

# BRAVE NEW MOTIVIC HOMOTOPY THEORY II: HOMOTOPY INVARIANT K-THEORY

DENIS-CHARLES CISINSKI AND ADEEL A. KHAN

ABSTRACT. This series of papers is dedicated to the study of motivic homotopy theory in the context of brave new or spectral algebraic geometry. In Part II we prove a comparison result with the classical motivic homotopy theory of Morel–Voevodsky. This comparison says roughly that any  $\mathbf{A}^1$ -homotopy invariant cohomology theory in spectral algebraic geometry is determined by its restriction to classical algebraic geometry. As an application we obtain a derived nilpotent invariance result for a brave new analogue of Weibel’s homotopy invariant K-theory.

## CONTENTS

1. Introduction	2
1.1. Main results	2
1.2. Why do we care?	2
1.3. Contents	3
1.4. Conventions and notation	4
1.5. Acknowledgments	4
2. Sch-fibred motivic homotopy theory	4
2.1. Sch-fibred spaces	4
2.2. Sm-fibred spaces	5
2.3. Functoriality	7
2.4. Direct image along closed immersions	8
2.5. Continuity	10
2.6. Sch-fibred spectra	12
3. Nil-localization	13
3.1. Nil-localization	14
3.2. Nil-localization of motivic spaces	15
3.3. Nil-descent of motivic spaces	17
3.4. The comparison	17
4. $\mathrm{Sm}^b$ -fibred motivic homotopy theory	18
4.1. Flat affine spaces	18
4.2. Fibre-smoothness	19
4.3. Flat homotopy invariance	20
4.4. $\mathrm{Sm}^b$ -fibred spaces	21
4.5. Sm-fibred vs. $\mathrm{Sm}^b$ -fibred motivic spaces	22
5. Homotopy invariant K-theory	23
5.1. Algebraic K-theory	23
5.2. Homotopy invariant K-theory	25
5.3. Nilpotent invariance	26
References	27

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## 1. INTRODUCTION

## 1.1. Main results.

1.1.1. In Part I [Kha16] we constructed a brave new analogue of the motivic homotopy category and studied some of its fundamental properties.

Our main goal in Part II is to prove (Theorem 3.4.2) that brave new motivic homotopy theory is equivalent to classical motivic homotopy theory. More precisely, there is a canonical equivalence of  $\infty$ -categories

$$(1.1) \quad \mathbf{H}^{\mathcal{E}_\infty}(S) \xrightarrow{\sim} \mathbf{H}^{\text{cl}}(S_{\text{cl}})$$

for any (quasi-compact quasi-separated) spectral scheme  $S$ . Here  $\mathbf{H}^{\mathcal{E}_\infty}(S)$  denotes the brave new motivic homotopy category over  $S$ , and  $\mathbf{H}^{\text{cl}}(S_{\text{cl}})$  denotes the classical motivic homotopy category of Morel–Voevodsky over  $S_{\text{cl}}$ , the underlying classical scheme of  $S$ .

In particular, for any (quasi-compact quasi-separated) classical scheme  $S$ , the brave new motivic homotopy category  $\mathbf{H}^{\mathcal{E}_\infty}(S)$  is equivalent to the classical motivic homotopy category  $\mathbf{H}^{\text{cl}}(S)$ .

1.1.2. Recall that in the construction of  $\mathbf{H}^{\mathcal{E}_\infty}(S)$ , we invert the “brave new affine line”  $\text{Spec}(\mathbf{S}\{t\})$ , where  $\mathbf{S}\{t\}$  denotes the free  $\mathcal{E}_\infty$ -algebra over the sphere spectrum  $\mathbf{S}$  on one generator  $t$  (in degree zero).

In Part II we also consider a variant of this construction, denoted  $\mathbf{H}^{\flat}(S)$ , where we invert the “flat affine line”  $\mathbf{A}_\flat^1$ , defined as the spectral scheme  $\text{Spec}(\mathbf{S}[t])$ , where  $\mathbf{S}[t]$  denotes the polynomial  $\mathcal{E}_\infty$ -algebra over  $\mathbf{S}$ . We show (Theorem 4.5.5) that the above equivalence (1.1) factors through equivalences

$$\mathbf{H}^{\mathcal{E}_\infty}(S) \xrightarrow{\sim} \mathbf{H}^{\flat}(S) \xrightarrow{\sim} \mathbf{H}^{\text{cl}}(S_{\text{cl}}).$$

1.1.3. The final subject of study in Part II is a brave new analogue of Weibel’s homotopy invariant K-theory, denoted  $\text{KH}^{\mathcal{E}_\infty}$ . This is obtained from algebraic K-theory by imposing homotopy invariance with respect to the brave new affine line.

We prove (Theorem 5.3.4) that this theory satisfies a derived nilpotent invariance property, i.e. for any connective  $\mathcal{E}_\infty$ -ring spectrum  $\mathbf{R}$  it computes the classical homotopy invariant K-theory of the ordinary ring  $\pi_0(\mathbf{R})$ :

$$\text{KH}^{\mathcal{E}_\infty}(\mathbf{R}) \approx \text{KH}^{\text{cl}}(\pi_0(\mathbf{R})).$$

There is also a variant  $\text{KH}^{\flat}$  where one imposes homotopy invariance with respect to the flat affine line. This theory also has the above-mentioned derived nilpotent invariance property. An independent proof of the latter fact, in the setting of connective  $\mathcal{E}_1$ -ring spectra, is due to B. Antieau, D. Gepner and J. Heller (private communication).

Let us note that there is another variant of algebraic K-theory which is known to satisfy derived nilpotent invariance. Namely, C. Barwick has proved that the G-theory (the K-theory of coherent sheaves) of locally noetherian spectral Deligne–Mumford stacks satisfies derived nilpotent invariance [Bar15, § 9].

## 1.2. Why do we care?

1.2.1. One consequence of the equivalence (1.1) is the existence, in any generalized motivic cohomology theory, of *virtual fundamental classes* associated to lci spectral schemes. This will be the subject of a future paper.

For *derived* schemes (we refer to the theory built out of simplicial commutative rings), this has been expected by experts. We found it a bit surprising that it remains true for spectral schemes, which at first glance seem less closely related to classical algebraic geometry.

1.2.2. Another way to interpret the equivalence (1.1) is as follows.

Suppose that we have constructed some hypothetical theory of motivic cohomology over the sphere spectrum  $\mathbf{S}$ . Whatever this is, we might expect that it defines contravariant functors  $X \mapsto \Gamma(X, \mathbf{Z}_{\mathrm{Spec}(\mathbf{S})}(p))$  ( $p \in \mathbf{Z}$ ), i.e. presheaves of spectra on the category  $\mathrm{Sm}_{/\mathrm{Spec}(\mathbf{S})}^{\mathcal{E}_\infty}$  of smooth spectral  $\mathbf{S}$ -schemes, whose homotopy groups compute integral motivic cohomology in weight  $p$ :

$$\mathbf{H}_{\mathrm{mot}}^q(X, \mathbf{Z}_{\mathrm{Spec}(\mathbf{S})}(p)) := \pi_{-q}(\Gamma(X, \mathbf{Z}_{\mathrm{Spec}(\mathbf{S})}(p))).$$

This presheaf should at least satisfy Nisnevich descent, so that one has Mayer–Vietoris long exact sequences for Nisnevich squares.

Now suppose that we also impose the condition of homotopy invariance with respect to the brave new affine line, i.e.

$$\mathbf{H}_{\mathrm{mot}}^q(X, \mathbf{Z}_{\mathrm{Spec}(\mathbf{S})}(p)) \xrightarrow{\sim} \mathbf{H}_{\mathrm{mot}}^q(X \times \mathbf{A}^1, \mathbf{Z}_{\mathrm{Spec}(\mathbf{S})}(p)) \quad (p, q \in \mathbf{Z}),$$

for all smooth spectral  $\mathbf{S}$ -schemes  $X$ , where  $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{S}\{t\})$  denotes the brave new affine line, whose ring of functions is the free  $\mathcal{E}_\infty$ -ring spectrum on one generator  $t$ .

Then the equivalence (1.1) tells us that we must have

$$(1.2) \quad \mathbf{H}_{\mathrm{mot}}^q(X, \mathbf{Z}_{\mathrm{Spec}(\mathbf{S})}(p)) \xrightarrow{\sim} \mathbf{H}_{\mathrm{mot}}^q(X_{\mathrm{cl}}, \mathbf{Z}_{\mathrm{Spec}(\mathbf{Z})}(p)) \quad (p, q \in \mathbf{Z})$$

for all  $X$ . For example, the brave new motivic cohomology of the sphere spectrum  $\mathrm{Spec}(\mathbf{S})$  is forced to be just the classical motivic cohomology of  $\mathrm{Spec}(\mathbf{Z})$ .

Thus in order to obtain a theory of motivic cohomology that is interesting from the perspective of chromatic stable homotopy theory, this means that it is necessary to consider a different approach.

**1.3. Contents.** In Sect. 2 we consider an enlargement of the motivic homotopy category, which is generated by spectral schemes that only satisfy a mild finiteness condition (but are not necessarily smooth). We call these “Sch-fibred” motivic spaces (as opposed to “Sm-fibred” motivic spaces). This will be necessary to make sense of the nil-localization process we use to prove the comparison theorem. We spend some time studying the relationship between Sm-fibred and Sch-fibred motivic homotopy theory. This material also applies to the setting of classical motivic homotopy theory, which may be of independent interest.

In Sect. 3 we prove the first main result, the comparison theorem. The proof uses the idea of nil-localization, which is the localization at the set of morphisms of the form  $\mathrm{h}_{\mathbf{S}}(X_{\mathrm{cl}}) \rightarrow \mathrm{h}_{\mathbf{S}}(X)$ , where  $X_{\mathrm{cl}}$  denotes the underlying classical scheme of  $X$ . We show that the nil-localization of the brave new motivic homotopy category  $\mathbf{H}^{\mathcal{E}_\infty}(\mathbf{S})$  is the classical motivic homotopy category  $\mathbf{H}^{\mathrm{cl}}(\mathbf{S}_{\mathrm{cl}})$ . The key observation is that the localization theorem implies that Sm-fibred motivic spaces are already nil-local, so nil-localization has no further effect.

Sect. 4 deals with the  $\mathrm{Sm}^b$ -variant of the theory, where we invert the flat affine line  $\mathbf{A}_b^1$ . We demonstrate the equivalence between the  $\mathrm{Sm}^b$ -fibred motivic spaces and Sm-fibred motivic spaces. The main point is the existence of a morphism  $\mathbf{A}_b^1 \rightarrow \mathbf{A}^1$ , which preserves the interval structures on both objects.

In Sect. 5 we apply our results to obtain the nilpotent invariance result for homotopy invariant K-theory.

**1.4. Conventions and notation.** We will use the language of  $\infty$ -categories freely throughout the text. The term “category” will mean “ $\infty$ -category” by default. Though we will use the language in a model-independent way, we fix for concreteness the model of quasi-categories as developed by A. Joyal and J. Lurie. Our main references are [Lur09] and [Lur16a].

By assumption, all spectral schemes will be quasi-compact and quasi-separated, and all smooth and étale morphisms will be of finite presentation.

**1.5. Acknowledgments.** We would like to thank Benjamin Antieau and David Gepner for their encouragement and interest in this paper.

## 2. SCH-FIBRED MOTIVIC HOMOTOPY THEORY

In this section we consider an enlarged version of the motivic homotopy category, which is generated by spectral schemes that only satisfy a mild finiteness condition over the base (i.e., they are not required to be smooth over the base). We show that the usual motivic homotopy category is equivalent to the full subcategory generated under colimits by smooth spectral schemes. This larger category will have the same formal properties as  $\mathbf{H}^{\varepsilon_\infty}(\mathbb{S})$ , but the localization theorem demonstrated in [Kha16] will not hold in this setting (see Remark 2.4.4).

Throughout this section,  $\mathbb{S}$  will be a quasi-compact quasi-separated spectral scheme.

**2.1. Sch-fibred spaces.** In this paragraph we define the category of  $\text{Sch}^{\varepsilon_\infty}$ -fibred motivic spaces. The construction fits into the general paradigm discussed in [Kha16, § 3].

**2.1.1.** We say that a morphism of affine spectral schemes  $\text{Spec}(\mathbb{B}) \rightarrow \text{Spec}(\mathbb{A})$  is *afp* if  $\mathbb{B}$  is almost of finite presentation as an  $\mathbb{A}$ -algebra, in the sense of [Lur16a, Def. 7.2.4.26].

This definition is globalized in the usual way:

**Definition 2.1.2.**

(i) *A morphism of spectral schemes is locally afp if there exist affine Zariski covers  $(Y_\alpha \hookrightarrow Y)_\alpha$  and  $(X_\beta \hookrightarrow X)_\beta$  such that, for each  $\alpha$ , there exists an index  $\beta$  and a morphism of affine spectral schemes  $Y_\alpha \rightarrow X_\beta$  which is almost of finite presentation and fits in a commutative square*

$$\begin{array}{ccc} Y_\alpha & \longrightarrow & X_\beta \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

(ii) *A morphism of spectral schemes  $Y \rightarrow X$  is afp if it is quasi-compact, quasi-separated and locally afp.*

We write  $\text{Sch}_{/\mathbb{S}}^{\varepsilon_\infty}$  for the category of afp spectral schemes over  $\mathbb{S}$ .

**2.1.3.** A  $\text{Sch}^{\varepsilon_\infty}$ -fibred space over  $\mathbb{S}$  is by definition a presheaf of spaces on the category  $\text{Sch}_{/\mathbb{S}}^{\varepsilon_\infty}$ .

We will write  $\underline{\text{Spc}}^{\varepsilon_\infty}(\mathbb{S})$  for the category of  $\text{Sch}^{\varepsilon_\infty}$ -fibred spaces.

The Yoneda embedding defines a fully faithful functor

$$h_{\mathbb{S}}(-) : \text{Sch}_{/\mathbb{S}}^{\varepsilon_\infty} \hookrightarrow \underline{\text{Spc}}^{\varepsilon_\infty}(\mathbb{S}).$$

2.1.4. Recall the notion of *Nisnevich square* from [Kha16, ¶ 4.1]. We say that a  $\text{Sch}^{\varepsilon_\infty}$ -fibred space  $\mathcal{F}$  satisfies *Nisnevich excision*, or is *Nisnevich-local*, if:

- (a) It is *reduced*, i.e. the space  $\Gamma(\emptyset, \mathcal{F})$  is contractible.
- (b) It sends any Nisnevich square of afp spectral S-schemes to a cartesian square of spaces.

