental class* [Z] such that the formula

$$[\mathrm{Z}_1] \cdot [\mathrm{Z}_2] = [\mathrm{Z}_1 \underset{\mathrm{X}}{\times} \mathrm{Z}_2]$$

always holds, where Z_1 and Z_2 are regularly immersed closed spectral subschemes of X, and the fibred product $\mathrm{Z}_1 \times_\mathrm{X} \mathrm{Z}_2$ is taken in the category of spectral schemes. This formula holds in any generalized motivic cohomology theory (in the above-mentioned sense) and can be viewed as a vast generalization of Bézout's theorem[4]. We will discuss this in detail in a sequel to this paper.

1.2. **Summary of construction.** Below we briefly describe our construction of the motivic homotopy category over a spectral scheme.

1.2.1. Let S be a spectral scheme (we will always assume that S is quasi-compact and quasi-separated). A *motivic space* over S will be a presheaf of spaces $\mathcal{F}$ on the ∞-category $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$ of smooth spectral S-schemes (of finite presentation), satisfying the following properties:

(Red) The space $\Gamma(\varnothing, \mathcal{F})$ is contractible.

(Nis) Let Q be a *Nisnevich square*, i.e. a cartesian square of smooth spectral S-schemes

$$\begin{array}{ccc} \mathrm{W} & \longrightarrow & \mathrm{V} \\ \downarrow & & \downarrow p \\ \mathrm{U} & \xrightarrow{j} & \mathrm{X} \end{array} \tag{1.1}$$

where j is an open immersion, p is étale, and the induced morphism $p^{-1}(\mathrm{X}-\mathrm{U}) \to \mathrm{X}-\mathrm{U}$ is an isomorphism of reduced classical schemes. Then the induced square of spaces

$$\begin{array}{ccc} \Gamma(\mathrm{X}, \mathcal{F}) & \xrightarrow{j^*} & \Gamma(\mathrm{U}, \mathcal{F}) \\ \downarrow p^* & & \downarrow \\ \Gamma(\mathrm{V}, \mathcal{F}) & \longrightarrow & \Gamma(\mathrm{W}, \mathcal{F}) \end{array}$$

is homotopy cartesian.

(Htp) Let $\mathbf{A}^1$ denote the brave new affine line, i.e. the affine spectral scheme $\mathrm{Spec}(\mathbf{S}\{t\})$, where $\mathbf{S}$ is the sphere spectrum, and $\mathbf{S}\{t\}$ denotes the free $\mathcal{E}_\infty$-algebra on one generator t (in degree zero). Then for every smooth spectral S-scheme, the canonical map

$$\Gamma(\mathrm{X}, \mathcal{F}) \to \Gamma(\mathrm{X} \times \mathbf{A}^1, \mathcal{F})$$

is invertible.

We write $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ for the ∞-category of motivic spaces over S. This is a left localization of the ∞-category of presheaves of spaces on $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$, and we write $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ for the localization functor.

Any smooth spectral S-scheme X represents a presheaf of spaces $\mathrm{h}_\mathrm{S}(\mathrm{X})$, whose localization we denote by $\mathrm{M}_\mathrm{S}(\mathrm{X}) := \mathrm{L}_{\mathrm{mot}}\, \mathrm{h}_\mathrm{S}(\mathrm{X})$.

1.2.2. The assignment $\mathrm{S} \mapsto \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ admits the following functorialities.

Given a morphism $f : \mathrm{T} \to \mathrm{S}$ of spectral schemes, there is an inverse image functor

$$f^*_{\mathbf{H}} : \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S}) \to \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{T})$$

which is given on representables by the assignment $\mathrm{M}_\mathrm{S}(\mathrm{X}) \mapsto \mathrm{M}_\mathrm{T}(\mathrm{X} \times_\mathrm{S} \mathrm{T})$. It is left adjoint to a direct image functor

$$f_*^{\mathbf{H}} : \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{T}) \to \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S}).$$

[4]For X a smooth projective variety over the field of complex numbers, a similar formula taking values in singular cohomology was announced by J. Lurie [Lur16b].

We also have the operations $(\otimes, \underline{\mathrm{Hom}})$, where the bifunctor $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \otimes \mathcal{G}$ which is given on representables by $\mathrm{M_S(X)} \otimes \mathrm{M_S(Y)} = \mathrm{M_S(X \times_S Y)}$.

If f is smooth, there is an "extra" operation

$$f_\sharp^{\mathbf{H}} : \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{T}) \to \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S}),$$

left adjoint to f^*, which is given on representables by the assignment $\mathrm{M_T(X)} \mapsto \mathrm{M_S(X)}$. Note that this operation is not one of the "six operations", but it is compatible with the four operations $(f^*, f_*, \otimes, \underline{\mathrm{Hom}})$ in the sense that it satisfies various base change and projection formulas, which are the subject of Sect. 6.

1.2.3. One consequence of the localization theorem is that there are well-defined operations $(i_!, i^!)$ (called *exceptional direct and inverse image*, respectively) for any closed immersion i, satisfying base change and projection formulas (see Sect. 7). In order to extend these operations to more general morphisms, it is necessary to pass to the *stable* version of motivic homotopy theory.

Classically, stable motivic homotopy theory over S is obtained from the unstable version by formally inverting the Thom spaces of all vector bundles over S, with respect to the smash product. Because of descent, it suffices in fact to invert the Thom space of the affine line $\mathbf{A}^1_\mathrm{S}$, which is identified up to $\mathbf{A}^1$-homotopy with the motivic space represented by the projective line $\mathbf{P}^1_\mathrm{S}$, pointed by the section at infinity. The latter can be further identified up to $\mathbf{A}^1$-homotopy with the S^1-suspension of the motivic space represented by the $\mathbf{G}_{m,\mathrm{S}}$, pointed by the unit section. Thus the process of stabilization can be done in two steps, first taking the S^1-stabilization and then $\mathbf{G}_m$-stabilization.

This whole discussion remains valid in the setting of spectral algebraic geometry. However we will defer our discussion of the $\mathbf{P}^1$-stable theory to Part III, since the localization theorem already holds in the unstable category.

1.3. **Contents.** In Sect. 2 we review the basics of spectral algebraic geometry. We follow the "functor of points" approach as in [TV08] or [AG14]; an alternative approach using locally ringed ∞-toposes can be found in [Lur16b]. Most of the material is well-known, so we generally omit proofs, except in the last few paragraphs where we include some geometric lemmas that are used in the proof of our main result.

Sect. 3 is an axiomatic exposition of Morel–Voevodsky homotopy theory in general settings. The framework encompasses even equivariant motivic homotopy theory [Hoy17] and noncommutative motivic homotopy theory [Rob15]. That said, there is no original mathematics in this section; it only serves to collect in one place some basic results which we will be referring to repeatedly elsewhere in the paper.

In Sect. 4 we specialize to give the construction of the brave new motivic homotopy category over a spectral scheme. We also define its pointed and S^1-stable variants. We defer discussion of the $\mathbf{P}^1$-stable variant to Part III.

In Sect. 5 we discuss the standard functorialities (f^*, f_*) that the assignment $\mathrm{S} \mapsto \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ is equipped with. We show in Sect. 6 that for any smooth morphism p, the functor p^* admits a left adjoint $p_\sharp$. We show that this is compatible with the operations f^* and $\otimes$ in the sense that it satisfies suitable base change and projection formulas.

Sect. 7 deals with the functor i_* of direct image along a closed immersion. Our first result here is that, in the pointed setting, it admits a right adjoint $i^!$. Next we state the main result of this paper, the localization theorem (Theorem 7.2.6), which says that i_* is fully faithful, and identifies its essential image. We then review some consequences, including nilpotent invariance, as well as some compatibilities between the operation i_* and the operations f^*, $p_\sharp$, and $\otimes$.

Sect. 8 is dedicated to the proof of the localization theorem. Our proof follows the same general outline used by Morel–Voevodsky [MV99] in the setting of classical base schemes. Following some ideas introduced by M. Hoyois [Hoy17] in the equivariant setting, we make the proof robust enough that it survives in the setting of spectral schemes, and without unnecessary noetherian assumptions made in the original proof of Morel–Voevodsky. Thus our proof, if interpreted in the setting of classical algebraic geometry, gives a proof of Morel–Voevodsky's theorem for classical schemes, over arbitrary quasi-compact quasi-separated base schemes.

Appendix A introduces an axiomatic framework which we find convenient for discussing base change and projection formulas in abstract categories of coefficients.

Appendix B deals with some technical but very elementary topos theory that is used in the proof that the direct image functor i_* commutes with contractible colimits.

1.4. **Conventions.** We will use the language of ∞-categories freely throughout the text. The term "category" will mean "∞-category" (= quasi-category) by default. Though we will use the language in a model-independent way, we fix for concreteness the model of quasi-categories as developed by A. Joyal and J. Lurie. Our main references are [Lur09] and [Lur16a].

Starting from § 4, all spectral schemes (and classical schemes) will be quasi-compact quasi-separated. Smooth morphisms will always be assumed to be of finite presentation.

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2. Spectral algebraic geometry

In this section we review the theory of spectral algebraic geometry. All the material covered is standard and can be found in [Lur16b] and [TV08], with the exception of Proposition 2.12.2.

2.1. Spectral prestacks.

2.1.1. Let $\mathcal{E}_\infty\text{-Alg}(\mathrm{Spt})_{\geqslant 0}$ denote the category of connective $\mathcal{E}_\infty$-ring spectra.

The Eilenberg–MacLane functor defines a fully faithful functor $\mathrm{CAlg} \to \mathcal{E}_\infty\text{-Alg}(\mathrm{Spt})_{\geqslant 0}$, whose essential image is spanned by the discrete (= 0-truncated) connective $\mathcal{E}_\infty$-ring spectra.

2.1.2. A *spectral prestack* is a presheaf of spaces on the category $\mathcal{E}_\infty\text{-Alg}(\mathrm{Spt})^{\mathrm{op}}_{\geqslant 0}$, i.e. a functor $\mathcal{E}_\infty\text{-Alg}(\mathrm{Spt})_{\geqslant 0} \to \mathrm{Spc}$.

We write PreStk for the category of spectral prestacks.

2.1.3. Given a connective $\mathcal{E}_\infty$-ring spectrum A, we will write Spec(A) for the spectral prestack represented by A. We say that a spectral prestack is an *affine spectral scheme* if it is represented by a connective $\mathcal{E}_\infty$-ring spectrum, and write $\mathrm{Sch}_{\mathrm{aff}}$ for the full subcategory of PreStk spanned by affine spectral schemes.

Let S be a spectral prestack. For a connective $\mathcal{E}_\infty$-ring spectrum A, we say that an A-*point of* S is a morphism $s : \mathrm{Spec}(\mathrm{A}) \to \mathrm{S}$, or equivalently a point of the space S(A).

2.1.4. A *classical*[5] *prestack* is a presheaf on the opposite of the ordinary category CAlg of commutative rings.

Given a spectral prestack S, let S_{cl} denote its *underlying classical prestack*, defined as the restriction to ordinary commutative rings. The functor $S \mapsto S_{cl}$ admits a fully faithful left adjoint, embedding the category of classical prestacks as a full subcategory of spectral prestacks.

We refer to spectral prestacks of the form Spec(A), with A an ordinary commutative ring, as *classical affine schemes*. The functor $S \mapsto S_{cl}$ sends spectral affine schemes to classical affine spectral schemes: we have $\mathrm{Spec}(A)_{cl} = \mathrm{Spec}(\pi_0(A))$ for any connective $\mathcal{E}_\infty$-ring spectrum A.

2.2. **Quasi-coherent sheaves.**

2.2.1. Let S = Spec(A) be an affine spectral scheme. A *quasi-coherent module* on S is the datum of an A-module. We write $\mathcal{O}_{\mathrm{Spec(A)}}$ for the quasi-coherent module given by A, viewed as a module over itself.

2.2.2. Let S be a spectral prestack. A *quasi-coherent* $\mathcal{O}_S$*-module* consists of the following data:

(1) For every affine spectral scheme Spec(A) and every morphism $s : \mathrm{Spec(A)} \to S$, a quasi-coherent module $\mathcal{F}_s$ on $\mathcal{O}_{\mathrm{Spec(A)}}$.

(2) For every pair of morphisms $s : \mathrm{Spec(A)} \to S$, $s' : \mathrm{Spec(B)} \to S$ fitting into a commutative triangle

$$\begin{array}{ccc} \mathrm{Spec(A)} & \xrightarrow{s} & S, \\ \downarrow f & \nearrow_{s'} & \\ \mathrm{Spec(B)} & & \end{array}$$

an isomorphism $f^*(\mathcal{F}_{s'}) \xrightarrow{\sim} \mathcal{F}_s$.

(3) A homotopy coherent system of compatibilities between all such isomorphisms.

2.2.3. More precisely, we define the category Qcoh(S) as the limit

$$\mathrm{Qcoh(S)} := \varprojlim_{\mathrm{Spec(A)} \to S} \mathrm{Qcoh(Spec(A))}$$

in the category of presentable ∞-categories.

This category is *stable*, as the property of stability is stable under limits of ∞-categories.

Better yet, we can define a presheaf of symmetric monoidal presentable ∞-categories $S \mapsto \mathrm{Qcoh(S)}$ as the right Kan extension of the presheaf $A \mapsto A\text{-mod}$ along the Yoneda embedding $\mathcal{E}_\infty\text{-Alg(Spt)}^{\mathrm{op}}_{\geqslant 0} \to \mathrm{PreStk}$.

In particular, for each morphism of spectral prestacks f, we have a symmetric monoidal colimit-preserving functor f^*, the inverse image functor, and its right adjoint f_*, the direct image functor.

[5]The adjective *classical* refers to the fact that they are defined on non-derived objects (ordinary commutative rings, not $\mathcal{E}_\infty$-ring spectra). In the literature they have been studied by C. Simpson and others under the name *higher prestacks*, since they may take values in arbitrary spaces, not just groupoids. In our terminology, prestacks are "higher" by default, and "non-higher" prestacks are 1-*truncated* prestacks.

2.2.4. Let S be a spectral prestack. We write $\mathcal{O}_{\mathrm{S}}$ for the quasi-coherent module defined by $\mathcal{O}_{\mathrm{S},s} = \mathcal{O}_{\mathrm{Spec(A)}}$ for each affine spectral scheme Spec(A) and each A-point $s : \mathrm{Spec(A)} \to \mathrm{S}$. This is the unit of the symmetric monoidal structure.

Given a quasi-coherent module $\mathcal{F}$ on S, we write $\Gamma(\mathrm{X}, \mathcal{F})$ for the space of sections over a spectral S-scheme X. This is by definition the mapping space $\mathrm{Maps}(\mathcal{O}_{\mathrm{X}}, p^*(\mathcal{F}))$, where $p : \mathrm{X} \to \mathrm{S}$ is the structural morphism.

2.3. **Spectral stacks.**

2.3.1. Let $f : \mathrm{T} \to \mathrm{S}$ be a morphism of affine spectral schemes, $\mathrm{S} = \mathrm{Spec(A)}$, $\mathrm{T} = \mathrm{Spec(B)}$.

Definition 2.3.2.

(i) *The morphism* f *is* locally of finite presentation *if* B *is compact in the category of* $\mathcal{E}_\infty$-A-*algebras.*

(ii) *The morphism* f *is* flat *if the underlying* A-*module* B *is a filtered colimit of finitely generated free* A-*modules. Equivalently, the morphism* $f_{\mathrm{cl}} : \mathrm{Spec}(\pi_0(\mathrm{B})) \to \mathrm{Spec}(\pi_0(\mathrm{A}))$ *of underlying classical schemes is flat (i.e.* $\pi_0(\mathrm{B})$ *is flat in the usual sense as an* $\pi_0(\mathrm{A})$-*module), and the canonical morphism* $\pi_i(\mathrm{A}) \otimes_{\pi_0(\mathrm{A})} \pi_0(\mathrm{B}) \to \pi_i(\mathrm{B})$ *is invertible for each* i.

(iii) *The morphism* f *is an* open immersion *if it is locally of finite presentation, flat, and a monomorphism*[6]*. Equivalently, it is flat and the morphism* $f_{\mathrm{cl}} : \mathrm{Spec}(\pi_0(\mathrm{B})) \to \mathrm{Spec}(\pi_0(\mathrm{A}))$ *of underlying classical schemes is an open immersion (in the classical sense).*

Remark 2.3.3. We will use the following observation repeatedly: let $f : \mathrm{T} \to \mathrm{S}$ be a morphism of affine spectral schemes; if f is flat and S is classical, then T is also classical. Similarly, if $f : \mathrm{T} \to \mathrm{S}$ is a flat morphism of affine spectral schemes, then the commutative square

$$\begin{array}{ccc} \mathrm{T}_{\mathrm{cl}} & \hookrightarrow & \mathrm{T} \\ \downarrow{\scriptstyle f_{\mathrm{cl}}} & & \downarrow{\scriptstyle f} \\ \mathrm{S}_{\mathrm{cl}} & \hookrightarrow & \mathrm{S} \end{array}$$

is cartesian.

2.3.4. The Zariski topology on $\mathrm{Sch}_{\mathrm{aff}}$ is the Grothendieck topology associated to the following pretopology. A family of morphisms of affine spectral schemes $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{X})_{\alpha \in \Lambda}$ is *Zariski covering* if and only if each j_α is an open immersion, and the family of functors $(j_\alpha)^* : \mathrm{Qcoh(X)} \to \mathrm{Qcoh}(\mathrm{U}_\alpha)$ is conservative.

Definition 2.3.5. *A* spectral stack *is a spectral prestack satisfying descent with respect to the Zariski topology.*

2.4. **Spectral schemes.**

2.4.1. Let $j : \mathrm{U} \to \mathrm{S}$ be a morphism of spectral stacks.

Definition 2.4.2.

(i) *If* S *is affine, then* j *is an* open immersion *if it is a monomorphism, and there exists a family* $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{S})_\alpha$, *with each* j_α *an open immersion of affine spectral schemes Definition 2.3.2 that factors through* U *and induces an effective epimorphism* $\sqcup_\alpha \mathrm{U}_\alpha \to \mathrm{U}$.

(ii) *For general* S, *the morphism* j *is an* open immersion *if for each connective* $\mathcal{E}_\infty$-*ring spectrum* A *and each* A-*point* $s : \mathrm{Spec(A)} \to \mathrm{S}$, *the base change* $\mathrm{U} \times_{\mathrm{S}} \mathrm{Spec(A)} \to \mathrm{Spec(A)}$ *is an open immersion in the sense of (i).*

[6] I.e. the canonical morphism $\mathrm{T} \to \mathrm{T} \times_{\mathrm{S}} \mathrm{T}$ is invertible.

2.4.3. Let S be a spectral stack. A *Zariski cover* of S is a small family of open immersions of spectral stacks $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{S})_\alpha$ such that the canonical morphism $\sqcup_\alpha \mathrm{U}_\alpha \to \mathrm{S}$ is surjective (i.e. an effective epimorphism in the topos of spectral stacks). If each U_α is an affine spectral scheme, we call this an *affine Zariski cover.*

We define:

Definition 2.4.4. *A* spectral scheme *is a spectral stack* S *which admits an affine Zariski cover.*

We write Sch for the category of spectral schemes, a full subcategory of the category of stacks. It admits coproducts and fibred products.

Definition 2.4.5.

(i) *A spectral scheme* S *is* quasi-compact *if for any Zariski cover* $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{S})_{\alpha\in\Lambda}$*, there exists a finite subset* $\Lambda_0 \subset \Lambda$ *such that the family* $(j_\alpha)_{\alpha\in\Lambda_0}$ *is still a Zariski cover.*

(ii) *A morphism of spectral schemes* $f : \mathrm{T} \to \mathrm{S}$ *is* quasi-compact *if for any connective* $\mathcal{E}_\infty$*-ring spectrum* A *and any* A*-point* $s : \mathrm{Spec}(\mathrm{A}) \to \mathrm{S}$*, the spectral scheme* $\mathrm{T} \times_\mathrm{S} \mathrm{Spec}(\mathrm{A})$ *is quasi-compact.*

(iii) *A spectral scheme* S *is* quasi-separated *if for any open immersions* $\mathrm{U} \hookrightarrow \mathrm{S}$ *and* $\mathrm{V} \hookrightarrow \mathrm{S}$*, with* U *and* V *affine, the intersection* $\mathrm{U} \times_\mathrm{S} \mathrm{V}$ *is quasi-compact.*

2.4.6. We define a *classical scheme* to be a Zariski sheaf of *sets* on the category $(\mathrm{CAlg})^{\mathrm{op}}$, admitting a Zariski affine cover. This is equivalent to the definition of scheme given in [GD71].

Given a spectral scheme S, the underlying classical prestack S_{cl} takes values in sets, and is a classical scheme. We therefore refer to S_{cl} as the *underlying classical scheme* of S.

2.5. **Closed immersions.**

2.5.1. Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of spectral schemes. We say that the morphism f is *affine* if, for any connective $\mathcal{E}_\infty$-ring spectrum A and A-point $x : \mathrm{Spec}(\mathrm{A}) \to \mathrm{X}$, the base change $\mathrm{Y} \times_\mathrm{X} \mathrm{Spec}(\mathrm{A})$ is an affine spectral scheme.

2.5.2. If $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ and $\mathrm{Y} = \mathrm{Spec}(\mathrm{B})$ are affine, a morphism $i : \mathrm{Y} \to \mathrm{X}$ is a *closed immersion* if the homomorphism $\mathrm{A} \to \mathrm{B}$ induces a surjection $\pi_0(\mathrm{A}) \to \pi_0(\mathrm{B})$.

In general a morphism $i : \mathrm{Y} \to \mathrm{X}$ is a *closed immersion* if it is affine, and for any connective $\mathcal{E}_\infty$-ring spectrum A and A-point $x : \mathrm{Spec}(\mathrm{A}) \to \mathrm{X}$, the base change $\mathrm{Y} \times_\mathrm{X} \mathrm{Spec}(\mathrm{A}) \to \mathrm{Spec}(\mathrm{A})$ is a closed immersion of affine spectral schemes.

Equivalently, i is a closed immersion if and only if it induces a closed immersion on underlying classical schemes.

2.5.3. A *nil-immersion* is a closed immersion $i : \mathrm{Y} \hookrightarrow \mathrm{X}$ which induces an isomorphism $\mathrm{Y}_{\mathrm{red}} \to \mathrm{X}_{\mathrm{red}}$ on underlying reduced classical schemes.

2.5.4. Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of spectral schemes. Let U be the spectral prestack defined as follows: for a connective $\mathcal{E}_\infty$-ring spectrum A, its A-points are A-points $s : \mathrm{Spec}(\mathrm{A}) \to \mathrm{S}$ such that the base change $\mathrm{Spec}(\mathrm{A}) \times_\mathrm{S} \mathrm{Z}$ is the empty spectral scheme. One can show that U is a spectral scheme, and that the canonical morphism $\mathrm{U} \to \mathrm{S}$ is an open immersion.

We call $j : \mathrm{U} \hookrightarrow \mathrm{S}$ the *complementary open immersion* to i.

2.6. **Vector bundles.**

2.6.1. Just as in Paragraph 2.2, we can define a notion of *quasi-coherent algebra* on a spectral prestack S, such that the category of quasi-coherent algebras on Spec(A) coincides with the category of A-algebras.

2.6.2. Let S be a spectral scheme and $\mathcal{A}$ a quasi-coherent algebra on S. Consider the presheaf $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})$ on the category of spectral schemes over S, which sends a spectral S-scheme X with structural morphism f to the space of quasi-coherent algebra homomorphisms $\mathrm{Maps}(f^*(\mathcal{A}), \mathcal{O}_{\mathrm{X}})$.

The presheaf $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})$ clearly satisfies Zariski descent. Hence it defines a spectral stack over S (there is a canonical equivalence $\mathrm{Sh}(\mathrm{Sch}_{/\mathrm{S}}) = \mathrm{Sh}(\mathrm{Sch})_{/\mathrm{S}} = \mathrm{Sh}(\mathrm{Sch}_{\mathrm{aff}})_{/\mathrm{S}}$), which we call the *relative spectrum* of the quasi-coherent algebra $\mathcal{A}$.

Further, we have:

Lemma 2.6.3. *Let* $\mathcal{A}$ *be a* connective *quasi-coherent algebra over a spectral scheme* S. *Then the spectral stack* $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})$ *is a spectral scheme.*

This follows from functoriality in S, and the fact that for S = Spec(A) affine, we have $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A}) = \mathrm{Spec}(\Gamma(\mathrm{S}, \mathcal{A}))$.

2.6.4. For a connective $\mathcal{E}_\infty$-ring spectrum A and an A-module M, we write $\mathrm{Sym}_{\mathrm{A}}(\mathrm{M})$ for the free A-algebra generated by M. The assignment $\mathrm{M} \mapsto \mathrm{Sym}_{\mathrm{A}}(\mathrm{M})$ defines a functor, left adjoint to the forgetful functor from A-algebras to A-modules, so that there are canonical isomorphisms

$$\mathrm{Maps}_{\text{A-alg}}(\mathrm{Sym}_{\mathrm{A}}(\mathrm{M}), \mathrm{B}) \xrightarrow{\sim} \mathrm{Maps}_{\text{A-mod}}(\mathrm{M}, \mathrm{B})$$

bifunctorial in M and B.

When M is free of rank n, we will also use the notation

$$\mathrm{A}\{t_1, \ldots, t_n\} := \mathrm{Sym}_{\mathrm{A}}(\mathrm{A}^{\oplus n}),$$

following [Lur16b].

2.6.5. Let $\mathcal{F}$ be a quasi-coherent module on S. The quasi-coherent algebra $\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{F})$ is defined by $\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{F})_s := \mathrm{Sym}_{\mathcal{O}_{\mathrm{S},s}}(\mathcal{F}_s)$ for each connective $\mathcal{E}_\infty$-ring spectrum A and A-point $s : \mathrm{Spec}(\mathrm{A}) \to \mathrm{S}$.

2.6.6. A connective quasi-coherent module $\mathcal{F}$ on S is *locally free* of rank n if there exists a Zariski cover $(j_\alpha : \mathrm{U}_\alpha \to \mathrm{S})_\alpha$ such that each inverse image $j_\alpha^*(\mathcal{F})$ is a free quasi-coherent $\mathcal{O}_{\mathrm{S}_\alpha}$-module of rank n, i.e. $j_\alpha^*(\mathcal{F}) \approx \mathcal{O}_{\mathrm{S}_\alpha}^{\oplus n}$.

Given a locally free module $\mathcal{E}$ of finite rank, we define:

Definition 2.6.7. *The* vector bundle *associated to* $\mathcal{E}$ *is the spectral* S*-scheme* $\mathrm{E} = \mathrm{Spec}_{\mathrm{S}}(\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{E}))$.

Note that any global section $s \in \Gamma(\mathrm{S}, \mathcal{E})$ defines a section $s : \mathrm{S} \hookrightarrow \mathrm{E}$ of the structural morphism, which is a closed immersion. In particular, every vector bundle admits a zero section.

2.6.8. For an integer $n \geqslant 0$, we define the *affine space* of dimension n over a spectral scheme S, to be the total space of the free $\mathcal{O}_{\mathrm{S}}$-module $\mathcal{O}_{\mathrm{S}}^{\oplus n}$:

$$\mathbf{A}^n_{\mathrm{S}} := \mathrm{Spec}_{\mathrm{S}}(\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{O}_{\mathrm{S}}^{\oplus n})).$$

We will write simply $\mathbf{A}^n := \mathbf{A}^n_{\mathbf{S}}$ for the affine spaces over the sphere spectrum $\mathbf{S}$. Thus $\mathbf{A}^n \approx \mathrm{Spec}(\mathbf{S}\{t_1, \ldots, t_n\})$.

2.6.9. For any morphism of spectral schemes $f : \mathrm{T} \to \mathrm{S}$, we have canonical isomorphisms $\mathbf{A}^n_{\mathrm{S}} \times_{\mathrm{S}} \mathrm{T} \approx \mathbf{A}^n_{\mathrm{T}}$.

2.6.10. The *affine line* $\mathbf{A}^1_S$ over S is the affine space of dimension 1. Since the quasi-coherent module $\mathcal{O}_S$ has a unit section (being a quasi-coherent *algebra*), the affine line admits both a zero and a unit section.

2.6.11. The underlying classical scheme of $\mathbf{A}^n_S$ is the classical affine space over S_{cl}. We note however that, even when S is classical, the spectral scheme $\mathbf{A}^n_S$ will not be classical, except in characteristic zero. We will look at another version of affine space in [CK17, Para. 4.2] that is flat, and does agree with the classical affine space.

2.7. Infinitesimal extensions.

2.7.1. Let $p : Y \to X$ be a morphism of affine spectral schemes, $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. Given a connective quasi-coherent module $\mathcal{F}$ on Y, we let

$$Y \hookrightarrow Y_{\mathcal{F}} := \mathrm{Spec}(B \oplus M)$$

denote the *trivial infinitesimal extension of* Y *along* $\mathcal{F}$, where $M = \Gamma(Y, \mathcal{F})$.

The morphism $Y \hookrightarrow Y_{\mathcal{F}}$ is the closed immersion induced by the homomorphism $B \oplus M \to B$, $(b, m) \mapsto b$.

