

# DESCENT BY QUASI-SMOOTH BLOW-UPS IN ALGEBRAIC K-THEORY

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ABSTRACT. We construct a semi-orthogonal decomposition on the category of perfect complexes on the blow-up of a derived Artin stack in a quasi-smooth centre. This gives a generalization of Thomason’s blow-up formula in algebraic K-theory to derived stacks. We also provide a new criterion for descent in Voevodsky’s cdh topology, which we use to give a direct proof of Cisinski’s theorem that Weibel’s homotopy invariant K-theory satisfies cdh descent.

## 1. INTRODUCTION

**1.1.** Let  $X$  be a scheme and  $i : Z \rightarrow X$  a regular closed immersion. This means that  $Z$  is, Zariski-locally on  $X$ , the zero-locus of some regular sequence of functions  $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$ . Then the blow-up  $\mathrm{Bl}_{Z/X}$  fits into a square

$$(1.1.a) \quad \begin{array}{ccc} \mathbf{P}(\mathcal{N}_{Z/X}) & \xrightarrow{i_D} & \mathrm{Bl}_{Z/X} \\ \downarrow q & & \downarrow p \\ Z & \xrightarrow{i} & X, \end{array}$$

where the exceptional divisor is the projective bundle associated to the conormal sheaf  $\mathcal{N}_{Z/X}$ , which under the assumptions is locally free of rank  $n$ . A result of Thomason [Tho93b] asserts that after taking algebraic K-theory, the induced square of spectra

$$\begin{array}{ccc} \mathrm{K}(X) & \xrightarrow{i^*} & \mathrm{K}(Z) \\ \downarrow p^* & & \downarrow \\ \mathrm{K}(\mathrm{Bl}_{Z/X}) & \longrightarrow & \mathrm{K}(\mathbf{P}(\mathcal{N}_{Z/X})) \end{array}$$

is homotopy cartesian. Here  $\mathrm{K}(X)$  denotes the Bass–Thomason–Trobaugh algebraic K-theory spectrum of perfect complexes on a scheme  $X$ . We may summarize this property by saying that algebraic K-theory satisfies *descent* with respect to blow-ups in regularly immersed centres.

Now suppose that  $i$  is more generally a *quasi-smooth* closed immersion of derived schemes. This means that  $Z$  is, Zariski-locally on  $X$ , the *derived* zero-locus of some arbitrary sequence of functions  $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$ . (When  $X$  is a classical scheme and the sequence is regular, this is the same as the classical zero-locus, and we are in the situation discussed above.) In the derived setting there is still a conormal sheaf  $\mathcal{N}_{Z/X}$  on  $Z$ , locally free of rank  $n$ , and one may still form the blow-up square (1.1.a), see [Kha18]. Our goal in this paper is to generalize Thomason’s result above to this situation. At the same time we also allow  $X$  to be a derived *Artin stack*, and consider any *additive invariant* of stable  $\infty$ -categories (see Definition 2.3.1). Examples of additive invariants include algebraic K-theory  $\mathrm{K}$ , connective algebraic K-theory  $\mathrm{K}^{\mathrm{cn}}$ , topological Hochschild homology  $\mathrm{THH}$ , and topological cyclic homology  $\mathrm{TC}$ .

**Theorem A.** *Let  $E$  be an additive invariant of stable  $\infty$ -categories. Then  $E$  satisfies descent by quasi-smooth blow-ups. That is, given a derived Artin stack  $X$  and a quasi-smooth closed immersion  $i : Z \rightarrow X$  of virtual codimension  $n \geq 1$ ,*

form the blow-up square (1.1.a). Then the induced commutative square

$$\begin{array}{ccc} E(X) & \xrightarrow{i^*} & E(Z) \\ \downarrow p^* & & \downarrow \\ E(\mathrm{Bl}_{Z/X}) & \longrightarrow & E(\mathbf{P}(\mathcal{N}_{Z/X})) \end{array}$$

is homotopy cartesian.

We deduce Theorem A from an analysis of the categories of perfect complexes on  $\mathrm{Bl}_{Z/X}$  and on the exceptional divisor  $\mathbf{P}(\mathcal{N}_{Z/X})$ . The relevant notion is that of a *semi-orthogonal decomposition*, see Definition 2.2.2.

**Theorem B.** *Let  $X$  be a derived Artin stack. For any locally free  $\mathcal{O}_X$ -module  $\mathcal{E}$  of rank  $n + 1$ ,  $n \geq 0$ , consider the projective bundle  $q : \mathbf{P}(\mathcal{E}) \rightarrow X$ . Then we have:*

- (i) *For each  $0 \leq k \leq n$ , the assignment  $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(-k)$  defines a fully faithful functor  $\mathrm{Perf}(X) \rightarrow \mathrm{Perf}(\mathbf{P}(\mathcal{E}))$ , whose essential image we denote  $\mathbf{A}(-k)$ .*
- (ii) *The sequence of full subcategories  $(\mathbf{A}(0), \dots, \mathbf{A}(-n))$  forms a semi-orthogonal decomposition of  $\mathrm{Perf}(\mathbf{P}(\mathcal{E}))$ .*

**Theorem C.** *Let  $X$  be a derived Artin stack. For any quasi-smooth closed immersion  $i : Z \rightarrow X$  of virtual codimension  $n \geq 1$ , form the blow-up square (1.1.a). Then we have:*

- (i) *The assignment  $\mathcal{F} \mapsto p^*(\mathcal{F})$  defines a fully faithful functor  $\mathrm{Perf}(X) \rightarrow \mathrm{Perf}(\mathrm{Bl}_{Z/X})$ , whose essential image we denote  $\mathbf{B}(0)$ .*
- (ii) *For each  $1 \leq k \leq n - 1$ , the assignment  $\mathcal{F} \mapsto (i_D)_*(q^*(\mathcal{F}) \otimes \mathcal{O}(-k))$  defines a fully faithful functor  $\mathrm{Perf}(Z) \rightarrow \mathrm{Perf}(\mathrm{Bl}_{Z/X})$ , whose essential image we denote  $\mathbf{B}(-k)$ .*
- (iii) *The sequence of full subcategories  $(\mathbf{B}(0), \dots, \mathbf{B}(-n + 1))$  forms a semi-orthogonal decomposition of  $\mathrm{Perf}(\mathrm{Bl}_{Z/X})$ .*

We immediately deduce the projective bundle and blow-up formulas

$$E(\mathbf{P}(\mathcal{E})) \simeq \bigoplus_{m=0}^n E(X), \quad E(\mathrm{Bl}_{Z/X}) \simeq E(X) \oplus \bigoplus_{k=1}^{n-1} E(Z),$$

for any additive invariant  $E$ , see Corollaries 3.4.1 and 4.5.2, from which Theorem A immediately follows (see Subsect. 4.5).

