§2. CONSTRUCTIBLE CATEGORIES AND LOCALIZATION

0.1. In these notes, I will use the term *constructible category of coefficients* to refer to a system of ∞-categories $\mathbf{D}(X)$, defined over derived Artin stacks X, equipped with the following six operations:

(a) *Basic functoriality.* For every morphism of derived Artin stacks $f : X \to Y$, a pair of adjoint functors
$$f^* : \mathbf{D}(Y) \to \mathbf{D}(X), \quad f_* : \mathbf{D}(X) \to \mathbf{D}(Y).$$
(b) *Exceptional functoriality.* For every locally of finite type (= lft) morphism of derived Artin stacks $f : X \to Y$, a pair of adjoint functors
$$f_! : \mathbf{D}(X) \to \mathbf{D}(Y), \quad f^! : \mathbf{D}(Y) \to \mathbf{D}(X).$$
(c) *Tensor and Hom.* For every derived Artin stack X, a pair of adjoint bifunctors
$$\otimes : \mathbf{D}(X) \times \mathbf{D}(X) \to \mathbf{D}(X), \quad \underline{\mathrm{Hom}} : \mathbf{D}(X) \times \mathbf{D}(X) \to \mathbf{D}(X).$$

This data is subject to the usual compatibilities and coherences. For example, the tensor and Hom are part of closed symmetric monoidal structures on $\mathbf{D}(X)$; the inverse image functor f^* is symmetric monoidal; there is a natural transformation $\alpha_f : f_! \to f_*$ which is invertible when f is proper and representable; and the functors $f_!$ satisfy base change and projection formulas.

Another important compatibility is the *localization* property. Suppose we have a closed/open pair, i.e., a diagram
$$Z \xhookrightarrow{i} S \xhookleftarrow{j} U$$
where i and j are complementary closed and open immersions. Then there is an "exact sequence"
$$\mathbf{D}(Z) \xrightarrow{i_*} \mathbf{D}(X) \xrightarrow{j^*} \mathbf{D}(U)$$
where i_* is fully faithful and its essential image is the kernel of j^*.

The localization property, along with $\mathbf{A}^1$-invariance, is what sets apart "constructible categories" from general six functor formalisms, such as the theory of quasi-coherent (or rather ind-coherent) sheaves. The choice of the adjective "constructible" is based on one of the consequences of localization, which is a kind of "constructible descent" property: namely, for any stratification of a derived Artin stack X by locally closed substacks $j_\alpha : X_\alpha \to X$, the inverse image functors
$$j_\alpha^* : \mathbf{D}(X) \to \mathbf{D}(X_\alpha)$$

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are jointly conservative.

Examples of constructible categories on derived stacks include various categories of motives and derived categories of ℓ-adic sheaves [LZ]. In the setting of derived C^∞-stacks, the derived category of usual sheaves of abelian groups is an example (though I don't know if it is fully documented).

0.2. Anyways, what I want to explain here is that for any closed immersion $i : \mathrm{Z} \to \mathrm{X}$, there is a natural transformation

$$\beta_i : i^! \to i^*.$$

Consider the diagram

$$\begin{array}{ccc} i^*i_!i^! & \xrightarrow{\text{counit}} & i^* \\ {\scriptstyle\alpha_i}\downarrow & & \downarrow{\scriptstyle\text{unit}} \\ i^*i_*i^! & \overset{\beta_i}{\nearrow} & i^!i_!i^* \\ {\scriptstyle\text{counit}}\downarrow & & \downarrow{\scriptstyle\alpha_i} \\ i^! & \xrightarrow{\text{unit}} & i^!i_*i^* \end{array} \tag{1}$$

Note that all vertical arrows are invertible. It turns out that the diagram (the solid part) commutes[1], so we get a unique diagonal arrow making both triangles commute.

0.3. From the localization property one can derive the exactness of the triangles

$$j_!j^* \xrightarrow{\text{counit}} \mathrm{id}_\mathrm{X} \xrightarrow{\text{unit}} i_*i^* \tag{2}$$

$$i_*i^! \xrightarrow{\text{counit}} \mathrm{id}_\mathrm{X} \xrightarrow{\text{unit}} j_*j^* \tag{3}$$

for any closed/open pair (i, j). We can use these to describe the cofibre of $\beta_i : i^! \to i^*$. Namely, apply i^* to (3) on the left, and simplify using commutativity of the upper triangle in (1). We get the exact triangle

$$i^! \xrightarrow{\beta_i} i^* \xrightarrow{\text{unit}} i^*j_*j^*. \tag{4}$$

We could have also done the dual thing: from (2) and the lower triangle in (1) we get the exact triangle

$$i^!j_!j^* \xrightarrow{\text{counit}} i^! \xrightarrow{\beta_i} i^*. \tag{5}$$

Combining both triangles, we deduce $i^*j_*j^* \simeq i^!j_!j^*[1]$, hence[2]

$$i^*j_* \simeq i^!j_![1] \tag{6}$$

by applying j_* on the right. Nice.

Next time we'll see what happens when we specialize to the case where i is the zero section of a vector bundle.

[1]Hint: apply the fully faithful functor i_* to the whole diagram and make use of the triangle identities by composing above/below by a unit/counit. Or, find an easier proof and let me know.

[2]Exercise: find a simpler derivation of (6).

References

[LZ] Y. Liu, W. Zheng, *Enhanced six operations and base change theorem for higher Artin stacks*. arXiv:1211.5948 (2012).