Let  $\underline{\text{Spc}}_{\text{Nis}}^{\varepsilon_\infty}(\text{S})$  denote the full subcategory of  $\underline{\text{Spc}}^{\varepsilon_\infty}(\text{S})$  spanned by Nisnevich-local spaces.

2.1.5. Let  $\mathbf{A}^1$  denote the spectral affine line (see [Kha16, 2.6.10]). We say that a  $\text{Sch}^{\varepsilon_\infty}$ -fibred space  $\mathcal{F}$  satisfies  $\mathbf{A}^1$ -*homotopy invariance*, or is  $\mathbf{A}^1$ -*local*, if, for every spectral S-scheme X, the canonical map

$$\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X \times \mathbf{A}^1, \mathcal{F}),$$

induced by the projection  $X \times \mathbf{A}^1 \rightarrow X$ , is invertible.

Let  $\underline{\text{Spc}}_{\mathbf{A}^1}^{\varepsilon_\infty}(\text{S})$  denote the full subcategory of  $\underline{\text{Spc}}^{\varepsilon_\infty}(\text{S})$  spanned by  $\mathbf{A}^1$ -local spaces.

2.1.6. A *motivic  $\text{Sch}^{\varepsilon_\infty}$ -fibred space* is a  $\text{Sch}^{\varepsilon_\infty}$ -fibred space satisfying Nisnevich excision and  $\mathbf{A}^1$ -homotopy invariance.

Let  $\underline{\mathbf{H}}^{\varepsilon_\infty}(\text{S})$  denote the full subcategory of  $\underline{\text{Spc}}^{\varepsilon_\infty}(\text{S})$  spanned by motivic  $\text{Sch}^{\varepsilon_\infty}$ -fibred spaces.

2.1.7. This construction fits into the general framework of [Kha16, § 3], and we have:

**Lemma 2.1.8.**

(i) *The category  $\underline{\text{Spc}}_{\text{Nis}}^{\varepsilon_\infty}(\text{S})$  is an accessible left localization of  $\underline{\text{Spc}}^{\varepsilon_\infty}(\text{S})$ , and the localization functor  $\mathcal{F} \mapsto \text{L}_{\text{Nis}}(\mathcal{F})$  is exact.*

(ii) *The category  $\underline{\text{Spc}}_{\mathbf{A}^1}^{\varepsilon_\infty}(\text{S})$  is an accessible left localization of  $\underline{\text{Spc}}^{\varepsilon_\infty}(\text{S})$ , and the localization functor  $\mathcal{F} \mapsto \text{L}_{\mathbf{A}^1}(\mathcal{F})$  admits the following description: for every  $\text{Sch}^{\varepsilon_\infty}$ -fibred space  $\mathcal{F}$ , there is a canonical isomorphism*

$$(2.1) \quad \Gamma(X, \text{L}_{\mathbf{A}^1}(\mathcal{F})) = \varinjlim_{(Y \rightarrow X) \in (\mathbf{A}_X)^{\text{op}}} \Gamma(Y, \mathcal{F})$$

for each afp spectral S-scheme X. Here  $(\mathbf{A}_X)^{\text{op}}$  is a sifted small category, opposite to the full subcategory of  $\text{Sm}_{/X}^{\varepsilon_\infty}$  spanned by compositions of  $\mathbf{A}^1$ -projections.

(iii) *The category  $\underline{\mathbf{H}}^{\varepsilon_\infty}(\text{S})$  is an accessible left localization of  $\underline{\text{Spc}}^{\varepsilon_\infty}(\text{S})$ . Further, the localization functor  $\mathcal{F} \mapsto \text{L}_{\text{mot}}(\mathcal{F})$  can be described as the transfinite composite*

$$(2.2) \quad \text{L}_{\text{mot}}(\mathcal{F}) = \varinjlim_{n \geq 0} (\text{L}_{\mathbf{A}^1} \circ \text{L}_{\text{Nis}})^{\circ n}(\mathcal{F}).$$

(vi) *The category  $\underline{\mathbf{H}}^{\varepsilon_\infty}(\text{S})$  is generated under sifted colimits by objects of the form  $\text{L}_{\text{mot}} \text{h}_S(X)$ , where  $X = \text{Spec}(A)$  is an affine afp spectral S-scheme.*

*Proof.* See [Kha16, § 4], replacing all instances of the word “smooth” by “afp”. □

2.2. **Sm-fibred spaces.** In this paragraph we show that the  $\text{Sch}^{\varepsilon_\infty}$ -fibred variant of the motivic homotopy category is indeed an enlargement of the usual ( $\text{Sm}^{\varepsilon_\infty}$ -fibred) version constructed in [Kha16, § 4].

This is tautological at the level of presheaves, and it is only necessary to verify that the inclusion functor  $\text{Sm}_{/S}^{\varepsilon_\infty} \hookrightarrow \text{Sch}_{/S}^{\varepsilon_\infty}$  is well-behaved with respect to the various localizations.

2.2.1. Let  $\mathrm{Sm}_{/S}^{\mathcal{E}^\infty}$  denote the category of smooth spectral  $S$ -schemes of finite presentation. In the sequel, “smooth” will always mean “smooth of finite presentation”.

Recall from [Kha16, § 4] that a  $\mathrm{Sm}^{\mathcal{E}^\infty}$ -fibre space over  $S$  is a presheaf of spaces on  $\mathrm{Sm}_{/S}^{\mathcal{E}^\infty}$ . We write  $\mathrm{Spc}^{\mathcal{E}^\infty}(S)$  for the category of  $\mathrm{Sm}^{\mathcal{E}^\infty}$ -fibre spaces.

2.2.2. Let  $\iota_{\mathrm{Sm}} : \mathrm{Sm}_{/S}^{\mathcal{E}^\infty} \hookrightarrow \mathrm{Sch}_{/S}^{\mathcal{E}^\infty}$  denote the inclusion functor. This induces a canonical fully faithful functor

$$(\iota_{\mathrm{Sm}})_! : \mathrm{Spc}^{\mathcal{E}^\infty}(S) \rightarrow \underline{\mathrm{Spc}}^{\mathcal{E}^\infty}(S),$$

left adjoint to the restriction functor  $(\iota_{\mathrm{Sm}})^*$ . Its essential image can be described as the full subcategory of  $\underline{\mathrm{Spc}}^{\mathcal{E}^\infty}(S)$  generated under colimits by objects of the form  $h_S(X)$ , with  $X$  a smooth spectral  $S$ -scheme.

By abuse of notation we will identify  $\mathrm{Spc}^{\mathcal{E}^\infty}(S)$  with its essential image in  $\underline{\mathrm{Spc}}^{\mathcal{E}^\infty}(S)$ .

2.2.3. We form localizations  $\mathrm{Spc}_{\mathrm{Nis}}^{\mathcal{E}^\infty}(S)$ ,  $\mathrm{Spc}_{\mathbf{A}^1}^{\mathcal{E}^\infty}(S)$ , and  $\mathbf{H}^{\mathcal{E}^\infty}(S)$  of the category  $\mathrm{Spc}^{\mathcal{E}^\infty}(S)$  (see [Kha16, § 4] for details).

We have:

**Lemma 2.2.4.**

- (i) *The functor  $(\iota_{\mathrm{Sm}})_!$  preserves Nisnevich-local equivalences. Its right adjoint  $(\iota_{\mathrm{Sm}})^*$  preserves Nisnevich-local spaces and Nisnevich-local equivalences.*
- (ii) *The functor  $(\iota_{\mathrm{Sm}})_!$  preserves  $\mathbf{A}^1$ -local equivalences. Its right adjoint  $(\iota_{\mathrm{Sm}})^*$  preserves  $\mathbf{A}^1$ -local spaces and  $\mathbf{A}^1$ -local equivalences.*

*Proof.*

(i) It is clear that  $(\iota_{\mathrm{Sm}})_!$  preserves Nisnevich-local equivalences, so its right adjoint preserves Nisnevich-local spaces by adjunction. To see that  $(\iota_{\mathrm{Sm}})^*$  preserves Nisnevich-local equivalences, it suffices to show that  $\iota_{\mathrm{Sm}}$  is topologically cocontinuous. This is clear because if  $X$  is smooth over  $S$ , and

$$\begin{array}{ccc} U \times_X V & \xleftarrow{k} & V \\ \downarrow & & \downarrow p \\ U & \xleftarrow{j} & X \end{array}$$

is a Nisnevich square over  $S$ , then both  $U$  and  $V$  will also be smooth over  $S$ .

(ii) It is clear that  $(\iota_{\mathrm{Sm}})_!$  preserves  $\mathbf{A}^1$ -local equivalences, so its right adjoint preserves  $\mathbf{A}^1$ -local spaces by adjunction. For the second claim it suffices to show that, for any spectral  $S$ -scheme  $X$ , the canonical morphism

$$i^* h_S(X \times \mathbf{A}^1) \rightarrow i^* h_S(X)$$

is an  $\mathbf{A}^1$ -local equivalence of  $\mathrm{Sm}^{\mathcal{E}^\infty}$ -fibre spaces. By universality of colimits it suffices to show that, for any smooth spectral  $S$ -scheme  $Y$  and morphism of presheaves  $\varphi : h_S(Y) \rightarrow i^* h_S(X)$ , the base change

$$i^* h_S(X \times \mathbf{A}^1) \times_{i^* h_S(X)} h_S(Y) \rightarrow h_S(Y)$$

is an  $\mathbf{A}^1$ -local equivalence. Since the morphism  $\varphi$  factors as  $h_S(Y) \rightarrow i^* h_S(Y) \rightarrow i^* h_S(X)$ , the morphism in question is obtained by base change from the morphism

$$i^* h_S(X \times \mathbf{A}^1) \times_{i^* h_S(X)} i^* h_S(Y) \rightarrow i^* h_S(Y),$$

which is identified with the canonical morphism

$$h_S(Y \times \mathbf{A}^1) \rightarrow h_S(Y),$$

since  $i^*$  and  $h_S$  commute with limits and  $i^*i_! = \text{id}$ . This is an  $\mathbf{A}^1$ -local equivalence, so the claim follows.  $\square$

2.2.5. It follows that the functor  $(\iota_{\text{Sm}})^*$  restricts to a well-defined functor

$$(\iota_{\text{Sm}})^* : \underline{\mathbf{H}}^{\mathcal{E}^\infty}(\mathbf{S}) \rightarrow \mathbf{H}^{\mathcal{E}^\infty}(\mathbf{S}),$$

which is right adjoint to  $L_{\text{mot}} \circ (\iota_{\text{Sm}})_!$ . Further, we have:

**Proposition 2.2.6.** *The functor  $L_{\text{mot}} \circ (\iota_{\text{Sm}})_! : \mathbf{H}^{\mathcal{E}^\infty}(\mathbf{S}) \rightarrow \underline{\mathbf{H}}^{\mathcal{E}^\infty}(\mathbf{S})$  is fully faithful. Its essential image is generated under colimits by the objects  $L_{\text{mot}} h_S(\mathbf{X})$ , for  $\mathbf{X}$  a smooth spectral  $\mathbf{S}$ -scheme.*

*Proof.* Given a functor  $u$  with a left adjoint  $u^L$  and a right adjoint  $u^R$ , it is a standard fact that  $u^L$  is fully faithful if and only if  $u^R$  is. Note that the functor  $(\iota_{\text{Sm}})^*$  does indeed have a right adjoint (at the level of motivic spaces): at the level of presheaves, it has a right adjoint  $(\iota_{\text{Sm}})_*$  given by right Kan extension; but since  $(\iota_{\text{Sm}})^*$  preserves motivic equivalences by Lemma 2.2.4,  $(\iota_{\text{Sm}})_*$  preserves motivic spaces (and hence restricts to a right adjoint).

Thus it suffices to show that  $(\iota_{\text{Sm}})_*$  is fully faithful. This follows directly from the fact that  $\iota_{\text{Sm}}$  is fully faithful.  $\square$

*Remark 2.2.7.* Note that the same applies for the localization functors  $L_{\text{Nis}}$  and  $L_{\mathbf{A}^1}$ .

2.2.8. For the record, let us also state the classical analogue of Proposition 2.2.6.

Let  $\text{Sch}_{/S_{\text{cl}}}$  denote the category of classical  $S_{\text{cl}}$ -schemes (of finite presentation), where  $S_{\text{cl}}$  denotes the underlying classical scheme of  $\mathbf{S}$ . Let  $\text{Sm}_{/S_{\text{cl}}}$  denote the full subcategory spanned by smooth classical  $S_{\text{cl}}$ -schemes (of finite presentation).

Let  $\text{Spc}^{\text{cl}}(S_{\text{cl}})$  (resp.  $\text{Spc}^{\text{cl}}(\mathbf{S})$ ) denote the category of  $\text{Sch}$ -fibred spaces (resp.  $\text{Sm}$ -fibred spaces) over  $S_{\text{cl}}$ , i.e. presheaves of spaces on  $\text{Sch}_{/S_{\text{cl}}}$  (resp. on  $\text{Sm}_{/S_{\text{cl}}}$ ).

Let  $\underline{\mathbf{H}}^{\text{cl}}(\mathbf{S})$  (resp.  $\mathbf{H}^{\text{cl}}(\mathbf{S})$ ) denote the full subcategory spanned by  $\text{Sch}$ -fibred motivic spaces (resp.  $\text{Sm}$ -fibred motivic spaces) over  $S_{\text{cl}}$ .

Then we have:

**Proposition 2.2.9.** *There is a canonical fully faithful functor*

$$L_{\text{mot}} \circ (\iota_{\text{Sm}})_! : \mathbf{H}^{\text{cl}}(S_{\text{cl}}) \rightarrow \underline{\mathbf{H}}^{\text{cl}}(S_{\text{cl}}),$$

*whose essential image is spanned by objects of the form  $L_{\text{mot}} h_{S_{\text{cl}}}(\mathbf{X})$ , where  $\mathbf{X}$  is a smooth classical  $S_{\text{cl}}$ -scheme.*

2.2.10. The following observation will be useful:

**Lemma 2.2.11.** *The localization functor  $\mathcal{F} \mapsto L_{\mathbf{A}^1}(\mathcal{F})$  commutes with the restriction functor  $(\iota_{\text{Sm}})^*$  on  $\text{Sch}^{\mathcal{E}^\infty}$ -fibred spaces.*

*Proof.* According to Lemma 2.1.8 and [Kha16, Lem. 4.2.3], both  $(\iota_{\text{Sm}})^*(L_{\mathbf{A}^1}(\mathcal{F}))$  and  $L_{\mathbf{A}^1}(\iota_{\text{Sm}})^*(\mathcal{F})$  are given section-wise by the same formula

$$\lim_{(Y \rightarrow X) \in (\mathbf{A}_X)^{\text{op}}} \Gamma(Y, \mathcal{F})$$

for each smooth spectral  $\mathbf{S}$ -scheme  $\mathbf{X}$ .  $\square$

**2.3. Functoriality.** The method described in [Kha16, §§ 5–6] evidently adapts to give the following basic<sup>1</sup> functorialities  $(f_{\sharp}, f^*, f_*)$  for each of the categories considered so far.