**1.2.** The results mentioned above admit the following interesting special cases:

- (a) Suppose that  $X$  is a smooth projective variety over the field of complex numbers. This case of Theorem B was proven by Orlov in [Orl92]. He also proved Theorem C for any *smooth* subvariety  $Z \hookrightarrow X$ .
- (b) More generally suppose that  $X$  is a quasi-compact quasi-separated classical scheme. Then the projective bundle formula (Corollary 3.4.1) for algebraic K-theory was proven by Thomason [TT90, Tho93a]. Similarly suppose that  $i : Z \rightarrow X$  is a quasi-smooth closed immersion of quasi-compact quasi-separated classical schemes. Then it is automatically a regular closed immersion, and in this case Thomason also proved Corollary 4.5.2 for algebraic K-theory [Tho93b]. In fact, the papers [Tho93a] and [Tho93b] essentially contain under these assumptions proofs of Theorems B and C, respectively, even if the term “semi-orthogonal decomposition” is not used explicitly. For THH and TC, these cases of Corollaries 3.4.1 and 4.5.2 were proven by Blumberg and Mandell [BM12].
- (c) More generally still, let  $X$  and  $Z$  be classical Artin stacks. These cases of Theorems B and C are proven by Bergh and Schnürer in [BS17]. However we note that Corollaries 3.4.1 and 4.5.2 were obtained earlier by Krishna and Ravi in [KR18], and their arguments in fact prove Theorems B and C for classical Artin stacks.

- (d) Let  $X$  be a noetherian affine classical scheme, and let  $Z$  be the derived zero-locus of some functions  $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$ . Then the canonical morphism  $i : Z \rightarrow X$  is a quasi-smooth closed immersion. In this case, Theorem A for algebraic K-theory was proven by Kerz–Strunk–Tamme [KST18] (where the blow-up  $\mathrm{Bl}_{Z/X}$  was explicitly modelled as the derived fibred product  $X \times_{\mathbf{A}^n} \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$ ), as part of their proof of Weibel’s conjecture on negative K-theory.

**1.3.** Let  $\mathrm{KH}$  denote homotopy invariant K-theory. Recall that this is the  $\mathbf{A}^1$ -localization of the presheaf  $X \mapsto \mathrm{K}(X)$ . That is, it is obtained by forcing the property of  $\mathbf{A}^1$ -homotopy invariance: for every quasi-compact quasi-separated algebraic space  $X$ , the map

$$\mathrm{KH}(X) \rightarrow \mathrm{KH}(X \times \mathbf{A}^1)$$

is invertible (see [Wei89, Cis13]). As an application of Theorem A, we give a new proof of the following theorem of Cisinski [Cis13]:

**Theorem D.** *The presheaf of spectra  $S \mapsto \mathrm{KH}(S)$  satisfies cdh descent on the site of quasi-compact quasi-separated algebraic spaces.*

This was first proven by Haesemeyer [Hae04] for schemes over a field of characteristic zero, using resolution of singularities. Cisinski’s proof over general bases (noetherian schemes of finite dimension) relies on Ayoub’s proper base change theorem in motivic homotopy theory. A different proof of Theorem D (also in the noetherian setting) was recently given by Kerz–Strunk–Tamme [KST18, Thm. C], as an application of pro-cdh descent and their resolution of Weibel’s conjecture on negative K-theory. The proof we give here is much more direct and uses a new criterion for cdh descent (Theorem 5.2.1) in terms of Nisnevich squares, quasi-smooth blow-up squares, and closed squares.

Theorem D was extended to certain nice Artin stacks recently by Hoyois and Hoyois–Krishna [Hoy16, HK17]. Our cdh descent criterion also applies in this setting and gives another potential approach to such results. In fact, a generalization of the Morel–Voevodsky localization theorem to derived quotient stacks (a common generalization of [Hoy17] and [Kha16]) would yield cdh descent in homotopy invariant K-theory for perfect Artin stacks satisfying the resolution property Nisnevich-locally. Similarly, the results here give a starting point for a potential generalization of the pro-cdh descent theorem of [KST18] to such stacks. For now, the missing ingredient is pro-excision for stacks.

A similar cdh descent criterion in a slightly different context has been noticed independently by Markus Land and Georg Tamme, see [LT18, Thm. A.2] (both criteria are directly inspired by [KST18]). We do not know if the Land–Tamme criterion can be applied here since we do not know that  $\mathrm{KH}$  can be extended to an invariant of stable  $\infty$ -categories which is truncating in the sense of *op. cit.* In fact the main new input here is the result of [CK17] which asserts that its extension to *derived schemes* (or algebraic spaces) does satisfy the property that  $\mathrm{KH}(X) \rightarrow \mathrm{KH}(X_{\mathrm{cl}})$  is invertible for all  $X$ .

**1.4.** The organization of this paper is as follows. We begin in Sect. 2 with some background on derived algebraic geometry and on semi-orthogonal decompositions of stable  $\infty$ -categories.

Sect. 3 is dedicated to the proof of Theorem B. We first show that the semi-orthogonal decomposition exists on the larger stable  $\infty$ -category  $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$  (Theorem 3.2.1). Then we show that it restricts to  $\mathrm{Perf}(\mathbf{P}(\mathcal{E}))$  (Subsect. 3.3), and deduce the projective bundle formula (Corollary 3.4.1) for any additive invariant.

We follow a similar pattern in Sect. 4 to prove Theorem C. There is a semi-orthogonal decomposition on  $\mathrm{Qcoh}(\mathrm{Bl}_{Z/X})$  (Theorem 4.3.1) which then restricts to  $\mathrm{Perf}(\mathrm{Bl}_{Z/X})$  (Subsect. 4.4). This gives both the blow-up formula (Corollary 4.5.2) as well as Theorem A (4.5.3) for additive invariants. As input we prove a Grothendieck

duality statement for virtual Cartier divisors (Proposition 4.2.1) that should be of independent interest.

Sect. 5 contains our results on cdh descent and KH. We first give the general cdh descent criterion (Theorem 5.2.1). We apply this criterion to KH to give our proof of Theorem D (5.4.2).

**1.5.** I would like to thank Marc Hoyois, Charanya Ravi, and David Rydh for helpful discussions and comments on previous revisions. I am especially grateful to David Rydh for pointing out the relevance of the resolution property in Sect. 5.

## 2. PRELIMINARIES

Throughout the paper we work with the language of  $\infty$ -categories as in [HTT, HA].

**2.1. Derived algebraic geometry.** This paper is set in the world of derived algebraic geometry, as in [TV08, SAG, GR17].