2.7.2. A *derivation* of Y over X with values in $\mathcal{F}$, is a retraction of the morphism $Y \hookrightarrow Y_{\mathcal{F}}$ (in the category of affine spectral schemes over X). There is a canonical retraction, the *trivial derivation* d_{triv}, defined by the morphism $B \to B \oplus M$, $b \mapsto (b, 0)$.

Let $\mathrm{Der}(Y/X, \mathcal{F})$ denote the space of derivations in $\mathcal{F}$.

2.7.3. Let $\mathcal{F}$ be a 0-*connected* quasi-coherent module on Y.

Any derivation d of Y/X valued in $\mathcal{F}$ gives rise to an *infinitesimal extension* $i : Y \hookrightarrow Y_d$. This is the closed immersion (in fact, nil-immersion[7]) defined as the cobase change of the trivial derivation along d, so that there is a cocartesian square

$$\begin{array}{ccc} Y_{\mathcal{F}} & \xrightarrow{d_{\mathrm{triv}}} & Y \\ \downarrow{\scriptstyle d} & & \downarrow \\ Y & \xrightarrow{i} & Y_d \end{array} \tag{2.1}$$

in the category of affine spectral schemes.

2.7.4. The following important fact often allows argument by induction along infinitesimal extensions (see e.g. [Lur16a, Prop. 7.1.3.19]).

Proposition 2.7.5. *Let* $S = \mathrm{Spec}(A)$ *be an affine spectral scheme. Then there exists a sequence of nil-immersions of affine spectral schemes*

$$S_{cl} = S_{\leqslant 0} \hookrightarrow S_{\leqslant 1} \hookrightarrow \cdots \hookrightarrow S_{\leqslant n} \hookrightarrow \cdots \hookrightarrow S, \tag{2.2}$$

with $S_{\leqslant n} = \mathrm{Spec}(A_{\leqslant n})$, *satisfying the following properties:*

(i) *For each* $n \geqslant 0$, *the homomorphism* $A \to A_{\leqslant n}$ *identifies* $A_{\leqslant n}$ *as the* n*-truncation of the connective* $\mathcal{E}_\infty$*-ring spectrum* A.

(ii) *The sequence is functorial in* A.

(iii) *The canonical morphism* $A \to \varprojlim_{n \geqslant 0} A_{\leqslant n}$ *is invertible.*

(iv) *Each morphism* $S_{\leqslant n} \hookrightarrow S_{\leqslant n+1}$ $(n \geqslant 0)$ *is an infinitesimal extension by a derivation valued in* $\pi_n(\mathcal{O}_S)[n+1]$.

[7] See (2.5.3).

Further, this sequence is uniquely characterized, up to isomorphism of diagrams indexed on the poset of nonnegative integers, by the property (i).

The sequence (2.2) is called the *Postnikov tower* of A.

2.8. **The cotangent sheaf.**

2.8.1. Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of affine spectral schemes. We have:

Proposition 2.8.2. *The functor* $\mathcal{F} \mapsto \mathrm{Der}(\mathrm{Y}/\mathrm{X}, \mathcal{F})$ *is representable by a connective quasi-coherent module* $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}} := \mathcal{T}^*_p$ *on* Y.

The quasi-coherent module $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}$ on Y is called the (relative) *cotangent sheaf* of the morphism $p : \mathrm{Y} \to \mathrm{X}$. We obtain the absolute cotangent sheaf $\mathcal{T}^*_{\mathrm{S}}$ by taking the relative cotangent sheaf of the unique morphism $\mathrm{S} \to \mathrm{Spec}(\mathbf{S})$.

2.8.3. The following lemma describes a transitivity property of the relative cotangent sheaf:

Lemma 2.8.4. *Let* $\mathrm{Z} \xrightarrow{g} \mathrm{Y} \xrightarrow{f} \mathrm{X}$ *be a sequence of morphisms of affine spectral schemes. Then there is a canonical exact triangle*

$$g^*(\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}) \to \mathcal{T}^*_{\mathrm{Z}/\mathrm{X}} \to \mathcal{T}^*_{\mathrm{Z}/\mathrm{Y}} \tag{2.3}$$

of quasi-coherent sheaves on Z.

In particular, we see that the relative cotangent sheaf $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X}}$ is the cofibre of the canonical morphism $f^*(\mathcal{T}^*_{\mathrm{X}}) \to \mathcal{T}^*_{\mathrm{Y}}$.

2.8.5. The cotangent sheaf of a vector bundle has a particularly simple description:

Lemma 2.8.6. *Let* S *be an affine spectral scheme. For any connective quasi-coherent module* $\mathcal{F}$ *on* S*, let* E *denote the spectral* S*-scheme* $\mathrm{Spec}_{\mathrm{S}}(\mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{F}))$*. Then we have a canonical isomorphism*

$$\mathcal{T}^*_{\mathrm{E}/\mathrm{S}} \xrightarrow{\sim} p^*(\mathcal{F}),$$

where p *denotes the structural morphism* $\mathrm{E} \to \mathrm{S}$.

2.8.7. Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of spectral schemes. For any connective $\mathcal{E}_\infty$-ring spectrum A, A-point $y : \mathrm{Spec}(\mathrm{A}) \to \mathrm{Y}$, and 0-connected quasi-coherent module $\mathcal{F}$ on Y, a *derivation at* y *of* p *with values in* $\mathcal{F}$ is a commutative triangle

$$\begin{array}{ccc} & \mathrm{Spec}(\mathrm{A}) & \\ \swarrow & & \searrow^{y} \\ \mathrm{Spec}(\mathrm{A})_{\mathcal{F}} & \xrightarrow{d} & \mathrm{Y} \end{array}$$

in the category of spectral X-schemes.

We write $\mathrm{Der}_y(\mathrm{Y}/\mathrm{X}, \mathcal{F})$ for the space of derivations at y.

2.8.8. The functor $\mathcal{F} \mapsto \mathrm{Der}_y(\mathrm{Y}/\mathrm{X}, \mathcal{F})$ is represented by a connective quasi-coherent module $\mathcal{T}^*_{\mathrm{Y}/\mathrm{X},y}$ on Spec(A), called the relative cotangent sheaf of p at y:

$$\mathrm{Der}_y(\mathrm{Y}/\mathrm{X}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Maps}_{\mathrm{Qcoh}(\mathrm{Spec}(\mathrm{A}))}(\mathcal{T}^*_{\mathrm{Y}/\mathrm{X},y}, \mathcal{F}).$$

2.8.9. Given a connective $\mathcal{E}_\infty$-ring spectrum A and an A-point $y : \mathrm{Spec(A)} \to \mathrm{Y}$, a connective $\mathcal{E}_\infty$-ring spectrum B and a B-point $y' : \mathrm{Spec(B)} \to \mathrm{Y}$, and a morphism of affine spectral schemes $f : \mathrm{Spec(B)} \to \mathrm{Spec(A)}$ with an identification $y \circ f \approx y'$, we obtain a canonical morphism of quasi-coherent modules on Spec(B)

$$f^*(\mathcal{T}^*_{\mathrm{Y/X},y}) \to \mathcal{T}^*_{\mathrm{Y/X},y'}$$

which is *invertible.*

Moreover, the data of the quasi-coherent modules $\mathcal{T}^*_{\mathrm{Y/X},y}$, as y varies over A-points of Y (with A an arbitrary connective $\mathcal{E}_\infty$-ring spectrum), together with the above isomorphisms, is compatible in a homotopy coherent way, and can therefore be refined to a connective quasi-coherent module $\mathcal{T}^*_{\mathrm{Y/X}}$ defined on the spectral scheme Y.

2.9. **Smooth and étale morphisms.** In this paragraph we review some standard material on smooth and étale morphisms in spectral algebraic geometry from [TV08], [Lur16a], and [Lur16b].

2.9.1. Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of affine spectral schemes, with $\mathrm{X} = \mathrm{Spec(A)}$ and $\mathrm{Y} = \mathrm{Spec(B)}$. We will write $\mathcal{T}^*_{\mathrm{Y/X}}$ for the relative cotangent sheaf, i.e. the connective quasi-coherent $\mathcal{O}_\mathrm{Y}$-module corresponding to the connective B-module $\mathbf{L}_{\mathrm{B/A}}$ (= the cotangent complex).

We define:

Definition 2.9.2.

(i) *The morphism p is* étale *if it is locally of finite presentation and the cotangent sheaf* $\mathcal{T}^*_{\mathrm{Y/X}}$ *is zero.*

(ii) *The morphism p is* smooth *if it is locally of finite presentation and the cotangent sheaf* $\mathcal{T}^*_{\mathrm{Y/X}}$ *is locally free of finite rank.*

Remark 2.9.3. Our use of the term *smooth* corresponds to *differentially smooth* in the sense of [Lur16b, Def. 11.2.2.2].

We have:

Lemma 2.9.4.

(i) *The set of étale (resp. smooth) morphisms is stable under composition and base change.*

(ii) *Open immersions are étale, and étale morphisms are smooth.*

(iii) *A morphism* $p : \mathrm{Y} \to \mathrm{X}$ *is étale if and only if it is flat and the underlying morphism of classical schemes* $p_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ *is étale in the sense of* [Gro67].

Remark 2.9.5. In [CK17, Para. 4.1] we will consider an alternative notion of smoothness called *fibre-smoothness*, which amounts to being flat, and smooth on underlying classical schemes. It is important to note that smoothness is not equivalent to fibre-smoothness. For example, the morphism $\mathrm{H}\mathbf{F}_p \to \mathrm{H}\mathbf{F}_p[t]$ has the latter property, but is not smooth (see [TV08, Prop. 2.4.1.5]).

2.9.6. We now extend the above definitions to morphisms of spectral schemes, by defining them Zariski-locally on the source. That is:

Definition 2.9.7. *A morphism of spectral schemes* $p : \mathrm{Y} \to \mathrm{X}$ *is smooth (resp. étale, flat, locally of finite presentation) if there exist affine Zariski covers* $(\mathrm{Y}_\alpha \hookrightarrow \mathrm{Y})_\alpha$ *and* $(\mathrm{X}_\beta \hookrightarrow \mathrm{X})_\beta$ *together with the data of, for each* α*, an index* β *and a morphism of affine spectral schemes* $\mathrm{Y}_\alpha \to \mathrm{X}_\beta$

which is smooth (resp. étale, flat, locally of finite presentation) and fits in a commutative square

$$\begin{array}{ccc} Y_\alpha & \longrightarrow & X_\beta \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

Example 2.9.8. For any vector bundle E over a spectral scheme S, the projection $\pi : E \to S$ is smooth. In particular, the affine space $\mathbf{A}^n_S$ is smooth over S for each $n \geqslant 0$.

2.9.9. The following is a brave new version of [Gro67, Thm. 17.11.4] (see [Lur16a, Prop. 11.2.2.1]).

Proposition 2.9.10. *A morphism $p : Y \to X$ is smooth if and only if, Zariski-locally on Y, there exists a factorization of p as a composite*

$$Y \xrightarrow{q} X \times \mathbf{A}^n \xrightarrow{r} X \tag{2.4}$$

for some integer $n \geqslant 0$, where q is étale and r is the canonical projection.

This implies (see [Lur16a, Props. 11.2.4.1 and 11.2.4.3]):

Lemma 2.9.11. *If a morphism $p : Y \to X$ is smooth, then the induced morphism $p_{cl} : Y_{cl} \to X_{cl}$ of underlying classical schemes is smooth.*

2.10. **Deformation along infinitesimal extensions.**

2.10.1. Let S be an affine spectral scheme and S′ the infinitesimal extension of S by a derivation $d : \mathcal{T}^*_S \to \mathcal{F}$, for some 0-connected quasi-coherent module $\mathcal{F}$. Let X be an affine spectral scheme over S with structural morphism p.

Definition 2.10.2. *A* deformation *of* X *along the infinitesimal extension* $S \hookrightarrow S'$ *is an affine spectral scheme* X′ *over* S′ *together with an isomorphism* $X \to X' \times_{S'} S$.

In other words, a deformation of X is a cartesian square:

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \downarrow{\scriptstyle p} & & \downarrow \\ S & \hookrightarrow & S' \end{array}$$

2.10.3. From [Lur16a, Prop. 7.4.2.5], we have:

Lemma 2.10.4. *The datum of a deformation of* X *along* $S \hookrightarrow S'$ *is equivalent to the datum of a null-homotopy of the composite*

$$\mathcal{T}^*_{X/S}[-1] \to p^*(\mathcal{T}^*_S) \to p^*(\mathcal{F}).$$

Given such a null-homotopy, one obtains a derivation $d' : \mathcal{T}^*_X \to p^*(\mathcal{F})$; the deformation X′ is constructed as the infinitesimal extension of X along d'.

2.10.5. For example, if p is *smooth*, then any morphism $\mathcal{T}^*_{X/S}[-1] \to p^*(\mathcal{F})$ must be null-homotopic; hence X admits a deformation along any infinitesimal extension $S \hookrightarrow S'$. If p is further *étale*, then this deformation is unique.

2.11. **Push-outs of closed immersions.**

2.11.1. Let $i_1 : \mathrm{Y} \hookrightarrow \mathrm{X}_1$ and $i_2 : \mathrm{Y} \hookrightarrow \mathrm{X}_2$ be closed immersions of spectral schemes.

The following is a straightforward variation on [Lur16b, Thm. 16.1.0.1] (cf. [Lur16b, Rem. 16.1.0.2]):

Lemma 2.11.2.

(i) *There a cocartesian square of spectral schemes*

$$\begin{array}{ccc} \mathrm{Y} & \xhookrightarrow{i_1} & \mathrm{X}_1 \\ \big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle k_2} \\ \mathrm{X}_2 & \xhookrightarrow{k_1} & \mathrm{X}, \end{array}$$

where k_1 and k_2 are closed immersions.

(ii) *If i_1 (resp. i_2) is a nil-immersion, then k_1 (resp. k_2) is a nil-immersion.*

2.12. Lifting smooth morphisms along closed immersions.

2.12.1. The following is a spectral version of [Gro67, Prop. 18.1.1]:

Proposition 2.12.2. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of spectral schemes. For any smooth (resp. étale) morphism $p : \mathrm{X} \to \mathrm{Z}$, there exists, Zariski-locally on X, a smooth (resp. étale) morphism $q : \mathrm{Y} \to \mathrm{S}$, and a cartesian square*

$$\begin{array}{ccc} \mathrm{X} & \longrightarrow & \mathrm{Y} \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ \mathrm{Z} & \hookrightarrow & \mathrm{S}. \end{array}$$

Proof. First we consider the étale case. The question being Zariski-local, we may assume that S, Z and X are affine. Consider the Postnikov towers (Proposition 2.7.5)

$$\mathrm{S}_{\mathrm{cl}} = \mathrm{S}_{\leqslant 0} \hookrightarrow \mathrm{S}_{\leqslant 1} \hookrightarrow \cdots \hookrightarrow \mathrm{S}_{\leqslant n} \hookrightarrow \cdots \hookrightarrow \mathrm{S}$$
$$\mathrm{Z}_{\mathrm{cl}} = \mathrm{Z}_{\leqslant 0} \hookrightarrow \mathrm{Z}_{\leqslant 1} \hookrightarrow \cdots \hookrightarrow \mathrm{Z}_{\leqslant n} \hookrightarrow \cdots \hookrightarrow \mathrm{Z}$$

for S and Z, respectively. Since p is flat, the Postnikov tower for X is identified with the base change of the Postnikov tower of Z.

For a fixed integer $n \geqslant 0$, consider the following claim:

$(*)$ There exists, Zariski-locally on $\mathrm{X}_{\leqslant n}$, an étale morphism $q_{\leqslant n} : \mathrm{Y}_{\leqslant n} \to \mathrm{S}_{\leqslant n}$ and a cartesian square

$$\begin{array}{ccc} \mathrm{X}_{\leqslant n} & \hookrightarrow & \mathrm{Y}_{\leqslant n} \\ \big\downarrow{\scriptstyle p_{\leqslant n}} & & \big\downarrow{\scriptstyle q_{\leqslant n}} \\ \mathrm{Z}^{\leqslant n} & \hookrightarrow & \mathrm{S}^{\leqslant n}. \end{array}$$

Note that it suffices to show that $(*)$ holds for each $n \geqslant 0$, since we can conclude by passing to filtered colimits. For $n = 0$, the claim is [Gro67, Prop. 18.1.1].

We proceed by induction; assume that the claim holds for a fixed n. We define $\mathrm{Y}_{\leqslant n+1}$ to be the deformation of $\mathrm{Y}_{\leqslant n}$ along the infinitesimal extension $\mathrm{S}_{\leqslant n} \hookrightarrow \mathrm{S}_{\leqslant n+1}$, which exists by Lemma 2.10.4. Note that $\mathrm{X}_{\leqslant n+1}$ is itself a deformation of $\mathrm{X}_{\leqslant n}$ along the infinitesimal extension $\mathrm{Z}_{\leqslant n} \hookrightarrow \mathrm{Z}_{\leqslant n+1}$. That the resulting square is cartesian is a straightforward verification.

In the smooth case, the claim follows from the étale case and from Proposition 2.9.10. □

2.13. Regular closed immersions.

2.13.1. Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a closed immersion of spectral schemes. We define:

Definition 2.13.2. *The closed immersion i is* regular *if the shifted cotangent sheaf $\mathcal{T}_{\mathrm{Z}/\mathrm{X}}[-1]$ is a locally free $\mathcal{O}_{\mathrm{Z}}$-module of finite rank.*

We have (cf. [AG15, Prop. 2.1.10]):

Lemma 2.13.3. *A closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ is regular if and only if, Zariski-locally on X, there exists a morphism $f : \mathrm{X} \to \mathbf{A}^n$ and a cartesian square*

$$\begin{array}{ccc} \mathrm{Z} & \hookrightarrow & \mathrm{X} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathbf{S}) & \overset{z}{\hookrightarrow} & \mathbf{A}^n, \end{array}$$

where z denotes the zero section.

2.13.4. Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a regular closed immersion.

We define the *conormal sheaf* $\mathcal{N}^*_{\mathrm{Z}/\mathrm{X}}$ as the shifted cotangent sheaf

$$\mathcal{N}^*_{\mathrm{Z}/\mathrm{X}} := \mathcal{T}^*_{\mathrm{Z}/\mathrm{X}}[-1].$$

By assumption, this is a locally free $\mathcal{O}_{\mathrm{Z}}$-module of finite rank. The associated vector bundle (Definition 2.6.7), which we denote $\mathbf{N}^*_{\mathrm{Z}/\mathrm{X}}$, is called the *conormal bundle* of i.

2.13.5. The *virtual codimension* of i, defined Zariski-locally on Z, is the rank of the locally free $\mathcal{O}_{\mathrm{Z}}$-module $\mathcal{N}^*_{\mathrm{Z}/\mathrm{X}}$.

2.13.6. By Lemma 2.8.4 we have:

Lemma 2.13.7. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a closed immersion of* smooth *spectral* S*-schemes. Then i is regular.*

2.13.8. The following is a slight variation on Lemma 2.13.3:

Lemma 2.13.9. *Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism of affine spectral schemes admitting a section $s : \mathrm{S} \hookrightarrow \mathrm{X}$. Then there exists an* S*-morphism $q : \mathrm{X} \to \mathbf{N}^*_{\mathrm{S}/\mathrm{X}}$ satisfying the following conditions:*

(i) *There is a cartesian square*

$$\begin{array}{ccc} \mathrm{S} & \overset{s}{\hookrightarrow} & \mathrm{X} \\ \| & & \downarrow q \\ \mathrm{S} & \overset{z}{\hookrightarrow} & \mathbf{N}^*_{\mathrm{S}/\mathrm{X}}, \end{array}$$

where z denotes the zero section.

(ii) *The morphism q is étale on some Zariski neighbourhood $\mathrm{X}_0 \subset \mathrm{X}$ of s. That is, the morphism s factors through an open immersion $j_0 : \mathrm{X}_0 \hookrightarrow \mathrm{X}$ with $q \circ j_0$ étale.*

Proof. Consider the closed immersion $s_{\mathrm{cl}} : \mathrm{S}_{\mathrm{cl}} \hookrightarrow \mathrm{X}_{\mathrm{cl}}$ of underlying classical schemes, which is defined by a (discrete) quasi-coherent sheaf of ideals $\mathcal{I}$. Since s_{cl} admits a retraction p_{cl}, which is smooth by Lemma 2.9.11, it is a regular closed immersion of classical schemes. In particular its conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is a (discrete) projective finitely generated $\mathcal{O}_{\mathrm{S}_{\mathrm{cl}}}$-module.

Therefore the epimorphism $(p_{\mathrm{cl}})_*(\mathcal{I}) \to \mathcal{I}/\mathcal{I}^2$ admits a section, which gives rise to a morphism $\mathcal{I}/\mathcal{I}^2 \to (p_{\mathrm{cl}})_*(\mathcal{O}_{\mathrm{X}_{\mathrm{cl}}})$. A lift $\mathcal{N}^*_s \to p_*(\mathcal{O}_{\mathrm{X}})$ of this morphism corresponds to a morphism of $\mathcal{O}_{\mathrm{S}}$-algebras

$$\varphi : \mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{N}^*_s) \to p_*(\mathcal{O}_{\mathrm{X}})$$

such that the commutative square of $\mathcal{O}_{\mathrm{S}}$-algebras

$$\begin{array}{ccc} \mathrm{Sym}_{\mathcal{O}_{\mathrm{S}}}(\mathcal{N}^*_s) & \xrightarrow{\zeta} & \mathcal{O}_{\mathrm{S}} \\ \downarrow{\scriptstyle\varphi} & & \| \\ p_*(\mathcal{O}_{\mathrm{X}}) & \xrightarrow{\sigma} & \mathcal{O}_{\mathrm{S}} \end{array}$$

is cocartesian. Here ζ corresponds to the zero section z and σ to s. We let $q : \mathrm{X} \to \mathbf{N}^*_s$ be the morphism of spectral S-schemes corresponding to φ.

For (ii), let j_0 be the étale locus of q. To show that s factors through j_0, it is sufficient to note that $s^*(\mathcal{T}^*_{\mathrm{X}/\mathbf{N}^*_s}) \approx \mathcal{T}^*_{\mathrm{S}/\mathrm{S}} \approx 0$ by (i). □

3. Abstract Morel–Voevodsky homotopy theory

In this section we collect some generalities about Morel–Voevodsky homotopy theory in an abstract setting. We claim no originality for this material.

Given any essentially small ∞-category $\mathbf{C}$, equipped with some additional structures, we construct unstable and stable homotopy categories associated to $\mathbf{C}$.

Aside from classical motivic homotopy theory and its generalizations to derived and spectral algebraic geometry, the setup also applies to interesting variants like equivariant motivic homotopy theory as developed in [Hoy17], and even noncommutative motivic homotopy theory as developed in [Rob15].

3.1. Presheaves.

In this short paragraph we fix our notations regarding ∞-categories of presheaves.

3.1.1. Let $\mathbf{C}$ be an essentially small ∞-category. We write $\mathrm{PSh}(\mathbf{C})$ for the ∞-category of presheaves of spaces on $\mathbf{C}$, i.e. functors $(\mathbf{C})^{\mathrm{op}} \to \mathrm{Spc}$.

We will write $\mathrm{h}_{\mathbf{C}} : \mathbf{C} \hookrightarrow \mathrm{PSh}(\mathbf{C})$ for the fully faithful functor defined by the Yoneda embedding.

3.1.2. Recall that $\mathrm{PSh}(\mathbf{C})$ is the free presentable ∞-category generated by the ∞-category $\mathbf{C}$.

More precisely, for any presentable ∞-category $\mathbf{D}$, let $\mathrm{Funct}_!(\mathrm{PSh}(\mathbf{C}), \mathbf{D})$ denote the ∞-category of colimit-preserving functors $\mathrm{PSh}(\mathbf{C}) \to \mathbf{D}$. Then we have the following universal property ([Lur09, Thm. 5.1.5.6]):

Theorem 3.1.3. *For any presentable ∞-category $\mathbf{D}$, the canonical functor*

$$\mathrm{Funct}_!(\mathrm{PSh}(\mathbf{C}), \mathbf{D}) \to \mathrm{Funct}(\mathbf{C}, \mathbf{D}), \tag{3.1}$$

given by restriction along the Yoneda embedding $\mathrm{h}_{\mathbf{C}} : \mathbf{C} \hookrightarrow \mathrm{PSh}(\mathbf{C})$, is an equivalence of ∞-categories.

In particular, any functor $u : \mathbf{C} \to \mathbf{D}$ admits a unique extension to a colimit-preserving functor $u_! : \mathrm{PSh}(\mathbf{C}) \to \mathbf{D}$, called the left Kan extension of u.

3.1.4. Recall that the cartesian product defines a canonical closed symmetric monoidal structure on the category $\mathrm{PSh}(\mathbf{C})$.

3.2. Left localizations.

We briefly recall the theory of left localizations of presentable ∞-categories, following [Lur09].

3.2.1. Let W be a set of morphisms in a presentable ∞-category $\mathbf{A}$. We define:

Definition 3.2.2.

(i) *An object* $c \in \mathbf{A}$ *is* W-local *if, for every morphism* $f : x \to y$ *in* W*, the induced morphism of spaces*

$$\mathrm{Maps}_{\mathbf{A}}(y, c) \to \mathrm{Maps}_{\mathbf{A}}(x, c)$$

is invertible.

(ii) *A morphism* $f : x \to y$ *is a* W-local equivalence *if for every* W*-local object* c*, the induced morphism of spaces*

$$\mathrm{Maps}_{\mathbf{A}}(y, c) \to \mathrm{Maps}_{\mathbf{A}}(x, c)$$

is invertible.

3.2.3. We use the following theorem repeatedly (see [Lur09, Prop. 5.5.4.15]):

Theorem 3.2.4. *Let* $\mathbf{A}$ *be a presentable* ∞*-category. For any essentially small*[8] *set* W *of morphisms in* $\mathbf{A}$*, the inclusion of the full subcategory* $\mathrm{L_W}\mathbf{A} \subset \mathbf{A}$ *of* W*-local objects admits a left adjoint* $\mathrm{L_W} : \mathbf{A} \to \mathrm{L_W}\mathbf{A}$*, which exhibits* $\mathrm{L_W}\mathbf{A}$ *as an accessible left localization of* $\mathbf{A}$*. Further, a morphism* f *in* $\mathbf{A}$ *induces an isomorphism* $\mathrm{L_W}(f)$ *if and only if* f *is a* W*-local equivalence, or equivalently, if and only if* f *belongs to the strongly saturated closure of the set* W*.*

Here, a *left localization* is by definition a functor that admits a fully faithful right adjoint. In fact, all accessible left localizations of a presentable ∞-category $\mathbf{A}$ arise in the above way. (Accessibility is a set-theoretic condition which we will not bother to explain here.)

3.2.5. Let $\mathbf{A}$ be a presentable ∞-category and W an essentially small set of morphisms.

Given a presentable ∞-category $\mathbf{B}$, let $\mathrm{Funct}_{!,\mathrm{W}}(\mathbf{A}, \mathbf{B})$ denote the full subcategory of $\mathrm{Funct}_!(\mathbf{A}, \mathbf{B})$ spanned by functors that send morphisms in W to isomorphisms in $\mathbf{B}$.

We have the following universal property of $\mathrm{L_W}\mathbf{A}$ (see [Lur09, Prop. 5.5.4.20]):

Theorem 3.2.6. *For any presentable* ∞*-category* $\mathbf{B}$*, the canonical morphism*

$$\mathrm{Funct}_!(\mathrm{L_W}\mathbf{A}, \mathbf{B}) \xrightarrow{\sim} \mathrm{Funct}_{!,\mathrm{W}}(\mathbf{A}, \mathbf{B})$$

given by restriction along the functor $\mathrm{L_W} : \mathbf{A} \to \mathrm{L_W}\mathbf{A}$*, is an equivalence.*

3.3. **Pointed objects.** We recall some standard facts about pointed objects in presentable ∞-categories.

3.3.1. Let $\mathbf{A}$ be an presentable ∞-category. We write $\mathbf{A}_\bullet$ for the presentable ∞-category of pointed objects in $\mathbf{A}$. Its objects are pairs (a, x), where a is an object of $\mathbf{A}$ and $x : \mathrm{pt} \to a$ is a morphism from the terminal object.

By [Lur16a, Ex. 4.8.1.20, Prop. 4.8.2.11], $\mathbf{A}_\bullet$ has a canonical structure of $\mathrm{Spc}_\bullet$-module category, and is canonically equivalent to the base change $\mathbf{A} \otimes_{\mathrm{Spc}} \mathrm{Spc}_\bullet$.

3.3.2. Consider the forgetful functor sending a pointed object (a, x) to its underlying object $a \in \mathbf{A}$. This admits a left adjoint, which freely adjoins a point to a; that is, it is given on objects by the assignment

$$a \mapsto a_+ := (a \sqcup \mathrm{pt}, x)$$

where x is the canonical morphism $\mathrm{pt} \to a \sqcup \mathrm{pt}$.

[8] A set of morphisms W is *essentially small* if there is a small subset $\mathrm{W}_0 \subset \mathrm{W}$ such that every morphism in W is isomorphic to a morphism of W_0 in the category of arrows of $\mathbf{A}$.

