VIRTUAL EXCESS INTERSECTION THEORY

ADEEL A. KHAN

To my little brother, on the occasion of his birthday.

ABSTRACT. We prove a K-theoretic excess intersection formula for derived Artin stacks. When restricted to classical schemes, it gives a refinement and new proof of R. Thomason's formula.

INTRODUCTION

Suppose given a commutative square of derived Artin stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where f and f' are quasi-smooth closed immersions. We call this an *excess intersection square* if the following conditions hold:

(i) The square is cartesian on underlying classical stacks, i.e., the canonical morphism $\mathcal{X}' \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ induces an isomorphism $\mathcal{X}'_{\mathrm{cl}} \simeq (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')_{\mathrm{cl}}$.

(ii) The canonical morphism of (shifted) relative cotangent complexes

$$p^* \mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1] \to \mathcal{L}_{\mathcal{X}'/\mathcal{Y}'}[-1]$$

is surjective (on π_0). In particular, its fibre Δ is locally free of finite rank.

For a derived stack $\mathcal{X}$, let $\mathrm{K}(\mathcal{X})$ denote the algebraic K-theory space of perfect complexes on $\mathcal{X}$. In this note we prove the following theorem:

Theorem 0.1. *For any excess intersection square as above, there is a canonical homotopy*

$$q^* f_*(-) \simeq f'_*(p^*(-) \cup e(\Delta))$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{Y}')$, where $e(\Delta) \in \mathrm{K}(\mathcal{X}')$ is the Euler class of the excess sheaf.

Applied to quotient stacks, this gives an equivariant excess intersection formula:

Corollary 0.2. *Let* G *be a flat group algebraic space of finite presentation over an algebraic space* S. *For any excess intersection square of* G-*equivariant derived algebraic spaces*

$$
\begin{array}{ccc}
X' & \xrightarrow{\;f'\;} & Y' \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
X & \xrightarrow{\;f\;} & Y,
\end{array}
$$

where f *and* f' *are regular closed immersions, there is a canonical homotopy*

$$q^* f_*(-) \simeq f'_*(p^*(-) \cup e(\Delta))$$

of maps $\mathrm{K}^{\mathrm{G}}(X) \to \mathrm{K}^{\mathrm{G}}(Y')$. *Here* Δ *is the excess sheaf as above and* K^{G} *is the algebraic K-theory of* G-*equivariant perfect complexes.*

Derived stacks arise naturally in moduli theory, especially in curve counting theories such as Gromov–Witten theory and Donaldson–Thomas theory. The moduli problems arising in these theories are typically singular, but are nevertheless *quasi-smooth* when regarded as derived stacks. As an example, Theorem 0.1 can be applied to derived moduli stacks of stable maps and allows a one-line K-theoretic proof of a conjecture of Cox, Katz and Lee [CKL] (cf. [Ke, Cor. 2.2.6, Eq. (4)]) relating the genus zero Gromov–Witten invariants of a smooth projective variety with those of the zero locus of a section of a convex vector bundle. Via the virtual Grothendieck–Riemann–Roch theorem of [Kh3] one also recovers the Chow group level formula proven in [KKP].

The present paper, together with [Kh2] and [Kh1], is part of a general program that sets up a K-theoretic formalism of intersection theory on derived schemes and stacks following [SGA6]. In particular, the current result was applied in [Kh1] to generalize the Grothendieck–Riemann–Roch theorem of [SGA6] to quasi-smooth projective morphisms of derived schemes. If X is a (possibly derived) scheme, then for any quasi-smooth closed immersion $i : Z \to X$, the class

$$i_*(1) = [i_*(\mathcal{O}_Z)] \in \mathrm{K}(X)$$

is the K-theoretic version of the cohomological virtual fundamental class [Z] constructed in [Kh3, (3.21)]. The GRR formula implies in particular that this class lives in the expected degree of the γ-filtration (determined by the relative virtual dimension). Thus one gets a map from the free abelian group on quasi-smooth closed subschemes ("derived cycles"), graded by virtual dimension, to the graded pieces of the γ-filtration on K-theory.

Both Theorem 0.1 and Corollary 0.2 seem to be new even in the setting of classical algebraic geometry. A closed immersion of classical stacks is quasi-smooth if and only if it is regular (a.k.a. a local complete intersection) in the sense of [SGA6, Exp. VII, Déf. 1.4] (see [KR, 2.3.6]), so in that case the statement becomes:

Corollary 0.3. *For any cartesian square of Artin stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\ f'\ } & \mathcal{Y}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y}, \end{array}$$

where f and f' are regular closed immersions, there is a canonical homotopy

$$q^* f_*(-) \simeq f'_*(p^*(-) \cup e(\Delta))$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{Y}')$, where Δ is the excess sheaf.