<sup>1</sup>As opposed to the “exceptional” functorialities  $(f_!, f^!)$ .

2.3.1. Let  $f : T \rightarrow S$  be a morphism of spectral schemes.

For  $\text{Sch}^{\varepsilon_\infty}$ -fibred spaces we have the operations  $(f_{\sharp}^{\text{Spc}}, f_{\text{Spc}}^*, f_*^{\text{Spc}})$ , as follows. The functor  $f_{\text{Spc}}^*$  sends a space of the form  $\text{h}_S(X)$ , with  $X$  a spectral  $S$ -scheme, to the space  $\text{h}_T(X \times_S T)$ . It commutes with colimits and admits a right adjoint  $f_*^{\text{Spc}}$ . When  $f$  is *afp*,  $f^*$  also admits a left adjoint  $f_{\sharp}^{\text{Spc}}$ , which sends a space of the form  $\text{h}_T(X)$  to the space  $\text{h}_S(X)$ , where  $X$  is viewed as an  $S$ -scheme by extending the structural map  $X \rightarrow T$  along  $f$ .

2.3.2. These operations induce operations  $(f_{\sharp}^{\mathbf{H}}, f_{\mathbf{H}}^*, f_*^{\mathbf{H}})$  motivic  $\text{Sch}^{\varepsilon_\infty}$ -fibred spaces as follows. First, the functor  $f_{\text{Spc}}^*$  preserves motivic spaces. Therefore we obtain an operation  $f_{\sharp}^{\mathbf{H}}$  on motivic spaces, right adjoint to  $f_{\mathbf{H}}^* := L_{\text{mot}} f_{\text{Spc}}^*$ . For  $f$  *afp*, the functor  $f_{\text{Spc}}^*$  already preserves motivic spaces and it will be right adjoint to  $f_{\sharp}^{\mathbf{H}} := L_{\text{mot}} f_{\sharp}$ .

2.3.3. On  $\text{Sm}^{\varepsilon_\infty}$ -fibred motivic spaces, the operation  $f_{\mathbf{H}}^*$  restricts to an operation  $f_{\mathbf{H}}^*$ , which is left adjoint to  $f_{\sharp}^{\mathbf{H}} := (\iota_{\text{Sm}})^* f_{\text{Spc}}^* L_{\text{mot}}(\iota_{\text{Sm}})!$ . When  $f$  is smooth, the operation  $f_{\sharp}^{\mathbf{H}}$  also restricts to an operation  $f_{\sharp}^{\mathbf{H}}$  which is left adjoint to  $f_{\mathbf{H}}^*$ . These operations  $(f_{\sharp}^{\mathbf{H}}, f_{\mathbf{H}}^*, f_*^{\mathbf{H}})$  are evidently the same as those constructed in [Kha16, §§ 5–6].

2.3.4. Whenever the functor  $f_{\sharp}$  exists, the operation  $f^*$  satisfies left base change [Kha16, Def. A.4.4] and the left projection formula [Kha16, Def. A.4.6] along it.

2.3.5. Since the base change functor  $\text{Sm}_{/S}^{\varepsilon_\infty} \rightarrow \text{Sm}_{/T}^{\varepsilon_\infty}$  commutes with the inclusion  $\iota_{\text{Sm}} : \text{Sm}_{/S}^{\varepsilon_\infty} \hookrightarrow \text{Sch}_{/S}^{\varepsilon_\infty}$ , we have:

**Lemma 2.3.6.**

(i) *There are canonical 2-isomorphisms*

$$\begin{aligned} (\iota_{\text{Sm}})! \circ f_{\text{Spc}}^* &\approx f_{\text{Spc}}^* \circ (\iota_{\text{Sm}})! \\ f_*^{\text{Spc}} \circ (\iota_{\text{Sm}})^* &\approx (\iota_{\text{Sm}})^* \circ f_{\text{Spc}}^*. \end{aligned}$$

(ii) *There are canonical 2-isomorphisms*

$$\begin{aligned} L_{\text{mot}}(\iota_{\text{Sm}})! \circ f_{\mathbf{H}}^* &\approx f_{\mathbf{H}}^* \circ L_{\text{mot}}(\iota_{\text{Sm}})! \\ f_{\sharp}^{\mathbf{H}} \circ (\iota_{\text{Sm}})^* &\approx (\iota_{\text{Sm}})^* \circ f_{\sharp}^{\mathbf{H}}. \end{aligned}$$

Similarly, when  $f$  is smooth, the forgetful functor  $\text{Sm}_{/T}^{\varepsilon_\infty} \rightarrow \text{Sm}_{/S}^{\varepsilon_\infty}$  commutes with the inclusion  $\iota_{\text{Sm}} : \text{Sm}_{/S}^{\varepsilon_\infty} \hookrightarrow \text{Sch}_{/S}^{\varepsilon_\infty}$ , so that we have:

**Lemma 2.3.7.**

(i) *There are canonical 2-isomorphisms*

$$\begin{aligned} (\iota_{\text{Sm}})! \circ f_{\sharp}^{\text{Spc}} &\approx f_{\sharp}^{\text{Spc}} \circ (\iota_{\text{Sm}})! \\ f_{\text{Spc}}^* \circ (\iota_{\text{Sm}})^* &\approx (\iota_{\text{Sm}})^* \circ f_{\text{Spc}}^*. \end{aligned}$$

(ii) *There are canonical 2-isomorphisms*

$$\begin{aligned} L_{\text{mot}}(\iota_{\text{Sm}})! \circ f_{\sharp}^{\mathbf{H}} &\approx f_{\sharp}^{\mathbf{H}} \circ L_{\text{mot}}(\iota_{\text{Sm}})! \\ f_{\mathbf{H}}^* \circ (\iota_{\text{Sm}})^* &\approx (\iota_{\text{Sm}})^* \circ f_{\mathbf{H}}^*. \end{aligned}$$

2.4. **Direct image along closed immersions.**



2.4.1. Just as in [Kha16, Prop. 7.1.2], we have:

**Proposition 2.4.2.** *Let  $i : Z \hookrightarrow S$  be a closed immersion of spectral schemes. The functor  $(i_*)^{\mathbf{H}}$  commutes with contractible<sup>2</sup> colimits.*

*Proof.* The proof of [Kha16, Prop. 7.1.2] also works on the afp site (i.e., the base change functor  $\mathrm{Sch}_{/S}^{\varepsilon_\infty} \rightarrow \mathrm{Sch}_{/Z}^{\varepsilon_\infty}$  is also topologically quasi-cocontinuous).  $\square$

We also have:

**Theorem 2.4.3.** *Let  $i : Z \hookrightarrow S$  be a closed immersion of spectral schemes with quasi-compact open complement  $j : U \hookrightarrow S$ . For any  $\mathrm{Sm}^{\varepsilon_\infty}$ -fibred motivic space  $\mathcal{F}$  over  $S$ , there is a cocartesian square*

$$\begin{array}{ccc} j_{\#}^{\mathbf{H}} j_{\mathbf{H}}^*(\mathcal{F}) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ j_{\#}^{\mathbf{H}}(\mathrm{pt}_U) & \longrightarrow & i_*^{\mathbf{H}} i_{\mathbf{H}}^*(\mathcal{F}) \end{array}$$

of  $\mathrm{Sm}^{\varepsilon_\infty}$ -fibred motivic spaces over  $S$ .

Note that this does not follow tautologically from the localization theorem as stated in [Kha16], since we do not (yet) know that the operation  $i_*$  commutes with the inclusion  $L_{\mathrm{mot}}(t_{\mathrm{Sm}})!$  (but see Corollary 2.4.6).

*Proof.* By Proposition 2.4.2 and Proposition 2.2.6, we may reduce to the case where  $\mathcal{F} = L_{\mathrm{mot}} h_S(X)$  for some smooth spectral  $S$ -scheme  $X$ . Then the proof of [Kha16, ¶ 8.5] applies *mutatis mutandis*.  $\square$

*Remark 2.4.4.* The localization theorem [Kha16, Thm. 7.2.6] does not hold for arbitrary  $\mathrm{Sch}^{\varepsilon_\infty}$ -fibred motivic spaces.

The point is that the category  $\mathbf{H}^{\varepsilon_\infty}(S)$  has “too many” base change formulas. Indeed, for any afp closed immersion  $i$ , we have a functor  $i_{\#}^{\mathbf{H}}$  satisfying a base change formula against  $f_{\mathbf{H}}^*$ . By adjunction this means that its right adjoint  $i_{\mathbf{H}}^*$  satisfies base change against  $f_{\#}^{\mathbf{H}}$ . Thus, if  $j : U \hookrightarrow S$  is the open complement of  $i : Z \hookrightarrow S$ , then associated to the cartesian square

$$\begin{array}{ccc} \emptyset & \hookrightarrow & Z \\ \downarrow & & \downarrow i \\ U & \xrightarrow{j} & S, \end{array}$$

we have for every  $\mathcal{F}_U \in \mathbf{H}^{\varepsilon_\infty}(U)$  an isomorphism

$$(2.3) \quad i_{\mathbf{H}}^* j_{\#}^{\mathbf{H}}(\mathcal{F}_U) \approx \mathrm{pt}_Z$$

in addition to the usual base change formulas  $i_{\mathbf{H}}^* j_{\#}^{\mathbf{H}} \approx \emptyset_Z$  and  $j_{\#}^{\mathbf{H}} i_{\mathbf{H}}^* \approx \mathrm{pt}_U$  (see [Kha16, Lem. 7.2.2 and 7.2.4]).

For example, let  $\mathcal{F} \in \mathbf{SH}^{\varepsilon_\infty}(S)_{S^1}$  be a  $\mathrm{Sch}^{\varepsilon_\infty}$ -fibred motivic spectrum over  $S$ . Consider the commutative diagram (we will omit the decoration  $\mathbf{H}$  from the notation in the remainder of this remark)

$$\begin{array}{ccccc} j_{\#} j^*(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & j_* j^*(\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & i_* i^*(\mathcal{F}) & \longrightarrow & \mathcal{K}, \end{array}$$

<sup>2</sup>Recall that an  $\infty$ -category is *contractible* if the  $\infty$ -groupoid formed by formally adding inverses to all morphisms is (weakly) contractible.

where  $\mathcal{K}$  is defined as the cofibre of the horizontal composite  $j_{\#}j^*(\mathcal{F}) \rightarrow j_*j^*(\mathcal{F})$ . Localization would say that the left-hand square is cocartesian, and therefore that the right-hand square is cocartesian (since the composite square is cocartesian by construction). Since we are in the setting of  $S^1$ -spectra, this means that the right-hand square is also cartesian.

The base change formulas mentioned above imply that both objects  $i^*(\mathcal{K})$  and  $j^*(\mathcal{K})$  are zero. Hence by localization,  $\mathcal{K}$  is itself zero, and we obtain a trivial splitting

$$\mathcal{F} \approx i_*i^*(\mathcal{F}) \times j_*j^*(\mathcal{F})$$

for any  $\text{Sch}^{\mathcal{E}^\infty}$ -fibred motivic spectrum  $\mathcal{F}$ .

In view of the previous theorem, we can interpret the above observation as saying that  $j_*$  does not generally commute with the inclusion  $L_{\text{mot}}(\iota_{\text{Sm}})!$  (i.e. the functors  $j_*^{\mathbf{H}}$  and  $j_{\#}^{\mathbf{H}}$  are not compatible), so that even if  $\mathcal{F}$  is  $\text{Sm}^{\mathcal{E}^\infty}$ -fibred, and thus satisfies localization, the object  $\mathcal{K}$  is in general not  $\text{Sm}^{\mathcal{E}^\infty}$ -fibred.

2.4.5. A corollary of Theorem 2.4.3 is that the functor  $i_*$  commutes with the inclusion  $L_{\text{mot}}(\iota_{\text{Sm}})!$ .

More precisely, the 2-isomorphism  $L_{\text{mot}}(\iota_{\text{Sm}})! \circ i_{\mathbf{H}}^* \approx i_{\mathbf{H}}^* \circ L_{\text{mot}}(\iota_{\text{Sm}})!$  (Lemma 2.3.6) induces, by the procedure described in [Kha16, ¶ A.2], a canonical 2-morphism

$$(2.4) \quad L_{\text{mot}}(\iota_{\text{Sm}})! \circ i_{\mathbf{H}}^* \rightarrow i_{\mathbf{H}}^* \circ L_{\text{mot}}(\iota_{\text{Sm}})!$$

which we claim is invertible:

**Corollary 2.4.6.** *Let  $i : Z \hookrightarrow S$  be a closed immersion of spectral schemes with quasi-compact open complement. Then the canonical 2-morphism (2.4) is invertible.*

*Proof.* Since the category  $\mathbf{H}^{\mathcal{E}^\infty}(Z)$  is generated under sifted colimits by objects of the form  $M_Z(X)$ , where  $X$  is a smooth spectral  $Z$ -scheme [Kha16, Lem. 4.3.5], it suffices to show that the canonical morphism

$$L_{\text{mot}}(\iota_{\text{Sm}})!(i_{\mathbf{H}}^*(M_Z(X))) \rightarrow i_{\mathbf{H}}^*(M_Z(X))$$

is invertible, where  $X$  is as above.

By [Kha16, Prop. 2.12.2] it follows that we may assume that  $X$  is of the form  $Y \times_S Z$ , where  $Y$  is a smooth spectral  $S$ -scheme.

By the localization theorem [Kha16, Thm. 7.2.6], we have a natural isomorphism

$$i_{\mathbf{H}}^*(M_Z(X)) \approx M_S(Y) \sqcup_{M_S(Y_U)} M_S(U),$$

where  $U$  is the open complement of  $i : Z \hookrightarrow S$ , and  $Y_U := Y \times_S U$ .

Applying Theorem 2.4.3 to  $\mathcal{F} = M_S(Y)$ , we obtain a natural isomorphism

$$i_{\mathbf{H}}^*(M_Z(X)) \approx M_S(Y) \sqcup_{M_S(Y_U)} M_S(U),$$

and the claim follows. □

2.5. **Continuity.** In this paragraph we demonstrate a continuity property. This will be useful for the elimination of noetherian hypotheses (see the proof of Theorem 3.4.2).