2.1.1. Let  $\text{SCRing}$  denote the  $\infty$ -category of simplicial commutative rings. A *derived stack* is an étale sheaf of spaces  $X : \text{SCRing} \rightarrow \text{Spc}$ . If  $X$  is corepresentable by a simplicial commutative ring  $A$ , we write  $X = \text{Spec}(A)$  and call  $X$  an *affine derived scheme*. A *derived scheme* is a derived stack  $X$  that admits a Zariski atlas by affine derived schemes, i.e., a jointly surjective family  $(U_i \rightarrow X)_i$  of Zariski open immersions with each  $U_i$  an affine derived scheme. Allowing Nisnevich, étale or smooth atlases, respectively, gives rise to the notions of *derived algebraic space*<sup>1</sup>, *derived Deligne–Mumford stack*, and *derived Artin stack*. The precise definition is slightly more involved, see e.g. [GR17, Vol. I, Sect. 4.1].

Any derived stack  $X$  admits an underlying classical stack which we denote  $X_{\text{cl}}$ . If  $X$  is a derived scheme, algebraic space, Deligne–Mumford or Artin stack, then  $X_{\text{cl}}$  is a classical such.

2.1.2. Let  $X$  be a derived scheme and let  $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$  be functions classifying a morphism  $f : X \rightarrow \mathbf{A}^n$  to affine space. The *derived zero-locus* of these functions is given by the derived fibred product

$$(2.1.a) \quad \begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \{0\} & \longrightarrow & \mathbf{A}^n. \end{array}$$

If  $X$  is classical, then  $Z$  is classical if and only if the sequence  $(f_1, \dots, f_n)$  is regular in the sense of [SGA 6], in which case  $Z$  is regularly immersed. A closed immersion of derived schemes  $i : Z \rightarrow X$  is called *quasi-smooth* (of virtual codimension  $n$ ) if it is cut out Zariski-locally as the derived zero-locus of  $n$  functions on  $X$ . Equivalently, this means that  $i$  is locally finitely presented and its shifted cotangent complex  $\mathcal{N}_{Z/X} := \mathcal{L}_{Z/X}[-1]$  is locally free (of rank  $n$ ). A closed immersion of derived Artin stacks is quasi-smooth if it satisfies this condition smooth-locally.

A morphism of derived schemes  $f : Y \rightarrow X$  is quasi-smooth if it can be factored, Zariski-locally on  $Y$ , through a quasi-smooth closed immersion  $i : Y \rightarrow X'$  and a smooth morphism  $X' \rightarrow X$ . A morphism of derived Artin stacks is quasi-smooth if it satisfies this condition smooth-locally on  $Y$ . We refer to [Kha18] for more details on quasi-smoothness.

<sup>1</sup>That this agrees with the classical notion of algebraic space (at least under quasi-compactness and quasi-separatedness hypotheses) follows from [RG71, Prop. 5.7.6]. That it agrees with Lurie's definition follows from [SAG, Ex. 3.7.1.5].

2.1.3. Important for us is the following construction from [Kha18]. Given any quasi-smooth closed immersion  $i : Z \rightarrow X$  of derived Artin stacks, there is an associated *quasi-smooth blow-up square*:

$$(2.1.b) \quad \begin{array}{ccc} D & \xrightarrow{i_D} & \mathrm{Bl}_{Z/X} \\ \downarrow q & & \downarrow p \\ Z & \xrightarrow{i} & X. \end{array}$$

Here  $\mathrm{Bl}_{Z/X}$  is the blow-up of  $X$  in  $Z$ , which is a quasi-smooth proper derived Artin stack over  $X$ , and  $D = \mathbf{P}(\mathcal{N}_{Z/X})$  is the projectivized normal bundle, which is a smooth proper derived Artin stack over  $X$ . This square is universal with the following properties: (a) the morphism  $i_D$  is a quasi-smooth closed immersion of virtual codimension 1, i.e., a virtual effective Cartier divisor; (b) the underlying square of classical Artin stacks is cartesian; and (c) the canonical map  $q^*\mathcal{N}_{Z/X} \rightarrow \mathcal{N}_{D/\mathrm{Bl}_{Z/X}}$  is surjective on  $\pi_0$ . When  $X$  is a derived scheme (resp. derived algebraic space, derived Deligne–Mumford stack), then so is  $\mathrm{Bl}_{Z/X}$ .

2.1.4. Given a derived stack  $X$ , the stable  $\infty$ -category of quasi-coherent sheaves  $\mathrm{Qcoh}(X)$  is the limit

$$\mathrm{Qcoh}(X) = \varprojlim_{\mathrm{Spec}(A) \rightarrow X} \mathrm{Qcoh}(\mathrm{Spec}(A))$$

taken over all morphisms  $\mathrm{Spec}(A) \rightarrow X$  with  $A \in \mathrm{SCRing}$ . Here  $\mathrm{Qcoh}(\mathrm{Spec}(A))$  is the stable  $\infty$ -category  $\mathrm{Mod}_A$  of  $A$ -modules<sup>2</sup> in the sense of Lurie. Informally speaking, a quasi-coherent sheaf  $\mathcal{F}$  on  $X$  is thus a collection of quasi-coherent sheaves  $x^*(\mathcal{F}) \in \mathrm{Qcoh}(\mathrm{Spec}(A))$ , for every simplicial commutative ring  $A$  and every  $A$ -point  $x : \mathrm{Spec}(A) \rightarrow X$ , together with a homotopy coherent system of compatibilities.

The full subcategory  $\mathrm{Perf}(X) \subset \mathrm{Qcoh}(X)$  is similarly the limit

$$\mathrm{Perf}(X) = \varprojlim_{\mathrm{Spec}(A) \rightarrow X} \mathrm{Perf}(\mathrm{Spec}(A)),$$

where  $\mathrm{Perf}(\mathrm{Spec}(A))$  is the stable  $\infty$ -category  $\mathrm{Mod}_A^{\mathrm{perf}}$  of *perfect*  $A$ -modules. In other words,  $\mathcal{F} \in \mathrm{Qcoh}(X)$  belongs to  $\mathrm{Perf}(X)$  if and only if  $x^*(\mathcal{F})$  is perfect for every simplicial commutative ring  $A$  and every morphism  $x : \mathrm{Spec}(A) \rightarrow X$ .