For quasi-compact quasi-separated schemes, the excess intersection formula was proven by Thomason [Th, Thm. 3.1]. Even in that case our result is more precise in that the proof provides an explicit chain of homotopies between the two sides.

For a homotopy cartesian[1] square, the excess sheaf Δ vanishes, so the excess intersection formula reduces to the base change formula. More interesting are the following two special cases:

Corollary 0.4 (Self-intersections). *For any quasi-smooth closed immersion $f : \mathcal{X} \hookrightarrow \mathcal{Y}$ of derived Artin stacks, there is a canonical homotopy*

$$f^* f_* \simeq (-) \cup e(\mathcal{N}_{\mathcal{X}/\mathcal{Y}})$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{X})$.

This is the result of applying Theorem 0.1 to the self-intersection square

$$\begin{array}{ccc} \mathcal{X} & = & \mathcal{X} \\ \| & & \downarrow{\scriptstyle f} \\ \mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y}, \end{array}$$

where the excess sheaf is the conormal sheaf $\mathcal{N}_{\mathcal{X}/\mathcal{Y}} = \mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1]$. A self-intersection formula for regular equivariant classical algebraic spaces was previously obtained by G. Vezzosi and A. Vistoli in [VV, Thm. 2.1][2]. Note that the equivariant version of this formula can be used to give a simple derivation of the virtual Atiyah–Bott formula for localizing to the fixed points of a torus action as in [Qu, Sect. 3] and [CFK, Sect. 5].

Similarly, we get a generalization of the "formule clef" (key formula) of [SGA6, Exp. VII]:

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[1]We remind the reader that a commutative square of *classical* stacks is homotopy cartesian (in the ∞-category of derived stacks) if and only if it is cartesian and Tor-independent.

[2]Moreover, the refined statement we prove here, identifying an explicit homotopy between the two sides of the formula, is strong enough to correct the gap in the proof of [VV, Thm. 3.2] that the authors point out in the erratum.

Corollary 0.5 (Blow-ups)**.** *For any quasi-smooth closed immersion $f : \mathcal{X} \hookrightarrow \mathcal{Y}$ of derived Artin stacks, consider the blow-up square*

$$
\begin{array}{ccc}
\mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) & \xrightarrow{\ i\ } & \mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
\mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y}.
\end{array}
$$

Then there is a canonical homotopy

$$
q^* f_* \simeq i_*(p^*(-) \cup e(\Delta))
$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}))$.

The blow-up square is the universal excess square over $f : \mathcal{X} \to \mathcal{Y}$ such that the upper morphism i is a virtual Cartier divisor; see [KR].

The proof of Theorem 0.1 is inspired by Fulton's proof of the Grothendieck–Riemann–Roch formula [Fu, Chap. 15], and uses a similar argument involving the deformation space $\mathrm{Bl}_{\mathcal{X} \times \{\infty\}}(\mathcal{Y} \times \mathbf{P}^1)$ to reduce to the case where f and f' are zero sections of (projective completions of) vector bundles. Due to the failure of $\mathbf{A}^1$-homotopy invariance in algebraic K-theory (of singular spaces), the reduction is much more subtle than for the analogous result in motivic Borel–Moore homology (= higher Chow groups, under suitable hypotheses), which was proven in [Kh3, Prop. 3.15].

The reader will find that the proof also goes through for any "additive invariant" in place of algebraic K-theory, i.e., for invariants of stable ∞-categories that satisfy additivity in the sense of Waldhausen.