For this purpose it will be convenient to form yet another enlargement of the category  $\mathbf{H}^{\mathcal{E}^\infty}(S)$ , generated by spectral  $S$ -schemes of finite presentation.

2.5.1. Recall that a morphism of affine spectral schemes  $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$  is of finite presentation if  $B$  is a compact object in the category of  $A$ -algebras. We globalize this definition in the same way as Definition 2.1.2. Write  $\mathrm{Sch}_{\mathrm{fp}/S}^{\varepsilon_\infty}$  for the category of spectral schemes of finite presentation over  $S$ .

Let  $\underline{\mathrm{Spc}}_{\mathrm{fp}}^{\varepsilon_\infty}(S)$  denote the category of  $\mathrm{Sch}_{\mathrm{fp}}^{\varepsilon_\infty}$ -fibred spaces, i.e. presheaves of spaces on  $\mathrm{Sch}_{\mathrm{fp}/S}^{\varepsilon_\infty}$ . Let  $\underline{\mathbf{H}}_{\mathrm{fp}}^{\varepsilon_\infty}(S)$  denote the category of  $\mathrm{Sch}_{\mathrm{fp}}^{\varepsilon_\infty}$ -fibred motivic spaces, i.e. the full subcategory of  $\underline{\mathrm{Spc}}_{\mathrm{fp}}^{\varepsilon_\infty}(S)$  spanned by Nisnevich-local  $\mathbf{A}^1$ -local spaces.

This construction has the same formal properties as  $\underline{\mathbf{H}}^{\varepsilon_\infty}(S)$ . In particular it is generated under sifted colimits by objects of the form  $L_{\mathrm{mot}} h_S(X)$ , where  $X$  is an affine spectral  $S$ -scheme of finite presentation (same proof as Lemma 2.1.8).

The procedure of Paragraph 2.3 equips  $\underline{\mathbf{H}}_{\mathrm{fp}}^{\varepsilon_\infty}$  with the basic functorialities  $(f_\sharp, f^*, f_*)$ . The operation  $f_\sharp$  is defined when  $f$  is of finite presentation.

2.5.2. The inclusion  $\iota_{\mathrm{Sm}} : \mathrm{Sm}_{/S}^{\varepsilon_\infty} \hookrightarrow \mathrm{Sch}_{/S}^{\varepsilon_\infty}$  factors through the full subcategory  $\mathrm{Sch}_{\mathrm{fp}/S}^{\varepsilon_\infty}$ . By abuse of notation we will write  $\iota_{\mathrm{Sm}}$  also for the inclusion  $\mathrm{Sm}_{/S}^{\varepsilon_\infty} \hookrightarrow \mathrm{Sch}_{\mathrm{fp}/S}^{\varepsilon_\infty}$ .

We have (same proof as Proposition 2.2.6):

**Proposition 2.5.3.** *The functor  $L_{\mathrm{mot}} \circ (\iota_{\mathrm{Sm}})_! : \mathbf{H}^{\varepsilon_\infty}(S) \rightarrow \underline{\mathbf{H}}_{\mathrm{fp}}^{\varepsilon_\infty}(S)$  is fully faithful. Its essential image is generated under colimits by the objects  $L_{\mathrm{mot}} h_S(X)$ , for  $X$  a smooth spectral  $S$ -scheme.*

2.5.4. Let  $S$  be an affine spectral scheme. Suppose that  $S$  can be written as the limit of a cofiltered diagram  $(S_\alpha)_\alpha$  of affine spectral schemes.

Consider the canonical functor

$$(2.5) \quad \varinjlim_\alpha \underline{\mathbf{H}}_{\mathrm{fp}}^{\varepsilon_\infty}(S_\alpha) \rightarrow \underline{\mathbf{H}}_{\mathrm{fp}}^{\varepsilon_\infty}(S)$$

induced by the inverse image functors  $(f_\alpha)^*$ , where  $f_\alpha : S \rightarrow S_\alpha$ . The colimit<sup>3</sup> here is taken the category of presentable  $\infty$ -categories and left adjoint functors, and the transition arrows in the filtered diagram are the inverse image functors  $(f_{\alpha,\beta})^*$ . We have:

**Proposition 2.5.5.** *The functor (2.5) is an equivalence.*

*Proof.* The main ingredient in this proof is a theorem of Lurie [Lur16b, Thm. 4.4.2.2] that provides an equivalence of categories

$$(2.6) \quad \varinjlim_\alpha \mathrm{Sch}_{\mathrm{aff},\mathrm{fp}/S_\alpha}^{\varepsilon_\infty} \xrightarrow{\sim} \mathrm{Sch}_{\mathrm{aff},\mathrm{fp}/S}^{\varepsilon_\infty}.$$

In order to apply this, it is convenient to consider a version of  $\underline{\mathbf{H}}^{\varepsilon_\infty}$  built out of the category  $\mathrm{Sch}_{\mathrm{aff},\mathrm{fp}}^{\varepsilon_\infty}$  of *affine* spectral  $S$ -schemes of finite presentation. Thus let  $\underline{\mathrm{Spc}}_{\mathrm{aff},\mathrm{fp}}^{\varepsilon_\infty}(S)$  denote the category of  $\mathrm{Sch}_{\mathrm{aff},\mathrm{fp}}^{\varepsilon_\infty}$ -fibred spaces, i.e. presheaves of spaces on  $\mathrm{Sch}_{\mathrm{aff},\mathrm{fp}/S}^{\varepsilon_\infty}$ . Let  $\underline{\mathbf{H}}_{\mathrm{aff},\mathrm{fp}}^{\varepsilon_\infty}(S)$  denote the category of  $\mathrm{Sch}_{\mathrm{aff},\mathrm{fp}}^{\varepsilon_\infty}$ -fibred motivic spaces, i.e. the full subcategory of  $\underline{\mathrm{Spc}}_{\mathrm{aff},\mathrm{fp}}^{\varepsilon_\infty}(S)$  spanned by Nisnevich-local  $\mathbf{A}^1$ -local spaces. By Zariski descent, the inclusion  $\mathrm{Sch}_{\mathrm{aff},\mathrm{fp}/S}^{\varepsilon_\infty} \hookrightarrow \mathrm{Sch}_{\mathrm{fp}/S}^{\varepsilon_\infty}$  induces an equivalence

$$\underline{\mathbf{H}}_{\mathrm{aff},\mathrm{fp}}^{\varepsilon_\infty}(S) \xrightarrow{\sim} \underline{\mathbf{H}}_{\mathrm{fp}}^{\varepsilon_\infty}(S).$$

Since this equivalence commutes with the inverse image functors  $(f_\alpha)^*$ , it will suffice to demonstrate the claim for  $\underline{\mathbf{H}}_{\mathrm{aff},\mathrm{fp}}^{\varepsilon_\infty}(S)$ .

<sup>3</sup>It can also be computed as a limit, where the transition arrows are the right adjoints  $(f_{\alpha,\beta})_*$ . This limit can be taken in the category of  $\infty$ -categories.

The equivalence (2.6) induces an equivalence

$$\varinjlim_{\alpha} \underline{\mathrm{Spc}}_{\mathrm{aff}, \mathrm{fp}}^{\varepsilon_{\infty}}(S_{\alpha}) \xrightarrow{\sim} \underline{\mathrm{Spc}}_{\mathrm{aff}, \mathrm{fp}}^{\varepsilon_{\infty}}(S).$$

It suffices to note that this equivalence preserves and detects  $\mathbf{A}^1$ -projections and Nisnevich covering families.

Indeed the equivalence (2.6) implies that for any affine spectral  $S$ -scheme  $X$  of finite presentation, there exists some index  $\alpha$  and an affine spectral  $S_{\alpha}$ -scheme  $X_{\alpha}$  such that the projection  $X \times \mathbf{A}^1 \rightarrow X$  is a base change of  $X_{\alpha} \times \mathbf{A}^1 \rightarrow X_{\alpha}$ . Similarly the results of [Lur16b, § 4.6] imply that any Nisnevich covering of  $X$  is a base change of a Nisnevich covering of some some affine spectral  $S_{\alpha}$ -scheme  $X_{\alpha}$ .  $\square$

**Corollary 2.5.6.** *The functor (2.5) restricts to an equivalence*

$$\varinjlim_{\alpha} \mathbf{H}^{\varepsilon_{\infty}}(S_{\alpha}) \xrightarrow{\sim} \mathbf{H}^{\varepsilon_{\infty}}(S).$$

*Proof.* It suffices to show that the equivalence of Proposition 2.5.5 preserves and detects  $\mathrm{Sm}^{\varepsilon_{\infty}}$ -fibred spaces.

Preservation is clear: it suffices to note that for any smooth spectral  $S_{\alpha}$ -scheme  $X_{\alpha}$ , the image  $(f_{\alpha})^*(M_{S_{\alpha}}(X_{\alpha})) = M_S(X_{\alpha} \times_{S_{\alpha}} S)$  is an  $\mathrm{Sm}^{\varepsilon_{\infty}}$ -fibred space over  $S$ .

Conversely, we need to show that every  $\mathrm{Sm}^{\varepsilon_{\infty}}$ -fibred space in the target comes from a  $\mathrm{Sm}^{\varepsilon_{\infty}}$ -fibred space in the source. It suffices to consider spaces of the form  $M_S(X)$ , where  $X$  is a smooth spectral  $S$ -scheme. By [Kha16, Prop. 2.9.10] and Zariski descent we may assume that  $X$  admits an étale morphism to an affine space  $\mathbf{A}_S^n$ . Since  $\mathbf{A}_S^n$  is the limit of the cofiltered diagram  $(\mathbf{A}_{S_{\alpha}}^n)_{\alpha}$ , it follows from the equivalence (2.6) that there exists an index  $\alpha$  and a spectral  $\mathbf{A}_{S_{\alpha}}^n$ -scheme  $X_{\alpha}$  of finite presentation, such that  $X = X_{\alpha} \times_{\mathbf{A}_{S_{\alpha}}^n} \mathbf{A}_S^n$ . By [Lur16b, Prop. 4.6.2.1] we may in fact assume that  $X_{\alpha}$  is étale over  $\mathbf{A}_{S_{\alpha}}^n$ , and hence smooth over  $S_{\alpha}$ .  $\square$

**2.6. Sch-fibred spectra.** In this paragraph we construct the  $S^1$ -stabilization of the theory of Sch-fibred spaces.

2.6.1. Let  $\underline{\mathrm{Spt}}^{\varepsilon_{\infty}}(S)_{S^1}$  denote the category  $\mathrm{Spt}_{S^1}(\mathrm{Sp}c^{\varepsilon_{\infty}}(S)_{\bullet})$  of  $S^1$ -spectrum objects in the category of pointed  $\mathrm{Sch}^{\varepsilon_{\infty}}$ -fibred spaces over  $S$ . This is equivalent to the category of presheaves of spectra on  $\mathrm{Sch}_{/S}^{\varepsilon_{\infty}}$ .

We define full subcategories  $\underline{\mathrm{Spt}}_{\mathrm{Nis}}^{\varepsilon_{\infty}}(S)_{S^1}$ ,  $\underline{\mathrm{Spt}}_{\mathbf{A}^1}^{\varepsilon_{\infty}}(S)_{S^1}$ , and  $\underline{\mathrm{SH}}^{\varepsilon_{\infty}}(S)_{S^1}$  of Nisnevich-local,  $\mathbf{A}^1$ -local, and motivic  $\mathrm{Sch}^{\varepsilon_{\infty}}$ -fibred spectra, respectively. Equivalently, we have

$$\begin{aligned} \underline{\mathrm{Spt}}_{\mathrm{Nis}}^{\varepsilon_{\infty}}(S)_{S^1} &= \mathrm{Spt}_{S^1}(\underline{\mathrm{Sp}}c_{\mathrm{Nis}}^{\varepsilon_{\infty}}(S)_{\bullet}), \\ \underline{\mathrm{Spt}}_{\mathbf{A}^1}^{\varepsilon_{\infty}}(S)_{S^1} &= \mathrm{Spt}_{S^1}(\underline{\mathrm{Sp}}c_{\mathbf{A}^1}^{\varepsilon_{\infty}}(S)_{\bullet}), \\ \underline{\mathrm{SH}}^{\varepsilon_{\infty}}(S)_{S^1} &= \mathrm{Spt}_{S^1}(\underline{\mathbf{H}}^{\varepsilon_{\infty}}(S)_{\bullet}). \end{aligned}$$

2.6.2. This construction fits into the general framework of [Kha16, § 3], and we have (cf. [Kha16, § 4]):

**Lemma 2.6.3.**

- (i) *The category  $\underline{\mathrm{Spt}}_{\mathrm{Nis}}^{\varepsilon_{\infty}}(S)_{S^1}$  is an accessible left localization of  $\underline{\mathrm{Spt}}^{\varepsilon_{\infty}}(S)_{S^1}$ .*
- (ii) *The category  $\underline{\mathrm{Spt}}_{\mathbf{A}^1}^{\varepsilon_{\infty}}(S)_{S^1}$  is an accessible left localization of  $\underline{\mathrm{Spt}}^{\varepsilon_{\infty}}(S)_{S^1}$ . Further, the localization functor  $\mathcal{F} \mapsto L_{\mathbf{A}^1}(\mathcal{F})$  preserves Nisnevich-local spectra, and admits the following*

description: for every  $\text{Sch}^{\varepsilon_\infty}$ -fibred spectrum  $\mathcal{F}$ , there is a canonical isomorphism

$$(2.7) \quad \Gamma(X, L_{\mathbf{A}^1}(\mathcal{F})) \approx \varinjlim_{(Y \rightarrow X) \in (\mathbf{A}_X)^{\text{op}}} \Gamma(Y, \mathcal{F})$$

for each spectral  $S$ -scheme  $X$ . Here  $(\mathbf{A}_X)^{\text{op}}$  is a sifted small category, opposite to the full subcategory of  $\text{Sm}_{/X}^{\varepsilon_\infty}$  spanned by composites of  $\mathbf{A}^1$ -projections.

(iii) The category  $\underline{\mathbf{SH}}^{\varepsilon_\infty}(S)_{S^1}$  is an accessible left localization of  $\underline{\text{Spt}}^{\varepsilon_\infty}(S)_{S^1}$ . Further, the localization functor  $\mathcal{F} \mapsto L_{\text{mot}}(\mathcal{F})$  can be described as the composite

$$(2.8) \quad L_{\text{mot}}(\mathcal{F}) = L_{\mathbf{A}^1} L_{\text{Nis}}(\mathcal{F}).$$

(vi) The category  $\underline{\mathbf{SH}}^{\varepsilon_\infty}(S)_{S^1}$  is generated under sifted colimits by objects of the form  $\Sigma_+^\infty L_{\text{mot}} h_S(X)$ , where  $X = \text{Spec}(A)$  is an affine *afp* spectral  $S$ -scheme.