2.1.5. There is an inverse image functor  $f^* : \mathrm{Qcoh}(X) \rightarrow \mathrm{Qcoh}(Y)$  for any morphism of derived stacks  $f : Y \rightarrow X$ . It preserves perfect complexes and induces a functor  $f^* : \mathrm{Perf}(X) \rightarrow \mathrm{Perf}(Y)$ . Regarded as presheaves of  $\infty$ -categories, the assignments  $X \mapsto \mathrm{Qcoh}(X)$  and  $X \mapsto \mathrm{Perf}(X)$  satisfy *descent* for the fpqc topology ([SAG, Cor. D.6.3.3], [GR17, Thm. 1.3.4]). This means in particular that given any fpqc covering family  $(f_\alpha : X_\alpha \rightarrow X)_\alpha$ , the family of inverse image functors  $f_\alpha^* : \mathrm{Qcoh}(X) \rightarrow \mathrm{Qcoh}(X_\alpha)$  is jointly conservative.

If  $f : Y \rightarrow X$  is quasi-compact and *schematic*, in the sense that its fibre over any affine derived scheme is a derived scheme, then there is a direct image functor  $f_*$ , right adjoint to  $f^*$ , which commutes with colimits and satisfies a base change formula against inverse images ([SAG, Prop. 2.5.4.5], [GR17, Vol. 1, Chap. 3, Prop. 2.2.2]). If  $f$  is proper, locally of finite presentation, and of finite tor-amplitude, then  $f_*$  also preserves perfect complexes [SAG, Thm. 6.1.3.2].

**2.2. Semi-orthogonal decompositions.** The following definitions were originally formulated by [BK89] in the language of triangulated categories and are standard.

<sup>2</sup>Note that if  $A$  is discrete (an ordinary commutative ring), then this is not the abelian category of discrete  $A$ -modules, but rather the derived  $\infty$ -category of this abelian category as in [HA, Chap. 1].

**Definition 2.2.1.** Let  $\mathbf{C}$  be a stable  $\infty$ -category and  $\mathbf{D}$  a stable full subcategory. An object  $x \in \mathbf{C}$  is *left orthogonal*, resp. *right orthogonal*, to  $\mathbf{D}$  if the mapping space  $\mathrm{Maps}_{\mathbf{C}}(x, d)$ , resp.  $\mathrm{Maps}_{\mathbf{C}}(d, x)$ , is contractible for all objects  $d \in \mathbf{D}$ . We let  ${}^{\perp}\mathbf{D} \subseteq \mathbf{C}$  and  $\mathbf{D}^{\perp} \subseteq \mathbf{C}$  denote the full subcategories of left orthogonal and right orthogonal objects, respectively.

**Definition 2.2.2.** Let  $\mathbf{C}$  be a stable  $\infty$ -category and let  $\mathbf{C}(0), \dots, \mathbf{C}(-n)$  be full stable subcategories. Suppose that the following conditions hold:

- (i) For all integers  $i > j$ , there is an inclusion  $\mathbf{C}(i) \subseteq {}^{\perp}\mathbf{C}(j)$ .
- (ii) The  $\infty$ -category  $\mathbf{C}$  is generated by the subcategories  $\mathbf{C}(0), \dots, \mathbf{C}(-n)$ , under finite limits and finite colimits.

Then we say that the sequence  $(\mathbf{C}(0), \dots, \mathbf{C}(-n))$  forms a *semi-orthogonal decomposition* of  $\mathbf{C}$ .

Semi-orthogonal decompositions of length 2 come from *split short exact sequences* of stable  $\infty$ -categories, as in [BGT13].

**Definition 2.2.3.**

- (i) A *short exact sequence* of small stable  $\infty$ -categories is a diagram

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'' ,$$

where  $i$  and  $p$  are exact, the composite  $p \circ i$  is null-homotopic,  $i$  is fully faithful, and  $p$  induces an equivalence  $(\mathbf{C}/\mathbf{C}')^{\mathrm{idem}} \simeq (\mathbf{C}'')^{\mathrm{idem}}$  (where  $(-)^{\mathrm{idem}}$  denotes idempotent completion).

- (ii) A short exact sequence of small stable  $\infty$ -categories

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}''$$

is *split* if there exist functors  $q : \mathbf{C} \rightarrow \mathbf{C}'$  and  $j : \mathbf{C}'' \rightarrow \mathbf{C}$ , right adjoint to  $i$  and  $p$ , respectively, such that the unit  $\mathrm{id} \rightarrow q \circ i$  and co-unit  $p \circ j \rightarrow \mathrm{id}$  are invertible.

**Remark 2.2.4.** Let  $\mathbf{C}$  be a small stable  $\infty$ -category, and let  $(\mathbf{C}(0), \mathbf{C}(-1))$  be a semi-orthogonal decomposition. Then for any object  $x \in \mathbf{C}$ , there exists an exact triangle

$$x(0) \rightarrow x \rightarrow x(-1),$$

where  $x(0) \in \mathbf{C}(0)$  and  $x(-1) \in \mathbf{C}(-1)$ . To see this, simply observe that the full subcategory spanned by objects  $x$  for which such a triangle exists, is closed under finite limits and colimits, and contains  $\mathbf{C}(0)$  and  $\mathbf{C}(-1)$ . Moreover, the assignments  $x \mapsto x(0)$  and  $x \mapsto x(-1)$  determine well-defined functors  $q : \mathbf{C} \rightarrow \mathbf{C}(0)$  and  $p : \mathbf{C} \rightarrow \mathbf{C}(-1)$ , respectively, which are right and left adjoint, respectively, to the inclusions (see e.g. [SAG, Rem. 7.2.0.2]). It follows from this that any semi-orthogonal decomposition  $(\mathbf{C}(0), \mathbf{C}(-1))$  induces a split short exact sequence

$$\mathbf{C}(0) \rightarrow \mathbf{C} \xrightarrow{p} \mathbf{C}(-1).$$

**Lemma 2.2.5.** *Let  $\mathbf{C}$  be a stable  $\infty$ -category, and let  $(\mathbf{C}(0), \dots, \mathbf{C}(-n))$  be a sequence of full stable subcategories forming a semi-orthogonal decomposition of  $\mathbf{C}$ . For each  $0 \leq m \leq n$ , let  $\mathbf{C}_{\leq -m} \subseteq \mathbf{C}$  denote the union  $\mathbf{C}(-m) \cup \dots \cup \mathbf{C}(-n)$ , and let  $\mathbf{C}_{\leq -n-1} \subseteq \mathbf{C}$  denote the full subcategory spanned by the zero object. Then there are split short exact sequences*

$$\mathbf{C}_{\leq -m-1} \hookrightarrow \mathbf{C}_{\leq -m} \rightarrow \mathbf{C}(-m)$$

for each  $0 \leq m \leq n$ .