1. DERIVED SYMMETRIC POWERS

Let $\mathcal{X}$ be a derived Artin stack. We denote by $\mathrm{Qcoh}(\mathcal{X})$ the stable presentable ∞-category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules, and by $\mathrm{Qcoh}(\mathcal{X})_{\geqslant 0}$ the full subcategory of connective objects. Recall that $\mathrm{Qcoh}(\mathcal{X})$ is the limit

$$
\varprojlim_{u:\mathrm{X} \to \mathcal{X}} \mathrm{Qcoh}(\mathrm{X})
$$

taken over the ∞-category $\mathrm{Lis}(\mathcal{X})$ of smooth morphisms $u : \mathrm{X} \to \mathcal{X}$, where X is an affine derived scheme. In other words, the sheaf $\mathcal{X} \mapsto \mathrm{Qcoh}(\mathcal{X})$ is right Kan extended from affines. For an affine $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$, $\mathrm{Qcoh}(\mathrm{X})$ is equivalent to the stable ∞-category of (nonconnective) modules over the simplicial commutative ring R, in the sense of [SAG, Not. 25.2.1.1]. Similarly for the connective and perfect subcategories $\mathrm{Qcoh}(-)_{\geqslant 0}$ and $\mathrm{Perf}(-)$, respectively. Recall that $\mathrm{K}(\mathcal{X})$ is defined as the algebraic K-theory space of the stable ∞-category $\mathrm{Perf}(\mathcal{X})$. See [Kh2] or [Kh4] for more details on these definitions. We will in particular make use of the base change and projection formulas (see [Kh4, Rem. 1.9] or [SAG, Cor. 3.4.2.2 (3), Rem. 3.4.2.6]).

By $\mathrm{QcohAlg}(\mathcal{X})$ we denote the presentable ∞-category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras. This admits a similar description as above, and for $\mathrm{Spec}(\mathrm{R})$ is equivalent to the ∞-category of R-algebras.

The *derived symmetric algebra* functor

$$\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}} : \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0} \to \mathrm{QcohAlg}(\mathcal{X})$$

is left adjoint to the forgetful functor. For any morphism $f : \mathcal{X}' \to \mathcal{X}$ there are natural isomorphisms

$$f^*(\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F})) \simeq \mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}'}}(f^*\mathcal{F})$$

for every $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0}$ (since the forgetful functors commute with f_*).

The derived symmetric algebra $\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F})$ can be described in terms of the derived symmetric powers $\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F})$. These are constructed in the affine case in [SAG, Sect. 25.2.2], and extend to stacks by descent. (We warn the reader that, even for ordinary commutative rings, the derived symmetric powers are different from the classical ones on non-flat modules; see [SAG, Prop. 25.2.3.4] or [Qui, Sect. 7] for a classical introduction.)

Construction 1.1. Let $q : (\mathrm{SCRMod})_{\geqslant 0} \to \mathrm{SCRing}$ denote the cocartesian fibration associated to the presheaf of ∞-categories $\mathrm{R} \mapsto (\mathrm{Mod}_{\mathrm{R}})_{\geqslant 0}$; its objects are pairs (R, M), where $\mathrm{R} \in \mathrm{SCRing}$ is a simplicial commutative ring and $\mathrm{M} \in (\mathrm{Mod}_{\mathrm{R}})_{\geqslant 0}$ is a connective R-module. By [SAG, Constr. 25.2.2.1], there is for each integer $n \geqslant 0$ a functor $(\mathrm{SCRMod})_{\geqslant 0} \to (\mathrm{SCRMod})_{\geqslant 0}$ given informally by the assignment

$$(\mathrm{R}, \mathrm{M}) \mapsto (\mathrm{R}, \mathrm{Sym}^n_{\mathrm{R}}(\mathrm{M})).$$

This functor preserves q-cocartesian morphisms [SAG, Prop. 25.2.3.1] and therefore induces functors

$$\mathrm{Sym}^n_{\mathrm{R}} : (\mathrm{Mod}_{\mathrm{R}})_{\geqslant 0} \to (\mathrm{Mod}_{\mathrm{R}})_{\geqslant 0},$$

which define a natural transformation of functors as R varies. By right Kan extension, this extends to a natural transformation on the ∞-category of derived Artin stacks. In other words for every derived Artin stack $\mathcal{X}$ we have functors

$$\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}} : \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0} \to \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0},$$

which commute with f^*.

Lemma 1.2. *Let $\mathcal{X}$ be a derived Artin stack. Then for every connective quasi-coherent sheaf $\mathcal{E} \in \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0}$, there is a canonical isomorphism*

$$\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}) \simeq \bigoplus_{n \geqslant 0} \mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$$

in $\mathrm{Qcoh}(\mathcal{X})$.