2.6.4. Recall from [Kha16, § 4] that a  $\text{Sm}^{\varepsilon_\infty}$ -fibred  $S^1$ -spectrum over  $S$  is a presheaf of spectra on the category  $\text{Sm}_{/S}^{\varepsilon_\infty}$ . We write  $\text{Spt}^{\varepsilon_\infty}(S)_{S^1}$  for the category of  $\text{Sm}^{\varepsilon_\infty}$ -fibred  $S^1$ -spectra over  $S$ .

Let  $\text{Spt}_{\text{Nis}}^{\varepsilon_\infty}(S)_{S^1}$ ,  $\text{Spt}_{\mathbf{A}^1}^{\varepsilon_\infty}(S)_{S^1}$ , and  $\underline{\mathbf{SH}}^{\varepsilon_\infty}(S)_{S^1}$  denote the respective localizations of  $\text{Spt}^{\varepsilon_\infty}(S)_{S^1}$ . Equivalently, each of these categories is the  $S^1$ -stabilization of its respective unstable counterpart.

Taking  $S^1$ -stabilizations, Proposition 2.2.6 gives:

**Lemma 2.6.5.** *The functor  $L_{\text{mot}} \circ (\iota_{\text{Sm}})!: \underline{\mathbf{SH}}^{\varepsilon_\infty}(S)_{S^1} \rightarrow \underline{\mathbf{SH}}^{\varepsilon_\infty}(S)_{S^1}$  is fully faithful.*

*Proof.* This follows from Proposition 2.2.6 and the fact that any limit of fully faithful functors is fully faithful (recall that the category of spectrum objects is a certain limit of the unstable categories, see [Kha16, ¶ 3.4]).  $\square$

2.6.6. Lemma 2.2.11 carries over to the stable setting:

**Lemma 2.6.7.** *The localization functor  $\mathcal{F} \mapsto L_{\mathbf{A}^1}(\mathcal{F})$  commutes with the restriction functor  $(\iota_{\text{Sm}})^*$  on  $\text{Sch}^{\varepsilon_\infty}$ -fibred  $S^1$ -spectra.*

2.6.8. From Corollary 2.5.6 we deduce:

**Corollary 2.6.9.** *Let  $S$  be an affine spectral scheme. Suppose that  $S$  can be written as the limit of a cofiltered diagram  $(S_\alpha)_\alpha$  of affine spectral schemes. Then there is a canonical equivalence*

$$\varinjlim_\alpha \underline{\mathbf{SH}}^{\varepsilon_\infty}(S_\alpha)_{S^1} \xrightarrow{\sim} \underline{\mathbf{SH}}^{\varepsilon_\infty}(S)_{S^1}.$$

### 3. NIL-LOCALIZATION

In this section we prove the comparison between the brave new motivic homotopy category  $\mathbf{H}^{\varepsilon_\infty}(S)$  and its classical analogue  $\mathbf{H}^{\text{cl}}(S_{\text{cl}})$ .

The first step will be to present the classical motivic homotopy category  $\mathbf{H}^{\text{cl}}(S_{\text{cl}})$  as a localization of  $\mathbf{H}^{\varepsilon_\infty}(S)$ , at a certain class of morphisms we call “nil-local equivalences”. In order to make sense of this nil-localization, we will first perform it at the level of  $\text{Sch}^{\varepsilon_\infty}$ -fibred spaces, and then show that it restricts to  $\text{Sm}^{\varepsilon_\infty}$ -fibred spaces.

Then we will re-interpret the localization theorem (Theorem 2.4.3) as the statement that motivic  $\text{Sm}^{\varepsilon_\infty}$ -fibred spaces are already nil-local.

For technical reasons (see Remark 3.1.3), it will be convenient for this argument to assume that the base scheme  $S$  is *noetherian*. Thus in this section,  $S$  will always denote a *noetherian* spectral scheme, except when stated otherwise. However, the comparison theorem (Theorem 3.4.2) itself will hold for general bases (we will explain how to reduce to the noetherian case).

### 3.1. Nil-localization.

3.1.1. Let  $S$  be a *noetherian* spectral scheme. This means that  $S$  is quasi-compact quasi-separated and locally noetherian, i.e. there exists an affine Zariski cover  $(S_\alpha \hookrightarrow S)_\alpha$  such that for each  $\alpha$ , the connective  $\mathcal{E}_\infty$ -ring spectrum  $A = \Gamma(S_\alpha, \mathcal{O}_S)$  is *noetherian* in the sense that  $\pi_0(A)$  is a noetherian commutative ring, and each  $\pi_0(A)$ -module  $\pi_i(A)$  is finitely generated.

When  $S$  is noetherian, the canonical morphism  $S_{\text{cl}} \hookrightarrow S$  is afp (this follows from [Lur16a, Prop. 7.2.4.31]).

3.1.2. Consider the canonical functor

$$v : \text{Sch}_{/S}^{\mathcal{E}_\infty} \rightarrow \text{Sch}_{/S_{\text{cl}}},$$

which sends an afp spectral  $S$ -scheme  $X$  to the classical  $S_{\text{cl}}$ -scheme  $X_{\text{cl}}$ .

The functor  $u$  is right adjoint to the fully faithful functor

$$u : \text{Sch}_{/S_{\text{cl}}} \rightarrow \text{Sch}_{/S}^{\mathcal{E}_\infty}$$

given by the assignment  $(X \rightarrow S_{\text{cl}}) \mapsto (X \rightarrow S_{\text{cl}} \hookrightarrow S)$ .

*Remark 3.1.3.* Note that the functor  $u$  is only well-defined because the morphism  $S_{\text{cl}} \hookrightarrow S$  is afp. This is in fact the only reason we assume that  $S$  is noetherian.

It is possible to remove this assumption by working with the site of all spectral  $S$ -schemes (without any finiteness assumptions), but this introduces size issues which we prefer to avoid here for the sake of simplicity of exposition. Instead, we will deduce the comparison theorem (Theorem 3.4.2) over general bases by a noetherian approximation argument.

3.1.4. The functor  $v$  induces a canonical colimit-preserving functor

$$v_! : \underline{\text{Spc}}^{\mathcal{E}_\infty}(S) \rightarrow \underline{\text{Spc}}^{\text{cl}}(S_{\text{cl}})$$

which sends the representable presheaf  $h_S(X)$  to  $h_{S_{\text{cl}}}(X_{\text{cl}})$ , for every afp spectral  $S$ -scheme  $X$ .

This is left adjoint to the restriction functor  $v^*$ .

3.1.5. Note that  $v_!$  is canonically identified with the functor  $u^*$  of restriction along  $u$ ; hence its right adjoint  $v^*$  is identified with  $u_*$ , the right adjoint of  $u^*$ . Since  $u$  is fully faithful, it follows that  $u_* = v^*$  is. Hence we have:

**Lemma 3.1.6.** *The functor  $v_! : \underline{\text{Spc}}^{\mathcal{E}_\infty}(S) \rightarrow \underline{\text{Spc}}^{\text{cl}}(S_{\text{cl}})$  is a left localization.*

We define:

**Definition 3.1.7.**

(i) A morphism of  $\text{Sch}^{\mathcal{E}_\infty}$ -fibred spaces over  $S$  is a nil-local equivalence if it induces an isomorphism after application of  $v_!$ .

(ii) A  $\text{Sch}^{\mathcal{E}_\infty}$ -fibred space  $\mathcal{F}$  over  $S$  is nil-local if it is contained in the essential image of  $v^*$ , or equivalently if the canonical morphism

$$\mathcal{F} \rightarrow v^*v_!(\mathcal{F})$$

is invertible.

Unravelling definitions, we have:

**Lemma 3.1.8.**

(i) A  $\text{Sch}^{\mathcal{E}_\infty}$ -fibred space  $\mathcal{F}$  is nil-local if and only if the canonical morphism

$$\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X_{\text{cl}}, \mathcal{F})$$

is invertible for every afp spectral  $S$ -scheme  $X$ .

(ii) The class of nil-local equivalences is the strongly saturated closure of the (essentially small) set of canonical morphisms

$$\text{h}_S(X_{\text{cl}}) \rightarrow \text{h}_S(X),$$

where  $X$  is an afp spectral  $S$ -scheme.

**3.2. Nil-localization of motivic spaces.**

3.2.1. We have:

**Lemma 3.2.2.**

- (i) The functor  $v_!$  preserves  $\mathbf{A}^1$ -local equivalences (resp. Nisnevich-local equivalences).
- (ii) The functor  $v^*$  preserves  $\mathbf{A}^1$ -local spaces (resp. Nisnevich-local spaces).

*Proof.* The first statement follows from the fact that  $v$  preserves  $\mathbf{A}^1$ -projections (resp. Nisnevich squares), and the second follows by adjunction.  $\square$

Hence the functor  $v^*$  restricts to a functor

$$v^* : \underline{\mathbf{H}}^{\text{cl}}(S_{\text{cl}}) \rightarrow \underline{\mathbf{H}}^{\mathcal{E}_\infty}(S),$$

which is right adjoint to  $L_{\text{mot}}v_!$ .

3.2.3. Since  $v^*$  is fully faithful, we have:

**Lemma 3.2.4.** *The functor  $L_{\text{mot}}v_! : \underline{\mathbf{H}}^{\mathcal{E}_\infty}(S) \rightarrow \underline{\mathbf{H}}^{\text{cl}}(S_{\text{cl}})$  is a left localization at the set of the canonical morphisms of the form*

$$L_{\text{mot}}\text{h}_S(X_{\text{cl}}) \rightarrow L_{\text{mot}}\text{h}_S(X),$$

where  $X$  is an afp spectral  $S$ -scheme.

3.2.5. Note that the functor  $u$  is also topologically continuous, i.e. it sends Nisnevich squares of classical  $S_{\text{cl}}$ -schemes to Nisnevich squares of spectral  $S$ -schemes. Hence we have:

**Lemma 3.2.6.**

- (i) The functor  $u_!$  preserves Nisnevich-local equivalences.
- (ii) The functor  $u^* = v_!$  preserves Nisnevich-local spaces.

*Remark 3.2.7.* Note that the same argument does not apply to  $\mathbf{A}^1$ -local equivalences, as  $u$  does not send  $\mathbf{A}_{\text{cl}}^1$  to  $\mathbf{A}^1$ .

3.2.8. We also have:

**Lemma 3.2.9.**

- (i) The functor  $v^*$  preserves Nisnevich-local equivalences.
- (ii) The functor  $v^*$  preserves  $\mathbf{A}^1$ -local equivalences.

*Proof.*

(i) The claim about Nisnevich-local equivalences follows from the fact that the functor  $v$  is topologically cocontinuous. Indeed, let  $X$  be a spectral scheme over  $S$ , and let  $Q$  be a classical Nisnevich square

$$\begin{array}{ccc} U \times_{X_{\text{cl}}} V & \hookrightarrow & V \\ \downarrow & & \downarrow \\ U & \hookrightarrow & X_{\text{cl}} \end{array}$$

over  $X_{\text{cl}}$ . The claim is that this lifts to a Nisnevich square  $\tilde{Q}$  over  $X$ . This follows directly from [Kha16, Lem. 2.10.4]: since both morphisms  $U \rightarrow X_{\text{cl}}$  and  $V \rightarrow X_{\text{cl}}$  are étale, both admit unique lifts  $\tilde{U} \rightarrow X_{\text{cl}}$  and  $\tilde{V} \rightarrow X$ , so that we obtain a Nisnevich square  $\tilde{Q}$

$$\begin{array}{ccc} \tilde{U} \times_X \tilde{V} & \hookrightarrow & \tilde{V} \\ \downarrow & & \downarrow \\ \tilde{U} & \hookrightarrow & X \end{array}$$

which refines  $Q$ .

(ii) It suffices to show that, for any classical  $S_{\text{cl}}$ -scheme  $X$ , the canonical morphism

$$v^* h_S(X \times \mathbf{A}_{\text{cl}}^1) \rightarrow v^* h_S(X)$$

is an  $\mathbf{A}^1$ -local equivalence.

By universality of colimits it suffices to show that, for any spectral  $S$ -scheme  $Y$  and morphism of presheaves  $\varphi : h_S(Y) \rightarrow v^* h_{S_{\text{cl}}}(X)$ , the base change

$$v^* h_S(X \times \mathbf{A}_{\text{cl}}^1) \times_{v^* h_{S_{\text{cl}}}(X)} h_S(Y) \rightarrow h_S(Y)$$

is an  $\mathbf{A}^1$ -local equivalence. Just as in the proof of Lemma 2.2.4, (ii), one observes that this morphism is a base change of the  $\mathbf{A}^1$ -local equivalence

$$h_S(Y \times \mathbf{A}^1) \rightarrow h_S(Y),$$

whence the claim. □

3.2.10. We have:

**Lemma 3.2.11.**

(i) *The functor*

$$L_{\text{mot}} v! : \underline{\mathbf{H}}^{\mathcal{E}\infty}(S) \rightarrow \underline{\mathbf{H}}^{\text{cl}}(S_{\text{cl}})$$

*sends  $\text{Sm}^{\mathcal{E}\infty}$ -fibred spaces to  $\text{Sm}$ -fibred spaces, and hence restricts to a functor*

$$(3.1) \quad L_{\text{mot}} v! : \mathbf{H}^{\mathcal{E}\infty}(S) \rightarrow \mathbf{H}^{\text{cl}}(S_{\text{cl}}).$$

(ii) *The fully faithful functor*

$$v^* : \underline{\mathbf{H}}^{\text{cl}}(S_{\text{cl}}) \hookrightarrow \underline{\mathbf{H}}^{\mathcal{E}\infty}(S)$$

*restricts to a fully faithful functor*

$$(3.2) \quad v^* : \mathbf{H}^{\text{cl}}(S_{\text{cl}}) \hookrightarrow \mathbf{H}^{\mathcal{E}\infty}(S).$$

(iii) *The functor (3.1) is the left localization at the set of morphisms  $\mathcal{F}_1 \rightarrow \mathcal{F}_2$  such that  $(\iota_{\text{Sm}})_!(\mathcal{F}_1) \rightarrow (\iota_{\text{Sm}})_!(\mathcal{F}_2)$  is a nil-local equivalence.*



*Proof.* It suffices to show that the analogous claim at the level of  $\text{Sch}^{\varepsilon_\infty}$ -fibrated spaces (i.e. before taking motivic localizations). That is, it suffices to show that the functor  $v_! : \underline{\text{Spc}}^{\varepsilon_\infty}(\mathbb{S}) \rightarrow \underline{\text{Spc}}^{\text{cl}}(\mathbb{S}_{\text{cl}})$  sends  $\text{Sm}^{\varepsilon_\infty}$ -fibrated spaces to  $\text{Sm}$ -fibrated spaces. This follows from the fact that, for any smooth spectral  $\mathbb{S}$ -scheme  $X$ , the underlying classical scheme  $v(X) = X_{\text{cl}}$  is smooth (in the classical sense) over  $\mathbb{S}_{\text{cl}}$  [Kha16, Lem. 2.9.11].  $\square$

Let  $\mathbf{H}_{\text{nil}}^{\varepsilon_\infty}(\mathbb{S})$  denote the essential image of  $v^*$ . That is, this is the full subcategory of  $\mathbf{H}^{\varepsilon_\infty}(\mathbb{S})$  spanned by motivic spaces  $\mathcal{F}$  such that  $(\iota_{\text{Sm}})_!(\mathcal{F})$  is nil-local.