*Proof.* It follows from the definitions that for each  $0 \leq m \leq n$ , the sequence  $(\mathbf{C}(-m), \mathbf{C}_{\leq -m-1})$  forms a semi-orthogonal decomposition of  $\mathbf{C}$ . Therefore the claim follows from Remark 2.2.4.  $\square$

**2.3. Additive and localizing invariants.** The following definition is from [BGT13], except that we do not require commutativity with filtered colimits.

**Definition 2.3.1.** Let  $\mathbf{A}$  be a stable presentable  $\infty$ -category. Let  $E$  be an  $\mathbf{A}$ -valued functor from the  $\infty$ -category of small stable  $\infty$ -categories and exact functors.

- (i) We say that  $E$  is an *additive invariant* if for any split short exact sequence

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'',$$

the induced map

$$E(\mathbf{C}') \oplus E(\mathbf{C}'') \xrightarrow{(i,j)} E(\mathbf{C})$$

is invertible, where  $j$  is a right adjoint to  $p$ .

- (ii) We say that  $E$  is a *localizing invariant* if for any short exact sequence

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'',$$

the induced diagram

$$E(\mathbf{C}') \rightarrow E(\mathbf{C}) \rightarrow E(\mathbf{C}'')$$

is an exact triangle.

**Remark 2.3.2.** Any localizing invariant is also additive.

**Lemma 2.3.3.** *Let  $\mathbf{C}$  be a stable  $\infty$ -category, and let  $(\mathbf{C}(0), \dots, \mathbf{C}(-n))$  be a sequence of full stable subcategories forming a semi-orthogonal decomposition of  $\mathbf{C}$ . Then for any additive invariant  $E$  there is a canonical isomorphism*

$$E(\mathbf{C}) \simeq \bigoplus_{m=0}^n E(\mathbf{C}(-m)).$$

*Proof.* Follows immediately from Lemma 2.2.5.  $\square$

### 3. THE PROJECTIVE BUNDLE FORMULA

**3.1. Projective bundles.** Let  $X$  be a derived stack and  $\mathcal{E}$  a locally free  $\mathcal{O}_X$ -module of finite rank. Recall that the *projective bundle* associated to  $\mathcal{E}$  is a derived stack  $\mathbf{P}(\mathcal{E})$  over  $X$  equipped with an invertible sheaf  $\mathcal{O}(1)$  together with a surjection  $\mathcal{E} \rightarrow \mathcal{O}(1)$ . More precisely, for any derived scheme  $S$  over  $X$ , with structural morphism  $x : S \rightarrow X$ , the space of  $S$ -points of  $\mathbf{P}(\mathcal{E})$  is the space of pairs  $(\mathcal{L}, u)$ , where  $\mathcal{L}$  is a locally free  $\mathcal{O}_S$ -module of rank 1, and  $u : x^*(\mathcal{E}) \rightarrow \mathcal{L}$  is surjective on  $\pi_0$ . We recall the standard properties of this construction:

**Proposition 3.1.1.**

- (i) *If  $f : X' \rightarrow X$  is a morphism of derived stacks, then there is a canonical isomorphism  $\mathbf{P}(f^*(\mathcal{E})) \rightarrow \mathbf{P}(\mathcal{E}) \times_X X'$  of derived stacks over  $X'$ .*
- (ii) *The projection  $\mathbf{P}(\mathcal{E}) \rightarrow X$  is proper and schematic. In particular, if  $X$  is a derived scheme (resp. derived algebraic space, derived Deligne–Mumford stack, derived Artin stack), then the same holds for the derived stack  $\mathbf{P}(\mathcal{E})$ .*
- (iii) *The relative cotangent complex  $\mathcal{L}_{\mathbf{P}(\mathcal{E})/X}$  is canonically isomorphic to  $\mathcal{F} \otimes \mathcal{O}(-1)$ , where the locally free sheaf  $\mathcal{F}$  is the fibre of the canonical map  $\mathcal{E} \rightarrow \mathcal{O}(1)$ . In particular, the morphism  $\mathbf{P}(\mathcal{E}) \rightarrow X$  is smooth of relative dimension equal to  $\mathrm{rk}(\mathcal{E}) - 1$ .*

**Proposition 3.1.2 (Serre).** *Let  $X$  be a derived Artin stack, and  $\mathcal{E}$  a locally free sheaf of rank  $n + 1$ ,  $n \geq 0$ . If  $q : \mathbf{P}(\mathcal{E}) \rightarrow X$  denotes the associated projective bundle, then we have canonical isomorphisms*

$$q_*(\mathcal{O}(0)) \simeq \mathcal{O}_X, \quad q_*(\mathcal{O}(-m)) \simeq 0 \quad (1 \leq m \leq n)$$

in  $\mathrm{Qcoh}(X)$ .

*Proof.* There is a canonical map  $\mathcal{O}_X \rightarrow q_*(\mathcal{O}(0))$ , the unit of the adjunction  $(q^*, q_*)$ , and there is a unique map  $0 \rightarrow q_*(\mathcal{O}(-m))$  for each  $m$ . To show that these are invertible, we may use fpqc descent and base change to the case where  $X$  is affine and  $\mathcal{E}$  is free. Then this is Serre's computation, as generalized to the derived setting by Lurie [SAG, Thm. 5.4.2.6].  $\square$

**3.2. Semi-orthogonal decomposition on  $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ .** In this subsection we will show that the stable  $\infty$ -category  $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$  admits a canonical semi-orthogonal decomposition.

**Theorem 3.2.1.** *Let  $X$  be a derived Artin stack. Let  $\mathcal{E}$  be a locally free  $\mathcal{O}_X$ -module of rank  $n + 1$ ,  $n \geq 0$ , and  $q : \mathbf{P}(\mathcal{E}) \rightarrow X$  the associated projective bundle. Then we have:*

- (i) *For every integer  $k \in \mathbf{Z}$ , the assignment  $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(k)$  defines a fully faithful functor  $\mathrm{Qcoh}(X) \rightarrow \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ .*
- (ii) *For every integer  $k \in \mathbf{Z}$ , let  $\mathbf{C}(k) \subset \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$  denote the essential image of the functor in (i). Then the subcategories  $\mathbf{C}(k), \dots, \mathbf{C}(k - n)$  form a semi-orthogonal decomposition of  $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ .*

We will need the following facts (see Lemmas 7.2.2.2 and 5.6.2.2 in [SAG]):

**Lemma 3.2.2.** *Let  $R$  be a simplicial commutative ring and  $X = \mathrm{Spec}(R)$ . Denote by  $\mathbf{P}_R^n = \mathbf{P}(\mathcal{O}_X^{n+1})$  the  $n$ -dimensional projective space over  $R$ . Then for every integer  $m \in \mathbf{Z}$ , there is a canonical isomorphism*

$$\varinjlim_{J \subsetneq [n]} \mathcal{O}(m + |J|) \xrightarrow{\sim} \mathcal{O}(m + n + 1)$$

in  $\mathrm{Qcoh}(\mathbf{P}_R^n)$ , where the colimit is taken over the proper subsets  $J$  of the set  $[n] = \{0, 1, \dots, n\}$ , and  $0 \leq |J| \leq n$  denotes the cardinality of such a subset.