Proof. As $\mathcal{X}$ varies, both $\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(-)$ and $\bigoplus_n \mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(-)$ define natural transformations $\mathrm{Qcoh}(-)_{\geqslant 0} \to \mathrm{Qcoh}(-)_{\geqslant 0}$ of presheaves on the ∞-category of derived Artin stacks. On the restrictions to affines, the two are canonically equivalent by [SAG, Constr. 25.2.2.6]. Since $\mathrm{Qcoh}(-)_{\geqslant 0}$ is right Kan extended from affines, the claim follows. $\square$

Lemma 1.3. *Let $\mathcal{X}$ be a derived Artin stack and $\mathcal{E}' \to \mathcal{E} \to \mathcal{E}''$ a cofibre sequence of connective perfect complexes. Then there are canonical equivalences*

$$[\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})] \simeq \bigoplus_{i+j=n} [\mathrm{Sym}^i_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}') \otimes_{\mathcal{O}_{\mathcal{X}}} \mathrm{Sym}^j_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}'')]$$

for every $n \geqslant 0$. *If* $\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}) = 0$ *for all sufficiently large* $n \gg 0$, *and similarly for* $\mathcal{E}'$ *and* $\mathcal{E}''$, *then also*

$$[\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})] \simeq [\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}') \otimes_{\mathcal{O}_{\mathcal{X}}} \mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}'')]$$

in $\mathrm{K}(\mathcal{X})$.

Proof. For every such cofibre sequence and every integer $n \geqslant 0$, there is a canonical filtration

$$\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}') = \mathrm{F}^{0,n} \to \mathrm{F}^{1,n} \to \cdots \to \mathrm{F}^{n,n} = \mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$$

together with cofibre sequences

$$\mathrm{F}^{i-1,n} \to \mathrm{F}^{i,n} \to \mathrm{Sym}^{n-i}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}') \otimes_{\mathcal{O}_{\mathcal{X}}} \mathrm{Sym}^i_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}'')$$

for each $0 < i \leqslant n$. This is constructed in [SAG, Constr. 25.2.5.4] for affines and the construction extends to stacks by a similar right Kan extension procedure. Since $\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$ is perfect for all n [SAG, Prop. 25.2.5.3] (and similarly for $\mathcal{E}'$ and $\mathcal{E}''$), these cofibre sequences give rise to the desired equivalences in $\mathrm{K}(\mathcal{X})$. The second claim follows from Lemma 1.2, since the assumption guarantees $\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$ is also perfect (and similarly for $\mathcal{E}'$ and $\mathcal{E}''$). $\qquad\square$

Definition 1.4. *The Euler class of a finite locally free sheaf* $\mathcal{E}$ *is defined by*

$$e(\mathcal{E}) = [\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1])] = \sum_{n \geqslant 0} (-1)^n \cdot [\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1])] \in \mathrm{K}(\mathcal{X}).$$

There are canonical isomorphisms $\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1]) \simeq \Lambda^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})[n]$ (cf. [SAG, Prop. 25.2.4.2]), where $\Lambda^n_{\mathcal{O}_{\mathcal{X}}}(-)$ denotes the derived exterior power, so this agrees with the usual definition of the Euler class (often denoted $\lambda_{-1}(\mathcal{E})$).

2. Projective bundles

Given a connective perfect complex $\mathcal{F}$ on a derived Artin stack $\mathcal{X}$, there is an associated "generalized vector bundle"

$$\mathbf{V}_{\mathcal{X}}(\mathcal{F}) = \mathrm{Spec}_{\mathcal{X}}(\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F})),$$

defined as the relative spectrum of its derived symmetric algebra. It is the moduli of cosections $\mathcal{F} \to \mathcal{O}_{\mathcal{X}}$; that is, for any derived scheme S over $\mathcal{X}$, the space of S-points of $\mathbf{V}_{\mathcal{X}}(\mathcal{F})$ over $\mathcal{X}$ is the space of $\mathcal{O}_\mathrm{S}$-linear homomorphisms $\mathcal{F}|_\mathrm{S} \to \mathcal{O}_\mathrm{S}$.