3.3.3. Note that $\mathbf{A}_\bullet$ is equivalent to the category of modules over the monad with underlying endofunctor $a \mapsto a \sqcup \mathrm{pt}$. Since the latter commutes with contractible colimits, this implies:

Lemma 3.3.4. *The forgetful functor $(a, x) \mapsto a$ is conservative, and preserves and reflects contractible colimits.*

This monadic description also implies that every pointed object can be written as the geometric realization of a simplicial diagram with each term in the essential image of $a \mapsto a_+$:

Lemma 3.3.5. *The ∞-category $\mathbf{A}_\bullet$ is generated under sifted colimits by objects of the form a_+, where a is an object of $\mathbf{A}$.*

3.3.6. We have the following universal property of $\mathbf{A}_\bullet$:

Lemma 3.3.7. *For any pointed presentable ∞-category $\mathbf{D}$, the canonical functor*

$$\mathrm{Funct}_!(\mathbf{A}_\bullet, \mathbf{D}) \to \mathrm{Funct}_!(\mathbf{A}, \mathbf{D}), \tag{3.2}$$

given by restriction along the functor $a \mapsto a_+$, is an equivalence.

3.3.8. Suppose that $\mathbf{A}$ is the ∞-category of presheaves on some essentially small ∞-category $\mathbf{C}$.

In this case, the ∞-category $\mathbf{A}_\bullet = \mathrm{PSh}(\mathbf{C})_\bullet$ can be described as the free pointed presentable ∞-category generated by the ∞-category $\mathbf{C}$. Indeed, combining Lemma 3.3.7 with Theorem 3.1.3 we obtain a canonical equivalence

$$\mathrm{Funct}_!(\mathrm{PSh}(\mathbf{C})_\bullet, \mathbf{D}) \xrightarrow{\sim} \mathrm{Funct}(\mathbf{C}, \mathbf{D}). \tag{3.3}$$

for any pointed presentable ∞-category $\mathbf{D}$.

3.3.9. Suppose that $\mathbf{A}$ admits a closed symmetric monoidal structure.

Then the category $\mathbf{A}_\bullet$ inherits a canonical closed symmetric monoidal structure, given by the "smash product" $\wedge$. The monoidal unit is the object pt_+.

This monoidal structure is uniquely characterized by the fact that the functor $a \mapsto a_+$ is symmetric monoidal. Further, we have the following universal property (see [Rob15, Cor. 2.32]):

Lemma 3.3.10. *For any pointed symmetric monoidal presentable ∞-category $\mathbf{D}$, the canonical functor*

$$\mathrm{Funct}_{!,\otimes}(\mathbf{A}_\bullet, \mathbf{D}) \to \mathrm{Funct}_{!,\otimes}(\mathbf{A}, \mathbf{D}), \tag{3.4}$$

given by restriction along the symmetric monoidal functor $a \mapsto a_+$, is an equivalence of ∞-categories.

3.3.11. Let $t \in \mathbf{A}_\bullet$ be a pointed object in $\mathbf{A}$.

The *t-suspension* endofunctor Σ_t on $\mathbf{A}_\bullet$ is defined by the assignment

$$(a, x) \mapsto (a, x) \wedge t.$$

Dually, the *t-loop space* endofunctor Ω_t is given by

$$(a, x) \mapsto \underline{\mathrm{Hom}}(t, (a, x)).$$

These endofunctors form an adjunction (Σ_t, Ω_t).

3.4. **Spectrum objects.** We recall some standard facts about spectrum objects in pointed presentable ∞-categories, following [Hoy17, §6.1].

3.4.1. Let $\mathbf{A}$ be a presentable ∞-category. We fix an (essentially small) set T of pointed objects of $\mathbf{A}$.

We write $\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)$ for the presentable ∞-category of T-spectrum objects in $\mathbf{A}_\bullet$. We now recall its construction.

Suppose that the set T contains a finite number of elements $t_1, \ldots, t_n$. Then write $\Sigma_{\mathrm{T}} := \Sigma_{t_1 \otimes \cdots \otimes t_n}$, and define $\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)$ as the colimit of the filtered diagram

$$\mathbf{A}_\bullet \xrightarrow{\Sigma_{\mathrm{T}}} \mathbf{A}_\bullet \xrightarrow{\Sigma_{\mathrm{T}}} \cdots \tag{3.5}$$

in the ∞-category of presentable ∞-categories. Equivalently, this is the limit of the cofiltered diagram

$$\cdots \xrightarrow{\Omega_{\mathrm{T}}} \mathbf{A}_\bullet \xrightarrow{\Omega_{\mathrm{T}}} \mathbf{A}_\bullet \tag{3.6}$$

in the ∞-category of ∞-categories, where we have written $\Omega_{\mathrm{T}} := \Omega_{t_1 \otimes \cdots \otimes t_n}$.

In the infinite case, we define $\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)$ as the filtered colimit

$$\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet) = \varinjlim_{\mathrm{T}_0 \subset \mathrm{T}} \mathrm{Spt}_{\mathrm{T}_0}(\mathbf{A}_\bullet),$$

indexed over finite subsets $\mathrm{T}_0 \subset \mathrm{T}$.

Remark 3.4.2. Note that, in the case where T is finite, a T-spectrum is the data of a sequence $(a_n)_{n \geqslant 0}$ of pointed presheaves and structural isomorphisms

$$\alpha_n : a_n \xrightarrow{\sim} \Omega_t(a_{n+1})$$

for each integer $n \geqslant 0$.

In the infinite case, we have, roughly speaking, deloopings with respect to finite tensor products of elements of T, and a homotopy coherent system of compatibilities.

3.4.3. By construction, the adjunctions (Σ_t, Ω_t) (3.3.11) at the level of pointed spaces give rise to *equivalences*

$$\Sigma_{\mathrm{T}_0} : \mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet) \rightleftarrows \mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet) : \Omega_{\mathrm{T}_0},$$

for each finite subset $\mathrm{T}_0 \subset \mathrm{T}$.

3.4.4. By construction, we have a canonical adjunction

$$\Sigma^\infty_{\mathrm{T}} : \mathbf{A}_\bullet \rightleftarrows \mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet) : \Omega^\infty_{\mathrm{T}}.$$

3.4.5. By [Lur09, Lem. 6.3.3.6], and the construction of $\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)$, we have[9]:

Lemma 3.4.6.

(i) *If* T *is finite, then the category* $\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)$ *is generated under filtered colimits by objects of the form* $\Omega^n_{\mathrm{T}}\Sigma^\infty_{\mathrm{T}}(a)$, *for* $a \in \mathbf{A}_\bullet$ *a pointed object and* $n \geqslant 0$.

(ii) *If* T *is infinite, then the category* $\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)$ *is generated under filtered colimits by objects of the form* $\Omega_{\mathrm{T}_n} \cdots \Omega_{\mathrm{T}_0}\Sigma^\infty_{\mathrm{T}}(a)$, *for* $a \in \mathbf{A}_\bullet$ *a pointed object,* $n \geqslant 0$, *and* $\mathrm{T}_0, \ldots, \mathrm{T}_n \subset \mathrm{T}$ *finite subsets.*

[9]For the reader's convenience, we state the finite case separately.

3.4.7. We have[10] the following universal property of the presentable ∞-category $\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)$:

Lemma 3.4.8. *Let* $\mathbf{D}$ *be a presentable* ∞*-category.*

(i) *If* T *is finite, then the canonical functor*

$$\mathrm{Funct}_!(\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet), \mathbf{D}) \to \varprojlim \mathrm{Funct}_!(\mathbf{A}_\bullet, \mathbf{D})$$

is an equivalence.

(ii) *If* T *is infinite, then the canonical functor*

$$\mathrm{Funct}_!(\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet), \mathbf{D}) \to \varprojlim_{\mathrm{T}_0 \subset \mathrm{T}} \varprojlim \mathrm{Funct}_!(\mathbf{A}_\bullet, \mathbf{D})$$

is an equivalence.

In the first statement, the limit is of the diagram

$$\cdots \xrightarrow{(\Sigma_{\mathrm{T}})^*} \mathrm{Funct}_!(\mathbf{A}_\bullet, \mathbf{D}) \xrightarrow{(\Sigma_{\mathrm{T}})^*} \mathrm{Funct}_!(\mathbf{A}_\bullet, \mathbf{D}).$$

In the second, the first limit is taken over finite subsets $\mathrm{T}_0 \subset \mathrm{T}$, and the second limit is of the diagram

$$\cdots \xrightarrow{(\Sigma_{\mathrm{T}_0})^*} \mathrm{Funct}_!(\mathbf{A}_\bullet, \mathbf{D}) \xrightarrow{(\Sigma_{\mathrm{T}_0})^*} \mathrm{Funct}_!(\mathbf{A}_\bullet, \mathbf{D}).$$

Proof. First consider the finite case. It suffices to show that, for every presentable ∞-category $\mathbf{E}$, the map of spaces obtained by applying the functor $\mathrm{Maps}(\mathbf{E}, -)$ is invertible. Recall that $\mathrm{Funct}_!$ computes the internal Hom in the closed symmetric monoidal ∞-category of presentable ∞-categories, i.e. we have $\mathrm{Maps}(\mathbf{E}, \mathrm{Funct}_!(\mathbf{E}', \mathbf{E}'')) \approx \mathrm{Maps}(\mathbf{E} \otimes \mathbf{E}', \mathbf{E}'')$ for any two presentable ∞-categories $\mathbf{E}'$ and $\mathbf{E}''$. Also, recall the tensor product of presentable ∞-categories commutes with colimits in each argument. It follows that, for each $\mathbf{E}$, the map in question is the canonical isomorphism

$$\mathrm{Maps}(\mathrm{Spt}_{\mathrm{T}}(\mathbf{E} \otimes \mathbf{A}_\bullet), \mathbf{D}) \xrightarrow{\sim} \varprojlim \mathrm{Maps}(\mathbf{E} \otimes \mathbf{A}_\bullet, \mathbf{D}).$$

In the infinite case, we use an argument similar to the one above to reduce to the finite case. □

3.4.9. Suppose that each pointed object $t \in \mathrm{T}$ is *k-symmetric*, i.e. the cyclic permutation of $t^{\otimes k}$ is homotopic to the identity morphism, for some $k \geqslant 2$.

In this case the main result of [Rob15] endows the presentable ∞-category $\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)$ with a canonical closed symmetric monoidal structure.

This monoidal structure is uniquely characterized by the fact that the functor $a \mapsto \Sigma^\infty_{\mathrm{T}}(a)$ lifts to a symmetric monoidal functor that sends each object $t \in \mathrm{T}$ to a monoidally invertible object of $\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)$.

The monoidal unit $\mathbf{1}_{\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet)}$ is the T-spectrum $\Sigma^\infty_{\mathrm{T}}(\mathrm{pt}_+)$.

Further, we have the following universal property:

Lemma 3.4.10. *If each object* $t \in \mathrm{T}$ *is* k*-symmetric for some* $k \geqslant 2$*, then the canonical functor*

$$\mathrm{Funct}_{!,\otimes}(\mathrm{Spt}_{\mathrm{T}}(\mathbf{A}_\bullet), \mathbf{D}) \to \mathrm{Funct}_{!,\otimes,\mathrm{T}}(\mathbf{A}_\bullet, \mathbf{D}),$$

given by restriction along the functor $a \mapsto \Sigma^\infty_{\mathrm{T}}(a)$*, is an equivalence.*

Here $\mathrm{Funct}_{!,\otimes,\mathrm{T}}(\mathbf{A}_\bullet, \mathbf{D})$ denotes the full sub-∞-category of $\mathrm{Funct}_{!,\otimes}(\mathbf{A}_\bullet, \mathbf{D})$ spanned by symmetric monoidal colimit-preserving functors that send each object $t \in \mathrm{T}$ to a monoidally invertible object of $\mathbf{D}$.

[10] For the reader's convenience, we state the finite case separately.

3.5. **Homotopy invariance.**

3.5.1. Let $\mathbb{A}$ be an (essentially small) set of morphisms in $\mathbf{C}$, which is stable under base change. That is, for each morphism $a : c' \to c$ in $\mathbb{A}$, and any morphism $f : d \to c$ in $\mathbf{C}$, the base change $c' \times_c d \to d$ exists and belongs to $\mathbb{A}$.

We define:

Definition 3.5.2. *A presheaf $\mathcal{F}$ is* $\mathbb{A}$-invariant *if for every morphism $a : c' \to c$ in $\mathbb{A}$, the induced morphism of spaces*

$$\mathcal{F}(a) : \mathcal{F}(c) \to \mathcal{F}(c')$$

is invertible.

Note that we have:

Lemma 3.5.3. *The condition of $\mathbb{A}$-invariance is stable by colimits.*

Proof. Let $\mathcal{F}$ be the colimit of a diagram $(\mathcal{F}_\alpha)_\alpha$, where each $\mathcal{F}_\alpha$ is $\mathbb{A}$-invariant. For any morphism $a \in \mathbb{A}$, the morphism $\mathcal{F}(a)$ is the colimit of the morphisms $\mathcal{F}_\alpha(a)$, since colimits of presheaves are computed section-wise. □

3.5.4. Let $\mathrm{PSh}_{\mathbb{A}}(\mathbf{C})$ denote the full sub-∞-category of $\mathrm{PSh}(\mathbf{C})$ spanned by presheaves that are $\mathbb{A}$-invariant. This is the left localization at the small set $\mathbb{A}$, and as such there exists a localization functor $\mathcal{F} \mapsto \mathrm{L}_{\mathbb{A}}(\mathcal{F})$, left adjoint to the inclusion.

We say that a morphism of presheaves is an $\mathbb{A}$-*local equivalence* if it becomes invertible after applying the functor $\mathrm{L}_{\mathbb{A}}$. The set of $\mathbb{A}$-local equivalences is equivalently the strongly saturated closure of the set $\mathbb{A}$.

3.5.5. According to [Hoy17, Prop. 3.3], the basic properties of $\mathrm{PSh}_{\mathbb{A}}(\mathbf{C})$ can be summarized as follows:

Lemma 3.5.6.

(i) *For each presheaf $\mathcal{F}$ on $\mathbf{C}$, there is a canonical isomorphism*

$$\mathrm{L}_{\mathbb{A}}(\mathcal{F})(c) \approx \varinjlim_{(d \to c) \in (\mathbb{A}_c)^{\mathrm{op}}} \mathcal{F}(d) \tag{3.7}$$

for each object $c \in \mathbf{C}$, where $\mathbb{A}_c$ denotes the full sub-∞-category of $\mathbf{C}_{/c}$ spanned by composites of morphisms in the set $\mathbb{A}$. Further, the category $(\mathbb{A}_c)^{\mathrm{op}}$ is sifted.

(ii) *The functor $\mathcal{F} \mapsto \mathrm{L}_{\mathbb{A}}(\mathcal{F})$ commutes with finite products.*

(iii) *The category $\mathrm{PSh}_{\mathbb{A}}(\mathbf{C})$ has universality of colimits.*

3.6. **Excision structures.** The notion of excision structure is a slight variation on *cd-structures* studied by Voevodsky [Voe10].

3.6.1. A *pre-excision structure* on $\mathbf{C}$ is an essentially small set $\mathbb{E}$ of cartesian squares in $\mathbf{C}$.

We may consider the following axioms:

(EXC0) There exists an initial object $\varnothing_{\mathbf{C}}$ of $\mathbf{C}$ such that every morphism $c \to \varnothing_{\mathbf{C}}$ is invertible.

(EXC1) The set $\mathbb{E}$ is stable under base change. That is, for any square Q in $\mathbb{E}$, the base change along any morphism in $\mathbf{C}$ exists and belongs to $\mathbb{E}$.

Definition 3.6.2. *We say that $\mathbb{E}$ is an* excision structure *if it satisfies the axioms (EXC0) and (EXC1).*

3.6.3. Let $\mathbb{E}$ be a pre-excision structure on $\mathbf{C}$. We define:

Definition 3.6.4. *A presheaf $\mathcal{F}$ on $\mathbf{C}$ is* $\mathbb{E}$-excisive *if it satisfies the following conditions:*

(i) *The presheaf $\mathcal{F}$ is* reduced, *i.e. the space $\mathcal{F}(\varnothing_{\mathbf{C}})$ is contractible.*

(ii) *For any cartesian square $\mathrm{Q} \in \mathbb{E}$ of the form*

$$\begin{array}{ccc} d' & \longrightarrow & d \\ \downarrow & & \downarrow g \\ c' & \xrightarrow{f} & c, \end{array} \tag{3.8}$$

the induced commutative square of spaces

$$\begin{array}{ccc} \mathcal{F}(c) & \longrightarrow & \mathcal{F}(c') \\ \downarrow & & \downarrow \\ \mathcal{F}(d) & \longrightarrow & \mathcal{F}(d') \end{array}$$

is cartesian.

3.6.5. Note that we have:

Lemma 3.6.6. *The condition of $\mathbb{E}$-excision is stable by filtered colimits.*

Proof. Condition (i) follows from the fact that filtered colimits of contractible spaces are contractible. Condition (ii) follows from the fact that finite limits commute with filtered colimits in $\mathrm{PSh}(\mathbf{C})$. □

3.6.7. Let $\mathrm{PSh}_{\mathbb{E}}(\mathbf{C})$ denote the full sub-∞-category of $\mathrm{PSh}(\mathbf{C})$ spanned by $\mathbb{E}$-excisive presheaves.

Note that this is a left localization of $\mathrm{PSh}(\mathbf{C})$ at the (essentially small) set containing the canonical morphism

$$e : \varnothing_{\mathrm{PSh}(\mathbf{C})} \to \mathrm{h}_{\mathbf{C}}(\varnothing_{\mathbf{C}})$$

and the morphisms

$$k_{\mathrm{Q}} : \mathrm{K}_{\mathrm{Q}} \to \mathrm{h}_{\mathbf{C}}(c) \tag{3.9}$$

for all squares $\mathrm{Q} \in \mathbb{E}$ of the form (3.8). Here K_{Q} denotes the presheaf

$$\mathrm{K}_{\mathrm{Q}} = \mathrm{h}_{\mathbf{C}}(c') \underset{\mathrm{h}_{\mathbf{C}}(d')}{\sqcup} \mathrm{h}_{\mathbf{C}}(d).$$

In particular, it follows that there exists a localization functor $\mathcal{F} \to \mathrm{L}_{\mathbb{E}}(\mathcal{F})$, left adjoint to the inclusion.

3.6.8. An $\mathbb{E}$-*local equivalence* is a morphism of presheaves which becomes invertible after applying the localization functor $\mathrm{L}_{\mathbb{E}}$. The set of $\mathbb{E}$-local equivalences is equivalently the strongly saturated closure of the set of morphisms containing e and k_{Q} for each $\mathrm{Q} \in \mathbb{E}$.

3.7. Topological excision structures.

3.7.1. Let $\mathbb{E}$ be an excision structure on $\mathbf{C}$. We consider the following axioms on $\mathbb{E}$:

(EXC2) For every square of the form (3.8) in $\mathbb{E}$, the lower horizontal morphism f is a monomorphism.

(EXC3) For every square in $\mathbb{E}$ of the form (3.8), the induced cartesian square

$$\begin{array}{ccc} d' & \longrightarrow & d \\ \downarrow & & \downarrow \\ d' \times_{c'} d' & \longrightarrow & d \times_c d, \end{array} \tag{3.10}$$

where the vertical arrows are the respective diagonal morphisms, belongs to $\mathbb{E}$.

We define:

Definition 3.7.2. *An excision structure* $\mathbb{E}$ *is* topological *if the axioms (EXC2) and (EXC3) are satisfied.*

This terminology will be explained by Theorem 3.7.9.

3.7.3. Let $\mathbb{E}$ be an excision structure on $\mathbf{C}$.

Consider the Grothendieck pretopology on $\mathbf{C}$ consisting of the following covering families:

(1) The empty family covering $\varnothing_{\mathbf{C}}$.

(2) For every square $\mathrm{Q} \in \mathbb{E}$ of the form

$$\begin{array}{ccc} d' & \longrightarrow & d \\ \downarrow & & \downarrow g \\ c' & \xrightarrow{f} & c, \end{array}$$

the family $\{f, g\}$ covering c.

We let $\tau_{\mathbb{E}}$ denote the Grothendieck topology generated by this pretopology.

3.7.4. Note that the axioms (EXC0) and (EXC1) imply that families of the form (1) and (2) are stable under pullback.

It follows from [Hoy15, Cor. C.2] that the condition of *descent* (Čech descent) with respect to the topology $\tau_{\mathbb{E}}$ can be described as follows.

Lemma 3.7.5. *A presheaf* $\mathcal{F}$ *on* $\mathbf{C}$ *is a* $\tau_{\mathbb{E}}$*-sheaf if it satisfies the following conditions:*

(i) *The space* $\mathcal{F}(\varnothing_{\mathbf{C}})$ *is contractible.*

(ii) *For every square* $\mathrm{Q} \in \mathbb{E}$ *of the form (3.8), the canonical morphism of spaces*

$$\mathcal{F}(c) \to \varprojlim_{[n]\in\mathbf{\Delta}} \mathrm{Maps}_{\mathrm{PSh}(\mathbf{C})}(\check{\mathrm{C}}(\tilde{c}/c)_n, \mathcal{F})$$

is invertible. Here the simplicial object $\check{\mathrm{C}}(\tilde{c}/c)_\bullet$ *is the Čech nerve of the morphism* $\tilde{c} = \mathrm{h}(c') \sqcup \mathrm{h}(d) \to \mathrm{h}(c)$.

3.7.6. Let $\mathrm{PSh}_{\tau_{\mathbb{E}}}(\mathbf{C})$ denote the full sub-∞-category of $\mathrm{PSh}(\mathbf{C})$ spanned by $\tau_{\mathbb{E}}$-sheaves, i.e. presheaves satisfying $\tau_{\mathbb{E}}$-descent.

Note that this is the left localization at the essentially small set containing the canonical morphism

$$e : \varnothing_{\mathrm{PSh}(\mathbf{C})} \to \mathrm{h}_{\mathbf{C}}(\varnothing_{\mathbf{C}})$$

and the morphisms

$$c_{\mathrm{Q}} : \mathrm{C}_{\mathrm{Q}} \to \mathrm{h}_{\mathbf{C}}(c), \tag{3.11}$$

for all squares $\mathrm{Q} \in \mathbb{E}$ of the form (3.8), where C_{Q} denotes the presheaf

$$\mathrm{C}_{\mathrm{Q}} = \varinjlim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \check{\mathrm{C}}(\tilde{c}/c)_n,$$

where $\tilde{c} = \mathrm{h}(c') \sqcup \mathrm{h}(d)$.

In particular, there is a localization functor $\mathcal{F} \mapsto \mathrm{L}_\tau(\mathcal{F})$, left adjoint to the inclusion. By ∞-topos theory, we have:

Proposition 3.7.7.

(i) *The localization functor* $\mathrm{L}_{\tau_{\mathbb{E}}}$ *is left exact, i.e. it commutes with finite limits.*

(ii) *The* ∞*-category* $\mathrm{Sh}_{\tau_{\mathbb{E}}}(\mathbf{C})$ *has universality of colimits.*

A $\tau_{\mathbb{E}}$*-local equivalence* is a morphism of presheaves which becomes invertible after applying the localization functor $\mathrm{L}_{\tau_{\mathbb{E}}}$. The set of $\tau_{\mathbb{E}}$-local equivalences is equivalently the strongly saturated closure of the set of morphisms containing e and c_{Q} for each $\mathrm{Q} \in \mathbb{E}$.

3.7.8. The following theorem of Voevodsky says that the localization defined by any topological excision structure coincides with the localization defined by the associated Grothendieck topology.

Theorem 3.7.9 (Voevodsky)**.** *If* $\mathbb{E}$ *is a topological excision structure, then for any presheaf* $\mathcal{F}$ *on* $\mathbf{C}$*, the condition of* $\mathbb{E}$*-excision is equivalent to* $\tau_{\mathbb{E}}$*-descent.*

This was proved in [Voe10, Thm. 5.10] (cf. [AHW15, Thm. 3.2.5]) in the case where the ∞-category $\mathbf{C}$ is an ordinary category. The reader will note that the proof generalizes *mutatis mutandis* to our setting.

In particular we obtain:

Corollary 3.7.10. *If* $\mathbb{E}$ *is a topological excision structure, then we have:*

(i) *The localization functor* $\mathrm{L}_{\mathbb{E}}$ *is left-exact, i.e. commutes with finite limits.*

(ii) *The* ∞*-category* $\mathrm{PSh}_{\mathbb{E}}(\mathbf{C})$ *has universality of colimits.*

3.8. **Unstable homotopy theory.**

3.8.1. Let $\mathbf{C}$ be an essentially small ∞-category, admitting an initial object $\varnothing_{\mathbf{C}}$.

We fix an excision structure $\mathbb{E}$ on $\mathbf{C}$, and an (essentially small) set $\mathbb{A}$ of morphisms in $\mathbf{C}$, which is stable under base change.

The *unstable homotopy theory* associated to the pair $(\mathbb{E}, \mathbb{A})$ is the ∞-category $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})$ defined as the full sub-∞-category of $\mathrm{PSh}(\mathbf{C})$ spanned by presheaves satisfying $\mathbb{E}$-excision and $\mathbb{A}$-invariance.

3.8.2. This is an accessible left localization of $\mathrm{PSh}(\mathbf{C})$, so $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})$ is a presentable ∞-category, and the inclusion admits a left adjoint $\mathcal{F} \mapsto \mathrm{L}_{\mathbb{E},\mathbb{A}}(\mathcal{F})$.

A $(\mathbb{E}, \mathbb{A})$*-local equivalence* is a morphism of presheaves that becomes invertible after applying $\mathrm{L}_{\mathbb{E},\mathbb{A}}$.

3.8.3. Let $\mathcal{F}$ be an $\mathbb{E}$-excisive $\mathbb{A}$-invariant presheaf. For any object $c \in \mathbf{C}$, we have canonical bifunctorial isomorphisms of spaces

$$\mathrm{Maps}_{\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})}(\mathrm{L}_{\mathbb{E},\mathbb{A}}(\mathrm{h}_{\mathbf{C}}(c)), \mathcal{F}) \approx \mathcal{F}(c).$$

3.8.4. Though the localization functors $\mathrm{L}_{\mathbb{E}}$ and $\mathrm{L}_{\mathbb{A}}$ do not commute, the functor $\mathrm{L}_{\mathbb{E},\mathbb{A}}$ can be described by the following transfinite composition:

Lemma 3.8.5. *For every presheaf $\mathcal{F}$ on $\mathbf{C}$, there is a canonical isomorphism*

$$\mathrm{L}_{\mathbb{E},\mathbb{A}}(\mathcal{F}) \approx \varinjlim_{n\geqslant 0} (\mathrm{L}_{\mathbb{A}} \circ \mathrm{L}_{\mathbb{E}})^{\circ n}(\mathcal{F}). \tag{3.12}$$

Proof. This follows from the fact that the properties of $\mathbb{E}$-excision and $\mathbb{A}$-invariance are stable under filtered colimits (Lemmas 3.6.6 and 3.5.3). □

3.8.6. We have:

Corollary 3.8.7. *If the excision structure $\mathbb{E}$ is topological, then the following hold.*

(i) *The localization functor $\mathrm{L}_{\mathbb{E},\mathbb{A}}$ commutes with finite products.*

(ii) *The ∞-category $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})$ has universality of colimits.*

Proof. The first claim follows from the formula given in Lemma 3.8.5. Indeed, the functors $\mathrm{L}_{\mathbb{E}}$ and $\mathrm{L}_{\mathbb{A}}$ both commute with finite products (Lemmas 3.7.10 and 3.5.6), and filtered colimits commute with finite products of presheaves.

The second claim follows directly from Corollary 3.7.10 and Lemma 3.5.6. □

3.8.8. We now formulate a universal property for $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})$.

For a presentable ∞-category $\mathbf{D}$, let $\mathrm{Funct}_{\mathbb{E},\mathbb{A}}(\mathbf{C},\mathbf{D})$ denote the full subcategory of $\mathrm{Funct}(\mathbf{C},\mathbf{D})$ spanned by functors $u : \mathbf{C} \to \mathbf{D}$ that satisfy $\mathbb{E}$-excision and $\mathbb{A}$-invariance, i.e. which send

(1) the initial object $\varnothing_{\mathbf{C}}$ to an initial object of $\mathbf{D}$,

(2) any square $\mathrm{Q} \in \mathbb{E}$ to a cocartesian square in $\mathbf{D}$,

(3) any morphism $a \in \mathbb{A}$ to an invertible morphism in $\mathbf{D}$.

Then we have:

Lemma 3.8.9. *For any presentable ∞-category $\mathbf{D}$, the canonical functor*

$$\mathrm{Funct}_!(\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C}),\mathbf{D}) \to \mathrm{Funct}_{\mathbb{E},\mathbb{A}}(\mathbf{C},\mathbf{D}), \tag{3.13}$$

given by restriction along the functor $\mathbf{C} \to \mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})$, is an equivalence of ∞-categories.

Proof. This follows immediately from Theorem 3.1.3 and the universal property of left localizations (Theorem 3.2.6). □

3.8.10. By point (i) of Corollary 3.8.7, we have:

Corollary 3.8.11. *Suppose that the excision structure $\mathbb{E}$ is topological. Then the cartesian monoidal structure on the presentable ∞-category $\mathrm{PSh}(\mathbf{C})$ restricts to a cartesian monoidal structure on the presentable ∞-category $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})$.*

3.9. Pointed homotopy theory.

3.9.1. Let $\mathrm{PSh}(\mathbf{C})_\bullet$ denote the presentable ∞-category of pointed objects in the presentable ∞-category $\mathrm{PSh}(\mathbf{C})$.