**Corollary 3.2.12.** *The restriction of the functor (3.1) defines a canonical equivalence of categories*

$$(3.3) \quad L_{\text{mot}}v_! : \mathbf{H}_{\text{nil}}^{\varepsilon_\infty}(\mathbb{S}) \xrightarrow{\sim} \mathbf{H}^{\text{cl}}(\mathbb{S}_{\text{cl}}).$$

### 3.3. Nil-descent of motivic spaces.

3.3.1. The following is a consequence of the localization theorem (Theorem 2.4.3):

**Proposition 3.3.2** (Nil-descent). *Every  $\text{Sm}^{\varepsilon_\infty}$ -fibrated motivic space  $\mathcal{F}$  over  $\mathbb{S}$  is nil-local. That is, the inclusion*

$$\mathbf{H}_{\text{nil}}^{\varepsilon_\infty}(\mathbb{S}) \hookrightarrow \mathbf{H}^{\varepsilon_\infty}(\mathbb{S})$$

*is an equivalence.*

*Proof.* Let  $\mathcal{F}$  be a  $\text{Sm}^{\varepsilon_\infty}$ -fibrated motivic space. Let  $X$  be a smooth spectral  $\mathbb{S}$ -scheme, and write  $i : X_{\text{cl}} \hookrightarrow X$  for the inclusion of its underlying classical scheme. The claim is that the canonical map

$$i^* : \Gamma(X, (\iota_{\text{Sm}})_!\mathcal{F}) \rightarrow \Gamma(X_{\text{cl}}, (\iota_{\text{Sm}})_!\mathcal{F})$$

is invertible.

By adjunction, this map is obtained by application of the functor  $\text{Maps}_{\mathbf{H}^{\varepsilon_\infty}(X)}(L_{\text{mot}}h_X(X), -)$  to the canonical map

$$f_{\mathbf{H}}^* \circ L_{\text{mot}}(\iota_{\text{Sm}})_!(\mathcal{F}) \rightarrow i_*^{\mathbf{H}} i_{\mathbf{H}}^* f_{\mathbf{H}}^* \circ L_{\text{mot}}(\iota_{\text{Sm}})_!(\mathcal{F}),$$

where  $f : X \rightarrow \mathbb{S}$  is the structural morphism.

By Lemma 2.3.6 and Corollary 2.4.6 this map is canonically identified with the image by  $L_{\text{mot}}(\iota_{\text{Sm}})_!$  of the map

$$f_{\mathbf{H}}^*(\mathcal{F}) \rightarrow i_*^{\mathbf{H}} i_{\mathbf{H}}^* f_{\mathbf{H}}^*(\mathcal{F}).$$

Since  $i$  is a closed immersion with empty complement, the localization theorem [Kha16, Thm. 7.2.6] implies that this map is invertible.  $\square$

**3.4. The comparison.** In this paragraph we deduce the main result of this section. The spectral scheme  $\mathbb{S}$  is no longer required to be noetherian.

3.4.1. Let  $\mathbb{S}$  be a (quasi-compact quasi-separated) spectral scheme. We have:

**Theorem 3.4.2** (Comparison). *The canonical adjunctions*

$$\begin{aligned} L_{\text{mot}}v_! : \mathbf{H}^{\varepsilon_\infty}(\mathbb{S}) &\rightleftarrows \mathbf{H}^{\text{cl}}(\mathbb{S}_{\text{cl}}) : v^*, \\ L_{\text{mot}}v_! : \mathbf{SH}^{\varepsilon_\infty}(\mathbb{S})_{\mathbb{S}^1} &\rightleftarrows \mathbf{SH}^{\text{cl}}(\mathbb{S}_{\text{cl}})_{\mathbb{S}^1} : v^* \end{aligned}$$

*are equivalences.*

*Proof.* Both categories in question have Zariski descent ([Kha16, Prop. 6.1.6] and [Hoy15, Appendix C]), so we may assume that  $S$  is affine. By [Lur16a, Prop. 7.2.4.27] we may write it as a cofiltered limit of affine spectral schemes which are of finite presentation over  $\mathrm{Spec}(\mathbf{S})$ , thus in particular noetherian. Then by continuity (Proposition 2.5.5), and the analogous property for the classical motivic homotopy category [Hoy15, Prop. C.7], we may reduce to the case where  $S$  is noetherian.

In this case, the first claim follows from Corollary 3.2.12, in view of Proposition 3.3.2. The second follows from the first by taking  $S^1$ -stabilizations.  $\square$

In particular, any classical scheme may be viewed as a spectral scheme (with discrete structure sheaf), and we have:

**Corollary 3.4.3.** *Let  $S$  be a (quasi-compact quasi-separated) classical scheme. The canonical adjunctions*

$$\begin{aligned} L_{\mathrm{mot}}v_! : \mathbf{H}^{\mathcal{E}^\infty}(S) &\rightleftarrows \mathbf{H}^{\mathrm{cl}}(S) : v^*, \\ L_{\mathrm{mot}}v_! : \mathbf{SH}^{\mathcal{E}^\infty}(S)_{S^1} &\rightleftarrows \mathbf{SH}^{\mathrm{cl}}(S)_{S^1} : v^* \end{aligned}$$

are equivalences.

3.4.4. Unwinding the definitions of the functors involved in the equivalence of Theorem 3.4.2, we arrive at the following formulation:

**Corollary 3.4.5.** *Let  $S$  be a (quasi-compact quasi-separated) spectral scheme. Let  $\mathcal{F}$  be an  $\mathrm{Sm}^{\mathcal{E}^\infty}$ -fibred Nisnevich-local  $S^1$ -spectrum over  $S$ . Then there is a canonical isomorphism*

$$L_{\mathbf{A}^1}(\mathcal{F}) \xrightarrow{\sim} v^*L_{\mathbf{A}_{\mathrm{cl}}^1}v_!(\mathcal{F})$$

of  $\mathrm{Sm}^{\mathcal{E}^\infty}$ -fibred motivic spectra over  $S$ . In particular, for each smooth spectral  $S$ -scheme  $X$ , there are canonical functorial isomorphisms of spectra

$$\Gamma(X, L_{\mathbf{A}^1}(\mathcal{F})) \xrightarrow{\sim} \Gamma(X_{\mathrm{cl}}, L_{\mathbf{A}_{\mathrm{cl}}^1}(\mathcal{F})).$$

Here we have written  $L_{\mathbf{A}_{\mathrm{cl}}^1}$  for the classical  $\mathbf{A}^1$ -localization functor  $\mathrm{Spt}^{\mathrm{cl}}(S_{\mathrm{cl}}) \rightarrow \mathrm{Spt}_{\mathbf{A}_{\mathrm{cl}}^1}^{\mathrm{cl}}(S_{\mathrm{cl}})$ .

*Proof.* It follows from Theorem 3.4.2 that there is a canonical isomorphism

$$L_{\mathbf{A}^1}(\mathcal{F}) \xrightarrow{\sim} v^*L_{\mathrm{mot}}v_!L_{\mathbf{A}^1}(\mathcal{F}).$$

Recall that  $L_{\mathbf{A}^1}$  preserves Nisnevich-local spectra (Lemma 2.6.3) and  $v_! = u^*$  preserves Nisnevich-local spaces (Lemma 3.2.6). Hence we have canonical functorial isomorphisms

$$L_{\mathrm{mot}}v_!L_{\mathbf{A}^1}(\mathcal{F}) \approx L_{\mathbf{A}_{\mathrm{cl}}^1}v_!L_{\mathbf{A}^1}(\mathcal{F}) \approx L_{\mathbf{A}_{\mathrm{cl}}^1}v_!(\mathcal{F})$$

by Lemma 3.2.2. The claim follows.  $\square$

## 4. $\mathrm{Sm}^b$ -FIBRED MOTIVIC HOMOTOPY THEORY

### 4.1. Flat affine spaces.

4.1.1. Given a connective  $\mathcal{E}_\infty$ -ring spectrum  $R$ , let  $R[t_1, \dots, t_n]$  denote the polynomial  $R$ -algebra in  $n$  variables ( $n \geq 1$ ).

This is by definition the monoid algebra  $R[\mathbf{N}^n] = R \otimes \Sigma^\infty(\mathbf{N}^n)_+$ , where  $\mathbf{N}$  is the set of natural numbers, viewed as a discrete (additive)  $\mathcal{E}_\infty$ -monoid space.

Note that the underlying spectrum of  $R[t_1, \dots, t_n]$  is the direct sum  $\bigoplus_{(k_1, \dots, k_n) \in \mathbf{N}^n} R$ . Further we have an isomorphism of ordinary commutative rings  $\pi_0(R[t_1, \dots, t_n]) \approx \pi_0(R)[t_1, \dots, t_n]$  for each  $n$ .

*Remark 4.1.2.* The notation is justified by the fact that, for an ordinary commutative ring  $R$ ,  $R[t_1, \dots, t_n]$  coincides with the usual polynomial algebra, viewed as a discrete  $\mathcal{E}_\infty$ -ring spectrum.

4.1.3. For any spectral scheme  $S$ , let  $\mathbf{A}_{b,S}^n$  denote the spectral scheme  $S \times \mathrm{Spec}(\mathbf{S}[t_1, \dots, t_n])$ . We call this the *flat affine space* of dimension  $n$  over  $S$ .

We will write  $\mathbf{A}_b^n := \mathbf{A}_{b, \mathrm{Spec}(\mathbf{S})}^n$ .

The underlying classical scheme  $(\mathbf{A}_{b,S}^n)_{\mathrm{cl}}$  coincides with the classical  $n$ -dimensional affine space over  $S_{\mathrm{cl}}$ .

4.1.4. As in [Kha16, 2.6.4] we write  $\mathbf{S}\{t_1, \dots, t_n\}$  for the free  $\mathcal{E}_\infty$ -algebra on  $n$  generators (in degree zero).

Consider the canonical morphism of  $\mathcal{E}_\infty$ -ring spectra

$$(4.1) \quad \mathbf{S}\{t_1, \dots, t_n\} \rightarrow \mathbf{S}[t_1, \dots, t_n]$$

which sends  $t_i \mapsto t_i$ . For any spectral scheme  $S$ , this gives rise to canonical morphisms

$$(4.2) \quad \mathbf{A}_{b,S}^n \rightarrow \mathbf{A}_S^n$$

for each  $n \geq 0$ . We have (see [Lur16a, Prop. 7.1.4.20]):

**Proposition 4.1.5.** *Let  $S$  be a spectral scheme of characteristic zero<sup>4</sup>. Then the canonical morphism (4.2) is invertible for each  $n \geq 0$ .*

4.1.6. There is a multiplication morphism

$$\mathbf{A}_{b,S}^1 \times \mathbf{A}_{b,S}^1 \rightarrow \mathbf{A}_{b,S}^1$$

induced by the diagonal map

$$\Sigma_+^\infty(\mathbf{N}) \rightarrow \Sigma_+^\infty(\mathbf{N} \times \mathbf{N}) \approx \Sigma_+^\infty(\mathbf{N}) \otimes \Sigma_+^\infty(\mathbf{N}).$$

4.1.7. The projection  $\mathbf{A}_{b,S}^1 \rightarrow S$  admits two sections  $i_0$  and  $i_1$ , the zero and unit sections, respectively.

One way to define the relevant morphisms

$$\mathbf{S}[t] \rightrightarrows \mathbf{S}$$

is by induction along the Postnikov tower [Kha16, Prop. 2.7.5] (note that each truncation  $(\mathbf{S}[t])_{\leq n}$  is canonically isomorphic to  $\mathbf{S}_{\leq n}[t]$ ).

## 4.2. Fibre-smoothness.

4.2.1. Let  $p : Y \rightarrow X$  be a morphism of spectral schemes.

**Definition 4.2.2.** *The morphism  $p$  is fibre-smooth if it satisfies the following conditions:*

- (i) *The morphism  $p$  is flat.*
- (ii) *The induced morphism  $p_{\mathrm{cl}} : Y_{\mathrm{cl}} \rightarrow X_{\mathrm{cl}}$  on underlying classical schemes is smooth (in the classical sense).*

Note that we have:

**Lemma 4.2.3.**

- (i) *The set of fibre-smooth morphisms is stable under composition and base change.*
- (ii) *Étale morphisms are fibre-smooth.*

<sup>4</sup>Here we say that  $S$  is of characteristic zero if it admits a morphism  $S \rightarrow \mathrm{Spec}(\mathbf{H}\mathbf{Q})$ , or equivalently, if the classical scheme  $S_{\mathrm{cl}}$  is of characteristic zero.

