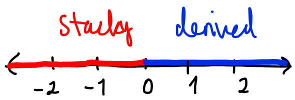
§1. OVERVIEW

The Fourier–Sato transform was introduced by Mikio Sato in 1969 as part of his microlocal sheaf theory program. It is a sheaf-theoretic analogue of the Fourier transform, which exchanges conic sheaves on a vector bundle with conic sheaves on its dual.

The purpose of these notes is to describe an extension of this construction from vector bundles to total spaces of perfect complexes. The total space of a perfect complex $\mathcal{E}$ exists naturally as a derived stack $\mathbf{V}(\mathcal{E})$, the moduli of cosections $\mathcal{E} \to \mathcal{O}$. Note that the geometry of the total space varies drastically depending on the amplitude of $\mathcal{E}$. If $\mathcal{E}$ is concentrated homologically in nonnegative degrees, then its total space is non-stacky but singular and derived; if it is concentrated homologically in nonpositive degrees, then $\mathbf{V}(\mathcal{E})$ is smooth and non-derived but stacky. Thus the Fourier–Sato transform for perfect complexes can be viewed as a kind of duality between derived and stacky phenomena.



My main motivation for this construction is the possibility of applying the Fourier–Sato transform to the (-1)-shifted cotangent complex of a possibly singular scheme. Combined with deformation to the 1-shifted tangent bundle, this gives rise to a theory of "microlocalization on the (-1)-shifted cotangent bundle". This allows one to apply the microlocal point of view—to work globally on the (shifted) cotangent bundle rather than locally on the space itself—to singular spaces such as those which arise, for instance, in cohomological Donaldson–Thomas theory. If this experiment lasts that long, then we should get to those applications in these notes eventually.

An exposition of the Fourier–Sato transform which is close to Sato's original formulation is given in [KS, §3.7]. In these notes I'll instead follow the setup of [Lau], since it is more independent of the geometric setting. In fact I will try to be agnostic about this throughout the notes: "derived stack" can be read as derived algebraic stack, derived C^∞-stack, derived complex analytic stack, etc.

Let $\mathbf{D}$ be any reasonable category of coefficients, such as the derived ∞-category of classical sheaves of abelian groups, or in the algebraic setting, the derived ∞-category of ℓ-adic sheaves (or étale motives; the Borel-type extension will suffice for our needs). Following Laumon, our replacement for conic sheaves will be sheaves equivariant with respect to an action of the multiplicative group $\mathbf{G}_m$. To compress notation, I will write

$$^{\wedge}\mathrm{X} := [\mathrm{X}/\mathbf{G}_m]$$

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for the stacky quotient of any X with $\mathbf{G}_m$-action and similarly $^\wedge f : {}^\wedge\mathrm{X} \to {}^\wedge\mathrm{Y}$ for the morphism induced by a $\mathbf{G}_m$-equivariant morphism $f : \mathrm{X} \to \mathrm{Y}$. Recall that $\mathbf{D}({}^\wedge\mathrm{X})$ is the ∞-category of $\mathbf{G}_m$-equivariant sheaves on X.

Let S be a derived Artin stack and $\mathcal{E}$ a finite locally free sheaf on S. Denote by $\mathbf{V}(\mathcal{E}) = \mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$ its total space. The Fourier–Sato transform is an equivalence of categories

$$\mathrm{FS}_{\mathcal{E}} : \mathbf{D}({}^\wedge\mathbf{V}(\mathcal{E})) \to \mathbf{D}({}^\wedge\mathbf{V}(\mathcal{E}^\vee))$$

satisfying the following properties.

First there is an isomorphism

$$\mathrm{FS}_{\mathcal{E}^\vee} \circ \mathrm{FS}_{\mathcal{E}} \simeq \mathrm{id} \tag{1}$$

up to a Tate twist.

The second main property of the Fourier–Sato duality is that it exchanges the operations of inverse images and exceptional direct image. For any homomorphism of finite locally free sheaves $\phi : \mathcal{E}' \to \mathcal{E}$ on X, let $f : \mathbf{V}(\mathcal{E}) \to \mathbf{V}(\mathcal{E}')$ denote the induced morphism of vector bundles and $f^\vee : \mathbf{V}(\mathcal{E}'^\vee) \to \mathbf{V}(\mathcal{E}^\vee)$ the dual (transpose). Then we have

$$\mathrm{FS}_{\mathcal{E}'} \circ {}^\wedge f_! \simeq {}^\wedge f^{\vee,*} \circ \mathrm{FS}_{\mathcal{E}} \tag{2}$$

up to a shift.

For example, write $p : \mathbf{V}(\mathcal{E}) \to \mathrm{S}$ for the projection and $s : \mathrm{S} \to \mathbf{V}(\mathcal{E})$ for the zero section. Then we have

$$\mathrm{FS}_{\mathcal{E}} \circ {}^\wedge s_! \simeq {}^\wedge p^{\vee,*} \circ \mathrm{FS}_0 \tag{3}$$

$$\mathrm{FS}_0 \circ {}^\wedge p_! \simeq {}^\wedge s^{\vee,*} \circ \mathrm{FS}_{\mathcal{E}} \tag{4}$$

again up to a shift.

These properties can all be found in [Lau], at least when S is a nice base scheme. Our first task in these notes, after reviewing some background, will be to prove these same facts when $\mathcal{E}$ is a perfect complex.

References

[Lau] G. Laumon, *Transformation de Fourier homogène*. Bull. Soc. Math. Fr. **131** (2003), no. 4, 527–551.

[KS] M. Kashiwara, P. Schapira, *Sheaves on manifolds* (1990).