For any pointed presheaf $(\mathcal{F},x)$ on $\mathbf{C}$, we define:

Definition 3.9.2. *The pointed presheaf $(\mathcal{F},x)$ is* $\mathbb{E}$-excisive *or* $\mathbb{A}$-invariant *if its underlying presheaf $\mathcal{F}$ has the respective property.*

Let $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})_\bullet$ denote the full sub-∞-category of $\mathrm{PSh}(\mathbf{C})_\bullet$ spanned by $\mathbb{E}$-excisive $\mathbb{A}$-invariant pointed presheaves. Note that this is equivalent to the presentable ∞-category of pointed objects in the presentable ∞-category $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})$.

We call this the *pointed homotopy theory* associated to the pair $(\mathbb{E}, \mathbb{A})$.

3.9.3. Note that $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})_\bullet$ is an accessible left localization of $\mathrm{PSh}(\mathbf{C})$ at the essentially small set of morphisms of the form

$$\mathrm{h}_{\mathbf{C}}(a)_+ : \mathrm{h}_{\mathbf{C}}(c)_+ \to \mathrm{h}_{\mathbf{C}}(c')_+, \tag{3.14}$$

for each morphism $a : c \to c'$ in $\mathbb{A}$,

$$(k_{\mathrm{Q}})_+ : (\mathrm{K}_{\mathrm{Q}})_+ \to \mathrm{h}_{\mathbf{C}}(c)_+, \tag{3.15}$$

for each square $\mathrm{Q} \in \mathbb{E}$ of the form (3.8), and

$$e_+ : \mathrm{pt} \to \mathrm{h}_{\mathbf{C}}(\varnothing_{\mathbf{C}})_+.$$

3.9.4. In particular, we have localization functors $\mathrm{L}_{\mathbb{E}}$, $\mathrm{L}_{\mathbb{A}}$, and

$$\mathrm{L}_{\mathbb{E},\mathbb{A}} : \mathrm{PSh}(\mathbf{C})_\bullet \to \mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})_\bullet \tag{3.16}$$

at the level of pointed objects.

By Lemma 3.3.10 it follows that these functors are symmetric monoidal, and are uniquely characterized by commutativity with the functor $\mathcal{F} \mapsto \mathcal{F}_+$.

3.9.5. Combining Lemma 3.8.9 with Lemma 3.3.7, we obtain the following universal property for the pointed presentable ∞-category $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})_\bullet$:

Lemma 3.9.6. *For any pointed presentable ∞-category $\mathbf{D}$, the canonical functor*

$$\mathrm{Funct}_!(\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})_\bullet, \mathbf{D}) \xrightarrow{\sim} \mathrm{Funct}_{\mathbb{E},\mathbb{A}}(\mathbf{C}, \mathbf{D}). \tag{3.17}$$

is an equivalence of ∞-categories.

3.9.7. By Lemma 3.3.10, we have a canonical closed symmetric monoidal structure on $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})_\bullet$, characterized by the following universal property:

Lemma 3.9.8. *For any pointed symmetric monoidal presentable ∞-category $\mathbf{D}$, the canonical functor*

$$\mathrm{Funct}_{!,\otimes}(\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})_\bullet, \mathbf{D}) \to \mathrm{Funct}_{!,\otimes}(\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C}), \mathbf{D}), \tag{3.18}$$

given by restriction along the functor $\mathcal{F} \mapsto \mathcal{F}_+$, is an equivalence of ∞-categories.

Here $\mathrm{Funct}_{!,\otimes}$ denotes the ∞-category of colimit-preserving symmetric monoidal functors.

3.10. **Stable homotopy theory.**

3.10.1. We fix an (essentially small) set $\mathbb{T}$ of pointed presheaves on $\mathbf{C}$.

We will always assume that each $\mathcal{T} \in \mathbb{T}$ is $\mathbb{E}$-excisive and $\mathbb{A}$-invariant, by replacing it with its $(\mathbb{E}, \mathbb{A})$-localization if necessary.

We will consider the following axioms on $\mathbb{T}$:

(STAB1) At least one of the pointed presheaves $\mathcal{T} \in \mathbb{T}$ can be written as an S^1-suspension $\Sigma_{\mathrm{S}^1}(\mathcal{T}')$ of some pointed presheaf $\mathcal{T}'$.

(STAB2) Each pointed presheaf $\mathcal{T} \in \mathbb{T}$ is k-symmetric, i.e. the cyclic permutation of $\mathcal{T}^{\otimes k}$ is homotopic to the identity morphism, for some $k \geqslant 2$.

3.10.2. Let $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_\bullet)$ denote the presentable ∞-category of $\mathbb{T}$-spectrum objects (Paragraph 3.4) in $\mathrm{PSh}(\mathbf{C})_\bullet$.

3.10.3. Since a pointed ∞-category is stable if and only if the S^1-suspension functor Σ_{S^1} is an equivalence, we have:

Lemma 3.10.4. *If the axiom (STAB1) holds, then the presentable ∞-category $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_\bullet)$ is stable.*

3.10.5. We define:

Definition 3.10.6. *We say that a $\mathbb{T}$-spectrum $\mathcal{F}$ satisfies $\mathbb{E}$-excision or $\mathbb{A}$-invariance if for each $n \geqslant 0$, its nth component $\Omega^{\infty-n}_{\mathbb{T}}(\mathcal{F})$ satisfies the respective property (as a pointed presheaf).*

Let $\mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C})$ denote the full sub-∞-category of $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_\bullet)$ spanned by $\mathbb{E}$-excisive $\mathbb{A}$-invariant $\mathcal{T}$-spectra. This is equivalent to the presentable ∞-category of $\mathbb{T}$-spectrum objects in the presentable ∞-category $\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C})_\bullet$.

We call the presentable ∞-category $\mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C})$ the *stable homotopy theory* of $\mathbf{C}$, with respect to $(\mathbb{E}, \mathbb{A}, \mathbb{T})$.

3.10.7. As in Lemma 3.10.4, we have:

Lemma 3.10.8. *If the axiom (STAB1) holds, then the presentable ∞-category $\mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C})$ is stable.*

3.10.9. Note that $\mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C})$ is the left localization at the essentially small set of morphisms of the form

$$\Sigma^\infty_{\mathbb{T}}\, \mathrm{h}_{\mathbf{C}}(a)_+ : \Sigma^\infty_{\mathbb{T}}\, \mathrm{h}_{\mathbf{C}}(c)_+ \to \Sigma^\infty_{\mathbb{T}}\, \mathrm{h}_{\mathbf{C}}(c')_+, \tag{3.19}$$

for each morphism $a : c \to c'$ in $\mathbb{A}$,

$$\Sigma^\infty_{\mathbb{T}}(k_{\mathrm{Q}})_+ : \Sigma^\infty_{\mathbb{T}}(\mathrm{K}_{\mathrm{Q}})_+ \to \Sigma^\infty_{\mathbb{T}}\, \mathrm{h}_{\mathbf{C}}(c)_+, \tag{3.20}$$

for each square $\mathrm{Q} \in \mathbb{E}$ of the form (3.8), and

$$\Sigma^\infty_{\mathbb{T}}(e_+) : 0 \to \Sigma^\infty_{\mathbb{T}}\, \mathrm{h}_{\mathbf{C}}(\varnothing_{\mathbf{C}})_+,$$

where 0 denotes the zero object of $\mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_\bullet)$.

In particular, we have localization functors $\mathrm{L}_{\mathbb{E}}$, $\mathrm{L}_{\mathbb{A}}$, and

$$\mathrm{L}_{\mathbb{E},\mathbb{A}} : \mathrm{Spt}_{\mathbb{T}}(\mathrm{PSh}(\mathbf{C})_\bullet) \to \mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C}), \tag{3.21}$$

at the level of $\mathbb{T}$-spectra.

3.10.10. Combining Lemma 3.9.6 with Lemma 3.4.8, we obtain the following universal property for the presentable ∞-category $\mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C})$:

Lemma 3.10.11. *For any pointed presentable ∞-category $\mathbf{D}$, the canonical functor*

$$\mathrm{Funct}_!(\mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C}), \mathbf{D}) \to \varprojlim_{\mathbb{T}_0 \subset \mathbb{T}} \varprojlim \mathrm{Funct}_{\mathbb{E},\mathbb{A}}(\mathbf{C}, \mathbf{D})$$

is an equivalence.

3.10.12. Suppose that the axiom (STAB2) holds. Then by Lemma 3.4.10, we have a canonical closed symmetric monoidal structure on $\mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C})$.

As a symmetric monoidal presentable ∞-category, $\mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C})$ is characterized by the following universal property:

Lemma 3.10.13. *Suppose the axiom (STAB2) holds. For any presentable ∞-category $\mathbf{D}$, the canonical functor*

$$\mathrm{Funct}_{!,\otimes}(\mathbf{SH}_{\mathbb{E},\mathbb{A},\mathbb{T}}(\mathbf{C}), \mathbf{D}) \to \mathrm{Funct}_{!,\otimes}(\mathbf{H}_{\mathbb{E},\mathbb{A}}(\mathbf{C}), \mathbf{D}),$$

given by restriction along the functor $\mathcal{F} \mapsto \Sigma^{\infty}_{\mathbb{T}}(\mathcal{F}_+)$, is an equivalence.

This follows by combining Lemma 3.4.10 with Lemma 3.9.8.

3.10.14. Using the universal property (Lemma 3.4.10), we see that the localization functors $\mathrm{L}_{\mathbb{E}}$, $\mathrm{L}_{\mathbb{A}}$, and $\mathrm{L}_{\mathbb{E},\mathbb{A}}$ are symmetric monoidal, and can be characterized uniquely by commutativity with the functor $\Sigma^{\infty}_{\mathbb{T}}$.

4. Brave new motivic homotopy theory

In this section, we construct the brave new motivic homotopy category over any (quasi-compact quasi-separated) spectral base scheme S.

We write $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$ for the (essentially small) category of smooth spectral schemes (of finite presentation) over S.

A $\mathrm{Sm}^{\mathcal{E}_\infty}$*-fibred space* over S is a presheaf of spaces on $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$. When there is no risk of confusion we will simply say *fibred space* or even *space* over S. We write $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$ for the category of $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces over S, and $\mathrm{h_S}$ for the Yoneda embedding $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}} \hookrightarrow \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$.

4.1. Nisnevich excision.

4.1.1. Let S be an spectral scheme. A *Nisnevich square* over S is a cartesian square of spectral schemes over S

$$\begin{array}{ccc} \mathrm{U} \times_{\mathrm{X}} \mathrm{V} & \xhookrightarrow{k} & \mathrm{V} \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle p} \\ \mathrm{U} & \xhookrightarrow{j} & \mathrm{X} \end{array} \tag{4.1}$$

such that j is an open immersion, p is étale, and there exists a closed immersion $\mathrm{Z} \hookrightarrow \mathrm{X}$ complementary to j such that the induced morphism $p^{-1}(\mathrm{Z}) \to \mathrm{Z}$ is invertible.

Remark 4.1.2. Note that a commutative square of classical schemes is Nisnevich in the usual sense if and only if it induces a Nisnevich square of spectral schemes. This follows from Lemma 2.9.4.

4.1.3. We let $\mathbb{E}^{\mathrm{Nis}}_{\mathrm{S}}$ denote the set of Nisnevich squares in the category $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$. It is clear that this defines an excision structure in the sense of Definition 3.6.2.

The following technical lemma will be useful:

Lemma 4.1.4. *The excision structure $\mathbb{E}^{\mathrm{Nis}}_{\mathrm{S}}$ is* topological *in the sense of Definition 3.7.2.*

Proof. The axiom (EXC2) is clear, since every open immersion of spectral schemes is a monomorphism.

For (EXC3), let Q be a Nisnevich square of the form (4.1) and consider the induced cartesian square

$$\begin{array}{ccc} W & \xhookrightarrow{k} & V \\ \downarrow{\scriptstyle \Delta_{W/U}} & & \downarrow{\scriptstyle \Delta_{V/X}} \\ W \times_U W & \xhookrightarrow{(k,k)} & V \times_X V. \end{array}$$

It is clear that the lower horizontal morphism is an open immersion. The morphism $\Delta_{V/X}$ is étale (in fact an open immersion) because p is étale, and it is clear that it induces an isomorphism on the complementary reduced closed subscheme. □

4.1.5. We say that a $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred space $\mathcal{F}$ is *Nisnevich-local*, or satisfies *Nisnevich excision*, if the following conditions hold:

(1) The presheaf $\mathcal{F}$ is *reduced*, i.e. the space $\mathcal{F}(\varnothing)$ is contractible, where $\varnothing$ is the empty spectral scheme.

(2) For every Nisnevich square of the form (4.1), the induced commutative square of spaces

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}) \\ \downarrow & & \downarrow \\ \Gamma(V, \mathcal{F}) & \longrightarrow & \Gamma(U \times_X V, \mathcal{F}) \end{array}$$

is cartesian.

We let $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathrm{Nis}}(S)$ denote the full subcategory of $\mathrm{Spc}^{\mathcal{E}_\infty}(S)$ spanned by Nisnevich-local spaces.

4.1.6. We have:

Proposition 4.1.7.

(i) *The category* $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathrm{Nis}}(S)$ *is an accessible left localization of the category* $\mathrm{Spc}^{\mathcal{E}_\infty}(S)$*, i.e. there is a localization functor* $L_{\mathrm{Nis}} : \mathrm{Spc}^{\mathcal{E}_\infty}(S) \to \mathrm{Spc}^{\mathcal{E}_\infty}_{\mathrm{Nis}}(S)$*, left adjoint to the inclusion. In particular,* $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathrm{Nis}}(S)$ *is a presentable* ∞*-category.*

(ii) *The localization functor* $\mathcal{F} \mapsto L_{\mathrm{Nis}}(\mathcal{F})$ *is left-exact, i.e. commutes with finite limits. In particular,* $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathrm{Nis}}(S)$ *is a topos, and has universality of colimits.*

(iii) *The full subcategory* $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathrm{Nis}}(S) \subset \mathrm{Spc}^{\mathcal{E}_\infty}(S)$ *is stable under filtered colimits.*

Proof. Claims (i) and (ii) follow from Corollary 3.7.10, in view of Lemma 4.1.4.

Claim (iii) follows from Lemma 3.6.6. □

We will say that a morphism of $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces is a *Nisnevich-local equivalence* if it becomes invertible after applying the localization functor L_{Nis}.

Remark 4.1.8. Theorem 3.7.9 implies that the property of Nisnevich excision is equivalent to Čech descent with respect to the Grothendieck topology generated by Nisnevich squares. It follows from [Lur16b, Thm. 3.7.5.1] that, over quasi-compact quasi-separated spectral schemes, this Grothendieck topology coincides with the Nisnevich topology as constructed by Lurie in [Lur16b, §3.7].

Since we do not assume that our spectral schemes are noetherian and finite-dimensional, this descent condition is in general much *weaker* than the condition of hyperdescent, i.e. descent with respect to arbitrary hypercovers. We refer to [Lur09, §6.5.4] for an explanation of this distinction.

4.1.9. The following lemma follows from [Lur09, Prop. 5.5.8.10, (3)]:

Lemma 4.1.10. *Let* $(\mathrm{X}_\alpha)_\alpha$ *be a finite family of smooth spectral schemes over* S*. Then the canonical morphism of presheaves*

$$\bigsqcup_\alpha \mathrm{h_S}(\mathrm{X}_\alpha) \to \mathrm{h_S}(\bigsqcup_\alpha \mathrm{X}_\alpha)$$

is a Nisnevich-local equivalence.

By [Lur09, Lem. 5.5.8.14] it follows that the category of Nisnevich sheaves is generated under sifted colimits by the representables. In fact, we can say even more:

Lemma 4.1.11. *The category of Nisnevich sheaves is generated under sifted colimits by the representable presheaves* $\mathrm{h_S}(\mathrm{X})$*, where* $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ *is an affine spectral scheme which is smooth over* S*.*

Proof. Let $\mathrm{h_S}(\mathrm{X})$ be a representable presheaf over S. Since $\mathrm{h_S}(\mathrm{X})$ satisfies Nisnevich descent, we can assume X is separated over S, by choosing an affine Zariski cover of X where the pairwise intersections are separated. Then we repeat the same argument to assume X is affine, by choosing an affine cover where the pairwise intersections are affine. □

4.2. Homotopy invariance.

4.2.1. We say that a $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred space $\mathcal{F}$ over S satisfies $\mathbf{A}^1$*-homotopy invariance*, or is $\mathbf{A}^1$*-local*, if for every smooth spectral scheme X over S, the morphism of spaces

$$\Gamma(\mathrm{X}, \mathcal{F}) \to \Gamma(\mathrm{X} \times \mathbf{A}^1, \mathcal{F}),$$

induced by the projection $\mathrm{X} \times \mathbf{A}^1 \to \mathrm{X}$, is invertible.

Here $\mathbf{A}^1$ denotes the spectral affine line over the sphere spectrum, i.e. $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{S}\{t\})$ (2.6.10).

4.2.2. Let $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathbf{A}^1}(\mathrm{S})$ denote the full subcategory of $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$ spanned by $\mathbf{A}^1$-homotopy invariant spaces.

We have:

Lemma 4.2.3.

(i) *The category* $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathbf{A}^1}(\mathrm{S})$ *is an accessible left localization of the category* $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$*, i.e. there is a localization functor* $\mathrm{L}_{\mathbf{A}^1} : \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S}) \to \mathrm{Spc}^{\mathcal{E}_\infty}_{\mathbf{A}^1}(\mathrm{S})$*, left adjoint to the inclusion. In particular,* $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathbf{A}^1}(\mathrm{S})$ *is a presentable* ∞*-category.*

(ii) *For every* $\mathrm{Sm}^{\mathcal{E}_\infty}$*-fibred space* $\mathcal{F}$*, there is a canonical isomorphism*

$$\Gamma(\mathrm{X}, \mathrm{L}_{\mathbf{A}^1}(\mathcal{F})) \approx \varinjlim_{(\mathrm{Y}\to\mathrm{X})\in(\mathbb{A}_\mathrm{X})^{\mathrm{op}}} \Gamma(\mathrm{Y}, \mathcal{F}) \tag{4.2}$$

for each smooth spectral scheme X *over* S*. Here* $(\mathbb{A}_\mathrm{X})^{\mathrm{op}}$ *is a sifted small category, opposite to the full subcategory of* $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{X}}$ *spanned by compositions of* $\mathbf{A}^1$*-projections.*

(iii) *The localization functor* $\mathcal{F} \mapsto \mathrm{L}_{\mathbf{A}^1}(\mathcal{F})$ *commutes with finite products.*

(iv) *The category* $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathbf{A}^1}(\mathrm{S})$ *has universality of colimits.*

(v) *The full subcategory* $\mathrm{Spc}^{\mathcal{E}_\infty}_{\mathbf{A}^1}(\mathrm{S}) \subset \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$ *is stable under colimits.*

Proof. Let $\mathbb{A}$ denote the set of projections $X \times \mathbf{A}^1 \to X$ for each smooth spectral scheme X over S; note that the condition of $\mathbf{A}^1$-homotopy invariance is nothing else than $\mathbb{A}$-*invariance* in the sense of Definition 3.5.2.

Hence the claims follow from Lemma 3.5.3 and Lemma 3.5.6. □

We will say that a morphism of $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces is an $\mathbf{A}^1$-homotopy equivalence if it becomes invertible after applying the $\mathbf{A}^1$-localization functor $\mathrm{L}_{\mathbf{A}^1}$.

4.2.4. Let $p : \mathbf{A}^1 \to \mathrm{Spec}(\mathbf{S})$ denote the projection to the terminal spectral scheme. This admits two sections i_0 and i_1, the zero and unit sections, which are closed immersions of spectral schemes. They correspond to the morphisms $\mathbf{S}\{t\} \to \mathbf{S}$ sending t to 0 and 1, respectively (which are defined, as usual, up to a contractible space of choices).

Remark 4.2.5. Following [MV99, §2.3], it is possible to use the sections i_0 and i_1 to construct a cosimplicial diagram of spectral schemes given degree-wise by

$$\Delta^p_\mathrm{S} \approx \mathrm{S} \times \mathbf{A}^p \approx \mathrm{S} \times \mathrm{Spec}(\mathbf{S}\{t_1, \ldots, t_p\}).$$

for each $[p] \in \mathbf{\Delta}$.

The localization functor $\mathrm{L}_{\mathbf{A}^1}$ can then be computed by the formula

$$\Gamma(\mathrm{X}, \mathrm{L}_{\mathbf{A}^1}(\mathcal{F})) \approx \varinjlim_{[p]\in\mathbf{\Delta}^{\mathrm{op}}} \Gamma(\mathrm{X} \times \Delta^p_\mathrm{S}, \mathcal{F}). \tag{4.3}$$

For our purposes, the formula (4.2) will suffice; both indexing categories $(\mathbb{A}_\mathrm{X})^{\mathrm{op}}$ and $\mathbf{\Delta}^{\mathrm{op}}$ are *sifted*, which is the only property we need.

4.2.6. Given two morphisms $f, g : \mathcal{F} \rightrightarrows \mathcal{G}$ of presheaves on $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$, an *elementary* $\mathbf{A}^1$*-homotopy* from f to g is a morphism

$$\mathrm{h}_\mathrm{S}(\mathbf{A}^1_\mathrm{S}) \times \mathcal{F} \to \mathcal{G}$$

whose restriction to $\mathcal{F} \approx \mathrm{h}_\mathrm{S}(\mathrm{S}) \times \mathcal{F}$ along i_0 (resp. i_1) is isomorphic to f (resp. g).

We say that f and g are $\mathbf{A}^1$*-homotopic* if there exists a sequence of elementary $\mathbf{A}^1$-homotopies connecting them. In this case the induced morphisms $\mathrm{L}_{\mathbf{A}^1}(\mathcal{F}) \rightrightarrows \mathrm{L}_{\mathbf{A}^1}(\mathcal{G})$ coincide.

4.2.7. A morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is called a *strict* $\mathbf{A}^1$*-homotopy equivalence* if there exists a morphism $\psi : \mathcal{G} \to \mathcal{F}$ such that the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are $\mathbf{A}^1$-homotopic to the identities.

Note that any strict $\mathbf{A}^1$-homotopy equivalence is an $\mathbf{A}^1$-homotopy equivalence.

4.3. **Motivic spaces.**

4.3.1. We define:

Definition 4.3.2. *A* motivic space *over* S *is a* $\mathrm{Sm}^{\mathcal{E}_\infty}$*-fibred space* $\mathcal{F}$ *satisfying Nisnevich excision and* $\mathbf{A}^1$*-homotopy invariance.*

Let $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ denote the full subcategory of $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$ spanned by motivic spaces.

4.3.3. For any smooth spectral scheme X over S, we let $\mathrm{M}_\mathrm{S}(\mathrm{X}) := \mathrm{L}_{\mathrm{mot}}(\mathrm{h}_\mathrm{S}(\mathrm{X}))$. We have canonical bifunctorial isomorphisms

$$\mathrm{Maps}_{\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})}(\mathrm{M}_\mathrm{S}(\mathrm{X}), \mathcal{F}) \approx \Gamma(\mathrm{X}, \mathcal{F})$$

for every motivic space $\mathcal{F}$ over S.

4.3.4. We have:

Lemma 4.3.5.

(i) *The category* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ *is an accessible left localization of* $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$*, i.e. there is a localization functor* $\mathrm{L}_{\mathrm{mot}} : \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S}) \to \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$*, left adjoint to the inclusion. In particular,* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ *is a presentable* ∞*-category.*

(ii) *The presentable* ∞*-category* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ *admits a cartesian monoidal structure, and the localization functor* $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ *lifts to a symmetric monoidal functor (i.e., it commutes with finite products).*

(iii) *The localization functor* $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ *can be described as the transfinite composite*

$$\mathrm{L}_{\mathrm{mot}}(\mathcal{F}) \approx \varinjlim_{n \geqslant 0} (\mathrm{L}_{\mathbf{A}^1} \circ \mathrm{L}_{\mathrm{Nis}})^{\circ n}(\mathcal{F}). \tag{4.4}$$

(iv) *The full subcategory* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S}) \subset \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$ *is stable under filtered colimits.*

(v) *The category* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ *has the property of universality of colimits.*

(vi) *The category* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ *is generated under sifted colimits by fibred spaces of the form* $\mathrm{M}_{\mathrm{S}}(\mathrm{X})$*, where* $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ *is an affine spectral scheme which is smooth over* S.

Proof. Note that the category $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ is nothing else than the category $\mathbf{H}_{\mathbb{E}_{\mathrm{S}}^{\mathrm{Nis}},\mathbb{A}_{\mathrm{S}}}(\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}})$, the unstable homotopy theory associated to the Nisnevich excision structure $\mathbb{E}_{\mathrm{S}}^{\mathrm{Nis}}$ and the set of $\mathbf{A}^1$-projections $\mathrm{X} \times \mathbf{A}^1 \to \mathrm{X}$ (for X a smooth spectral S-scheme). Hence claims (i)–(v) follow from the general results collected in Sect. 3.

Claim (vi) follows directly from Lemma 4.1.11. □

We will say that a morphism of $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces is a *motivic equivalence* if it induces an isomorphism after applying the motivic localization functor $\mathrm{L}_{\mathrm{mot}}$.

4.4. **Pointed motivic spaces.**

4.4.1. Let $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$ denote the presentable ∞-category of pointed objects in $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$, and $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$ the presentable ∞-category of pointed objects in $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$.

We have:

Lemma 4.4.2.

(i) *The category* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$ *is an accessible left localization of* $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$*, i.e. there is a localization functor* $\mathrm{L}_{\mathrm{mot}} : \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet \to \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$*, left adjoint to the inclusion.*

(ii) *The presentable* ∞*-category* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$ *admits a symmetric monoidal structure. The functors* $\mathcal{F} \mapsto \mathcal{F}_+$ *and* $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ *lift to symmetric monoidal functors.*

(iii) *The localization functor* $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ *satisfies the formula*

$$\mathrm{L}_{\mathrm{mot}}(\mathcal{F}_+) \approx \mathrm{L}_{\mathrm{mot}}(\mathcal{F})_+ \tag{4.5}$$

for every space $\mathcal{F}$ *over* S.

(iv) *The category* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$ *is generated under sifted colimits by objects of the form* $\mathrm{M}_{\mathrm{S}}(\mathrm{X})_+$*, where* $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ *is an affine spectral scheme which is smooth over* S.

Proof. The claims (i)–(iii) follow from the general statements of Paragraph 3.9.

Claim (iv) follows from point (vi) of Lemma 4.3.5 and Lemma 3.3.5. □

We will write $\wedge_{\mathrm{S}}$ for the monoidal product on $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$.

4.5. **Motivic spectra.**

4.5.1. Let $\mathrm{Spt}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$ denote the stable presentable ∞-category $\mathrm{Spt}_{\mathrm{S}^1}(\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet)$ of S^1-spectrum objects in the presentable ∞-category of pointed $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces over S. (Here we view S^1 as a constant pointed presheaf.) This is equivalent to the presentable ∞-category of presheaves of spectra on $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$.

The objects of $\mathrm{Spt}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$, which we will call ($\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred) *spectra* over S, are sequences $(\mathcal{F}_n)_{n\geqslant 0}$ of pointed $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces $\mathcal{F}_n$, together with isomorphisms

$$\mathcal{F}_n \xrightarrow{\sim} \Omega_{\mathrm{S}^1}(\mathcal{F}_{n+1}).$$

4.5.2. Let $\mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$ denote the presentable ∞-category $\mathrm{Spt}_{\mathrm{S}^1}(\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet)$. This is equivalent to the presentable ∞-category of presheaves of spectra on $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$ satisfying Nisnevich excision and $\mathbf{A}^1$-homotopy invariance.

There is a canonical adjunction

$$\Sigma^\infty_{\mathrm{S}^1} : \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet \rightleftarrows \mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1} : \Omega^\infty_{\mathrm{S}^1}. \tag{4.6}$$

We have:

Lemma 4.5.3.

(i) *The category* $\mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$ *is an accessible left localization of* $\mathrm{Spt}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$*, i.e. there is a localization functor* $\mathrm{L}_{\mathrm{mot}} : \mathrm{Spt}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1} \to \mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$*, left adjoint to the inclusion. In particular it is a presentable ∞-category.*

(ii) *The presentable ∞-category* $\mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$ *is symmetric monoidal, and the functors* $\Sigma^\infty_{\mathbf{T}}$ *and* $\mathrm{L}_{\mathrm{mot}}$ *lift to symmetric monoidal functors.*

(iii) *The presentable ∞-category* $\mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$ *is stable, and the localization functor* $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ *is exact.*

(iv) *The localization functor* $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ *satisfies the formula*

$$\mathrm{L}_{\mathrm{mot}}(\Sigma^{\infty-k}_{\mathrm{S}^1}(\mathrm{h}_{\mathrm{S}}(\mathrm{X})_+)) \approx \Sigma^{\infty-k}_{\mathrm{S}^1}(\mathrm{M}_{\mathrm{S}}(\mathrm{X})_+)$$

for each smooth spectral scheme X *over* S.