4.2.4. The notion of fibre-smoothness has also been considered in [Lur16b]. The following lemma shows that our definition coincides with that of the latter, in view of Corollary 11.2.4.2 in *loc. cit.*

**Lemma 4.2.5.** *A morphism  $p : Y \rightarrow X$  of affine spectral schemes is fibre-smooth if and only if it satisfies the following conditions:*

- (a) *The morphism  $p$  is flat.*
- (b) *The underlying morphism of classical schemes  $p_{\text{cl}} : Y_{\text{cl}} \rightarrow X_{\text{cl}}$  is locally of finite presentation.*
- (c) *Let  $R \rightarrow R'$  be a surjective homomorphism of ordinary commutative rings with nilpotent kernel. For every commutative square of the form*

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X) & \longrightarrow & R \\ \downarrow & & \downarrow \\ \Gamma(Y, \mathcal{O}_Y) & \longrightarrow & R', \end{array}$$

*there exists a diagonal arrow  $\Gamma(Y, \mathcal{O}_Y) \rightarrow R$  and homotopies up to which both triangles commute.*

*Proof.* According to [Gro67, § 17.3], the condition that  $p_{\text{cl}}$  is smooth is equivalent (by definition) to the condition that it is locally of finite presentation, and that we have the lifting condition of (c) for the morphism of ordinary rings  $\pi_0(\Gamma(X, \mathcal{O}_X)) \rightarrow \pi_0(\Gamma(Y, \mathcal{O}_Y))$ . But since  $R$  and  $R'$  are ordinary rings, it follows by adjunction that this is equivalent to the lifting condition for the morphism  $\Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(Y, \mathcal{O}_Y)$ .  $\square$

4.2.6. We also have the following characterization:

**Proposition 4.2.7.** *A morphism  $p : Y \rightarrow X$  is fibre-smooth if and only if, Zariski-locally on  $Y$ , there exists a factorization of  $p$  as a composite*

$$(4.3) \quad Y \xrightarrow{q} X \times \mathbf{A}_b^n \xrightarrow{r} X$$

*for some integer  $n \geq 0$ , where  $q$  is étale and  $r$  is the canonical projection.*

*Proof.* This follows from [Lur16b, Rem. 11.2.3.5 and 11.2.4.1].  $\square$

**4.3. Flat homotopy invariance.** In this paragraph we show that the category of  $\mathbf{A}_b^1$ -local  $\text{Sch}^{\varepsilon_\infty}$ -fibred spaces is a localization of the category of  $\mathbf{A}^1$ -local  $\text{Sch}^{\varepsilon_\infty}$ -fibred spaces.

4.3.1. Let  $\underline{\text{Spc}}_{\mathbf{A}_b^1}^{\varepsilon_\infty}(\mathcal{S})$  denote the full subcategory of  $\underline{\text{Spc}}^{\varepsilon_\infty}(\mathcal{S})$  spanned by  $\text{Sch}^{\varepsilon_\infty}$ -fibred spaces  $\mathcal{F}$  that satisfy homotopy invariance with respect to the flat affine line  $\mathbf{A}_b^1$ , i.e. for each afd spectral  $\mathcal{S}$ -scheme  $X$ , the canonical morphism

$$\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X \times \mathbf{A}^1, \mathcal{F})$$

is invertible.

Let  $\underline{\mathbf{H}}^b(\mathcal{S})$  denote the full subcategory of  $\underline{\text{Spc}}^{\varepsilon_\infty}(\mathcal{S})$  spanned by  $\text{Sch}^{\varepsilon_\infty}$ -fibred spaces that satisfy Nisnevich excision and  $\mathbf{A}_b^1$ -homotopy invariance.

We have, by the general results of [Kha16, § 3]:

**Lemma 4.3.2.**

- (i) *The category  $\underline{\text{Spc}}_{\mathbf{A}_b^1}^{\varepsilon_\infty}(\mathcal{S})$  is an accessible left localization of  $\underline{\text{Spc}}^{\varepsilon_\infty}(\mathcal{S})$ . The localization functor  $\mathcal{F} \mapsto \mathbf{L}_{\mathbf{A}_b^1}(\mathcal{F})$  commutes with finite products, and admits the following description: for every*

$\text{Sch}^{\mathcal{E}_\infty}$ -fibred space  $\mathcal{F}$ , there is a canonical isomorphism

$$(4.4) \quad \Gamma(X, L_{\mathbf{A}_b^1}(\mathcal{F})) = \varinjlim_{(Y \rightarrow X) \in (\mathbf{A}_X^b)^{\text{op}}} \Gamma(Y, \mathcal{F})$$

for each afp spectral  $S$ -scheme  $X$ . Here  $(\mathbf{A}_X^b)^{\text{op}}$  is a sifted small category, opposite to the full subcategory of  $\text{Sm}_{/X}^b$  spanned by compositions of  $\mathbf{A}_b^1$ -projections.

(ii) The category  $\underline{\mathbf{H}}^b(S)$  is an accessible left localization of  $\underline{\text{Spc}}^{\mathcal{E}_\infty}(S)$ . Further, the localization functor  $\mathcal{F} \mapsto L_{\text{mot}}^b(\mathcal{F})$  can be described as the transfinite composite

$$(4.5) \quad L_{\text{mot}}^b(\mathcal{F}) = \varinjlim_{n \geq 0} (L_{\mathbf{A}_b^1} \circ L_{\text{Nis}})^{\circ n}(\mathcal{F}).$$

(iii) The category  $\underline{\mathbf{H}}^b(S)$  is generated under sifted colimits by objects of the form  $L_{\text{mot}}^b h_S(X)$ , where  $X$  is an affine spectral  $S$ -scheme.

4.3.3. Note that the canonical morphism of spectral schemes (see 4.1.4)

$$(4.6) \quad \mathbf{A}_b^1 \rightarrow \mathbf{A}^1$$

preserves the zero and unit sections as well as the multiplicative structures of both intervals.

It follows that any  $\mathbf{A}^1$ -homotopy gives rise to an  $\mathbf{A}_b^1$ -homotopy, which implies that  $\mathbf{A}^1$ -local equivalences are  $\mathbf{A}_b^1$ -local equivalences. Thus we have:

**Lemma 4.3.4.** *The category  $\underline{\text{Spc}}_{\mathbf{A}_b^1}^{\mathcal{E}_\infty}(S)$  (resp.  $\underline{\mathbf{H}}^b(S)$ ) is a left localization of  $\underline{\text{Spc}}_{\mathbf{A}^1}^{\mathcal{E}_\infty}(S)$  (resp.  $\underline{\mathbf{H}}^{\mathcal{E}_\infty}(S)$ ).*

This means that the localization

$$L_{\text{mot}}^b : \underline{\text{Spc}}^{\mathcal{E}_\infty}(S) \rightarrow \underline{\mathbf{H}}^b(S)$$

factors through the localization

$$L_{\text{mot}} : \underline{\text{Spc}}^{\mathcal{E}_\infty}(S) \rightarrow \underline{\mathbf{H}}^{\mathcal{E}_\infty}(S)$$

and a further localization

$$L_{\mathbf{A}_b^1} : \underline{\mathbf{H}}^{\mathcal{E}_\infty}(S) \rightarrow \underline{\mathbf{H}}^b(S).$$

#### 4.4. $\text{Sm}^b$ -fibred spaces.

4.4.1. Let  $\text{Sm}_{/S}^b$  denote the full subcategory of  $\text{Sch}_{/S}^{\mathcal{E}_\infty}$  spanned by fibre-smooth spectral  $S$ -schemes.

A  $\text{Sm}^b$ -fibred space over  $S$  is a presheaf of spaces on  $\text{Sm}_{/S}^b$ .

We write  $\text{Spc}^b(S)$  for the category of  $\text{Sm}^b$ -fibred spaces.

4.4.2. Let  $\iota_{\text{Sm}}^b : \text{Sm}_{/S}^b \hookrightarrow \text{Sch}_{/S}^{\mathcal{E}_\infty}$  denote the inclusion functor. This induces a canonical fully faithful functor

$$(\iota_{\text{Sm}}^b)_! : \text{Spc}^b(S) \rightarrow \underline{\text{Spc}}^{\mathcal{E}_\infty}(S),$$

left adjoint to the restriction functor  $(\iota_{\text{Sm}}^b)^*$ . Its essential image can be described as the full subcategory of  $\underline{\text{Spc}}^{\mathcal{E}_\infty}(S)$  generated under colimits by objects of the form  $h_S(X)$ , with  $X$  a fibre-smooth spectral  $S$ -scheme.

We will abuse notation by identifying  $\text{Spc}^b(S)$  with its essential image in  $\underline{\text{Spc}}^{\mathcal{E}_\infty}(S)$ .

4.4.3. Just as in [Kha16, § 4], we can form localizations  $\mathrm{Spc}_{\mathrm{Nis}}^b(S)$ ,  $\mathrm{Spc}_{\mathbf{A}_b^1}^b(S)$ , and  $\mathbf{H}^b(S)$  of the category  $\mathrm{Spc}^b(S)$ . To be more precise, the latter is obtained by left localization with respect to Nisnevich covers and the canonical morphisms

$$h_S(X \times \mathbf{A}_b^1) \rightarrow h_S(X)$$

for all fibre-smooth spectral  $S$ -schemes  $X$ .

We have (same proof as Lemma 2.2.4):

**Lemma 4.4.4.**

- (i) *The functor  $(\iota_{\mathrm{Sm}}^b)_!$  preserves Nisnevich-local equivalences. Its right adjoint  $(\iota_{\mathrm{Sm}}^b)^*$  preserves Nisnevich-local spaces and Nisnevich-local equivalences.*
- (ii) *The functor  $(\iota_{\mathrm{Sm}}^b)_!$  preserves  $\mathbf{A}_b^1$ -local equivalences. Its right adjoint  $(\iota_{\mathrm{Sm}}^b)^*$  preserves  $\mathbf{A}_b^1$ -local spaces and  $\mathbf{A}_b^1$ -local equivalences.*

4.4.5. It follows that the functor  $(\iota_{\mathrm{Sm}}^b)^*$  restricts to a well-defined functor

$$(\iota_{\mathrm{Sm}}^b)^* : \underline{\mathbf{H}}^{\mathcal{E}\infty}(S) \rightarrow \mathbf{H}^b(S),$$

which is right adjoint to  $L_{\mathrm{mot}} \circ (\iota_{\mathrm{Sm}}^b)_!$ . Further, we have (same proof as Proposition 2.2.6):

**Lemma 4.4.6.** *The functor  $L_{\mathrm{mot}} \circ (\iota_{\mathrm{Sm}}^b)_! : \mathbf{H}^b(S) \rightarrow \underline{\mathbf{H}}^{\mathcal{E}\infty}(S)$  is fully faithful. Its essential image is generated under colimits by the objects  $L_{\mathrm{mot}} h_S(X)$ , for  $X$  a fibre-smooth spectral  $S$ -scheme.*

4.4.7. A  $\mathrm{Sm}^b$ -fibred  $S^1$ -spectrum over  $S$  is a presheaf of spectra on  $\mathrm{Sm}_{S^1}^b$ , or equivalently an  $S^1$ -spectrum object in the category  $\mathrm{Spc}^b(S)_\bullet$ . We write  $\mathrm{Spt}^b(S)_{S^1}$  for the category of  $\mathrm{Sm}^b$ -fibred  $S^1$ -spectra.

Let  $\mathrm{Spt}_{\mathrm{Nis}}^b(S)_{S^1}$ ,  $\mathrm{Spt}_{\mathbf{A}^1}^b(S)_{S^1}$ , and  $\mathbf{SH}^b(S)_{S^1}$  denote the respective localizations of  $\mathrm{Spt}^b(S)$ . Each of these categories is the  $S^1$ -stabilization of its respective unstable counterpart.

Taking  $S^1$ -stabilizations, Lemma 4.4.6 gives:

**Lemma 4.4.8.** *The functor  $L_{\mathrm{mot}} \circ (\iota_{\mathrm{Sm}}^b)_! : \mathbf{SH}^b(S)_{S^1} \rightarrow \underline{\mathbf{SH}}^{\mathcal{E}\infty}(S)_{S^1}$  is fully faithful.*

4.5. **Sm-fibred vs.  $\mathrm{Sm}^b$ -fibred motivic spaces.** In this paragraph we will show that there is a canonical equivalence of categories

$$\mathbf{H}^{\mathcal{E}\infty}(S) \xrightarrow{\sim} \mathbf{H}^b(S).$$

4.5.1. Recall from Lemma 4.3.4 that the localization functor  $L_{\mathrm{mot}}^b : \underline{\mathrm{Spc}}^{\mathcal{E}\infty}(S) \rightarrow \underline{\mathbf{H}}^b(S)$  factors through an intermediate localization

$$(4.7) \quad L_{\mathbf{A}_b^1} : \underline{\mathbf{H}}^{\mathcal{E}\infty}(S) \rightarrow \underline{\mathbf{H}}^b(S).$$

We have:

**Lemma 4.5.2.** *The functor (4.7) sends  $\mathrm{Sm}^{\mathcal{E}\infty}$ -fibred spaces to  $\mathrm{Sm}^b$ -fibred spaces, and restricts to a localization functor*

$$L_{\mathbf{A}_b^1} : \mathbf{H}^{\mathcal{E}\infty}(S) \rightarrow \mathbf{H}^b(S).$$

*Proof.* Since  $\mathbf{H}^{\mathcal{E}\infty}(S)$  is generated under colimits by objects of the form  $L_{\mathrm{mot}} h_S(X)$ , where  $X$  is a smooth spectral  $S$ -scheme, it suffices to show that  $L_{\mathrm{mot}}^b h_S(X)$  is contained in  $\mathbf{H}^b(S)$  for each such  $X$ .

By [Kha16, Prop. 2.9.10], there exists a Zariski cover  $(X_\alpha \hookrightarrow X)_\alpha$  such that each morphism  $X_\alpha \rightarrow S$  may be factorized as a composite  $X_\alpha \xrightarrow{p_\alpha} \mathbf{A}_S^{n_\alpha} \rightarrow S$ , where  $p_\alpha$  is étale and  $n_\alpha \geq 0$ .

Since  $X$  is isomorphic to the colimit of the Čech nerve  $\check{C}(X_\alpha/X)_\bullet$ , it suffices to assume that  $X$  admits an étale morphism to some affine space  $\mathbf{A}_S^n$ .

In this case, we may form the base change

$$\begin{array}{ccc} X^b & \longrightarrow & X \\ \downarrow q & & \downarrow p \\ \mathbf{A}_{b,S}^n & \longrightarrow & \mathbf{A}_S^n \end{array}$$

along the canonical morphism  $\mathbf{A}_{b,S}^n \rightarrow \mathbf{A}_S^n$  (4.1.4). This induces an isomorphism

$$L_{\text{mot}} h_S(X^b) \approx L_{\text{mot}} h_S(X) \times L_{\text{mot}} h_S(\mathbf{A}_{b,S}^n),$$

and thus an isomorphism

$$L_{\text{mot}}^b h_S(X^b) \approx L_{\text{mot}}^b h_S(X)$$

after application of  $L_{\mathbf{A}_b^1}$ . The spectral scheme  $X^b$  is fibre-smooth over  $S$  by Proposition 4.2.7, so the claim follows.  $\square$

4.5.3. By definition, the functor  $v : \text{Sch}_{/S}^{\mathcal{E}^\infty} \rightarrow \text{Sch}_{/S_{\text{cl}}}$  (Paragraph 3.1) sends fibre-smooth spectral  $S$ -schemes to smooth classical  $S_{\text{cl}}$ -schemes.