**Lemma 3.2.3.** *Let  $R$  be a simplicial commutative ring and  $X = \mathrm{Spec}(R)$ . Denote by  $\mathbf{P}_R^n = \mathbf{P}(\mathcal{O}_X^{n+1})$  the  $n$ -dimensional projective space over  $R$ . Then for any connective quasi-coherent sheaf  $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}_R^n)$ , there exists a map*

$$\bigoplus_{\alpha} \mathcal{O}(d_{\alpha}) \rightarrow \mathcal{F},$$

with  $d_{\alpha} \in \mathbf{Z}$ , which is surjective on  $\pi_0$ .

*Proof of Theorem 3.2.1.* Since the functors  $-\otimes \mathcal{O}(k)$  are equivalences, it will suffice to take  $k = 0$  in both claims. For claim (i) we want to show that the unit map  $\mathcal{F} \rightarrow q_* q^*(\mathcal{F})$  is invertible for all  $\mathcal{F} \in \mathrm{Qcoh}(X)$ . By fpqc descent and base change (2.1.5), we may reduce to the case where  $X = \mathrm{Spec}(R)$  is affine and  $\mathcal{E} = \mathcal{O}_{\mathbb{S}^1}^{n+1}$  is free. Now both functors  $q^*$  and  $q_*$  are exact and moreover commute with arbitrary colimits (the latter by 2.1.5 since  $q$  is quasi-compact and schematic), and  $\mathrm{Qcoh}(X) \simeq \mathrm{Mod}_R$  is generated by  $\mathcal{O}_X$  under colimits and finite limits. Therefore we may assume  $\mathcal{F} = \mathcal{O}_X$ , in which case the claim holds by Proposition 3.1.2.

For claim (ii), let us first check the orthogonality condition in Definition 2.2.2. Thus take  $\mathcal{F}, \mathcal{G} \in \mathrm{Qcoh}(X)$  and consider the mapping space

$$\mathrm{Maps}(q^*(\mathcal{F}), q^*(\mathcal{G}) \otimes \mathcal{O}(-m)) \simeq \mathrm{Maps}(\mathcal{F}, q_*(\mathcal{O}(-m)) \otimes \mathcal{G})$$

for  $1 \leq m \leq n$ , where the identification results from the projection formula. Since  $q_*(\mathcal{O}(-m)) \simeq 0$  by Proposition 3.1.2, this space is contractible.

It now remains to show that every  $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$  belongs to the full subcategory  $\langle \mathbf{C}(0), \dots, \mathbf{C}(-n) \rangle \subseteq \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$  generated under finite colimits and limits by the subcategories  $\mathbf{C}(0), \dots, \mathbf{C}(-n)$ . Set  $\mathcal{G}_{-1} = \mathcal{F} \otimes \mathcal{O}(-1)$  and define  $\mathcal{G}_m$ , for  $m \geq 0$ , so that we have exact triangles

$$(3.2.a) \quad q^* q_*(\mathcal{G}_{m-1} \otimes \mathcal{O}(1)) \xrightarrow{\mathrm{counit}} \mathcal{G}_{m-1} \otimes \mathcal{O}(1) \rightarrow \mathcal{G}_m.$$



For each  $m \geq -1$ , we claim that  $\mathcal{G}_m$  is right orthogonal to each of the subcategories  $\mathbf{C}(0), \mathbf{C}(1), \dots, \mathbf{C}(m)$ . For  $m = -1$  the claim is vacuous, so take  $m \geq 0$  and assume by induction that it holds for  $m-1$ . Since  $q^*q_*(\mathcal{G}_{m-1} \otimes \mathcal{O}(1))$  is contained in  $\mathbf{C}(0)$ , it follows that  $\mathcal{G}_m$  is right orthogonal to  $\mathbf{C}(0)$ . To show that  $\mathcal{G}_m$  is right orthogonal to  $\mathbf{C}(i)$ , for  $1 \leq i \leq m$ , it will suffice to show that the left-hand and middle terms of the exact triangle (3.2.a) are both right orthogonal to  $\mathbf{C}(i)$ . For the left-hand term this follows from the inclusion  $\mathbf{C}(0) \subset \mathbf{C}(i)^\perp$ , demonstrated above. For the middle term  $\mathcal{G}_{m-1} \otimes \mathcal{O}(1)$ , the claim follows by the induction hypothesis.

Now we claim that  $\mathcal{G}_n$  is zero. Using fpqc descent again, we may assume that  $X = \text{Spec}(\mathbf{R})$  and  $\mathcal{E} = \mathcal{O}_X^{\oplus n+1}$  is free (since the sequence  $(\mathcal{G}_{-1}, \mathcal{G}_0, \dots, \mathcal{G}_n)$  is stable under base change). Using Lemma 3.2.3 we can build a map

$$\varphi : \bigoplus_{\alpha} \mathcal{O}(m_{\alpha})[k_{\alpha}] \rightarrow \mathcal{G}_n$$

which is surjective on all homotopy groups. From Lemma 3.2.2 it follows that  $\mathcal{G}_n$  is right orthogonal to all  $\mathbf{C}(i)$ ,  $i \in \mathbf{Z}$ . Thus  $\varphi$  must be null-homotopic, so  $\mathcal{G}_n \simeq 0$  as claimed. Working backwards, we deduce that  $\mathcal{G}_{n-1} \in \langle \mathbf{C}(-1), \dots, \mathcal{G}_0 \rangle \subset \langle \mathbf{C}(-1), \dots, \mathbf{C}(-n) \rangle$ , and then finally that  $\mathcal{F} \in \langle \mathbf{C}(0), \mathbf{C}(-1), \dots, \mathbf{C}(-n) \rangle$  as claimed.  $\square$

**3.3. Proof of Theorem B.** We now deduce Theorem B from Theorem 3.2.1. First note that the fully faithful functor  $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(k)$  of Theorem 3.2.1(i) restricts to a fully faithful functor  $\text{Perf}(X) \rightarrow \text{Perf}(\mathbf{P}(\mathcal{E}))$ , since  $q^*$  preserves perfect complexes. This shows Theorem B(i).

For part (ii) we argue again as in the proof of Theorem 3.2.1. The point is that if  $\mathcal{F} \in \text{Qcoh}(\mathbf{P}(\mathcal{E}))$  is perfect, then so is each  $\mathcal{G}_m \in \text{Qcoh}(\mathbf{P}(\mathcal{E}))$ , since  $q^*$  and  $q_*$  preserve perfect complexes (the latter because  $q$  is smooth and proper).