(v) *The localization functor* $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{mot}}(\mathcal{F})$ *can be computed as the composite*

$$\mathrm{L}_{\mathrm{mot}} \approx \mathrm{L}_{\mathbf{A}^1} \circ \mathrm{L}_{\mathrm{Nis}}. \tag{4.7}$$

(vi) *The presentable ∞-category* $\mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$ *is generated under sifted colimits by objects of the form* $\Sigma^{\infty-n}_{\mathrm{S}^1}(\mathrm{M}_{\mathrm{S}}(\mathrm{X})_+)$*, where* $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ *is an affine spectral scheme which is smooth over* S*, and* $n \geqslant 0$.

Proof. The claims follow from the general results of Paragraph 3.10 and point (iv) of Lemma 4.4.2. □

5. Inverse and direct image functoriality

5.1. **For motivic spaces.**

5.1.1. Let $f : \mathrm{T} \to \mathrm{S}$ be a morphism of spectral schemes. The direct image functor

$$f_*^{\mathrm{Spc}} : \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{T}) \to \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S}) \tag{5.1}$$

is defined as restriction along the base change functor $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}} \to \mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{T}}$.

According to Theorem 3.1.3, its left adjoint f^*_{Spc}, the inverse image functor, is uniquely characterized by commutativity with colimits and the formula

$$f^*_{\mathrm{Spc}}(\mathrm{h}_\mathrm{S}(\mathrm{X})) \approx \mathrm{h}_\mathrm{T}(\mathrm{X} \underset{\mathrm{S}}{\times} \mathrm{T}) \tag{5.2}$$

for smooth spectral schemes X over S.

5.1.2. Note that the base change functor $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}} \to \mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{T}}$ preserves Nisnevich covering families and $\mathbf{A}^1$-projections. It follows that the inverse image functor f^*_{Spc} preserves Nisnevich-local equivalences and $\mathbf{A}^1$-homotopy equivalences.

By adjunction, its right adjoint f_*^{Spc} preserves Nisnevich sheaves and $\mathbf{A}^1$-homotopy invariant spaces and induces a functor $f_*^{\mathbf{H}} : \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{T}) \to \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$. This admits a left adjoint $f^*_{\mathbf{H}}$ given by the formula

$$f^*_{\mathbf{H}}(\mathcal{F}) \approx \mathrm{L}_{\mathrm{mot}}(f^*_{\mathrm{Spc}}(\mathcal{F})).$$

According to Lemma 3.8.9, this is characterized by commutativity with colimits and the formula

$$f^*_{\mathbf{H}}(\mathrm{M}_\mathrm{S}(\mathrm{X})) \approx \mathrm{M}_\mathrm{T}(\mathrm{X} \underset{\mathrm{S}}{\times} \mathrm{T}). \tag{5.3}$$

5.1.3. Both the direct and inverse image functors are symmetric monoidal:

Lemma 5.1.4. *The functor f_*^{Spc} (resp. $f_*^{\mathbf{H}}$) admits a canonical symmetric monoidal structure.*

Proof. Since the respective symmetric monoidal structures are cartesian, it suffices to show that f_* commutes with finite products. In fact, it commutes with arbitrary limits since it is a right adjoint. □

Lemma 5.1.5. *The functor f^*_{Spc} (resp. $f^*_{\mathbf{H}}$) admits a canonical symmetric monoidal structure.*

Proof. By adjunction from Lemma 5.1.4, we obtain a canonical structure of colax symmetric monoidal functor on f^*. That is, there are canonical morphisms

$$f^*(\mathcal{F} \underset{\mathrm{S}}{\times} \mathcal{G}) \to f^*(\mathcal{F}) \underset{\mathrm{T}}{\times} f^*(\mathcal{G}) \tag{5.4}$$

for any two spaces $\mathcal{F}$ and $\mathcal{G}$ over S. It suffices to show that these morphisms are invertible.

Since f^*_{Spc} commutes with colimits, and the cartesian product commutes with colimits in each argument, one reduces to the case of representables, in which case the claim is clear. For $f^*_{\mathbf{H}}$, the claim follows from the first because motivic localization commutes with finite products. □

5.2. For pointed motivic spaces.

5.2.1. Let $f : \mathrm{T} \to \mathrm{S}$ be a morphism of spectral schemes. Since f_*^{Spc} preserves the terminal object, it induces a functor $f_*^{\mathrm{Spc}_\bullet} : \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{T})_\bullet \to \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$ given on objects by the formula

$$f_*^{\mathrm{Spc}_\bullet}(\mathcal{G}, y) \approx (f_*^{\mathrm{Spc}}(\mathcal{G}), f_*^{\mathrm{Spc}}(y)).$$

Its left adjoint $f^*_{\mathrm{Spc}_\bullet} : \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{T})_\bullet \to \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$ is uniquely characterized, according to Lemma 3.3.5, by the fact that it commutes with sifted colimits and with the functor $\mathcal{F} \mapsto \mathcal{F}_+$:

$$f^*_{\mathrm{Spc}_\bullet}(\mathcal{F}_+) \approx f^*_{\mathrm{Spc}}(\mathcal{F})_+ \tag{5.5}$$

for any space $\mathcal{F}$ over S.

Explicitly, it is given on objects by the formula

$$f^*_{\mathrm{Spc}_\bullet}(\mathcal{F}, x) \approx (f^*_{\mathrm{Spc}}(\mathcal{F}), f^*_{\mathrm{Spc}}(x))$$

for each pointed space $(\mathcal{F}, x)$ over S.

5.2.2. The direct image functor $f^{\mathrm{Spc}_\bullet}_*$ preserves the properties of Nisnevich descent and $\mathbf{A}^1$-homotopy invariance, and induces a functor $f^{\mathbf{H}}_*$. Its left adjoint $f^*_{\mathbf{H}_\bullet}$ is given by composing $f^*_{\mathrm{Spc}_\bullet}$ with the motivic localization functor:

$$f^*_{\mathbf{H}_\bullet} := \mathrm{L}_{\mathrm{mot}} f^*_{\mathrm{Spc}_\bullet}. \tag{5.6}$$

It is uniquely characterized, according to Lemma 3.3.5, by commutativity with sifted colimits and the formula

$$f^*_{\mathbf{H}_\bullet}(\mathcal{F}_+) \approx f^*_{\mathbf{H}}(\mathcal{F})_+. \tag{5.7}$$

5.2.3.

Lemma 5.2.4. *The inverse image functor $f^*_{\mathrm{Spc}_\bullet}$ (resp. $f^*_{\mathbf{H}_\bullet}$) admits a canonical symmetric monoidal structure.*

Proof. For $f^*_{\mathrm{Spc}_\bullet}$, this follows directly from the universal property of Lemma 3.3.10 and the formula (5.5). For $f^*_{\mathbf{H}_\bullet}$ it follows from the universal property of Lemma 3.9.8 and the formula (5.7). □

5.3. For motivic spectra.

5.3.1. Let $f : \mathrm{T} \to \mathrm{S}$ be a morphism of spectral schemes.

By construction of $\mathrm{Spt}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$, there exists a unique functor $f^*_{\mathrm{Spt}} : \mathrm{Spt}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1} \to \mathrm{Spt}^{\mathcal{E}_\infty}(\mathrm{T})_{\mathrm{S}^1}$ which commutes with colimits and with the functor $\Sigma^\infty_{\mathrm{S}^1}$, i.e.:

$$f^*_{\mathrm{Spt}} \Sigma^\infty_{\mathrm{S}^1} \approx \Sigma^\infty_{\mathrm{S}^1} f^*_{\mathrm{Spc}_\bullet}. \tag{5.8}$$

5.3.2. Let f^{Spt}_* be the right adjoint of f^*_{Spt}. This can be described as the unique functor which commutes with limits and with the functor Ω^∞, i.e.:

$$\Omega^\infty f^{\mathrm{Spt}}_* \approx f^{\mathrm{Spc}_\bullet}_* \Omega^\infty \tag{5.9}$$

It is given on objects by the assignment

$$\mathbb{E} = (\mathcal{F}_n)_n \mapsto f_*(\mathbb{E}) = (f^{\mathrm{Spc}_\bullet}_*(\mathcal{F}_n))_n.$$

5.3.3. The direct image functor f^{Spt}_* preserves motivic spectra and induces a functor $f^{\mathbf{SH}}_*$.

We let $f^*_{\mathbf{SH}}$ be its left adjoint, the symmetric monoidal functor $\mathrm{L}_{\mathrm{mot}} f^*_{\mathrm{Spt}}$. This is the unique functor which commutes with colimits and with the functor $\Sigma^\infty_{\mathrm{S}^1}$, i.e.:

$$f^*_{\mathbf{SH}} \Sigma^\infty_{\mathrm{S}^1} \approx \Sigma^\infty_{\mathrm{S}^1} f^*_{\mathbf{H}_\bullet}. \tag{5.10}$$

5.3.4. Using the universal properties of Lemma 3.4.10 and Lemma 3.10.13, we get:

Lemma 5.3.5. *The functor f^*_{Spt} (resp. $f^*_{\mathbf{SH}}$) admits a canonical symmetric monoidal structure.*

6. Functoriality along smooth morphisms

As per our conventions, smooth morphisms will be assumed to be of finite presentation.

6.1. The functor $p_\sharp$.

6.1.1. Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism of spectral schemes. In this case the base change functor admits a right adjoint, the forgetful functor $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{X}} \to \mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$:

$$(\mathrm{Y} \to \mathrm{X}) \mapsto (\mathrm{Y} \to \mathrm{X} \xrightarrow{p} \mathrm{S}).$$

It follows that the functor p^*_{Spc} coincides with restriction along the forgetful functor, and admits a left adjoint

$$p^{\mathrm{Spc}}_\sharp : \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{T}) \to \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S}).$$

This is uniquely characterized, according to Theorem 3.1.3, by commutativity with colimits and the formula

$$p^{\mathrm{Spc}}_\sharp(\mathrm{h}_\mathrm{X}(\mathrm{Y})) \approx \mathrm{h}_\mathrm{S}(\mathrm{Y}). \tag{6.1}$$

for smooth spectral schemes Y over X.

6.1.2. Since the forgetful functor $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{X}} \to \mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}}$ preserves Nisnevich covering families and $\mathbf{A}^1$-projections, it follows that $p^{\mathrm{Spc}}_\sharp$ preserves Nisnevich-local equivalences and $\mathbf{A}^1$-homotopy equivalences.

In particular its right adjoint p^*_{Spc} preserves Nisnevich descent and $\mathbf{A}^1$-homotopy invariance, and induces a morphism $p^*_{\mathbf{H}}$ on motivic spaces.

Its left adjoint $p^{\mathbf{H}}_\sharp$ is given by applying $p^{\mathrm{Spc}}_\sharp$ and then the localization functor $\mathrm{L}_{\mathrm{mot}}$:

$$p^{\mathbf{H}}_\sharp(\mathcal{F}) \approx \mathrm{L}_{\mathrm{mot}}(p^{\mathrm{Spc}}_\sharp(\mathcal{F})).$$

It is uniquely characterized, according to Lemma 3.8.9, by commutativity with colimits and the formula

$$p^{\mathbf{H}}_\sharp(\mathrm{M}_\mathrm{X}(\mathrm{Y})) \approx \mathrm{M}_\mathrm{S}(\mathrm{Y}), \tag{6.2}$$

for smooth spectral schemes Y over X.

6.1.3. Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. By Lemma 5.1.5, the functors p^*_{Spc} and $p^*_{\mathbf{H}}$ admit canonical symmetric monoidal structures, so that their respective left adjoints $p^{\mathrm{Spc}}_\sharp$ and $p^{\mathbf{H}}_\sharp$ admit colax symmetric monoidal structures.

If p is an *open immersion*, then these monoidal structures are strict:

Lemma 6.1.4. *Let $j : \mathrm{U} \hookrightarrow \mathrm{X}$ be a quasi-compact open immersion. Then the canonical colax symmetric monoidal structure on the functor $j^{\mathrm{Spc}}_\sharp$ (resp. $j^{\mathbf{H}}_\sharp$) is strict.*

Proof. It suffices to show that the canonical morphisms

$$j^{\mathrm{Spc}}_\sharp(\mathcal{F} \underset{\mathrm{U}}{\times} \mathcal{G}) \to j^{\mathrm{Spc}}_\sharp(\mathcal{F}) \underset{\mathrm{S}}{\times} j^{\mathrm{Spc}}_\sharp(\mathcal{G})$$

are invertible for all spaces $\mathcal{F}$ and $\mathcal{G}$ on U. Since $j^{\mathrm{Spc}}_\sharp$ commutes with colimits, and the cartesian product commutes with colimits in each argument, one reduces to the case of representables.

Then the claim follows from the fact that the fibred products $\mathrm{X} \times_\mathrm{U} \mathrm{Y}$ and $\mathrm{X} \times_\mathrm{S} \mathrm{Y}$ are canonically identified (since j is a monomorphism, i.e. its diagonal morphism is invertible). The claim for $j^{\mathbf{H}}_\sharp$ follows from the fact that motivic localization commutes with finite products. ☐

6.1.5. Let $(f_\alpha : \mathrm{S}_\alpha \to \mathrm{S})_\alpha$ be a Nisnevich covering family. Given a morphism of motivic spaces over S, the following proposition says that it is invertible if and only if its inverse image on each S_α is invertible:

Proposition 6.1.6 (Nisnevich separation)**.** *Let* S *be a spectral scheme. For any Nisnevich covering family* $(p_\alpha : \mathrm{S}_\alpha \to \mathrm{S})_\alpha$*, the family of inverse image functors* $(p_\alpha)^*_{\mathbf{H}} : \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S}) \to \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S}_\alpha)$ *is conservative.*

This is in fact true at the level of Nisnevich sheaves, which is what we will prove.

Proof. Let $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of Nisnevich sheaves on S, and suppose that the following condition holds:

(∗) For each α, the morphism $(p_\alpha)^*_{\mathrm{Nis}}(\mathcal{F}_1) \to (p_\alpha)^*_{\mathrm{Nis}}(\mathcal{F}_2)$ is invertible.

The claim is that under this assumption, the morphism

$$\Gamma(\mathrm{X}, \mathcal{F}_1) \to \Gamma(\mathrm{X}, \mathcal{F}_2)$$

is invertible for every smooth spectral S-scheme X.

Since $\mathcal{F}_i$ satisfy Nisnevich descent, it suffices to show that the morphism

$$\Gamma(\mathrm{X}_\alpha, \mathcal{F}_1) \to \Gamma(\mathrm{X}_\alpha, \mathcal{F}_2) \tag{6.3}$$

is invertible for each α, where X_α is the base change of X along p_α.

Since $\mathrm{h}_\mathrm{S}(\mathrm{X}_\alpha) \approx (p_\alpha)_\sharp (p_\alpha)^*(\mathrm{h}_\mathrm{S}(\mathrm{X}))$, we have by adjunction

$$\Gamma(\mathrm{X}_\alpha, \mathcal{F}_i) \approx \Gamma(\mathrm{X}, (p_\alpha)_\sharp (p_\alpha)^* \mathcal{F}_i)$$

for each α and i.

Hence the claim follows from the assumption (∗). □

6.1.7. As in Paragraph 5.2, the functor $p_\sharp$ extends immediately to pointed spaces.

That is, we obtain functors

$$p_\sharp^{\mathrm{Spc}_\bullet} : \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{X})_\bullet \to \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet,$$
$$p_\sharp^{\mathbf{H}_\bullet} : \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{X})_\bullet \to \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$$

left adjoint to $p^*_{\mathrm{Spc}_\bullet}$ and $p^*_{\mathbf{H}_\bullet}$, respectively. These are uniquely characterized by commutativity with colimits and the formulas

$$p_\sharp^{\mathrm{Spc}_\bullet}(\mathrm{h}_\mathrm{X}(\mathrm{Y})_+) \approx \mathrm{h}_\mathrm{S}(\mathrm{Y})_+, \qquad p_\sharp^{\mathbf{H}_\bullet}(\mathrm{M}_\mathrm{X}(\mathrm{Y})_+) \approx \mathrm{M}_\mathrm{S}(\mathrm{Y})_+. \tag{6.4}$$

6.1.8. Similarly, as in Paragraph 5.3, the functor $p_\sharp$ extends immediately to spectra.

That is, we obtain functors

$$p_\sharp^{\mathrm{Spt}} : \mathrm{Spt}^{\mathcal{E}_\infty}(\mathrm{X})_{\mathrm{S}^1} \to \mathrm{Spt}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1},$$
$$p_\sharp^{\mathbf{SH}} : \mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{X})_{\mathrm{S}^1} \to \mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$$

left adjoint to p^*_{Spt} and $p^*_{\mathbf{SH}}$, respectively. These are uniquely characterized by commutativity with colimits and the formulas

$$p_\sharp^{\mathrm{Spt}}(\Sigma^\infty_{\mathrm{S}^1} \mathrm{h}_\mathrm{X}(\mathrm{Y})_+) \approx \Sigma^\infty_{\mathrm{S}^1} \mathrm{h}_\mathrm{S}(\mathrm{Y})_+, \qquad p_\sharp^{\mathbf{SH}}(\Sigma^\infty_{\mathrm{S}^1} \mathrm{M}_\mathrm{X}(\mathrm{Y})_+) \approx \Sigma^\infty_{\mathrm{S}^1} \mathrm{M}_\mathrm{S}(\mathrm{Y})_+. \tag{6.5}$$

6.2. Smooth base change formulas.

6.2.1. Suppose we have a cartesian square

$$\begin{array}{ccc} \mathrm{T}' & \xrightarrow{f'} & \mathrm{S}' \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \mathrm{T} & \xrightarrow{f} & \mathrm{S} \end{array}$$

of spectral schemes.

At the level of (motivic) spaces, pointed spaces, and spectra, there are canonical 2-morphisms

$$(p')_\sharp (f')^* \to f^* p_\sharp, \tag{6.6}$$

$$p^* f_* \to (f')_* (p')^*, \tag{6.7}$$

constructed in Paragraph A.4.

The following says that $\mathrm{Spc}^{\mathcal{E}_\infty}$ and $\mathbf{H}^{\mathcal{E}_\infty}$ satisfy the *left base change property* along smooth morphisms (see *loc. cit.*):

Proposition 6.2.2. *If p and p' are smooth, then the 2-morphisms (6.6) and (6.7) are invertible at the level of spaces and motivic spaces.*

Proof. It suffices to consider (6.6); the morphism (6.7) is its right transpose.

At the level of fibred spaces, we note that the functors in question commute with colimits, so that we may reduce to representable spaces, in which case the claim is obvious.

Similarly, for motivic spaces we may reduce to the case of motivic localizations of representable spaces. □

6.2.3. Next we consider the case of pointed spaces. Then we have:

Proposition 6.2.4. *If p and p' are smooth, then the 2-morphisms (6.6) and (6.7) are invertible at the level of pointed spaces and pointed motivic spaces.*

Proof. By transposition it suffices to consider (6.6). Since the functors in question commute with colimits and with the functor $\mathcal{F} \mapsto \mathcal{F}_+$, the claim follows from Lemma 3.3.5 and smooth base change for unpointed spaces (Proposition 6.2.2). □

6.2.5. At the level of spectra, we have:

Proposition 6.2.6. *If p and p' are smooth, then the 2-morphisms (6.6) and (6.7) are invertible at the level of spectra and motivic spectra.*

Proof. This follows from Lemma 3.4.6 and smooth base change for pointed spaces (Proposition 6.2.4). □

6.3. **Smooth projection formulas.** Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism.

As recalled in Definition A.4.6, there are canonical morphisms

$$p_\sharp^{\mathrm{Spc}}(\mathcal{F} \times p^*_{\mathrm{Spc}}(\mathcal{G})) \to p_\sharp^{\mathrm{Spc}}(\mathcal{F}) \times \mathcal{G} \tag{6.8}$$

for any pair of $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces $\mathcal{F} \in \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{X})$ and $\mathcal{G} \in \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$; by adjunction these correspond to the canonical morphisms

$$\mathcal{F} \times p^*_{\mathrm{Spc}}(\mathcal{G}) \to p^*_{\mathrm{Spc}} p_\sharp^{\mathrm{Spc}}(\mathcal{F}) \times p^*_{\mathrm{Spc}}(\mathcal{G}) \xrightarrow{\sim} p^*_{\mathrm{Spc}}(p_\sharp^{\mathrm{Spc}}(\mathcal{F}) \times \mathcal{G})$$

induced by the counit of the adjunction $(p_\sharp^{\mathrm{Spc}}, p^*_{\mathrm{Spc}})$ and the monoidality of the functor p^*_{Spc}.

By right transposition we also obtain two canonical morphisms

$$p^*_{\mathrm{Spc}}\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) \to \underline{\mathrm{Hom}}(p^*_{\mathrm{Spc}}\mathcal{F}, p^*_{\mathrm{Spc}}\mathcal{G}) \tag{6.9}$$

for $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces $\mathcal{F},\mathcal{G} \in \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$ and

$$\underline{\mathrm{Hom}}(p^{\mathrm{Spc}}_{\sharp}\mathcal{F},\mathcal{G}) \to p^{\mathrm{Spc}}_{*}\underline{\mathrm{Hom}}(\mathcal{F}, p^*_{\mathrm{Spc}}\mathcal{G}) \tag{6.10}$$

for $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces $\mathcal{F} \in \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{X})$ and $\mathcal{G} \in \mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$.

The following verifies the *left projection formula* for $\mathrm{Spc}^{\mathcal{E}_\infty}$ along smooth morphisms, in the sense of Paragraph A.4:

Proposition 6.3.1.

(i) *The canonical morphism (6.8) is invertible for all* $\mathrm{Sm}^{\mathcal{E}_\infty}$*-fibred spaces* $\mathcal{F}$ *over* X *and* $\mathcal{G}$ *over* S*.*

(ii) *The canonical morphism (6.9) is invertible for all* $\mathrm{Sm}^{\mathcal{E}_\infty}$*-fibred spaces (resp. motivic spaces)* $\mathcal{F}$ *and* $\mathcal{G}$ *over* S*.*

(iii) *The canonical morphism (6.10) is invertible for all* $\mathrm{Sm}^{\mathcal{E}_\infty}$*-fibred spaces* $\mathcal{F}$ *over* X *and* $\mathcal{G}$ *over* S*.*

Proof. The claims (ii) and (iii) follow by adjunction from (i). To show that the canonical morphism (6.8) is invertible, we may reduce to the case where the spaces $\mathcal{F}$ and $\mathcal{G}$ are representable, since the functions involved commute with colimits; in this case the claim is clear. □

As recalled in Paragraph A.5, the symmetric monoidal functor p^*_{Spc} endows $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{X})$ with a structure of $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$-module category, and the statement of Proposition 6.3.1 is equivalent to the following:

Corollary 6.3.2. *The functor* $p^{\mathrm{Spc}}_{\sharp}$ *lifts to a morphism of* $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$*-module categories.*

6.3.3. Next we verify the corresponding statement at the level of *motivic* spaces:

Proposition 6.3.4.

(i) *The canonical morphism*

$$p^{\mathbf{H}}_{\sharp}(\mathcal{F} \times p^*_{\mathbf{H}}(\mathcal{G})) \to p^{\mathbf{H}}_{\sharp}(\mathcal{F}) \times \mathcal{G} \tag{6.11}$$

is invertible for all motivic spaces $\mathcal{F}$ *over* X *and* $\mathcal{G}$ *over* S*.*

(ii) *The canonical morphism*

$$p^*_{\mathbf{H}}\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) \to \underline{\mathrm{Hom}}(p^*_{\mathbf{H}}\mathcal{F}, p^*_{\mathbf{H}}\mathcal{G}) \tag{6.12}$$

is invertible for all motivic spaces $\mathcal{F}$ *and* $\mathcal{G}$ *over* S*.*

(iii) *The canonical morphism*

$$\underline{\mathrm{Hom}}(p^{\mathbf{H}}_{\sharp}\mathcal{F},\mathcal{G}) \to p^{\mathbf{H}}_{*}\underline{\mathrm{Hom}}(\mathcal{F}, p^*_{\mathbf{H}}\mathcal{G}) \tag{6.13}$$

is invertible for all motivic spaces $\mathcal{F}$ *over* X *and* $\mathcal{G}$ *over* S*.*

Proof. As above, we only need to show claim (i), and only for $\mathcal{F}$ and $\mathcal{G}$ which are motivic localizations of representable spaces; in this case the claim is clear using the formula (6.2) and the fact that motivic localization commutes with finite products (Lemma 4.3.5). □

As above, we also have the following reformulation:

Corollary 6.3.5. *The functor* $p^{\mathbf{H}}_{\sharp}$ *lifts to a morphism of* $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$*-module categories.*

6.3.6. The following slightly more general formula, proved in exactly the same way, will also be useful:

Lemma 6.3.7. *Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. Let $\mathcal{F}$ be a $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred space (resp. motivic space) over* X*, and $\mathcal{G} \to \mathcal{G}'$ a morphism of $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces (resp. motivic spaces) over* S*. Then there is a canonical isomorphism*

$$p_\sharp(\mathcal{F} \underset{p^*(\mathcal{G}')}{\times} p^*(\mathcal{G})) \xrightarrow{\sim} p_\sharp(\mathcal{F}) \underset{\mathcal{G}'}{\times} \mathcal{G} \tag{6.14}$$

of $\mathrm{Sm}^{\mathcal{E}_\infty}$-fibred spaces (resp. motivic spaces) over S.

6.3.8. At the level of pointed motivic spaces, we get:

Proposition 6.3.9.

(i) *The canonical morphism*

$$p_\sharp^{\mathbf{H}_\bullet}(\mathcal{F} \wedge_{\mathrm{X}} p^*_{\mathbf{H}_\bullet}(\mathcal{G})) \to p_\sharp^{\mathbf{H}_\bullet}(\mathcal{F}) \wedge_{\mathrm{S}} \mathcal{G} \tag{6.15}$$

is invertible for all pointed motivic spaces $\mathcal{F}$ over X *and $\mathcal{G}$ over* S.

(ii) *The canonical morphism*

$$p^*_{\mathbf{H}_\bullet}\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}(p^*_{\mathbf{H}_\bullet}\mathcal{F}, p^*_{\mathbf{H}_\bullet}\mathcal{G}) \tag{6.16}$$

is invertible for all pointed motivic spaces $\mathcal{F}$ and $\mathcal{G}$ over S.

(iii) *The canonical morphism*

$$\underline{\mathrm{Hom}}(p_\sharp^{\mathbf{H}_\bullet}\mathcal{F}, \mathcal{G}) \to p_*^{\mathbf{H}_\bullet}\underline{\mathrm{Hom}}(\mathcal{F}, p^*_{\mathbf{H}_\bullet}\mathcal{G}) \tag{6.17}$$

is invertible for all pointed motivic spaces $\mathcal{F}$ over X *and $\mathcal{G}$ over* S.

Proof. This follows from Lemma 3.3.5 and the smooth projection formula for unpointed spaces (Proposition 6.3.4). □

As above, we have the equivalent reformulation:

Corollary 6.3.10. *The functor $p_\sharp^{\mathbf{H}_\bullet}$ lifts to a morphism of $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$-module categories.*

6.3.11. At the level of motivic spectra, we get:

Proposition 6.3.12.

(i) *The canonical morphism*

$$p_\sharp^{\mathbf{SH}}(\mathbb{E} \otimes_{\mathrm{X}} p^*_{\mathbf{SH}}(\mathbb{E}')) \to p_\sharp^{\mathbf{SH}}(\mathbb{E}) \otimes_{\mathrm{S}} \mathbb{E}' \tag{6.18}$$

is invertible for all motivic spectra $\mathbb{E}$ over X *and $\mathcal{G}$ over* S.

(ii) *The canonical morphism*

$$p^*_{\mathbf{SH}}\underline{\mathrm{Hom}}(\mathbb{E}, \mathbb{E}') \to \underline{\mathrm{Hom}}(p^*_{\mathbf{SH}}\mathbb{E}, p^*_{\mathbf{SH}}\mathbb{E}') \tag{6.19}$$

is invertible for all motivic spectra $\mathbb{E}$ and $\mathbb{E}'$ over S.

(iii) *The canonical morphism*

$$\underline{\mathrm{Hom}}(p_\sharp^{\mathbf{SH}}\mathbb{E}, \mathbb{E}') \to p_*^{\mathbf{SH}}\underline{\mathrm{Hom}}(\mathbb{E}, p^*_{\mathbf{SH}}\mathbb{E}') \tag{6.20}$$

is invertible for all motivic spectra $\mathbb{E}$ over X *and $\mathbb{E}'$ over* S.

Proof. This follows from Lemma 3.4.6 and the smooth projection formula for pointed motivic spaces (Proposition 6.3.9). □

As above, we have the equivalent reformulation:

Corollary 6.3.13. *The functor* $p_\sharp^{\mathbf{SH}}$ *lifts to a morphism of* $\mathbf{SH}(\mathrm{S})_{\mathrm{S}^1}$*-module categories.*

7. Functoriality along closed immersions

In this section we study some properties of the functor i_* of direct image along a closed immersion i. In particular it turns out to admit a right adjoint $i^!$. We also state the localization theorem and deduce some of its interesting consequences, which include base change and projection formulas involving $i^!$.