This construction can exhibit some surprising behaviour. For example, if $\mathcal{E}$ is a finite locally free sheaf, consider the morphism $p : \mathbf{V}_{\mathcal{X}}(\mathcal{E}[1]) \to \mathcal{X}$. This is a quasi-smooth closed immersion that fits in the homotopy cartesian square

$$\begin{array}{ccc} \mathbf{V}_{\mathcal{X}}(\mathcal{E}[1]) & \xrightarrow{\ p\ } & \mathcal{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s} \\ \mathcal{X} & \xrightarrow{\ s\ } & \mathbf{V}_{\mathcal{X}}(\mathcal{E}), \end{array}$$

where s is the zero section. By the base change formula we obtain a canonical isomorphism

$$s^* s_* \simeq p_* p^* \simeq (-) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1])$$

of functors $\mathrm{Perf}(\mathcal{X}) \to \mathrm{Perf}(\mathcal{X})$, where the second isomorphism follows from $p_*(\mathcal{O}) = \mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1])$ and the projection formula. We have just proven the following lemma in the special case where $t = s$:

Lemma 2.1. *Let $\mathcal{X}$ be a derived Artin stack. Given a finite locally free sheaf $\mathcal{E}$ and a cosection $t : \mathcal{E} \to \mathcal{O}_{\mathcal{X}}$, we let $i : \mathcal{Z} \to \mathcal{X}$ denote its derived zero locus, so that there is a homotopy cartesian square*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\ i\ } & \mathcal{X} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle t} \\ \mathcal{X} & \xrightarrow{\ s\ } & \mathbf{V}_{\mathcal{X}}(\mathcal{E}). \end{array}$$

Then there is an essentially unique homotopy

$$i_* i^* \simeq (-) \cup e(\mathcal{E})$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{X})$.

Proof. The result of composing such a square with the open immersion $\mathbf{V}_{\mathcal{X}}(\mathcal{E}) \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ is still homotopy cartesian (see [Kh2, Subsect. 3.1] for background on projective bundles in this setting). Let $\bar{s} : \mathcal{X} \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ and $\bar{t} : \mathcal{X} \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$, respectively, denote the induced morphisms. By the base change formula we have

$$i_* i^* \simeq \bar{t}^* \bar{s}_*$$

as functors $\mathrm{Perf}(\mathcal{X}) \to \mathrm{Perf}(\mathcal{X})$. In the case $t = s$, we have as above $\bar{s}^* \bar{s}_* \simeq (-) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1])$, so in particular $\bar{s}^* \bar{s}_* \simeq (-) \cup e(\mathcal{E})$ in K-theory. Thus it will suffice to exhibit an (essentially unique) homotopy between the two maps $\mathrm{K}(\mathbf{P}(\mathcal{E} \oplus \mathcal{O})) \to \mathrm{K}(\mathcal{X})$ induced by $\bar{s}^*$ and $\bar{t}^*$. But from the projective bundle formula [Kh2, Cor. 3.4.1] it follows that there is an exact triangle

$$\mathrm{K}(\mathbf{P}_{\mathcal{X}}(\mathcal{E})) \xrightarrow{\ \infty_*\ } \mathrm{K}(\mathbf{P}_{\mathcal{X}}(\mathcal{E} \oplus \mathcal{O})) \xrightarrow{\ \bar{u}^*\ } \mathrm{K}(\mathcal{X}),$$

for *any* $\bar{u} : \mathcal{X} \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ induced by a section $u : \mathcal{X} \to \mathbf{V}_{\mathcal{X}}(\mathcal{E})$. In other words, both maps in question are the cofibre of the same map ∞_*, whence the desired homotopy. $\qquad\square$

Let $\pi : \mathbf{P}(\mathcal{E} \oplus \mathcal{O}) \to \mathcal{X}$ denote the projection. We have on $\mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ the canonical exact triangle of locally free sheaves

$$\mathcal{Q} \to \pi^*(\mathcal{E}) \oplus \mathcal{O} \to \mathcal{O}(1).$$

Recall that the zero section $\bar{s} : \mathcal{X} \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ can be written as the derived zero locus of the canonical cosection

$$\mathcal{Q} \to \pi^*(\mathcal{E}) \oplus \mathcal{O} \xrightarrow{\ \mathrm{pr}_2\ } \mathcal{O}.$$

Thus we get:

Corollary 2.2. *There is a canonical homotopy*

$$\bar{s}_*(-) \simeq e(\mathcal{Q}) \cup \pi^*(-)$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathbf{P}(\mathcal{E} \oplus \mathcal{O}))$.