**3.4. Projective bundle formula.** From Theorem B and Lemma 2.3.3 we deduce:

**Corollary 3.4.1.** *Let  $X$  be a derived Artin stack,  $\mathcal{E}$  a locally free  $\mathcal{O}_X$ -module of rank  $n+1$ ,  $n \geq 0$ , and  $q : \mathbf{P}(\mathcal{E}) \rightarrow X$  the associated projective bundle. Then for any additive invariant  $E$ , there is a canonical isomorphism*

$$E(\mathbf{P}(\mathcal{E})) \simeq \bigoplus_{k=0}^n E(X)$$

induced by the functors  $q^*(-) \otimes \mathcal{O}(-k) : \text{Perf}(X) \rightarrow \text{Perf}(\mathbf{P}(\mathcal{E}))$ .

## 4. THE BLOW-UP FORMULA

**4.1. Virtual Cartier divisors.** Recall from [Kha18] that a *virtual (effective) Cartier divisor* on a derived Artin stack  $X$  is a quasi-smooth closed immersion  $i : D \rightarrow X$  of virtual codimension 1. For any such  $i : D \rightarrow X$ , there is a canonical exact triangle

$$\mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow i_*(\mathcal{O}_D),$$

where  $\mathcal{O}_X(-D)$  is a locally free sheaf of rank 1, equipped with a canonical isomorphism  $i^*(\mathcal{O}_X(-D)) \simeq \mathcal{N}_{D/X}$ .

**Lemma 4.1.1.** *Let  $X$  be a derived Artin stack and  $i : D \rightarrow X$  a virtual Cartier divisor. Then there is a canonical isomorphism*

$$i^*i_*(\mathcal{O}_D) \simeq \mathcal{O}_D \oplus \mathcal{N}_{D/X}[1].$$

*Proof.* Applying  $i^*$  to the exact triangle above (and rotating), we get the exact triangle

$$\mathcal{O}_D \rightarrow i^*i_*(\mathcal{O}_D) \rightarrow \mathcal{N}_{D/X}[1].$$

The map  $\mathcal{O}_D \rightarrow i^*i_*(\mathcal{O}_D)$  is induced by the natural transformation  $i^*(\eta) : i^* \rightarrow i^*i_*i^*$  (where  $\eta$  is the adjunction unit), so by the triangle identities it has a retraction given by the co-unit map  $i^*i_*(\mathcal{O}_D) \rightarrow \mathcal{O}_D$ . In other words, the triangle splits.  $\square$

**4.2. Grothendieck duality.** Let  $i : Z \rightarrow X$  be a quasi-smooth closed immersion of derived Artin stacks. The functor  $i_*$  admits a right adjoint  $i^!$ , which for formal reasons can be computed by the formula

$$i^!(-) \simeq i^*(-) \otimes \omega_{D/X},$$

where  $\omega_{D/X} := i^!(\mathcal{O}_X)$  is called the *relative dualizing sheaf*. See [SAG, Cor. 6.4.2.7]. When  $i$  is a virtual Cartier divisor,  $\omega_{D/X}$  can be computed as follows:

**Proposition 4.2.1** (Grothendieck duality). *Let  $X$  be a derived Artin stack. Then for any virtual Cartier divisor  $i : D \rightarrow X$ , there is a canonical isomorphism*

$$\mathcal{N}_{D/X}^\vee[-1] \xrightarrow{\sim} \omega_{D/X}$$

*of perfect complexes on  $D$ . In particular, there is a canonical identification  $i^! \simeq i^*(-) \otimes \mathcal{N}_{D/X}^\vee[-1]$ .*

*Proof.* Write  $\mathcal{L} := \mathcal{O}_X(-D)$  and consider again the exact triangle  $\mathcal{L} \rightarrow \mathcal{O}_X \rightarrow i_*(\mathcal{O}_D)$ . By the projection formula, this can be refined to an exact triangle of natural transformations  $\text{id} \otimes \mathcal{L} \rightarrow \text{id} \rightarrow i_*i^*$ , or, passing to right adjoints, an exact triangle  $i_*i^! \rightarrow \text{id} \rightarrow \text{id} \otimes \mathcal{L}^\vee$ . In particular we get the exact triangle

$$(4.2.a) \quad i_*i^!(\mathcal{O}_X) \rightarrow \mathcal{O}_X \rightarrow \mathcal{L}^\vee.$$

The associated map  $\mathcal{L}^\vee[-1] \rightarrow i_*i^!(\mathcal{O}_X)$  gives by adjunction a canonical morphism

$$\mathcal{N}_{D/X}^\vee[-1] \simeq i^*(\mathcal{L}^\vee)[-1] \rightarrow i^!(\mathcal{O}_X),$$

which we claim is invertible. By fpqc descent and the fact that  $i^!$  commutes with the operation  $f^*$ , for any morphism  $f$  [SAG, Prop. 6.4.2.1], we may assume that  $X$  is affine. In this case the functor  $i_*$  is conservative, so it will suffice to show that the canonical map

$$i_*(\mathcal{N}_{D/X}^\vee[-1]) \rightarrow i_*i^!(\mathcal{O}_X)$$

is invertible. Considering again the triangle  $\mathcal{F} \otimes \mathcal{L} \rightarrow \mathcal{F} \rightarrow i_*i^*(\mathcal{F})$  above and taking  $\mathcal{F} = \mathcal{L}^\vee$ , we get the exact triangle

$$\mathcal{O}_X \rightarrow \mathcal{L}^\vee \rightarrow i_*i^*(\mathcal{L}^\vee) \simeq i_*(\mathcal{N}_{D/X}^\vee),$$

since  $\mathcal{L}$  is invertible. Comparing with (4.2.a) yields the claim.  $\square$

**4.3. Semi-orthogonal decomposition on  $\text{Qcoh}(\text{Bl}_{Z/X})$ .** In this subsection we prove:

**Theorem 4.3.1.** *Let  $X$  be a derived Artin stack and  $i : Z \rightarrow X$  a quasi-smooth closed immersion of virtual codimension  $n \geq 1$ . Let  $\tilde{X} = \text{Bl}_{Z/X}$  and consider the quasi-smooth blow-up square (2.1.b)*

$$\begin{array}{ccc} D & \xrightarrow{i_D} & \tilde{X} \\ \downarrow q & & \downarrow p \\ Z & \xrightarrow{i} & X \end{array}$$