7.1. The exceptional inverse image functor $i^!$.

7.1.1. Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion. Note that if the base change functor $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}} \to \mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{Z}}$ were topologically cocontinuous (see Paragraph B.2), then the direct image functor i_* on Nisnevich sheaves would commute with arbitrary small colimits. Though this is not quite true, it turns out that the weaker condition of *quasi-cocontinuity* (see *loc. cit.*) holds, which implies that i_* commutes with contractible colimits:

Theorem 7.1.2. *Let* $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ *be a closed immersion. Then the direct image functor* $i_*^{\mathbf{H}}$ *commutes with contractible colimits.*

Proof. By Lemma B.2.6 it suffices to show that the base change functor $\mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{S}} \to \mathrm{Sm}^{\mathcal{E}_\infty}_{/\mathrm{Z}}$ is topologically quasi-cocontinuous in the sense of Lemma B.2.2. This amounts to the following:

($*$) For any smooth spectral S-scheme X and any Nisnevich covering sieve R' of X_Z, the sieve R of X generated by morphisms $\mathrm{X}' \to \mathrm{X}$ such that either (i) the empty sieve on X'_Z is Nisnevich-covering, or (ii) $\mathrm{X}'_\mathrm{Z} \to \mathrm{X}_\mathrm{Z}$ factors through R', is Nisnevich-covering.

This condition follows directly from Proposition 2.12.2, which says that étale morphisms can be lifted (Zariski-locally) along i. □

At the level of pointed spaces or spectra, we get:

Corollary 7.1.3. *The direct image functor* $i_*^{\mathbf{H}_\bullet}$ *(resp.* $i_*^{\mathbf{SH}}$*) commutes with small colimits.*

Proof. It suffices to show that i_* commutes with contractible colimits and preserves the initial object. Indeed, given any diagram indexed on an ∞-category $\mathbf{I}$, the initial object defines a cone over this diagram, which is a new diagram that is indexed on a *contractible* ∞-category and which evidently has the same colimit.

On pointed spaces or spectra, it is obvious that i_* preserves the initial object (= terminal object). Hence it suffices to note that i_* commutes with contractible colimits, which follows directly from the unpointed statement (Theorem 7.1.2). □

By the adjoint functor theorem we obtain a right adjoint $i^!_{\mathbf{H}_\bullet}$ (resp. $i^!_{\mathbf{SH}}$), called the *exceptional inverse image* functor.

7.2. The localization theorem.

In this paragraph, we formulate the statement of the localization theorem and explore several of its consequences. The proof will be deferred to the next section.

Throughout the paragraph we will work in the category of motivic spaces, and will omit the decoration $\mathbf{H}^{\mathcal{E}_\infty}$ from the notation for simplicity.

7.2.1. Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of spectral schemes with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. We deduce some immediate consequences of smooth base change in this situation.

Considering the commutative square

$$\begin{array}{ccc} \mathrm{U} & = & \mathrm{U} \\ \| & & \downarrow{\scriptstyle j} \\ \mathrm{U} & \xhookrightarrow{j} & \mathrm{S}, \end{array}$$

which is cartesian because j is a monomorphism, we get:

Lemma 7.2.2. *For any quasi-compact open immersion $j : \mathrm{U} \hookrightarrow \mathrm{S}$, the canonical morphisms*

$$\mathrm{id} \to j^* j_\sharp, \tag{7.1}$$

$$j^* j_* \to \mathrm{id}, \tag{7.2}$$

are invertible.

In other words, the functors $j_\sharp$ and j_* are fully faithful.

7.2.3. Considering the cartesian square

$$\begin{array}{ccc} \varnothing & \hookrightarrow & \mathrm{Z} \\ \downarrow & & \downarrow{\scriptstyle i} \\ \mathrm{U} & \xhookrightarrow{j} & \mathrm{S}, \end{array}$$

we get:

Lemma 7.2.4. *For any closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$, the canonical morphisms*

$$\varnothing_{\mathrm{Z}} \to i^* j_\sharp(\mathcal{F}_{\mathrm{U}}),$$
$$j^* i_*(\mathcal{F}_{\mathrm{Z}}) \to \mathrm{pt}_{\mathrm{U}},$$

are invertible, for $\mathcal{F}_{\mathrm{U}}$ (resp. $\mathcal{F}_{\mathrm{Z}}$) a motivic space over U *(resp.* Z*).*

7.2.5. Consider the canonical commutative square

$$\begin{array}{ccc} j_\sharp j^*(\mathcal{F}) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ j_\sharp j^* i_* i^*(\mathcal{F}) & \longrightarrow & i_* i^*(\mathcal{F}) \end{array}$$

for any motivic space $\mathcal{F}$ over S.

By Lemma 7.2.4 this induces a canonical commutative square

$$\begin{array}{ccc} j_\sharp j^*(\mathcal{F}) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ j_\sharp(\mathrm{pt}_{\mathrm{U}}) & \longrightarrow & i_* i^*(\mathcal{F}) \end{array} \tag{7.3}$$

which we call the *localization square* associated to the pair (i, j).

The main result in this paper is the following, due to [MV99] in the setting of classical algebraic geometry:

Theorem 7.2.6 (Localization)**.** *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of spectral schemes with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. Then for every motivic space $\mathcal{F}$ over* S*, the localization square (7.3) is cocartesian.*

The proof will occupy Sect. 8.

7.2.7. We can deduce from Theorem 7.2.6 a pointed version:

Corollary 7.2.8. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of spectral schemes with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. For any pointed motivic space $(\mathcal{F}, s)$ over S, there is a canonical cofibre sequence*

$$j_\sharp j^*(\mathcal{F}, s) \to (\mathcal{F}, s) \to i_* i^*(\mathcal{F}, s). \tag{7.4}$$

and dually, a canonical fibre sequence

$$i_* i^!(\mathcal{F}, s) \to (\mathcal{F}, s) \to j_* j^*(\mathcal{F}, s) \tag{7.5}$$

of motivic spaces over S.

Proof. We want to show that the commutative square of pointed motivic spaces

$$\begin{array}{ccc} j_\sharp j^*(\mathcal{F}, x) & \longrightarrow & (\mathcal{F}, x) \\ \downarrow & & \downarrow \\ \mathrm{pt}_{\mathrm{S}}^{\mathrm{Spc}_\bullet} & \longrightarrow & i_* i^*(\mathcal{F}, x) \end{array}$$

is cocartesian.

Since the forgetful functor $(\mathcal{F}, x) \mapsto \mathcal{F}$ reflects contractible colimits (Lemma 3.3.4), it suffices to show that the induced square of underlying motivic spaces

$$\begin{array}{ccc} j_\sharp j^*\mathcal{F} \sqcup_{j_\sharp j^*(\mathrm{pt}_{\mathrm{S}})} \mathrm{pt}_{\mathrm{S}} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathrm{pt}_{\mathrm{S}}^{\mathbf{H}} & \longrightarrow & i_* i^*\mathcal{F}. \end{array}$$

is cocartesian.

Consider the composite square

$$\begin{array}{ccccc} j_\sharp j^*\mathcal{F} & \longrightarrow & (j_\sharp j^*\mathcal{F}) \sqcup_{j_\sharp j^*(\mathrm{pt}_{\mathrm{S}})} \mathrm{pt}_{\mathrm{S}} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow \\ j_\sharp j^*(\mathrm{pt}_{\mathrm{S}}) & \longrightarrow & \mathrm{pt}_{\mathrm{S}} & \longrightarrow & i_* i^*\mathcal{F}. \end{array}$$

which is cocartesian by Theorem 7.2.6.

Since the left-hand square is evidently cocartesian, it follows that the right-hand square is also cocartesian. □

7.2.9. Similarly we also deduce localization for motivic S^1-spectra:

Corollary 7.2.10. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion, with quasi-compact open complement $j : \mathrm{U} \hookrightarrow \mathrm{S}$. For any motivic S^1-spectrum $\mathbb{E}$ over S, there are canonical exact triangles*

$$j_\sharp j^*(\mathbb{E}) \to \mathbb{E} \to i_* i^*(\mathbb{E}), \tag{7.6}$$

$$i_* i^!(\mathbb{E}) \to \mathbb{E} \to j_* j^*(\mathbb{E}), \tag{7.7}$$

of motivic S^1-spectra over S.

Proof. It suffices to show the first sequence is a cofibre sequence. Since the functors in question commute with small colimits, Lemma 4.5.3 allows us to the reduce to the case of pointed motivic spaces, which is Corollary 7.2.8. □

7.3. A reformulation of localization.

7.3.1. An immediate corollary of Theorem 7.2.6 is:

Corollary 7.3.2. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of spectral schemes with quasi-compact open complement. Then the direct image functor $i_*^{\mathbf{H}}$ (resp. $i_*^{\mathbf{H}\bullet}$, $i_*^{\mathbf{SH}}$) is fully faithful, and its essential image is spanned by motivic spaces $\mathcal{F}$ for which the restriction $j_{\mathbf{H}}^*(\mathcal{F})$ (resp. $j_{\mathbf{H}_\bullet}^*(\mathcal{F})$, $j_{\mathbf{SH}}^*(\mathcal{F})$) is contractible.*

Proof. The claims for $i_*^{\mathbf{H}\bullet}$ and $i_*^{\mathbf{SH}}$ follow directly from that of $i_*^{\mathbf{H}}$.

First we show that $i_*^{\mathbf{H}}$ is fully faithful. Considering the localization square for $i_*^{\mathbf{H}}(\mathcal{F})$, we see that the co-unit morphism $i_*^{\mathbf{H}} i_{\mathbf{H}}^* i_*^{\mathbf{H}} \to i_*^{\mathbf{H}}$ is invertible. By a standard argument it therefore suffices to show that $i_*^{\mathbf{H}}$ is conservative.

For this, let $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of motivic spaces over Z such that $i_*^{\mathbf{H}}(\varphi)$ is invertible. To show that φ is invertible, it suffices to show that

$$\Gamma(\mathrm{X}, \mathcal{F}_1) \to \Gamma(\mathrm{X}, \mathcal{F}_2)$$

is invertible for each smooth spectral Z-scheme X.

By Proposition 2.12.2, we may assume that X is the base change of a smooth spectral S-scheme Y. In this case the claim follows by assumption, since $\Gamma(\mathrm{X}, \mathcal{F}_k) \approx \Gamma(\mathrm{Y}, i_*^{\mathbf{H}}(\mathcal{F}_k))$ for each k, by adjunction.

Next we consider the claim about the essential image. We already know from Lemma 7.2.4 that, for any object $\mathcal{F}$ in the essential image, the restriction $j_{\mathbf{H}}^*(\mathcal{F})$ is contractible. Conversely, given a motivic space $\mathcal{F}$ over S, as soon as $j_{\mathbf{H}}^*(\mathcal{F})$ is contractible, we obtain from the localization square that the canonical morphism

$$\mathcal{F} \to i_*^{\mathbf{H}} i_{\mathbf{H}}^*(\mathcal{F})$$

is invertible. □

7.4. Nilpotent invariance.

A particularly important consequence of the localization theorem will be formulated in this paragraph.

7.4.1. Let S be a spectral scheme. The canonical morphism $i : \mathrm{S}_{\mathrm{cl}} \hookrightarrow \mathrm{S}$ is a closed immersion with empty complement $j : \varnothing \hookrightarrow \mathrm{S}$. In this situation, the localization theorem immediately gives:

Corollary 7.4.2 (Nilpotent invariance). *Let S be a spectral scheme. Then the adjunctions*

$$i^* : \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S}) \rightleftarrows \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S}_{\mathrm{cl}}) : i_*,$$
$$i^* : \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet \rightleftarrows \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S}_{\mathrm{cl}})_\bullet : i_*,$$
$$i^* : \mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1} \rightleftarrows \mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S}_{\mathrm{cl}})_{\mathrm{S}^1} : i_*$$

are equivalences of categories.

Proof. This follows immediately from Corollary 7.3.2. □

Note that the same argument also applies to the closed immersion $(\mathrm{S}_{\mathrm{cl}})_{\mathrm{red}} \hookrightarrow \mathrm{S}_{\mathrm{cl}}$, so we also have invariance with respect to classical nilpotents.

7.5. Closed base change formula.

7.5.1. Consider a cartesian square

$$\begin{array}{ccc} X_Z & \xhookrightarrow{k} & X \\ \downarrow g & & \downarrow f \\ Z & \xhookrightarrow{i} & S, \end{array} \tag{7.8}$$

of spectral schemes, with i and k closed immersions with quasi-compact open complements.

At the level of motivic spaces, there is a canonical 2-morphism

$$f^* i_* \to k_* g^* \tag{7.9}$$

constructed in Paragraph A.5.

The following says that $\mathbf{H}^{\mathcal{E}_\infty}$ satisfies the *right base change property* along closed immersions (see *loc. cit.*):

Corollary 7.5.2. *The* 2*-morphism (7.9) is invertible at the level of motivic spaces.*

Proof. By Corollary 7.3.2 it suffices to show that the 2-morphism

$$f^* i_* i^* \to k_* g^* i^*$$

is invertible. This follows by considering the localization squares associated to the closed immersions i and k, respectively, and using the smooth base change formula (Proposition 6.2.2). □

7.5.3. In the pointed setting, the functor i_* admits a right adjoint $i^!$ (Corollary 7.1.3), so we obtain another 2-morphism by right transposition from (7.9). Hence we have:

Corollary 7.5.4. *Given a cartesian square of the form (7.8), the canonical* 2*-morphisms*

$$k_* g^* \to f^* i_* \tag{7.10}$$

$$i^! f_* \to g_* k^! \tag{7.11}$$

are invertible at the level of pointed motivic spaces.

7.5.5. At the level of spectra, we have:

Corollary 7.5.6. *Given a cartesian square of the form (7.8), the canonical* 2*-morphisms*

$$k_* g^* \to f^* i_* \tag{7.12}$$

$$i^! f_* \to g_* k^! \tag{7.13}$$

are invertible at the level of motivic spectra.

7.6. **Closed projection formula.** Let $i : Z \hookrightarrow S$ be a closed immersion with quasi-compact open complement.

As recalled in Definition A.5.8, there are canonical morphisms

$$i_*(\mathcal{F}) \times \mathcal{G} \to i_*(\mathcal{F} \times i^*(\mathcal{G})) \tag{7.14}$$

for any pair of motivic spaces $\mathcal{F} \in \mathbf{H}^{\mathcal{E}_\infty}(Z)$ and $\mathcal{G} \in \mathbf{H}^{\mathcal{E}_\infty}(S)$.

The following verifies the *right projection formula* for $\mathbf{H}^{\mathcal{E}_\infty}$ along closed immersions, in the sense of Paragraph A.5:

Corollary 7.6.1. *The canonical morphism (7.14) is invertible, for all motivic spaces* $\mathcal{F} \in \mathbf{H}^{\mathcal{E}_\infty}(Z)$ *and* $\mathcal{G} \in \mathbf{H}^{\mathcal{E}_\infty}(S)$.

Proof. This follows from the localization theorem (Theorem 7.2.6) and the smooth projection formula (Proposition 6.3.4). □

As recalled in Paragraph A.5, the symmetric monoidal functor $i^*_{\mathbf{H}}$ endows $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{Z})$ with a structure of $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$-module category, and the statement of Corollary 7.6.1 is equivalent to the following:

Corollary 7.6.2. *The functor $i^{\mathbf{H}}_*$ lifts to a morphism of $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$-module categories.*

7.6.3. In the pointed setting, the functor i_* admits a right adjoint $i^!$ (Corollary 7.1.3), so we obtain a dual form of the isomorphism (7.14) by right transposition:

$$i^!\underline{\mathrm{Hom}}_{\mathrm{S}}(\mathcal{G},\mathcal{F}) \to \underline{\mathrm{Hom}}_{\mathrm{Z}}(i^*\mathcal{G}, i^!\mathcal{F}) \tag{7.15}$$

for any pair of pointed motivic spaces $\mathcal{F}$ and $\mathcal{G}$ over S.

Just as above, we obtain:

Corollary 7.6.4. *The canonical morphism (7.14) is invertible, for all pointed motivic spaces $\mathcal{F} \in \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{Z})_\bullet$ and $\mathcal{G} \in \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$. Dually, the canonical morphism (7.15) is invertible, for all pointed motivic spaces $\mathcal{F}, \mathcal{G} \in \mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$.*

As above, this is equivalent to:

Corollary 7.6.5. *The functor $i^{\mathbf{H}_\bullet}_*$ lifts to a morphism of $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})_\bullet$-module categories.*

7.6.6. At the level of motivic spectra, we get:

Corollary 7.6.7. *The canonical morphism (7.14) is invertible, for all motivic spectra $\mathcal{F} \in \mathbf{SH}(\mathrm{Z})_{\mathrm{S}^1}$ and $\mathcal{G} \in \mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$. Dually, the canonical morphism (7.15) is invertible, for all motivic spectra $\mathcal{F}, \mathcal{G} \in \mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$.*

As above, this is equivalent to:

Corollary 7.6.8. *The functor $i^{\mathbf{SH}}_*$ lifts to a morphism of $\mathbf{SH}^{\mathcal{E}_\infty}(\mathrm{S})_{\mathrm{S}^1}$-module categories.*

7.7. Smooth-closed base change formula.

7.7.1. Consider a cartesian square of spectral schemes

$$\begin{array}{ccc} \mathrm{X_Z} & \xhookrightarrow{k} & \mathrm{X} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \mathrm{Z} & \xhookrightarrow{i} & \mathrm{S}, \end{array} \tag{7.16}$$

where i and j are closed immersions with quasi-compact open complements, and p and q are smooth.

There is a canonical 2-morphism

$$p_\sharp k_* \to i_* q_\sharp \tag{7.17}$$

at the level of motivic spaces, constructed in Paragraph A.6.

The following verifies the *bidirectional base change property* for $\mathbf{H}^{\mathcal{E}_\infty}$ with respect to smooth morphisms and closed immersions:

Corollary 7.7.2 (Smooth-closed base change)**.** *Given a cartesian square of the form (7.16), the 2-morphism (7.17) is invertible at the level of motivic spaces.*

Proof. Since the direct image functor k_* is fully faithful (Corollary 7.3.2), it suffices to show that the transformation

$$p_\sharp k_* k^* \to i_* q_\sharp k^*,$$

obtained by pre-composition with k^*, is invertible. This follows directly from the localization theorem (Theorem 7.2.6) and the smooth base change formula (Proposition 6.2.2). □

7.7.3. In the pointed setting, we have canonical 2-morphisms

$$p_\sharp k_* \to i_* q_\sharp \tag{7.18}$$

$$q^* i^! \to k^! p^* \tag{7.19}$$

where the second is obtained by right transposition from the first.

The same argument as above shows:

Corollary 7.7.4 (Smooth-closed base change). *Given a cartesian square of the form (7.16), the* 2*-morphisms (7.18) and (7.19) are invertible at the level of pointed motivic spaces.*

7.7.5. At the level of motivic spectra, we have:

Corollary 7.7.6 (Smooth-closed base change). *Given a cartesian square of the form (7.16), the* 2*-morphisms (7.18) and (7.19) are invertible at the level of motivic spectra.*

8. The localization theorem

This section is dedicated to the proof of the localization theorem (see Paragraph 7.2).

Throughout the section, we let $i : \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion of spectral schemes, such that the complementary open immersion $j : \mathrm{U} \hookrightarrow \mathrm{S}$ is quasi-compact.

8.1. **The space of** Z**-trivialized maps.** In this paragraph we formulate Proposition 8.1.9, which aside from Theorem 7.1.2 is the main input that goes into the proof of the localization theorem; most of the remainder of this section will be dedicated to its proof.

Let X be a smooth spectral S-scheme and let $t : \mathrm{Z} \hookrightarrow \mathrm{X}_\mathrm{Z}$ be a section of the base change $\mathrm{X}_\mathrm{Z} := \mathrm{X} \times_\mathrm{S} \mathrm{Z}$. Proposition 8.1.9 asserts the motivic contractibility of a certain fibred space $\mathrm{h}_\mathrm{S}(\mathrm{X}, t)$; the latter can be thought of as a modified version of the representable space $\mathrm{h}_\mathrm{S}(\mathrm{X})$ whose sections we refer to informally as Z*-trivialized morphisms* to X.

8.1.1. Given a smooth spectral S-scheme X, let $\mathrm{X}_\mathrm{U} := \mathrm{X} \times_\mathrm{S} \mathrm{U}$ denote its base change along j, and $\mathrm{X}_\mathrm{Z} := \mathrm{X} \times_\mathrm{S} \mathrm{Z}$ its base change along i.

We will write $\mathrm{h}^\mathrm{Z}_\mathrm{S}(\mathrm{X})$ for the space over S defined by the cocartesian square

$$\begin{array}{ccc} \mathrm{h}_\mathrm{S}(\mathrm{X}_\mathrm{U}) & \longrightarrow & \mathrm{h}_\mathrm{S}(\mathrm{X}) \\ \downarrow & & \downarrow \\ \mathrm{h}_\mathrm{S}(\mathrm{U}) & \longrightarrow & \mathrm{h}^\mathrm{Z}_\mathrm{S}(\mathrm{X}). \end{array} \tag{8.1}$$

Note that there is a canonical isomorphism

$$i^*_{\mathrm{Spc}}(\mathrm{h}^\mathrm{Z}_\mathrm{S}(\mathrm{X})) \approx \mathrm{h}_\mathrm{Z}(\mathrm{X}_\mathrm{Z}) \tag{8.2}$$

of spaces over Z, since i^*_{Spc} commutes with colimits.

Since colimits in $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$ are computed section-wise, we can describe the spaces of sections of $\mathrm{h}^\mathrm{Z}_\mathrm{S}(\mathrm{X})$ explicitly:

Lemma 8.1.2. *Let* Y *be a smooth spectral* S*-scheme. If* Y_Z *is the empty spectral scheme, then the space* $\Gamma(\mathrm{Y}, \mathrm{h}^\mathrm{Z}_\mathrm{S}(\mathrm{X}))$ *is contractible. Otherwise, there is a canonical isomorphism of spaces*

$$\Gamma(\mathrm{Y}, \mathrm{h}^\mathrm{Z}_\mathrm{S}(\mathrm{X})) \approx \Gamma(\mathrm{Y}, \mathrm{h}_\mathrm{S}(\mathrm{X})) \approx \mathrm{Maps}_\mathrm{S}(\mathrm{Y}, \mathrm{X}). \tag{8.3}$$

8.1.3. Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. Let $t : \mathrm{Z} \hookrightarrow \mathrm{X}$ be an S-morphism, i.e. a partially defined section of p.

Consider the canonical morphism

$$\varepsilon : \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}) \to i_*^{\mathrm{Spc}} i_{\mathrm{Spc}}^*(\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X})) \approx i_*^{\mathrm{Spc}}(\mathrm{h}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}})) \tag{8.4}$$

induced by the counit of the adjunction $(i_{\mathrm{Spc}}^*, i_*^{\mathrm{Spc}})$.

The morphism t corresponds by adjunction to a morphism $\tau : \mathrm{h}_{\mathrm{S}}(\mathrm{S}) \to i_*^{\mathrm{Spc}}(\mathrm{h}_{\mathrm{S}}(\mathrm{X}_{\mathrm{Z}}))$. We define a space $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ over S as the fibre of ε at the point τ, so that we have a cartesian square

$$\begin{array}{ccc} \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t) & \longrightarrow & \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}) \\ \downarrow & & \downarrow \varepsilon \\ \mathrm{h}_{\mathrm{S}}(\mathrm{S}) & \xrightarrow{\tau} & i_*^{\mathrm{Spc}}(\mathrm{h}_{\mathrm{S}}(\mathrm{X}_{\mathrm{Z}})) \end{array}$$

of spaces over S.

8.1.4. Sections of the fibred space $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ can be described as follows, using the fact that limits in $\mathrm{Spc}^{\mathcal{E}_\infty}(\mathrm{S})$ are computed section-wise:

Lemma 8.1.5. *Let* Y *be a smooth spectral* S*-scheme. If* Y_{Z} *is the empty spectral scheme, then the space* $\Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t))$ *is contractible. Otherwise,* $\Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t))$ *is canonically identified with the fibre of the restriction map*

$$\mathrm{Maps}_{\mathrm{S}}(\mathrm{Y}, \mathrm{X}) \to \mathrm{Maps}_{\mathrm{Z}}(\mathrm{Y}_{\mathrm{Z}}, \mathrm{X}_{\mathrm{Z}})$$

at the point defined by the composite $\mathrm{Y}_{\mathrm{Z}} \to \mathrm{Z} \overset{t}{\hookrightarrow} \mathrm{X}_{\mathrm{Z}}$.

In other words, points of the space $\Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t))$ (when $\mathrm{Y}_{\mathrm{Z}} \neq \varnothing$) are pairs (f, α), with $f : \mathrm{Y} \to \mathrm{X}$ an S-morphism and α a commutative triangle

$$\begin{array}{ccc} \mathrm{Y}_{\mathrm{Z}} & \xrightarrow{f_{\mathrm{Z}}} & \mathrm{X}_{\mathrm{Z}}. \\ \downarrow & \nearrow_{t} & \\ \mathrm{Z} & & \end{array} \tag{8.5}$$

We refer to such a pair (f, α) informally as a Z*-trivialized morphism* from Y to X.

8.1.6. If p is a smooth morphism, then since p_{Spc}^* commutes with both limits and colimits, we have:

Lemma 8.1.7. *Let* X *be a smooth spectral* S*-scheme and* $t : \mathrm{Z} \hookrightarrow \mathrm{X}$ *an* S*-morphism. If* $p : \mathrm{T} \to \mathrm{S}$ *is a smooth morphism, then there is a canonical isomorphism of fibred spaces*

$$p_{\mathrm{Spc}}^*(\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)) \approx \mathrm{h}_{\mathrm{T}}(\mathrm{X}_{\mathrm{T}}, t_{\mathrm{T}}),$$

where $t_{\mathrm{T}} : \mathrm{Z}_{\mathrm{T}} \hookrightarrow \mathrm{X}_{\mathrm{T}}$ *is obtained from* t *by base change along* p.

8.1.8. Our main result about the fibred space $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ is as follows:

Proposition 8.1.9. *Let* $p : \mathrm{X} \to \mathrm{S}$ *be a smooth morphism of affine spectral schemes. Then for every* S*-morphism* $t : \mathrm{Z} \hookrightarrow \mathrm{X}$*, the space* $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ *is motivically contractible, i.e. the morphism* $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t) \to \mathrm{pt}_{\mathrm{S}}$ *is a motivic equivalence.*

The proof will occupy the rest of this section.

8.1.10. We first consider the case of vector bundles:

Lemma 8.1.11. *Let* E *be a vector bundle over* S *with zero section* $s : \mathrm{S} \hookrightarrow \mathrm{E}$. *Then the space* $\mathrm{h_S}(\mathrm{E}, s_\mathrm{Z})$ *is motivically contractible, where* $s_\mathrm{Z} : \mathrm{Z} \hookrightarrow \mathrm{E_Z}$ *denotes the base change of* s *along* $i : \mathrm{Z} \hookrightarrow \mathrm{S}$.

Proof. It suffices to construct a section σ of the morphism

$$\varphi : \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z}) \to \mathrm{h_S}(\mathrm{S}),$$

and provide an $\mathbf{A}^1$-homotopy between the composite $\sigma \circ \varphi$ and the identity.

The section

$$\sigma : \mathrm{h_S}(\mathrm{S}) \to \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z})$$

is induced by the composite $\mathrm{h_S}(\mathrm{S}) \xrightarrow{s} \mathrm{h_S}(\mathrm{E}) \to \mathrm{h_S^Z}(\mathrm{E})$.

It remains to define the $\mathbf{A}^1$-homotopy

$$\vartheta : \mathrm{h_S}(\mathbf{A}^1_\mathrm{S}) \times \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z}) \to \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z}).$$

For each smooth spectral S-scheme Y with $\mathrm{Y_Z} \neq \varnothing$, the map

$$\Gamma(\mathrm{Y}, \vartheta) : \Gamma(\mathrm{Y}, \mathrm{h_S}(\mathbf{A}^1_\mathrm{S})) \times \Gamma(\mathrm{Y}, \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z})) \to \Gamma(\mathrm{Y}, \mathrm{h_S}(\mathrm{E}, s_\mathrm{Z}))$$

is given by the assignment

$$(a : \mathrm{Y} \to \mathbf{A}^1_\mathrm{S}, f : \mathrm{Y} \to \mathrm{E}) \mapsto (a \cdot f : \mathrm{Y} \to \mathrm{E}).$$

□

8.2. **Étale base change.** In this paragraph we show that the space $\mathrm{h_S}(\mathrm{X}, t)$ is invariant under étale base change.

8.2.1. The assignment $(\mathrm{X}, t) \mapsto \mathrm{h_S}(\mathrm{X}, t)$ is functorial in the following sense.

Let (X, t) and (X', t') be pairs, with X (resp. X′) a smooth spectral S-scheme, and $t : \mathrm{Z} \hookrightarrow \mathrm{X}$ (resp. $t' : \mathrm{Z} \hookrightarrow \mathrm{X}'$) a partially defined section. Suppose $f : \mathrm{X}' \to \mathrm{X}$ is an S-morphism such that the square

$$\begin{array}{ccc} \mathrm{Z} & \xhookrightarrow{t'} & \mathrm{X}'_\mathrm{Z} \\ \| & & \downarrow \\ \mathrm{Z} & \xhookrightarrow{t} & \mathrm{X_Z} \end{array}$$

is cartesian. Then there is a canonical morphism of spaces over S

$$\mathrm{h_S}(\mathrm{X}', t') \to \mathrm{h_S}(\mathrm{X}, t). \tag{8.6}$$

8.2.2. Let (X, t) and (X', t') be as above. Suppose that $p : \mathrm{X}' \to \mathrm{X}$ is an *étale* morphism, such that the above square is cartesian. Then we have:

Lemma 8.2.3. *The canonical morphism*

$$\mathrm{h_S}(\mathrm{X}', t') \to \mathrm{h_S}(\mathrm{X}, t)$$

is a Nisnevich-local equivalence.