Proof. By Lemma 2.1, $\bar{s}_*(\mathcal{O}) \simeq e(\mathcal{Q})$. By the projection formula, $\bar{s}_*(-) \simeq \bar{s}_*(\mathcal{O}) \cup \pi^*(-) \simeq e(\mathcal{Q}) \cup \pi^*(-)$. $\qquad\square$

We are now ready to prove a special case of Theorem 0.1. Let $p : \mathcal{X}' \to \mathcal{X}$ be a morphism of derived Artin stacks. Let $\mathcal{E}$ and $\mathcal{E}'$ be finite locally free sheaves on $\mathcal{X}$ and $\mathcal{X}'$, respectively, together with a surjection

$$p^*(\mathcal{E}) \twoheadrightarrow \mathcal{E}'$$

whose fibre we denote Δ. This induces an excess intersection square

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\bar{s}'} & \mathbf{P}(\mathcal{E}' \oplus \mathcal{O}) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\mathcal{X} & \xrightarrow{\bar{s}} & \mathbf{P}(\mathcal{E} \oplus \mathcal{O}),
\end{array}
$$

where $\bar{s}$ and $\bar{s}'$ are the zero sections.

Claim 2.3. *The excess intersection formula*

$$q^* \bar{s}_* \simeq \bar{s}'_*(p^*(-) \cup e(\Delta))$$

holds for the above square.

Proof. Let $\pi : \mathbf{P}(\mathcal{E} \oplus \mathcal{O}) \to \mathcal{X}$ and $\pi' : \mathbf{P}(\mathcal{E}' \oplus \mathcal{O}) \to \mathcal{X}'$ denote the respective projections. Let $\mathcal{Q}$ and $\mathcal{Q}'$ denote the respective universal hyperplane sheaves on $\mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ and $\mathbf{P}(\mathcal{E}' \oplus \mathcal{O})$. The surjection $p^*(\mathcal{E}) \to \mathcal{E}'$ gives rise to a canonical morphism $q^*\mathcal{Q} \to \mathcal{Q}'$, whose fibre is $(\pi')^*(\Delta)$. Thus Lemma 1.3 provides a canonical homotopy

$$e(q^*\mathcal{Q}) \simeq e((\pi')^*\Delta) \cup e(\mathcal{Q}')$$

in $\mathrm{K}(\mathbf{P}(\mathcal{E}' \oplus \mathcal{O}))$. Now two applications of Corollary 2.2 give:

$$
\begin{aligned}
q^* \bar{s}_* &\simeq q^*(\pi^*(-) \cup e(\mathcal{Q})) \\
&\simeq (\pi')^* p^*(-) \cup e(q^*\mathcal{Q}) \\
&\simeq (\pi')^* p^*(-) \cup e((\pi')^*\Delta) \cup e(\mathcal{Q}') \\
&\simeq (\pi')^*(p^*(-) \cup e(\Delta)) \cup e(\mathcal{Q}') \\
&\simeq \bar{s}'_*(p^*(-) \cup e(\Delta)),
\end{aligned}
$$

as desired. $\qquad\square$

3. Deformation space

Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth closed immersion of derived Artin stacks. Write M for the blow-up $\mathrm{Bl}_{\mathcal{X} \times \{\infty\}}(\mathcal{Y} \times \mathbf{P}^1)$ as in [KR]. It fits in a commutative

diagram

$$
\begin{array}{ccccc}
\mathcal{X} & \xrightarrow{s_0} & \mathcal{X} \times \mathbf{P}^1 & \xleftarrow{s_\infty} & \mathcal{X} \\
{\scriptstyle f}\downarrow & & {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle f_\infty} \\
\mathcal{Y} & \xrightarrow{\sigma_0} & \mathrm{M} & \xleftarrow{\sigma_\infty} & \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}} \oplus \mathcal{O}) \\
{\scriptstyle \pi_0}\downarrow & & {\scriptstyle \widehat{\pi}}\downarrow & & \downarrow{\scriptstyle \pi_\infty} \\
\{0\} & \longrightarrow & \mathbf{P}^1 & \longleftarrow & \{\infty\}
\end{array}
$$

The two left-hand squares and upper right-hand square are homotopy cartesian. The morphism $\hat{f}$ is

$$
\mathcal{X} \times \mathbf{P}^1 = \mathrm{Bl}_{\mathcal{X}\times\{\infty\}}(\mathcal{X} \times \mathbf{P}^1) \to \mathrm{Bl}_{\mathcal{X}\times\{\infty\}}(\mathcal{Y} \times \mathbf{P}^1),
$$

induced by $f \times \mathrm{id} : \mathcal{X} \times \mathbf{P}^1 \to \mathcal{Y} \times \mathbf{P}^1$, and the morphism f_∞ is the zero section.