Therefore, exactly as in Lemma 3.2.11, we have:

**Lemma 4.5.4.** *The functor  $v$  induces a left localization*

$$(4.8) \quad L_{\text{mot}} v! : \mathbf{H}^b(S) \rightarrow \mathbf{H}^{\text{cl}}(S_{\text{cl}}).$$

By construction, this is compatible with the localization  $L_{\text{mot}} v! : \mathbf{H}^{\mathcal{E}^\infty}(S) \rightarrow \mathbf{H}^{\text{cl}}(S_{\text{cl}})$  in the sense that we have a commutative triangle of localizations

$$\begin{array}{ccc} \mathbf{H}^{\mathcal{E}^\infty}(S) & \xrightarrow{L_{\text{mot}} v!} & \mathbf{H}^{\text{cl}}(S_{\text{cl}}) \\ & \searrow L_{\mathbf{A}_b^1} & \nearrow L_{\text{mot}} v! \\ & \mathbf{H}^b(S) & \end{array}$$

Since the horizontal functor is an equivalence (Theorem 3.4.2), both intermediate localizations must be equivalences. In particular:

**Theorem 4.5.5.** *The canonical functors*

$$\begin{aligned} L_{\mathbf{A}_b^1} : \mathbf{H}^{\mathcal{E}^\infty}(S) &\rightarrow \mathbf{H}^b(S), \\ L_{\text{mot}} v! : \mathbf{H}^b(S) &\rightarrow \mathbf{H}^{\text{cl}}(S_{\text{cl}}) \end{aligned}$$

*are equivalences.*

## 5. HOMOTOPY INVARIANT K-THEORY

In this section we will define a cohomology theory  $\text{KH}^{\mathcal{E}^\infty}$ , a brave new analogue of Weibel's homotopy invariant K-theory. We will show that it is representable as a motivic  $\text{Sm}^{\mathcal{E}^\infty}$ -fibre  $S^1$ -spectrum. As an application of our comparison result (Theorem 3.4.2) we will deduce a derived nilpotent invariance property for  $\text{KH}^{\mathcal{E}^\infty}$ .

We continue the convention that all spectral schemes are quasi-compact quasi-separated.

**5.1. Algebraic K-theory.** We recall a few standard facts about the algebraic K-theory of spectral schemes.

5.1.1. Let  $S$  be a spectral scheme. Write  $\mathrm{Qcoh}(S)$  for the stable  $\infty$ -category of quasi-coherent sheaves on  $S$ , and  $\mathrm{Perf}(S)$  for the full exact subcategory of perfect complexes (see [Lur16b, Chap. 2]).

Recall:

**Theorem 5.1.2.** *The assignments  $S \mapsto \mathrm{Qcoh}(S)$  and  $S \mapsto \mathrm{Perf}(S)$ , viewed as presheaves of  $\infty$ -categories on the  $\infty$ -category of quasi-compact quasi-separated spectral schemes, satisfy descent with respect to the fpqc topology.*

*Proof.* The claim for  $\mathrm{Qcoh}$  follows immediately from [Lur16b, Thm. D.6.3.3]. The second claim follows from the first in view of the fact that perfect complexes can be described as the dualizable objects with respect to the canonical symmetric monoidal structure on  $\mathrm{Qcoh}(S)$  [Lur16b, Prop. 6.2.6.2], and taking dualizable objects commutes with taking limits of symmetric monoidal  $\infty$ -categories.  $\square$

5.1.3. Let  $j : U \hookrightarrow X$  be a quasi-compact open immersion between spectral schemes. Let  $\mathrm{Qcoh}(X)_U$  denote the full subcategory of  $\mathrm{Qcoh}(X)$  spanned by quasi-coherent sheaves  $\mathcal{F}$  which vanish on  $U$ , i.e.  $j^*(\mathcal{F}) = 0$ . Similarly let  $\mathrm{Perf}(X)_U$  denote the full subcategory of  $\mathrm{Perf}(X)$  spanned by perfect complexes which vanish on  $U$ .

**Theorem 5.1.4.** *Let  $j : U \hookrightarrow X$  be a quasi-compact open immersion between spectral schemes.*

- (i) *An object of  $\mathrm{Qcoh}(X)_U$  is compact if and only if it is a perfect complex.*
- (ii) *The category  $\mathrm{Qcoh}(X)_U$  is compactly generated (by a single object).*
- (iii) *The sequence of small stable  $\infty$ -categories*

$$\mathrm{Perf}(X)_U \rightarrow \mathrm{Perf}(X) \xrightarrow{j^*} \mathrm{Perf}(U),$$

*is exact in the sense of [BGT13].*

*Proof.* In the case  $U = \emptyset$ , claims (i) and (ii) are [Lur16b, Thm. 9.6.1.1 and Cor. 9.6.3.2]; the general case can be proved by an adaptation of these arguments. Alternatively, these claims follow from [AG14, Prop. 6.9 and Thm. 6.11].

For claim (iii), it suffices by (i) and (ii) to pass to ind-completions and note that the sequence

$$\mathrm{Qcoh}(X)_U \rightarrow \mathrm{Qcoh}(X) \xrightarrow{j^*} \mathrm{Qcoh}(U)$$

is an exact sequence of stable compactly generated  $\infty$ -categories.  $\square$

5.1.5. Given a spectral scheme  $X$ , let  $K(X)$  denote its (nonconnective) algebraic K-theory (i.e. the nonconnective algebraic K-theory of the stable  $\infty$ -category  $\mathrm{Perf}(X)$ ; see [BGT13]). More generally, let  $K(X)_U$  denote the nonconnective algebraic K-theory of  $\mathrm{Perf}(X)_U$ .

A celebrated result of [TT07] says that, in the classical setting,  $K(X)_U$  can be identified with the homotopy fibre of the map  $K(X) \rightarrow K(U)$ . This remains true in spectral algebraic geometry:

**Theorem 5.1.6** (Thomason). *Let  $j : U \hookrightarrow X$  be a quasi-compact open immersion of spectral schemes. There is a canonical exact triangle of spectra*

$$K(X)_U \rightarrow K(X) \rightarrow K(U).$$

*Proof.* According to [BGT13, Sect. 9], nonconnective K-theory sends exact sequence of stable  $\infty$ -categories to exact triangles. Hence the result follows from Theorem 5.1.4.  $\square$

We obtain the following as a consequence:



**Theorem 5.1.7** (Thomason). *The presheaf of spectra  $S \mapsto K(S)$ , on the category of quasi-compact quasi-separated spectral schemes, satisfies Nisnevich descent.*

*Proof.* According to [Kha16, Rem. 4.1.8], it suffices to show that it satisfies Nisnevich excision. It is clear that  $K(\emptyset) = 0$ , since  $\text{Perf}(\emptyset) = 0$ . It remains to show that for every Nisnevich square

$$\begin{array}{ccc} W & \hookrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

with  $j : U \rightarrow X$  an open immersion and  $p : V \rightarrow X$  étale, the induced commutative square of spectra

$$\begin{array}{ccc} K(X) & \xrightarrow{j^*} & K(U) \\ \downarrow p^* & & \downarrow \\ K(V) & \longrightarrow & K(W) \end{array}$$

is cartesian.

This is equivalent to the invertibility of the induced morphism on fibres,

$$(5.1) \quad \text{Fib}(K(X) \rightarrow K(U)) \rightarrow \text{Fib}(K(V) \rightarrow K(W)).$$

By Theorem 5.1.6, this is identified with the canonical morphism

$$K(X)_U \rightarrow K(V)_W$$

which is induced by the canonical functor

$$\text{Perf}(X)_U \rightarrow \text{Perf}(V)_W,$$

which is an equivalence by Nisnevich excision for the presheaf  $X \mapsto \text{Perf}(X)$  (Theorem 5.1.2).  $\square$

## 5.2. Homotopy invariant K-theory.

5.2.1. For each spectral scheme  $S$ , the presheaf of spectra

$$K : (\text{Sch}_S^{\text{ét}\infty})^{\text{op}} \rightarrow \text{Spt}$$

defines a  $\text{Sch}^{\text{ét}\infty}$ -fibred  $S^1$ -spectrum over  $S$ , which by Theorem 5.1.7 is Nisnevich-local.

We will denote this object by  $\underline{K}_S$  to emphasize that we consider it as a  $\text{Sch}^{\text{ét}\infty}$ -fibred spectrum over  $S$ .

5.2.2. Let  $\underline{KH}_S^{\text{ét}\infty}$  denote the  $\mathbf{A}^1$ -localization of the fibred spectrum  $\underline{K}_S$ . According to Lemma 2.6.3, this is a motivic  $S^1$ -spectrum which can be computed by the formula

$$\underline{KH}_S^{\text{ét}\infty}(X) = \varinjlim_{(Y \rightarrow X) \in (\mathbf{A}_X)^{\text{op}}}$$

for each smooth spectral  $S$ -scheme  $X$  (see *loc. cit.* for the notation  $\mathbf{A}_X$ ).

5.2.3. Let  $\text{KH}_S^{\text{ét}\infty}$  denote the  $\text{Sm}^{\text{ét}\infty}$ -fibred  $S^1$ -spectrum obtained by restriction of  $\underline{KH}_S^{\text{ét}\infty}$  to the site  $\text{Sm}_S^{\text{ét}\infty}$ , i.e.

$$\text{KH}_S^{\text{ét}\infty} := (\iota_{\text{Sm}})^*(\underline{KH}_S^{\text{ét}\infty}),$$

in the notation of Paragraph 2.2.

By Lemma 2.2.4, this is a motivic  $\text{Sm}^{\text{ét}\infty}$ -fibred  $S^1$ -spectrum.

5.2.4. Let  $\underline{K}_S$  denote the restriction  $(\iota_{\text{Sm}})^*(\underline{K}_S)$ . By Lemma 2.2.11, the  $\text{Sm}^{\mathcal{E}_\infty}$ -fibred  $S^1$ -spectrum  $\text{KH}_S^{\mathcal{E}_\infty}$  can be described equivalently as the  $\mathbf{A}^1$ -localization of  $\underline{K}_S$ , i.e.:

$$\text{KH}_S^{\mathcal{E}_\infty} = L_{\mathbf{A}^1}(\underline{K}_S).$$

### 5.3. Nilpotent invariance.

5.3.1. Let  $\underline{K}_{S_{\text{cl}}}^{\text{cl}}$  denote the restriction of the K-theory presheaf

$$\mathbf{K} : (\text{Sch}_{/S_{\text{cl}}}^{\mathcal{E}_\infty})^{\text{op}} \rightarrow \text{Spt}$$

to the site  $\text{Sch}_{/S_{\text{cl}}}$  of classical  $S_{\text{cl}}$ -schemes. That is,

$$(5.2) \quad \underline{K}_{S_{\text{cl}}}^{\text{cl}} := u^*(\underline{K}_S)$$

in the notation of Paragraph 3.1.

This defines a classical Sch-fibred  $S^1$ -spectrum over  $S_{\text{cl}}$ , which is Nisnevich-local by Lemma 3.2.6.

5.3.2. Let  $\underline{\text{KH}}_{S_{\text{cl}}}^{\text{cl}} := L_{\mathbf{A}_{\text{cl}}^1}(\underline{K}_{S_{\text{cl}}}^{\text{cl}})$  denote the  $\mathbf{A}_{\text{cl}}^1$ -localization of  $\underline{K}_{S_{\text{cl}}}^{\text{cl}}$ . Here  $\mathbf{A}_{\text{cl}}^1$  denotes the classical affine line (over  $\text{Spec}(\mathbf{Z})$ ).

This is a classical motivic  $S^1$ -spectrum over  $S_{\text{cl}}$ , which computes Weibel's homotopy K-theory:

$$\underline{\text{KH}}_{S_{\text{cl}}}^{\text{cl}}(X) \approx \varinjlim_{[n] \in \Delta^{\text{op}}} \mathbf{K}(X \times \Delta_{S_{\text{cl}}}^n),$$

for any smooth classical  $S_{\text{cl}}$ -scheme  $X$ , where  $\Delta_{S_{\text{cl}}}^\bullet$  is the cosimplicial classical  $S_{\text{cl}}$ -scheme whose  $n$ th term is the algebraic  $n$ -simplex over  $S_{\text{cl}}$ . See [Cis13, § 2].

5.3.3. Let  $\text{KH}_{S_{\text{cl}}}^{\text{cl}}$  denote the restriction of the Sch-fibred motivic  $S^1$ -spectrum  $\underline{\text{KH}}_{S_{\text{cl}}}^{\text{cl}}$  to the site  $\text{Sm}_{/S_{\text{cl}}}$  of classical smooth  $S_{\text{cl}}$ -schemes. This is a  $\text{Sm}^{\mathcal{E}_\infty}$ -fibred motivic  $S^1$ -spectrum.

We have:

**Theorem 5.3.4** (Nilpotent invariance). *For each spectral scheme  $S$ , there is a canonical isomorphism*

$$\text{KH}_S^{\mathcal{E}_\infty} \xrightarrow{\sim} v^*(\text{KH}_{S_{\text{cl}}}^{\text{cl}})$$

of  $\text{Sm}^{\mathcal{E}_\infty}$ -fibred motivic  $S^1$ -spectra over  $S$ . In particular, for each smooth spectral  $S$ -scheme  $X$ , there are canonical functorial isomorphisms of spectra

$$\text{KH}_S^{\mathcal{E}_\infty}(X) \xrightarrow{\sim} \text{KH}_{S_{\text{cl}}}^{\text{cl}}(X_{\text{cl}}).$$

*Proof.* Since the functor  $v_! = u^*$  sends  $\underline{K}_S \mapsto \underline{K}_{S_{\text{cl}}}^{\text{cl}}$  (5.2), this follows from Corollary 3.4.5.  $\square$

*Remark 5.3.5.* One can also consider the flat version  $\text{KH}_S^{\flat} \in \mathbf{H}^{\flat}(S)$ . By Theorem 4.5.5 it follows that it will also have nilpotent invariance.

This variant  $\text{KH}_S^{\flat}$  has also been considered by Antieau–Gepner–Heller, who have an alternative proof of nilpotent invariance, which applies to connective  $\mathcal{E}_1$ -ring spectra (private communication).

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FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT REGENSBURG, 93040 REGENSBURG, GERMANY

*E-mail address:* denis-charles.cisinski@mathematik.uni-regensburg.de

*URL:* <http://www.mathematik.uni-regensburg.de/cisinski/>

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT REGENSBURG, 93040 REGENSBURG, GERMANY

*E-mail address:* adeel.khan@mathematik.uni-regensburg.de

*URL:* <http://www.preschema.com>