Let φ denote the morphism in question. The claim is that the induced morphism of Nisnevich sheaves $\mathrm{L_{Nis}}(\varphi)$ is invertible. It suffices to show that it is 0-truncated (i.e. its diagonal is a monomorphism) and 0-connected (i.e. it is an effective epimorphism and so is its diagonal).

8.2.4. *Proof of Lemma 8.2.3, step 1.* To show that $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is 0-truncated, it suffices to show that φ is 0-truncated (since $\mathrm{L}_{\mathrm{Nis}}$ is exact). For this, it suffices to show that for each smooth spectral S-scheme Y, the induced morphism of spaces of Y-sections

$$\Gamma(\mathrm{Y},\varphi) : \Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')) \to \Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}, t))$$

is 0-truncated.

We may assume Y_{Z} is not empty; then this is the morphism induced on fibres in the diagram

$$\begin{array}{ccccc}
\Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')) & \longrightarrow & \mathrm{Maps}_{\mathrm{S}}(\mathrm{Y}, \mathrm{X}') & \longrightarrow & \mathrm{Maps}_{\mathrm{Z}}(\mathrm{Y}_{\mathrm{Z}}, \mathrm{X}'_{\mathrm{Z}}) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(\mathrm{Y}, \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}, t)) & \longrightarrow & \mathrm{Maps}_{\mathrm{S}}(\mathrm{Y}, \mathrm{X}) & \longrightarrow & \mathrm{Maps}_{\mathrm{Z}}(\mathrm{Y}_{\mathrm{Z}}, \mathrm{X}_{\mathrm{Z}})
\end{array}$$

Note that the two right-hand vertical morphisms are 0-truncated: p is itself 0-truncated since it is étale, and since the Yoneda embedding commutes with limits, the induced morphism $\mathrm{h}_{\mathrm{S}}(\mathrm{X}') \to \mathrm{h}_{\mathrm{S}}(\mathrm{X})$ is also 0-truncated. It follows that the left-hand vertical morphism is also 0-truncated for each Y, and therefore so is φ.

8.2.5. *Proof of Lemma 8.2.3, step 2.* To show that $\mathrm{L}_{\mathrm{Nis}}(\varphi)$ is an effective epimorphism, it suffices to show that for each smooth spectral S-scheme Y (with Y_{Z} not empty), any Y-section of $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}, t)$ can be lifted Nisnevich-locally along φ.

Let f be a Y-section of $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}, t)$, i.e. a Z-trivialized morphism $f : \mathrm{Y} \to \mathrm{X}$. Let $q : \mathrm{Y}' \to \mathrm{Y}$ denote the base change of $p : \mathrm{X}' \to \mathrm{X}$ along f:

$$\begin{array}{ccc}
\mathrm{Y}' & \xrightarrow{q} & \mathrm{Y} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
\mathrm{X}' & \xrightarrow{p} & \mathrm{X}.
\end{array}$$

Then note that

$$\begin{array}{ccc}
q^{-1}(\mathrm{Y}_{\mathrm{U}}) & \hookrightarrow & \mathrm{Y}' \\
\downarrow & & \downarrow{\scriptstyle q} \\
\mathrm{Y}_{\mathrm{U}} & \hookrightarrow & \mathrm{Y}
\end{array}$$

is a Nisnevich square. Indeed, the closed immersion $\mathrm{Y}_{\mathrm{Z}} \hookrightarrow \mathrm{Y}$ is complementary to $\mathrm{Y}_{\mathrm{U}} \hookrightarrow \mathrm{Y}$, and it is clear that $q^{-1}(\mathrm{Y}_{\mathrm{Z}}) \to \mathrm{Y}_{\mathrm{Z}}$ is invertible because in the diagram

$$\begin{array}{ccc}
q^{-1}(\mathrm{Y}_{\mathrm{Z}}) & \longrightarrow & \mathrm{Y}_{\mathrm{Z}} \\
\downarrow & & \downarrow \\
\mathrm{Z} & \xrightarrow{\mathrm{id}_{\mathrm{Z}}} & \mathrm{Z} \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} \\
\mathrm{X}'_{\mathrm{Z}} & \xrightarrow{p_{\mathrm{Z}}} & \mathrm{X}_{\mathrm{Z}}
\end{array}$$

the lower square and the composite square are cartesian, and hence so is the upper square.

Hence it suffices to show that the restriction of f to either component of this Nisnevich cover lifts to $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')$. The restriction $f|\mathrm{Y}'$ lifts to a section of $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')$ given by $g : \mathrm{Y}' \to \mathrm{X}'$. The restriction $f|\mathrm{Y}_{\mathrm{U}}$ admits a lift trivially: since $(\mathrm{Y}_{\mathrm{U}}) \times_{\mathrm{S}} \mathrm{Z} = \varnothing$, the spaces $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}, t)(\mathrm{Y}_{\mathrm{U}})$ and $\mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}', t')(\mathrm{Y}_{\mathrm{U}})$ are both contractible.

8.2.6. *Proof of Lemma 8.2.3, step 3.* It remains to show that the diagonal $\Delta_{\mathrm{L_{Nis}}(\varphi)}$ of $\mathrm{L_{Nis}}(\varphi)$ is an effective epimorphism, or equivalently that $\mathrm{L_{Nis}}(\Delta_\varphi)$ is.

For each smooth spectral S-scheme Y, the diagonal induces a morphism of spaces

$$\Gamma(\mathrm{Y}, \mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}', t')) \to \Gamma(\mathrm{Y}, \mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}', t')) \underset{\Gamma(\mathrm{Y}, \mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}, t))}{\times} \Gamma(\mathrm{Y}, \mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}', t')).$$

It suffices to show that for each Y (with $\mathrm{Y_Z}$ not empty), any point of the target lifts Nisnevich-locally to a point of the source. Choose a point of the target, given by two Z-trivialized morphisms $f : \mathrm{Y} \to \mathrm{X}'$ and $g : \mathrm{Y} \to \mathrm{X}'$, and an identification $\alpha : p \circ f \approx p \circ g$.

Let $\mathrm{Y}_0 \hookrightarrow \mathrm{Y}$ denote the open immersion defined as the equalizer of the pair (f, g); note that the closed immersion $\mathrm{Y_Z} \hookrightarrow \mathrm{Y}$ factors through Y_0. Thus the open immersions $\mathrm{Y}_0 \hookrightarrow \mathrm{Y}$ and $\mathrm{Y_U} \hookrightarrow \mathrm{Y}$ form a Zariski cover of Y. It is clear that the point (f, g, α) lifts after restriction to Y_0 by definition, and after restriction to $\mathrm{Y_U}$ since $\mathrm{Y_U} \times_{\mathrm{S}} \mathrm{Z} = \varnothing$, so the claim follows.

8.3. **Reduction to the case of vector bundles.** In this paragraph we show that, Nisnevich-locally, any partially defined section of a smooth spectral S-scheme can be lifted to a globally defined section.

Together with the étale base change property demonstrated in the last paragraph, and Lemma 2.13.9, this will allow us to to reduce to the case of vector bundles.

8.3.1. The following lemma will allow us to reduce to the situation where the Z-section t lifts to an S-section $s : \mathrm{S} \hookrightarrow \mathrm{X}$.

Lemma 8.3.2. *Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. Given an S-morphism $t : \mathrm{Z} \hookrightarrow \mathrm{X}$, there exists a Nisnevich square*

$$\begin{array}{ccc} \mathrm{Y_U} & \hookrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow q \\ \mathrm{U} & \overset{j}{\hookrightarrow} & \mathrm{S} \end{array} \tag{8.7}$$

such that q factors through p.

Proof. We will construct a commutative square

$$\begin{array}{ccc} \mathrm{Z} & \dashrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow \\ \mathrm{X_Z} & \hookrightarrow & \mathrm{X}. \end{array} \tag{8.8}$$

with the following properties:

(i) The induced square of underlying classical schemes

$$\begin{array}{ccc} \mathrm{Z_{cl}} & \hookrightarrow & \mathrm{Y_{cl}} \\ \downarrow & & \downarrow \\ (\mathrm{X_Z})_{\mathrm{cl}} & \hookrightarrow & \mathrm{X_{cl}} \end{array}$$

is cartesian.

(ii) The composite morphism $\mathrm{Y} \to \mathrm{X} \to \mathrm{S}$ is étale.

Given such a square (8.8), it is clear that we get a Nisnevich square (8.7) as claimed, by taking q to be the composite $\mathrm{Y} \to \mathrm{X} \to \mathrm{S}$: indeed, the closed immersion $\mathrm{Z_{cl}} \hookrightarrow \mathrm{S}$ is complementary to

j, and the squares

$$\begin{array}{ccccc} \mathrm{Z}_{\mathrm{cl}} & \hookrightarrow & \mathrm{Y}_{\mathrm{cl}} & \hookrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow & & \downarrow q \\ \mathrm{Z}_{\mathrm{cl}} & \hookrightarrow & \mathrm{S}_{\mathrm{cl}} & \hookrightarrow & \mathrm{S} \end{array}$$

are cartesian: the left-hand one by (i), and the right-hand one by flatness of q, which is implied by (ii).

In the classical case, the existence of the square (8.8) is known (this is a non-equivariant version of [Hoy17, Thm. 2.21], for instance).

Hence one obtains a cartesian square

$$\begin{array}{ccc} \mathrm{Z}_{\mathrm{cl}} & \dashrightarrow & \mathrm{Y}_0 \\ \downarrow & & \downarrow \\ (\mathrm{X}_{\mathrm{Z}})_{\mathrm{cl}} & \hookrightarrow & \mathrm{X}_{\mathrm{cl}} \end{array}$$

of classical schemes. Then one defines Y by the cocartesian square of closed immersions

$$\begin{array}{ccc} \mathrm{Z}_{\mathrm{cl}} & \hookrightarrow & \mathrm{Y}_0 \\ \downarrow & & \downarrow \\ \mathrm{Z} & \hookrightarrow & \mathrm{Y}. \end{array}$$

By Lemma 2.11.2 this exists, and the morphism $\mathrm{Y}_0 \hookrightarrow \mathrm{Y}$ is a closed immersion identifying Y_0 with the classical scheme underlying Y. The existence of the desired commutative square

$$\begin{array}{ccc} \mathrm{Z} & \hookrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow \\ \mathrm{X}_{\mathrm{Z}} & \hookrightarrow & \mathrm{X} \end{array}$$

follows by construction. □

8.4. **Motivic contractibility of** $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$**.** In this paragraph we prove Proposition 8.1.9.

Let $p : \mathrm{X} \to \mathrm{S}$ be a smooth morphism of affine spectral schemes. Recall the statement of Proposition 8.1.9: we want to show that for any S-morphism $t : \mathrm{Z} \hookrightarrow \mathrm{X}$, the fibred space $\mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ is motivically contractible.

8.4.1. By Lemma 8.3.2 there exists a Nisnevich square

$$\begin{array}{ccc} \mathrm{Y}_{\mathrm{U}} & \hookrightarrow & \mathrm{Y} \\ \downarrow & & \downarrow q \\ \mathrm{U} & \overset{j}{\hookrightarrow} & \mathrm{S} \end{array} \tag{8.9}$$

where q factors through $p : \mathrm{X} \to \mathrm{S}$. It suffices then by the Nisnevich separation property (Proposition 6.1.6) to show that $j^* \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ and $q^* \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t) \approx \mathrm{h}_{\mathrm{Y}}(\mathrm{Y} \times_{\mathrm{S}} \mathrm{X}, t')$ are contractible, where $t' : \mathrm{Y}_{\mathrm{Z}} \hookrightarrow (\mathrm{Y} \times_{\mathrm{S}} \mathrm{X})_{\mathrm{Z}}$ is the base change of t.

8.4.2. The case of $j^* \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$ is clear, since j is complementary to $i : \mathrm{Z} \hookrightarrow \mathrm{S}$.

8.4.3. For $q^* \mathrm{h}_{\mathrm{S}}(\mathrm{X}, t)$, note that by construction there exists a section $t'' : \mathrm{Y} \hookrightarrow \mathrm{Y} \times_{\mathrm{S}} \mathrm{X}$ which lifts t' (since q factors through X):

$$\begin{array}{ccc} (\mathrm{Y} \times_{\mathrm{S}} \mathrm{X})_{\mathrm{Z}} & \hookrightarrow & \mathrm{Y} \times_{\mathrm{S}} \mathrm{X} \\ {\scriptstyle t'}\uparrow & & \uparrow{\scriptstyle t''} \\ \mathrm{Y}_{\mathrm{Z}} & \hookrightarrow & \mathrm{Y} \end{array}$$

Hence by Lemma 2.13.9, Lemma 8.2.3 and Lemma 8.1.11, we have motivic equivalences

$$\mathrm{h}_{\mathrm{S}}(\mathrm{Y} \underset{\mathrm{S}}{\times} \mathrm{X}, t') \approx \mathrm{h}_{\mathrm{S}}(\mathbf{N}^*_{t''}, z) \approx \mathrm{h}_{\mathrm{S}}(\mathrm{S}),$$

where $\mathbf{N}^*_{t''}$ is the conormal bundle, and z is the base change of its zero section.

8.5. **Proof of the localization theorem.** We conclude this section by proving the localization theorem.

Recall that our goal is to show that the canonical morphism

$$\mathcal{F} \underset{j_\sharp j^*(\mathcal{F})}{\sqcup} \mathrm{M}_{\mathrm{S}}(\mathrm{U}) \to i_* i^*(\mathcal{F}) \tag{8.10}$$

is invertible for each motivic space $\mathcal{F}$ over S.

After a series of reductions, we will see that the main ingredient in the proof is Proposition 8.1.9 (which was proved in Paragraph 8.4).

8.5.1. First of all, note that we may use Zariski separation (Proposition 6.1.6) and the smooth base change formula (Proposition 6.2.2) to reduce to the case where S is affine.

Next note that we may reduce to the case where $\mathcal{F}$ is a motivic localization $\mathrm{M}_{\mathrm{S}}(\mathrm{X})$ of an affine smooth spectral S-scheme X. Indeed, we have seen that the category $\mathbf{H}^{\mathcal{E}_\infty}(\mathrm{S})$ is generated under sifted colimits by such objects (Lemma 4.3.5) and that each of the functors $j_\sharp$, j^*, i^*, and i_* commutes with contractible colimits (Theorem 7.1.2).

In this case the morphism (8.10) is canonically identified with the morphism

$$\mathrm{M}_{\mathrm{S}}(\mathrm{X}) \underset{\mathrm{M}_{\mathrm{S}}(\mathrm{X}_{\mathrm{U}})}{\sqcup} \mathrm{M}_{\mathrm{S}}(\mathrm{U}) \to i_* \mathrm{M}_{\mathrm{S}}(\mathrm{X}_{\mathrm{Z}}) \tag{8.11}$$

where we write $\mathrm{X}_{\mathrm{U}} = \mathrm{X} \times_{\mathrm{S}} \mathrm{U}$ and $\mathrm{X}_{\mathrm{Z}} = \mathrm{X} \times_{\mathrm{S}} \mathrm{Z}$.

8.5.2. Note that the source of the morphism (8.11) is the motivic localization of the space $\mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X})$, and that the target $i^{\mathbf{H}}_*(\mathrm{M}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}}))$ is the motivic localization of $i^{\mathrm{Spc}}_*(\mathrm{h}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}}))$.

Hence it suffices to show that the morphism

$$\mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}) \to i^{\mathrm{Spc}}_* \mathrm{h}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}}) \tag{8.12}$$

is a motivic equivalence.

8.5.3. By universality of colimits, it suffices to show that for every smooth spectral S-scheme Y and every morphism $\mathrm{h}_{\mathrm{S}}(\mathrm{Y}) \to i^{\mathrm{Spc}}_* \mathrm{h}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}})$, corresponding to an S-morphism $t : \mathrm{Z} \to \mathrm{X}$, the base change

$$\mathrm{h}^{\mathrm{Z}}_{\mathrm{S}}(\mathrm{X}) \underset{i^{\mathrm{Spc}}_* \mathrm{h}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}})}{\times} \mathrm{h}_{\mathrm{S}}(\mathrm{Y}) \to \mathrm{h}_{\mathrm{S}}(\mathrm{Y}) \tag{8.13}$$

is invertible.

8.5.4. Let $p : \mathrm{Y} \to \mathrm{S}$ be the structural morphism of Y. Then since $\mathrm{h}_{\mathrm{S}}(\mathrm{Y}) \approx p_{\sharp}^{\mathrm{Spc}} \mathrm{h}_{\mathrm{Y}}(\mathrm{Y})$, one sees that (8.13) is identified, by the smooth projection formula (Lemma 6.3.7), with a morphism

$$p_{\sharp}^{\mathrm{Spc}}(p_{\mathrm{Spc}}^{*} \mathrm{h}_{\mathrm{S}}^{\mathrm{Z}}(\mathrm{X}) \underset{p_{\mathrm{Spc}}^{*} i_{*}^{\mathrm{Spc}} \mathrm{h}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}})}{\times} \mathrm{h}_{\mathrm{Y}}(\mathrm{Y})) \to p_{\sharp}^{\mathrm{Spc}} \mathrm{h}_{\mathrm{Y}}(\mathrm{Y}). \tag{8.14}$$

8.5.5. Note that we have $p^{*} i_{*} \approx k_{*} q^{*}$ (Proposition 6.2.2), where k (resp. q) is the base change of i (resp. p) along p (resp. i). Hence the morphism (8.14) is identified with the image by $p_{\sharp}$ of

$$\mathrm{h}_{\mathrm{Y}}^{\mathrm{Y}_{\mathrm{Z}}}(\mathrm{X} \underset{\mathrm{S}}{\times} \mathrm{Y}) \underset{k_{*}^{\mathrm{Spc}} \mathrm{h}_{\mathrm{Y}_{\mathrm{Z}}}((\mathrm{X} \times_{\mathrm{S}} \mathrm{Y})_{\mathrm{Z}})}{\times} \mathrm{h}_{\mathrm{Y}}(\mathrm{Y}) \to \mathrm{h}_{\mathrm{Y}}(\mathrm{Y}). \tag{8.15}$$

8.5.6. The source of the morphism (8.15) is nothing else than the space $\mathrm{h}_{\mathrm{Y}}(\mathrm{X} \times_{\mathrm{S}} \mathrm{Y}, t_{\mathrm{Y}})$, where $t_{\mathrm{Y}} : \mathrm{Z} \times_{\mathrm{S}} \mathrm{Y} \to \mathrm{X} \times_{\mathrm{S}} \mathrm{Y}$ is the base change of t along p. Hence we conclude by Proposition 8.1.9.

Appendix A. Coefficient systems

In this section we will introduce a formalism for working with categories of coefficients. These are systems of categories $\mathbf{D}(\mathrm{S})$, indexed by objects in some category $\mathbf{C}$ (e.g. schemes), equipped with some basic functorialities f^{*} and f_{*} associated to any morphism f in $\mathbf{C}$. This is a natural framework for stating base change and projection formulas.

We will use the language of $(\infty, 2)$-categories as a convenient way to formulate some of the results in this section, even though we only apply them in the $(\infty, 2)$-category of presentable ∞-categories. Following [GR16], our definition of $(\infty, 2)$-category will be a complete Segal space in the ∞-category of ∞-categories.

A.1. Passage to right/left adjoints.

A.1.1. Recall that in an $(\infty, 2)$-category $\mathbb{C}$, there is a notion of adjunction between two objects x and y.

A pair $(f : x \to y, g : y \to x)$ forms an adjunction if and only if it defines an adjunction in the underlying ordinary 2-category $(\mathbb{C})^{\text{2-ordn}}$.

A.1.2. Let $\mathbb{C}$ and $\mathbb{D}$ be $(\infty, 2)$-categories. We say that an $(\infty, 2)$-functor $u : \mathbb{C} \to \mathbb{D}$ is *right-adjointable* (resp. *left-adjointable*) if, for each 1-morphism $f : x \to y$ in $\mathbb{C}$, its image $u(f)$ admits a right adjoint (resp. a left adjoint) in the 2-category $\mathbb{D}$.

A.1.3. Let $\mathrm{Maps}_{!}(\mathbb{S}, \mathbb{T})$ denote the space of right-adjointable $(\infty, 2)$-functors. Let $\mathrm{Maps}_{*}(\mathbb{S}, \mathbb{T})$ denote the space of left-adjointable $(\infty, 2)$-functors.

Lemma A.1.4. *There is a canonical isomorphism of spaces*

$$\mathrm{Maps}_{!}(\mathbb{S}, \mathbb{T}) = \mathrm{Maps}_{*}((\mathbb{S})^{1\&2\text{-op}}, \mathbb{T}),$$

where $(\mathbb{C})^{1\&2\text{-op}}$ *denotes the* $(\infty, 2)$*-category obtained by flipping the directions of* 1- *and* 2-*morphisms in* $\mathbb{S}$.

See [GR16, Cor. 1.3.4].

Given a right-adjointable $(\infty, 2)$-functor $u : \mathbb{S} \to \mathbb{T}$, we will call the corresponding functor $u_{*} : (\mathbb{S})^{1\&2\text{-op}} \to \mathbb{T}$ the functor obtained from u by *passage to right adjoints*.

Dually, given a left-adjointable $(\infty, 2)$-functor $u : \mathbb{S} \to \mathbb{T}$, we will call the corresponding functor $u_{!} : (\mathbb{S})^{1\&2\text{-op}} \to \mathbb{T}$ the functor obtained from u by *passage to left adjoints*.

A.2. **Adjointable squares.** In this section we will formulate the notion of *horizontally/vertically left/right-adjointable square* in a 2-category, which we will be used in the text to express base change formulas.

A.2.1. We fix an $(\infty, 2)$-category $\mathbb{C}$.

Let Θ be a square in $\mathbb{C}$

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{u} & \mathbf{C}' \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} \\ \mathbf{D} & \xrightarrow{u'} & \mathbf{D}' \end{array} \tag{A.1}$$

which commutes up to an invertible 2-morphism

$$v'u \xrightarrow{\sim} u'v.$$

Suppose that v (resp. v') admits a right adjoint v^{R} (resp. $(v')^{\mathrm{R}}$) in $\mathbb{C}$. Then the square $\Theta^{\mathrm{vert:R}}$

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{u} & \mathbf{C}' \\ {\scriptstyle v^{\mathrm{R}}}\uparrow & & {\scriptstyle (v')^{\mathrm{R}}}\uparrow \\ \mathbf{D} & \xrightarrow{u'} & \mathbf{D}' \end{array} \tag{A.2}$$

commutes up to the 2-morphism

$$uv^{\mathrm{R}} \to (v')^{\mathrm{R}}v'uv^{\mathrm{R}} \xrightarrow{\sim} (v')^{\mathrm{R}}u'vv^{\mathrm{R}} \to (v')^{\mathrm{R}}u', \tag{A.3}$$

where the first morphism is obtained by precomposition with the counit of the adjunction, the isomorphism in the middle is given by the commutativity of the square Θ, and the final morphism is given by the unit of the adjunction.

If this 2-morphism is invertible, then we say that the square Θ is *vertically right-adjointable.*

A.2.2. Similarly if v (resp. v') admits a *left* adjoint v^{L} (resp. $(v')^{\mathrm{L}}$), then the square $\Theta^{\mathrm{vert:L}}$

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{u} & \mathbf{C}' \\ {\scriptstyle v^{\mathrm{L}}}\uparrow & & {\scriptstyle (v')^{\mathrm{L}}}\uparrow \\ \mathbf{D} & \xrightarrow{u'} & \mathbf{D}' \end{array} \tag{A.4}$$

commutes up to the 2-morphism

$$(v')^{\mathrm{L}}u' \to (v')^{\mathrm{L}}u'vv^{\mathrm{L}} \xleftarrow{\sim} (v')^{\mathrm{L}}v'uv^{\mathrm{L}} \to uv^{\mathrm{L}}. \tag{A.5}$$

If this is invertible, we say that the square Θ is *vertically left-adjointable.*

A.2.3. If u (resp. u') admits a right adjoint u^{R} (resp. $(u')^{\mathrm{R}}$), then the square $\Theta^{\mathrm{horiz:R}}$

$$\begin{array}{ccc} \mathbf{C} & \xleftarrow[u^{\mathrm{R}}]{} & \mathbf{C}' \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} \\ \mathbf{D} & \xleftarrow[(u')^{\mathrm{R}}]{} & \mathbf{D} \end{array} \tag{A.6}$$

commutes up to a 2-morphism

$$vu^{\mathrm{R}} \to (u')^{\mathrm{R}}v'. \tag{A.7}$$

If it is invertible, we say that Θ is *horizontally right-adjointable.*

Similarly, if u (resp. u') admits a left adjoint u^{L} (resp. $(u')^{\mathrm{L}}$), then the square $\Theta^{\text{horiz:L}}$

(A.8)
$$\begin{array}{ccc} \mathbf{C} & \xleftarrow[u^{\mathrm{L}}]{} & \mathbf{C}' \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} \\ \mathbf{D} & \xleftarrow[(u')^{\mathrm{L}}]{} & \mathbf{D} \end{array}$$

commutes up to a 2-morphism

(A.9)
$$(u')^{\mathrm{L}}v' \to vu^{\mathrm{L}}.$$

If it is invertible, we say that Θ is *horizontally left-adjointable.*

A.2.4. We have:

Lemma A.2.5. *Suppose that in the square Θ (A.1), u (resp. u') admits a right adjoint u^{R} (resp. $(u')^{\mathrm{R}}$), and v (resp. v') admits a left adjoint v^{L} (resp. $(v')^{\mathrm{L}}$). Then Θ is vertically left-adjointable if and only if it is horizontally right-adjointable.*

Proof. The square $\Theta^{\text{vert:L}}$ (resp. $\Theta^{\text{horiz:R}}$) commutes up to a 2-morphism $\alpha : (v')^{\mathrm{L}}u' \to uv^{\mathrm{L}}$ (resp. $\beta : vu^{\mathrm{R}} \to (u')^{\mathrm{R}}v'$). The category of left adjoint functors $\mathbf{C} \to \mathbf{D}$ is equivalent to the category of right adjoint functors $\mathbf{D} \to \mathbf{C}$ (see [GR16, Chap. A.3, Cor. 3.1.9]), and under this equivalence the morphism α corresponds to the morphism β. ☐

A.3. **Coefficient systems.** Let Pres denote the symmetric monoidal ∞-category of presentable ∞-categories. Recall that a presentable ∞-category is an accessible left localization of a presheaf category $\mathrm{PSh}(\mathbf{C})$ with $\mathbf{C}$ essentially small, and a morphism of presentable ∞-categories is a colimit-preserving functor.

For the rest of this section, we fix an essentially small ∞-category $\mathbf{C}$ which admits fibred products. Note that the cartesian monoidal structure on $\mathbf{C}$ induces a canonical cocartesian monoidal structure on $(\mathbf{C})^{\mathrm{op}}$.

A.3.1.

Definition A.3.2. *A* coefficient system *(defined on $\mathbf{C}$) is a symmetric monoidal functor*

$$\mathbf{D}^* : (\mathbf{C})^{\mathrm{op}} \to \mathrm{Pres}.$$

Given a coefficient system $\mathbf{D}^*$, we will write

$$\mathbf{D}(\mathrm{S}) := \mathbf{D}^*(\mathrm{S})$$

for the presentable ∞-category associated to an object $\mathrm{S} \in \mathbf{C}$. We will write $\varnothing_{\mathrm{S}}$ (resp. e_{S}) for the initial (resp. terminal) object of $\mathbf{D}(\mathrm{S})$. We will often refer to the objects of $\mathbf{D}(\mathrm{S})$ as *sheaves* on S.

For a morphism $f : \mathrm{T} \to \mathrm{S}$, we will write

$$f^* := \mathbf{D}^*(f) : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{T})$$

for the induced functor, which we call the functor of *inverse image* along f. It is cocontinuous, and admits (by the adjoint functor theorem) a right adjoint

$$f_* : \mathbf{D}(\mathrm{T}) \to \mathbf{D}(\mathrm{S})$$

which we call the functor of *direct image* along f.

A.3.3. By passing to right adjoints (see Paragraph A.1), $\mathbf{D}^*$ gives rise to a unique functor

$$\mathbf{D}_* : \mathbf{C} \to \infty\text{-Cat}$$

such that each functor

$$\mathbf{D}_*(f) : \mathbf{D}(\mathrm{T}) \to \mathbf{D}(\mathrm{S})$$

is the right adjoint f_*.

A.3.4. Since the functor $\mathbf{D}^*$ underlying a coefficient system is symmetric monoidal, it sends cocommutative comonoids in $\mathbf{C}$ to commutative monoids in Pres. Note that every object $\mathrm{S} \in \mathbf{C}$ has a canonical structure of cocommutative comonoid (with respect to the cartesian monoidal structure).