Denote by $\mathrm{M}_\infty := \mathrm{M} \times^{\mathbf{R}}_{\mathbf{P}^1} \{\infty\}$ the special fibre, and by $i_\infty : \mathrm{M}_\infty \to \mathrm{M}$ the inclusion. Then we have a canonical homotopy

$$
(3.1) \qquad\qquad (\sigma_0)_* (\sigma_0)^* \simeq (i_\infty)_* (i_\infty)^*
$$

of maps $\mathrm{K}(\mathrm{M}) \to \mathrm{K}(\mathrm{M})$. Indeed, we have $0_*(\mathcal{O}) \simeq \infty_*(\mathcal{O})$ in $\mathrm{K}(\mathbf{P}^1)$, so by the base change formula there is a canonical identification

$$
(\sigma_0)_*(\mathcal{O}) \simeq (\sigma_0)_*(\pi_0)^*(\mathcal{O}) \simeq \widehat{\pi}^* 0_*(\mathcal{O})
$$
$$
\simeq \widehat{\pi}^* \infty_*(\mathcal{O}) \simeq (i_\infty)_*(\pi_\infty)^*(\mathcal{O}) \simeq (i_\infty)_*(\mathcal{O})
$$

in $\mathrm{K}(\mathrm{M})$. Thus the claim follows from the projection formula.

The fibre M_∞ fits in a homotopy cartesian and cocartesian square

$$
\begin{array}{ccc}
\mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) & \longrightarrow & \mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \\
\downarrow & & \downarrow \\
\mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}} \oplus \mathcal{O}) & \longrightarrow & \mathrm{M}_\infty.
\end{array}
$$

That is, M_∞ is the sum of the two virtual Cartier divisors $\mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}} \oplus \mathcal{O})$ and $\mathrm{Bl}_{\mathcal{X}}(\mathcal{Y})$ on M. We denote by $i_\infty : \mathrm{M}_\infty \to \mathrm{M}$ the inclusion, and by $b : \mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{M}$, and $c : \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) \to \mathrm{M}$ the composites with i_∞. We have a canonical homotopy

$$
(3.2) \qquad\qquad (i_\infty)_*(i_\infty)^* \simeq (\sigma_\infty)_*(\sigma_\infty)^* + b_* b^* - c_* c^*
$$

of maps $\mathrm{K}(\mathrm{M}) \to \mathrm{K}(\mathrm{M})$, by the following lemma.

Lemma 3.3. *Let* $\mathrm{D} \hookrightarrow \mathcal{X}$ *and* $\mathrm{D}' \hookrightarrow \mathcal{X}$ *be virtual Cartier divisors on a derived Artin stack* $\mathcal{X}$. *Denote by* $\mathrm{D} \cap \mathrm{D}' = \mathrm{D} \times^{\mathbf{R}}_{\mathcal{X}} \mathrm{D}'$ *their intersection and by*

$$
\mathrm{D} + \mathrm{D}' = \mathrm{D} \underset{\mathrm{D} \cap \mathrm{D}'}{\sqcup} \mathrm{D}'
$$

their sum. Then we have a canonical homotopy

$$
(i_{\mathrm{D}+\mathrm{D}'})_*(i_{\mathrm{D}+\mathrm{D}'})^* \simeq (i_{\mathrm{D}})_*(i_{\mathrm{D}})^* + (i_{\mathrm{D}'})_*(i_{\mathrm{D}'})^* - (i_{\mathrm{D}\cap\mathrm{D}'})_*(i_{\mathrm{D}\cap\mathrm{D}'})^*
$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{X})$.

Proof. By definition of $D + D'$ we have

$$(i_{D+D'})_*(\mathcal{O}_{D+D'}) \simeq (i_D)_*(\mathcal{O}_D) \underset{(i_{D\cap D'})_*(\mathcal{O}_{D\cap D'})}{\times} (i_{D'})_*(\mathcal{O}_{D'})$$

in $\mathrm{Perf}(\mathcal{X})$. This induces in $\mathrm{K}(\mathcal{X})$ a canonical homotopy

$$(i_{D+D'})_*(\mathcal{O}_{D+D'}) \simeq (i_D)_*(\mathcal{O}_D) + (i_{D'})_*(\mathcal{O}_{D'}) - (i_{D\cap D'})_*(\mathcal{O}_{D\cap D'}).$$