Hence for each object $\mathrm{S} \in \mathbf{C}$, the presentable ∞-category $\mathbf{D}(\mathrm{S})$ has a canonical symmetric monoidal structure, and for each morphism f in $\mathbf{C}$, the inverse image functor f^* has a canonical symmetric monoidal structure (giving by adjunction a lax monoidal structure on its right adjoint f_*).

We will write $\otimes_{\mathrm{S}}$ for the monoidal product of $\mathbf{D}(\mathrm{S})$, $\mathbf{1}_{\mathrm{S}}$ for the monoidal unit, and $\underline{\mathrm{Hom}}_{\mathrm{S}}$ for the internal hom.

A.3.5. Dually, suppose we are given a functor (not necessarily symmetric monoidal)

$$\mathbf{D}_! : \mathbf{C} \to \mathrm{Pres}.$$

We will write $\mathbf{D}(\mathrm{S}) := \mathbf{D}_!(\mathrm{S})$ for the presentable ∞-category associated to an object $\mathrm{S} \in \mathbf{C}$. For each morphism $f : \mathrm{T} \to \mathrm{S}$ in $\mathbf{C}$, we write

$$f_! := \mathbf{D}_!(f) : \mathbf{D}(\mathrm{T}) \to \mathbf{D}(\mathrm{S})$$

for the induced functor, which commutes with colimits and admits a right adjoint $f^!$.

As above, we can pass to right adjoints to obtain a functor $\mathbf{D}^! : (\mathbf{C})^{\mathrm{op}} \to \mathrm{Pres}$.

A.3.6. For future use, we make the following definition:

Definition A.3.7. *A coefficient system* $\mathbf{D}^*$ *is* pointed *(resp.* S^1-stable, compactly generated*) if the functor* $\mathbf{D}^* : (\mathbf{C})^{\mathrm{op}} \to \mathrm{Pres}$ *factors through the full subcategory spanned by pointed (resp. stable, compactly generated) presentable* ∞*-categories.*

A.4. **Left-adjointability.**

A.4.1. Let us fix a class *left* of *left-admissible* morphisms in $\mathbf{C}$, containing all isomorphisms, closed under composition and base change, and satisfying the 2-out-of-3 property. Let $\mathbf{C}^{left}$ denote the (non-full) subcategory of $\mathbf{C}$ spanned by left-admissible morphisms.

Definition A.4.2. *We say that the coefficient system* $\mathbf{D}^*$ *is* weakly left-adjointable *along a morphism* $p : \mathrm{T} \to \mathrm{S}$ *if it satisfies the following property:*

(Adj^p) *The functor* p^* *admits a left adjoint* $p_\sharp$.

We say that $\mathbf{D}^*$ *is* weakly left-adjointable *along the class left if it satisfies the following property:*

(Adj^{left}) *For every left-admissible morphism p, the property* (Adj^p) *holds.*

Note that if $\mathbf{D}^*$ is weakly left-adjointable along *left*, then one obtains a canonical functor

$$\mathbf{D}_\sharp : \mathbf{C}^{left} \to \mathrm{Pres}$$

by passage to left adjoints (see Paragraph A.1).

A.4.3. Recall the notion of adjointability of squares from Paragraph A.2.

Definition A.4.4. *The coefficient system* $\mathbf{D}^*$ *satisfies* left base change *along a morphism* $p : \mathrm{S}' \to \mathrm{S}$ *if it is weakly left-adjointable along left, and the following property holds:*

(BC^p) *For all cartesian squares* Θ

(A.10)
$$\begin{array}{ccc} \mathrm{T}' & \xrightarrow{f'} & \mathrm{S}' \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ \mathrm{T} & \xrightarrow{f} & \mathrm{S}, \end{array}$$

the induced commutative square Θ^*

(A.11)
$$\begin{array}{ccc} \mathbf{D}(\mathrm{S}) & \xrightarrow{f^*} & \mathbf{D}(\mathrm{T}) \\ \big\downarrow{\scriptstyle p^*} & & \big\downarrow{\scriptstyle (p')^*} \\ \mathbf{D}(\mathrm{S}') & \xrightarrow{(f')^*} & \mathbf{D}(\mathrm{T}'). \end{array}$$

is vertically left-adjointable.

We say that $\mathbf{D}^*$ *satisfies* left base change *along the class left if it is weakly left-adjointable along left, and the following property holds:*

(BC^{left}) *For every left-admissible morphism* p*, the property* (BC^p) *holds.*

In other words, $\mathbf{D}^*$ satisfies left base change along a morphism p if for every such cartesian square Θ, the exchange 2-morphism

(A.12)
$$(p')_\sharp (f')^* \to f^* p_\sharp$$

is invertible.

According to Lemma A.2.5, this is equivalent to the condition that its right transpose

(A.13)
$$p^* f_* \to (f')_* (p')^*$$

is invertible.

A.4.5. For any morphism $f : \mathrm{T} \to \mathrm{S}$, the symmetric monoidal functor $f^* : \mathbf{D}(\mathrm{S}) \to \mathbf{D}(\mathrm{T})$ gives $\mathbf{D}(\mathrm{T})$ a structure of $\mathbf{D}(\mathrm{S})$-module category. If $f_\sharp$ is left adjoint to f^*, then it admits a canonical structure of *colax* morphism of $\mathbf{D}(\mathrm{S})$-modules. In particular there are canonical morphisms

$$f_\sharp(\mathcal{F} \otimes_{\mathrm{T}} f^*(\mathcal{G})) \to f_\sharp(\mathcal{F}) \otimes_{\mathrm{S}} \mathcal{G} \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{T}), \mathcal{G} \in \mathbf{D}(\mathrm{S})).$$

Definition A.4.6. *The coefficient system* $\mathbf{D}^*$ *satisfies the* left projection formula *along a morphism* $p : \mathrm{T} \to \mathrm{S}$ *if it is weakly left-adjointable along* p*, and the following property holds:*

(Proj^p) *The colax morphism of* $\mathbf{D}(\mathrm{S})$*-modules* $p_\sharp$ *is* strict*.*

We say that $\mathbf{D}^*$ *satisfies the* left projection formula *along the class left if it is weakly left-adjointable along left, and the following property holds:*

(Proj^{left}) *For every left-admissible morphism* p*, the property* (Proj^{left}) *holds.*

In other words, $\mathbf{D}^*$ satisfies the left projection formula along $p : \mathrm{T} \to \mathrm{S}$ if the canonical morphisms

(A.14)
$$p_\sharp(\mathcal{F} \otimes_{\mathrm{T}} p^*(\mathcal{G})) \to p_\sharp(\mathcal{F}) \otimes_{\mathrm{S}} \mathcal{G} \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{T}), \mathcal{G} \in \mathbf{D}(\mathrm{S}))$$

are invertible.

A.4.7.

Definition A.4.8. *A coefficient system* $\mathbf{D}^*$ *is* left-adjointable *along the class left if it is weakly left-adjointable along left* (Adj^{left})*, satisfies left base change along left* (BC^{left})*, and satisfies the left projection formula along left* (Proj^{left})*.*

A.5. **Right-adjointability.**

A.5.1. Let us fix a class *right* of *right-admissible* morphisms in $\mathbf{C}$, containing all isomorphisms, and closed under composition and base change. Let $\mathbf{C}^{right}$ denote the (non-full) subcategory of $\mathbf{C}$ spanned by right-admissible morphisms.

In this paragraph we consider the dual versions of the definitions given in Paragraph A.4.

A.5.2. Let $\mathbf{D}^*$ be a coefficient system.

Definition A.5.3. *The coefficient system* $\mathbf{D}^*$ *is* weakly right-adjointable *along a morphism* q *if it satisfies the following property:*

(Adj_q) *The direct image functor* q_* *admits a right adjoint.*

We say that $\mathbf{D}^*$ *is* weakly right-adjointable *along a class right if it satisfies the following property:*

(Adj_{right}) *For each right-admissible morphism* q*, the property* (Adj_q) *holds.*

Remark A.5.4. Note that the situation is not completely symmetric: by requiring that our coefficient system $\mathbf{D}^*$ takes values in Pres, we have imposed (by the adjoint functor theorem) that the inverse image functor f^* *always* admits a right adjoint f_*, for arbitrary morphisms f. In the above definition of *weak right-adjointability*, we are instead imposing the condition that the right adjoint f_* itself admits a further right adjoint $f^!$ (when the morphism f is right-admissible).

A.5.5. We define:

Definition A.5.6. *The coefficient system* $\mathbf{D}^*$ *satisfies* right base change *along a morphism* $q : \mathrm{S}' \to \mathrm{S}$ *if the following property holds:*

(BC_{right}) *For all cartesian squares* Θ *in* $\mathbf{C}$

$$\begin{array}{ccc} \mathrm{T}' & \xrightarrow{f'} & \mathrm{S}' \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ \mathrm{T} & \xrightarrow{f} & \mathrm{S} \end{array}$$

with q *and* q' *right-admissible, the induced commutative square* Θ^*

$$\begin{array}{ccc} \mathbf{D}(\mathrm{S}) & \xrightarrow{f^*} & \mathbf{D}(\mathrm{T}) \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle (q')^*} \\ \mathbf{D}(\mathrm{S}') & \xrightarrow{(f')^*} & \mathbf{D}(\mathrm{T}'). \end{array}$$

is vertically right-adjointable.

We say that $\mathbf{D}^*$ *satisfies* right base change *along the class right if the following property holds:*

(BC_{right}) *For every right-admissible morphism* q*, the property* (BC_q) *holds.*

In other words, $\mathbf{D}^*$ satisfies right base change along a morphism q if for every such cartesian square Θ, the exchange 2-morphism

$$f^* q_* \to (q')_* (f')^*$$

is invertible.

A.5.7. As already mentioned, for any morphism $f : \mathrm{T} \to \mathrm{S}$, the functor f^* endows $\mathbf{D}(\mathrm{T})$ with a canonical structure of $\mathbf{D}(\mathrm{S})$-module category. It follows by adjunction that the right adjoint f_* admits a canonical structure of *lax* morphism of $\mathbf{D}(\mathrm{S})$-modules. In particular there are canonical morphisms

$$f_\sharp(\mathcal{F}) \otimes_{\mathrm{S}} \mathcal{G} \to f_\sharp(\mathcal{F} \otimes_{\mathrm{T}} f^*(\mathcal{G})) \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{T}), \mathcal{G} \in \mathbf{D}(\mathrm{S})).$$

We define:

Definition A.5.8. *The coefficient system* $\mathbf{D}^*$ *satisfies the* right projection formula *along a morphism* $q : \mathrm{T} \to \mathrm{S}$ *if the following property holds:*

(Proj_q) *The canonical structure of lax morphism of* $\mathbf{D}(\mathrm{S})$*-modules on* q_* *is* strict.

We say that $\mathbf{D}^*$ *satisfies the* right projection formula *along the class right if the following property holds:*

(Proj_{right}) *For every right-admissible morphism q, the property* (Proj_q) *holds.*

In other words, $\mathbf{D}^*$ satisfies the right projection formula along $q : \mathrm{T} \to \mathrm{S}$ if the canonical morphisms

$$q_*(\mathcal{F}) \otimes_{\mathrm{S}} \mathcal{G} \to q_*(\mathcal{F} \otimes_{\mathrm{T}} q^*(\mathcal{G})) \qquad (\mathcal{F} \in \mathbf{D}(\mathrm{T}), \mathcal{G} \in \mathbf{D}(\mathrm{S})). \tag{A.15}$$

are invertible.

A.5.9. Finally, we define:

Definition A.5.10. *A coefficient system* $\mathbf{D}^*$ *is* right-adjointable *along the class right if it is weakly right-adjointable along right* (Adj_{right})*, satisfies right base change along right* (BC_{right})*, and satisfies the right projection formula along right* (Proj_{right}).

A.6. **Biadjointability.**

A.6.1. Let $\mathbf{D}^*$ be a coefficient system which satisfies left base change along *left*.

This means that, for every cartesian square Θ in $\mathbf{C}$

$$\begin{array}{ccc} \mathrm{T}' & \xrightarrow{q'} & \mathrm{S}' \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \mathrm{T} & \xrightarrow{q} & \mathrm{S} \end{array} \tag{A.16}$$

with p and p' left-admissible, the square

$$\begin{array}{ccc} \mathbf{D}(\mathrm{S}) & \xrightarrow{q^*} & \mathbf{D}(\mathrm{T}) \\ {\scriptstyle p_\sharp}\uparrow & & \uparrow{\scriptstyle (p')_\sharp} \\ \mathbf{D}(\mathrm{S}') & \xrightarrow{(q')^*} & \mathbf{D}(\mathrm{T}'). \end{array} \tag{A.17}$$

commutes up to a canonical invertible 2-morphism.

One can then ask whether this resulting square is further *horizontally right-adjointable*, i.e. whether the square

$$\begin{array}{ccc} \mathbf{D}(\mathrm{S}) & \xleftarrow{q_*} & \mathbf{D}(\mathrm{T}) \\ {\scriptstyle p_\sharp}\uparrow & & \uparrow{\scriptstyle (p')_\sharp} \\ \mathbf{D}(\mathrm{S}') & \xleftarrow[(q')_*]{} & \mathbf{D}(\mathrm{T}') \end{array} \tag{A.18}$$

commutes via the 2-morphism

$$p_\sharp(q')_* \to q_*q^*p_\sharp(q')_* \simeq q_*(p')_\sharp(q')^*(q')_* \to q_*(p')_\sharp. \tag{A.19}$$

We define:

Definition A.6.2. *The coefficient system* $\mathbf{D}^*$ *satisfies* bidirectional base change *along the pair* $(\mathit{left}, \mathit{right})$ *if it satisfies left base change along left, right base change along right, and the following property holds:*

($\mathrm{BC}^{\mathit{left}}_{\mathit{right}}$) *For all cartesian squares* Θ *in* $\mathbf{C}$ *of the form (A.16), with* p *and* p' *left-admissible (resp.* q *and* q' *right-admissible), the square (A.18) commutes.*

In other words, we require that for all cartesian squares Θ as above, the 2-morphism

$$p_\sharp(q')_* \to q_*(p')_\sharp. \tag{A.20}$$

is invertible.

According to Lemma A.2.5, this is equivalent to the condition that the right transpose

$$(p')^*q^! \to (q')^!p^* \tag{A.21}$$

is invertible.

A.6.3. Finally, we define:

Definition A.6.4. *A coefficient system* $\mathbf{D}^*$ *is* $(\mathit{left}, \mathit{right})$-biadjointable *if it left-adjointable along left, right-adjointable along right, and satisfies bidirectional base change along* $(\mathit{left}, \mathit{right})$ ($\mathrm{BC}^{\mathit{left}}_{\mathit{right}}$).

Appendix B. Topologically quasi-cocontinuous functors

B.1. Reduced presheaves.

B.1.1. Let $\mathbf{C}$ be an essentially small ∞-category admitting an initial object $\varnothing_{\mathbf{C}}$.

Recall from Definition 3.6.4 that a presheaf $\mathcal{F}$ is *reduced* if it satisfies the condition that the space $\mathcal{F}(\varnothing_{\mathbf{C}})$ is contractible.

B.1.2. Let $\mathbb{E}_{\mathrm{red}}$ denote the empty pre-excision structure on $\mathbf{C}$, containing no cartesian squares.

Assume that the axiom (EXC0) holds, i.e. every morphism $c \to \varnothing_{\mathbf{C}}$ is invertible. The axioms (EXC1), (EXC2) and (EXC3) are vacuous, so $\mathbb{E}_{\mathrm{red}}$ is a topological excision structure. The corresponding topology is denoted τ_{red}; it is defined by the single covering sieve $\varnothing \hookrightarrow h(\varnothing_{\mathbf{C}})$, where $\varnothing$ is the initial presheaf.

The following lemma is tautological (it can be viewed as a trivial special case of Theorem 3.7.9):

Lemma B.1.3. *A presheaf* $\mathcal{F}$ *is reduced if and only if it is* $\mathbb{E}_{\mathrm{red}}$*-excisive, or if and only if it satisfies* τ_{red}*-descent.*

B.1.4. Let $\mathrm{PSh}_{\mathrm{red}}(\mathbf{C})$ denote the ∞-category of reduced presheaves. It is well-known that $\mathrm{PSh}_{\mathrm{red}}(\mathbf{C})$ is the ∞-category freely generated by $\mathbf{C}$ under contractible colimits.

(Recall that a small ∞-category is *contractible* if the ∞-groupoid formed by formally adding inverses to all morphisms is (weakly) contractible.)

B.1.5. Let $\mathrm{L}_{\mathrm{red}}$ denote the τ_{red}-localization functor, the exact left adjoint to the inclusion $\mathrm{inc}_{\mathrm{red}} : \mathrm{PSh}_{\mathrm{red}}(\mathbf{C}) \hookrightarrow \mathrm{PSh}(\mathbf{C})$.

For a presheaf $\mathcal{F}$ on $\mathbf{C}$, $\mathrm{L}_{\mathrm{red}}(\mathcal{F})$ can be described as the unique reduced presheaf for which the space $\mathrm{L}_{\mathrm{red}}(\mathcal{F})(c)$ is identified with $\mathcal{F}(c)$ whenever c is not initial.

B.2. **Topologically quasi-cocontinuous functors.** Recall that a functor u between sites is *topologically cocontinuous*[11] if the restriction of presheaves functor u^* preserves local equivalences.

In this paragraph we introduce a slightly weaker version of this condition, which implies that u^* preserves local equivalences between presheaves $\mathcal{F}$ satisfying the mild condition that $\mathcal{F}(\varnothing)$ is contractible. This will have the useful consequence that u^* commutes with contractible colimits at the level of sheaves.

B.2.1. Let $\mathbf{C}$ and $\mathbf{D}$ be essentially small ∞-categories, admitting initial objects $\varnothing_{\mathbf{C}}$ and $\varnothing_{\mathbf{D}}$, respectively. Let $\mathbb{E}_{\mathbf{C}}$ (resp. $\mathbb{E}_{\mathbf{D}}$) be a topological excision structure on $\mathbf{C}$ (resp. $\mathbf{D}$), and let $\tau_{\mathbf{C}}$ (resp. $\tau_{\mathbf{D}}$) denote the associated Grothendieck topology.

Recall that a functor $u : \mathbf{C} \to \mathbf{D}$ is *topologically cocontinuous* if the following condition is satisfied:

(COC) For every τ'-covering sieve $\mathrm{R}' \hookrightarrow h(u(c))$, the sieve $\mathrm{R} \hookrightarrow h(c)$, generated by morphisms $c' \to c$ such that $h(u(c')) \to h(u(c))$ factors through R', is τ-covering.

We define:

Definition B.2.2. *A functor $u : \mathbf{C} \to \mathbf{D}$ is* topologically quasi-cocontinuous *if it satisfies the following condition:*

(COC') For every τ'-covering sieve $\mathrm{R}' \hookrightarrow h(u(c))$, the sieve $\mathrm{R} \hookrightarrow h(c)$, generated by morphisms $c' \to c$ such that either $h(u(c')) \to h(u(c))$ factors through $\mathrm{R}' \hookrightarrow h(u(c))$ or $u(c')$ is initial, is τ-covering.

B.2.3. Let $\mathbf{C}$ and $\mathbf{D}$ be as above. We have:

Lemma B.2.4. *Let $u : \mathbf{C} \to \mathbf{D}$ be a topologically quasi-cocontinuous functor. Then the composite functor*

$$u^*_{\mathrm{red}} : \mathrm{PSh}_{\mathrm{red}}(\mathbf{D}) \hookrightarrow \mathrm{PSh}(\mathbf{D}) \xrightarrow{u^*} \mathrm{PSh}(\mathbf{C})$$

sends $\tau_{\mathbf{D}}$-local equivalences to $\tau_{\mathbf{C}}$-local equivalences.

Proof. Recall that the set of $\tau_{\mathbf{D}}$-local equivalences in $\mathrm{PSh}(\mathbf{D})$ is the strongly saturated closure of the set $\mathrm{S}_{\mathbb{E}_{\mathbf{D}}}$ containing the morphism

$$\varnothing \hookrightarrow h(\varnothing_{\mathbf{D}})$$

and the morphisms

$$c_{\mathrm{Q}} : \mathrm{C}_{\mathrm{Q}} \to h(d),$$

[11] The term *cocontinuous* is used in [AGV73]. Nowadays this term is often used for colimit-preserving functors, so we have slightly deviated from the classical terminology in order to avoid confusion.

for every object $d \in \mathbf{D}$ and every square $\mathrm{Q} \in \mathbb{E}_{\mathbf{D}}$ over d (see (3.11) for notation). In other words, the set of $\tau_{\mathbf{D}}$-local equivalences is the closure of $\mathrm{S}_{\mathbb{E}_{\mathbf{D}}}$ under the 2-of-3 property, cobase change, and colimits.

It follows that the set of $\tau_{\mathbf{D}}$-local equivalences in $\mathrm{PSh}_{\mathrm{red}}(\mathbf{D})$ is the closure of the set of the morphisms $\mathrm{L}_{\mathrm{red}}(s)$ for $s \in \mathrm{S}_{\mathbb{E}_{\mathbf{D}}}$, under 2-of-3, cobase change, and *contractible* colimits.

Note that the functor u^*_{red} commutes with contractible colimits, since u^* commutes with small colimits and the inclusion $\mathrm{inc}_{\mathrm{red}}$ commutes with contractible colimits. This implies that it is sufficient to show that u^*_{red} sends every morphism $\mathrm{L}_{\mathrm{red}}(s)$, for $s \in \mathrm{S}_{\mathbb{E}_{\mathbf{D}}}$, to a $\tau_{\mathbf{C}}$-local equivalence.

Note that $\mathrm{L}_{\mathrm{red}}(\varnothing) \to \mathrm{L}_{\mathrm{red}}(h(\varnothing_{\mathbf{D}}))$ is invertible, and the morphisms

$$\mathrm{L}_{\mathrm{red}}(c_{\mathrm{Q}}) : \mathrm{L}_{\mathrm{red}}(\mathrm{C}_{\mathrm{Q}}) \to \mathrm{L}_{\mathrm{red}}(h(d))$$

are canonically identified with $c_{\mathrm{Q}} : \mathrm{C}_{\mathrm{Q}} \to h(d)$ for every Q and d. The latter fact follows from the fact that representable presheaves are reduced, and $\mathrm{L}_{\mathrm{red}}$ is exact. This means that we only need to show that u^*_{red} sends each morphism c_{Q} to a $\tau_{\mathbf{C}}$-local equivalence.

For each square $\mathrm{Q} \in \mathbb{E}_{\mathbf{D}}$ of the form

$$\begin{array}{ccc} e' & \longrightarrow & e \\ \downarrow & & \downarrow g \\ d' & \xrightarrow{f} & d, \end{array} \tag{B.1}$$

recall that C_{Q} is identified with the $\tau_{\mathbf{D}}$-covering sieve $\mathrm{R}' \hookrightarrow h(d)$ generated by the family $\{d' \to d, e \to d\}$.

To show that $\vartheta : u^*_{\mathrm{red}}(\mathrm{R}') \hookrightarrow u^*_{\mathrm{red}}(h(d))$ is a $\tau_{\mathbf{C}}$-local equivalence, it suffices by universality of colimits to show that, for every object c of $\mathbf{C}$ and every morphism $\varphi : h(c) \to u^*_{\mathrm{red}}h(d)$, the base change

$$u^*_{\mathrm{red}}\mathrm{R}' \underset{u^*_{\mathrm{red}}h(d)}{\times} h(c) \hookrightarrow h(c)$$

is a $\tau_{\mathbf{C}}$-covering sieve.

Note that u^*_{red} admits a left adjoint

$$u^{\mathrm{red}}_! : \mathrm{PSh}(\mathbf{C}) \xrightarrow{u_!} \mathrm{PSh}(\mathbf{D}) \xrightarrow{\mathrm{L}_{\mathrm{red}}} \mathrm{PSh}_{\mathrm{red}}(\mathbf{D})$$

by construction. The morphism φ factors canonically through the unit morphism $h(c) \to u^*_{\mathrm{red}}u^{\mathrm{red}}_!h(c) = u^*_{\mathrm{red}}h(u(c))$ and the canonical morphism $u^*_{\mathrm{red}}u^{\mathrm{red}}_!h(c) = u^*_{\mathrm{red}}h(u(c)) \to u^*_{\mathrm{red}}h(d)$ (obtained by adjunction from φ). The base change of ϑ by $u^*_{\mathrm{red}}h(u(c)) \to u^*_{\mathrm{red}}h(d)$ is identified, since u^*_{red} commutes with limits, with the canonical morphism

$$u^*(\mathrm{R}' \underset{h(d)}{\times} h(u(c))) \to u^*_{\mathrm{red}}h(u(c)).$$

Since the sieve $\mathrm{R}' \times_{h(d)} h(u(c)) \hookrightarrow h(u(c))$ is $\tau_{\mathbf{D}}$-covering, as the base change of a $\tau_{\mathbf{D}}$-covering sieve, the conclusion follows directly from the condition (COC'). □

B.2.5. Let $u : \mathbf{C} \to \mathbf{D}$ be as above. The following lemma is a formal consequence of Lemma B.2.4, and the fact that reduced presheaves are stable by contractible colimits:

Lemma B.2.6. *Suppose that u is topologically quasi-cocontinuous. Then the composite functor*

$$\mathrm{Sh}_{\tau_{\mathbf{D}}}(\mathbf{D}) \xhookrightarrow{\mathrm{inc}_{\tau_{\mathbf{D}}}} \mathrm{PSh}(\mathbf{D}) \xrightarrow{u^*} \mathrm{PSh}(\mathbf{C}) \xrightarrow{\mathrm{L}_{\tau_{\mathbf{C}}}} \mathrm{Sh}_{\tau_{\mathbf{C}}}(\mathbf{C})$$

commutes with contractible colimits.

Proof. The inclusion $\mathrm{inc}_{\tau_{\mathbf{D}}}$ factors as

$$\mathrm{inc}_{\tau_{\mathbf{D}}} : \mathrm{Sh}_{\tau_{\mathbf{D}}}(\mathbf{D}) \xhookrightarrow{\mathrm{inc}^{\mathrm{red}}_{\tau_{\mathbf{D}}}} \mathrm{PSh}_{\mathrm{red}}(\mathbf{D}) \xhookrightarrow{\mathrm{inc}_{\mathrm{red}}} \mathrm{PSh}(\mathbf{D}),$$

and its left adjoint $\mathrm{L}_{\tau_{\mathbf{D}}}$ factors as

$$\mathrm{L}_{\tau_{\mathbf{D}}} : \mathrm{PSh}(\mathbf{D}) \xrightarrow{\mathrm{L}_{\mathrm{red}}} \mathrm{PSh}_{\mathrm{red}}(\mathbf{D}) \xrightarrow{\mathrm{L}^{\mathrm{red}}_{\tau_{\mathbf{D}}}} \mathrm{Sh}_{\tau_{\mathbf{D}}}(\mathbf{D}),$$

with $\mathrm{L}^{\mathrm{red}}_{\tau_{\mathbf{D}}}$ left adjoint to $\mathrm{inc}^{\mathrm{red}}_{\tau_{\mathbf{D}}}$.

Given a diagram $(\mathcal{F}_i)_{i\in\mathbf{I}}$ of $\tau_{\mathbf{D}}$-sheaves indexed by a contractible ∞-category $\mathbf{I}$, consider the canonical morphism

$$\varinjlim_{i\in\mathbf{I}} \mathrm{inc}^{\mathrm{red}}_{\tau_{\mathbf{D}}}(\mathcal{F}_i) \longrightarrow \mathrm{inc}^{\mathrm{red}}_{\tau_{\mathbf{D}}} \mathrm{L}^{\mathrm{red}}_{\tau_{\mathbf{D}}} \varinjlim_{i\in\mathbf{I}} \mathrm{inc}^{\mathrm{red}}_{\tau_{\mathbf{D}}}(\mathcal{F}_i),$$

which is clearly a $\tau_{\mathbf{D}}$-local equivalence (where the colimits are taken in $\mathrm{PSh}_{\mathrm{red}}(\mathbf{D})$). By applying $u^*_{\mathrm{red}} = u^* \mathrm{inc}_{\mathrm{red}}$ this induces a morphism

$$u^*_{\mathrm{red}} \varinjlim_{i\in\mathbf{I}} \mathrm{inc}^{\mathrm{red}}_{\tau_{\mathbf{D}}}(\mathcal{F}_i) \longrightarrow u^*_{\mathrm{red}} \mathrm{inc}^{\mathrm{red}}_{\tau_{\mathbf{D}}} \mathrm{L}^{\mathrm{red}}_{\tau_{\mathbf{D}}} \varinjlim_{i\in\mathbf{I}} \mathrm{inc}^{\mathrm{red}}_{\tau_{\mathbf{D}}}(\mathcal{F}_i),$$

which is identified with a canonical morphism

$$\varinjlim_{i\in\mathbf{I}} u^* \mathrm{inc}_{\tau_{\mathbf{D}}}(\mathcal{F}_i) \longrightarrow u^* \mathrm{inc}_{\tau_{\mathbf{C}}} \varinjlim_{i\in\mathbf{I}} \mathcal{F}_i$$

since the inclusion $\mathrm{inc}_{\mathrm{red}}$ commutes with contractible colimits. By Lemma B.2.4, this is a $\tau_{\mathbf{C}}$-local equivalence, so the claim follows. □

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