We conclude using the projection formula. $\qquad\square$

Since the intersection

$$\mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \underset{\mathrm{M}}{\times} (\mathcal{X} \times \mathbf{P}^1) = \mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \underset{\mathrm{M}_\infty}{\times} \mathcal{X} = \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) \underset{\mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}\oplus\mathcal{O}})}{\times} \mathcal{X}$$

is empty, we have $b^*\hat{f}_* = 0$ and $c^*\hat{f}_* = 0$ by the base change formula. Thus (3.1) and (3.2) induce the homotopy

$$(3.4) \qquad (\sigma_0)_*(\sigma_0)^*\hat{f}_* \simeq (i_\infty)_*(i_\infty)^*\hat{f}_* \simeq (\sigma_\infty)_*(\sigma_\infty)^*\hat{f}_*$$

of maps $\mathrm{K}(\mathcal{X} \times \mathbf{P}^1) \to \mathrm{K}(\mathrm{M})$.

4. Proof

Consider an excess intersection square of the form

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

We keep the notation of the previous section, so $\mathrm{M} = \mathrm{Bl}_{\mathcal{X}\times\{\infty\}}(\mathcal{Y} \times \mathbf{P}^1)$, etc. We consider all the same constructions for $f' : \mathcal{X}' \to \mathcal{Y}'$, with notation decorated by primes: $\mathrm{M}' = \mathrm{Bl}_{\mathcal{X}'\times\{\infty\}}(\mathcal{Y}'\times\mathbf{P}^1)$, and so on. We have morphisms of excess intersection squares

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array} \quad\hookrightarrow\quad \begin{array}{ccc} \mathcal{X}'\times\mathbf{P}^1 & \xrightarrow{\hat{f}'} & \mathrm{M}' \\ {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle \hat{q}} \\ \mathcal{X}\times\mathbf{P}^1 & \xrightarrow{\hat{f}} & \mathrm{M} \end{array} \quad\hookleftarrow\quad \begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'_\infty} & \mathbf{P}(\mathcal{N}_{\mathcal{X}'/\mathcal{Y}'}\oplus\mathcal{O}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q_\infty} \\ \mathcal{X} & \xrightarrow{f_\infty} & \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}\oplus\mathcal{O}). \end{array}$$

That the middle square is an excess intersection square is clear from the observation that the surjectivity condition (ii) can be checked on the fibres of $\mathcal{X}' \times \mathbf{P}^1$.

Consider the canonical morphisms $r : \mathcal{X} \times \mathbf{P}^1 \to \mathcal{X}$, $\rho : \mathrm{M} \to \mathcal{Y}$ (as well as their primed versions), retractions of $s_0 : \mathcal{X} \to \mathcal{X} \times \mathbf{P}^1$ and $\sigma_0 : \mathcal{Y} \to \mathrm{M}$, respectively. Using $r \circ s_0 = \mathrm{id}$ and the base change formula, we have

$$\begin{aligned} q^*f_* &\simeq q^*f_*(s_0)^*r^* \\ &\simeq q^*(\sigma_0)^*\hat{f}_*r^* \\ &\simeq (\sigma'_0)^*\hat{q}^*\hat{f}_*r^* \end{aligned}$$

and similarly

$$(q_\infty)^*(f_\infty)_* \simeq (q_\infty)^*(f_\infty)_*(s_\infty)^* r^*$$
$$\simeq (q_\infty)^*(\sigma_\infty)^* \hat{f}_* r^*$$
$$\simeq (\sigma'_\infty)^* \hat{q}^* \hat{f}_* r^*.$$

Thus (3.4) induces an equivalence

$$(\sigma'_0)_* q^* f_* \simeq (\sigma'_0)_*(\sigma'_0)^* \hat{q}^* \hat{f}_* r^* \simeq (\sigma'_\infty)_*(\sigma'_\infty)^* \hat{q}^* \hat{f}_* r^* \simeq (\sigma'_\infty)_*(q_\infty)^*(f_\infty)_*.$$

Applying ρ'_* gives

$$q^* f_* \simeq \rho'_*(\sigma'_\infty)_*(q_\infty)^*(f_\infty)_*$$

since $\rho' \circ \sigma'_0 =$ id. Finally, Claim 2.3 yields the desired equivalence

$$q^* f_* \simeq \rho'_*(\sigma'_\infty)_*(f'_\infty)_*(p^*(-) \cup e(\Delta)) \simeq f'_*(p^*(-) \cup e(\Delta)).$$

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khan@ihes.fr

IHES
35 route de Chartres
Bures-sur-Yvette
91440 France

Institute of Mathematics
Academia Sinica
Taipei
10617 Taiwan