*Then we have:*

- (i) *The functor  $p^* : \text{Qcoh}(X) \rightarrow \text{Qcoh}(\tilde{X})$  is fully faithful. We denote its essential image by  $\mathbf{D}(0) \subset \text{Qcoh}(\tilde{X})$ .*
- (ii) *The functor  $(i_D)_*(q^*(-) \otimes \mathcal{O}(-k)) : \text{Qcoh}(Z) \rightarrow \text{Qcoh}(\tilde{X})$  is fully faithful, for each  $1 \leq k \leq n-1$ . We denote its essential image by  $\mathbf{D}(-k) \subset \text{Qcoh}(\tilde{X})$ .*
- (iii) *For each  $1 \leq k \leq n-1$ , the full stable subcategory  $\mathbf{D}(-k) \subset \text{Qcoh}(\tilde{X})$  is right orthogonal to each of  $\mathbf{D}(0), \dots, \mathbf{D}(-k+1)$ .*

- (iv) The stable  $\infty$ -category  $\mathrm{Qcoh}(\tilde{X})$  is generated by the full subcategories  $\mathbf{D}(0)$ ,  $\mathbf{D}(-1)$ ,  $\dots$ ,  $\mathbf{D}(-n+1)$  under finite colimits and finite limits. In particular, the sequence  $(\mathbf{D}(0), \mathbf{D}(-1), \dots, \mathbf{D}(-n+1))$  forms a semi-orthogonal decomposition of  $\mathrm{Qcoh}(\tilde{X})$ .

4.3.2. *Proof of (i).* The claim is that for any  $\mathcal{F} \in \mathrm{Qcoh}(X)$ , the unit map  $\mathcal{F} \rightarrow p_*p^*(\mathcal{F})$  is invertible. By fpqc descent we may reduce to the case where  $X$  is affine and  $i$  fits in a cartesian square of the form (2.1.a). Since  $\mathrm{Qcoh}(X)$  is then generated under colimits and finite limits by  $\mathcal{O}_X$ , and  $p_*$  commutes with colimits since  $p$  is quasi-compact and schematic (2.1.5), we may assume that  $\mathcal{F} = \mathcal{O}_X$ . In other words, it suffices to show that the canonical map  $\mathcal{O}_X \rightarrow p_*(\mathcal{O}_{\tilde{X}})$  is invertible.

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{i_D} & \tilde{X} & & \mathbf{P}^{n-1} & \longrightarrow & \mathrm{Bl}_{\{0\}/\mathbf{A}^n} \\ \downarrow q & & \downarrow p & & \downarrow & & \downarrow p_0 \\ \mathbf{Z} & \xrightarrow{i} & X & & \{0\} & \xrightarrow{i_0} & \mathbf{A}^n, \end{array}$$

Since the left-hand square is the (derived) base change of the right-hand square along the morphism  $f : X \rightarrow \mathbf{A}^n$ , it follows that the map  $\mathcal{O}_X \rightarrow p_*(\mathcal{O}_{\tilde{X}})$  is the inverse image of the canonical map  $\mathcal{O}_{\mathbf{A}^n} \rightarrow (p_0)_*(\mathcal{O}_{\mathrm{Bl}_{\{0\}/\mathbf{A}^n}})$ . Thus we reduce to the case where  $i$  is the immersion  $\{0\} \rightarrow \mathbf{A}^n$ . This is well-known, see [SGA 6, Exp. VII].

4.3.3. *Proof of (ii).* It suffices to show the unit map  $\mathcal{F} \rightarrow q_*(i_D)^!(i_D)_*q^*(\mathcal{F})$  is invertible for all  $\mathcal{F} \in \mathrm{Qcoh}(Z)$ . As in the previous claim we may assume  $X$  is affine and that  $\mathcal{F} = \mathcal{O}_Z$ . Using Proposition 4.2.1, the canonical identification  $\mathcal{N}_{\mathbf{D}/\tilde{X}} \simeq \mathcal{O}_{\mathbf{D}}(1)$ , and Lemma 4.1.1, the unit map is identified with

$$\mathcal{O}_Z \rightarrow q_*((i_D)^*(i_D)_*(\mathcal{O}_{\mathbf{D}}) \otimes \mathcal{O}_{\mathbf{D}}(-1))[-1] \simeq q_*(\mathcal{O}_{\mathbf{D}}(-1)) \oplus q_*(\mathcal{O}_{\mathbf{D}}).$$

Since  $q : \mathbf{D} \rightarrow Z$  is the projection of the projective bundle  $\mathbf{P}(\mathcal{N}_{Z/X})$ , it follows from Proposition 3.1.2 that we have identifications  $q_*(\mathcal{O}_{\mathbf{D}}(-1)) \simeq 0$  and  $q_*(\mathcal{O}_{\mathbf{D}}) \simeq \mathcal{O}_Z$ , under which the map in question is the identity.

4.3.4. *Proof of (iii).* To see that  $\mathbf{D}(-k)$  is right orthogonal to  $\mathbf{D}(0)$ , observe that by Theorem 3.2.1, the mapping space

$$\mathrm{Maps}(p^*(\mathcal{F}_X), (i_D)_*(q^*(\mathcal{F}_Z) \otimes \mathcal{O}(-k))) \simeq \mathrm{Maps}(q^*i^*(\mathcal{F}_X), q^*(\mathcal{F}_Z) \otimes \mathcal{O}(-k))$$

is contractible for every  $\mathcal{F}_X \in \mathrm{Qcoh}(X)$  and  $\mathcal{F}_Z \in \mathrm{Qcoh}(Z)$ .

To see that  $\mathbf{D}(-k)$  is right orthogonal to  $\mathbf{D}(-k')$ , for  $1 \leq k' < k$ , consider the mapping space

$$\mathrm{Maps}((i_D)_*(q^*(\mathcal{F}_Z) \otimes \mathcal{O}(-k')), (i_D)_*(q^*(\mathcal{F}'_Z) \otimes \mathcal{O}(-k))),$$

for  $\mathcal{F}_Z, \mathcal{F}'_Z \in \mathrm{Qcoh}(Z)$ . Using fpqc descent and base change for  $(i_D)_*$  against  $f^*$  for any morphism  $f : U \rightarrow \tilde{X}$ , we may reduce to the case where  $X$  is affine. Since  $\mathrm{Qcoh}(Z)$  is then generated under colimits and finite limits by  $\mathcal{O}_Z$ , we may assume that  $\mathcal{F}_Z = \mathcal{F}'_Z = \mathcal{O}_Z$ . Then we have

$$\begin{aligned} \mathrm{Maps}((i_D)_*(\mathcal{O}(-k')), (i_D)_*(\mathcal{O}(-k))) &\simeq \mathrm{Maps}((i_D)^*(i_D)_*(\mathcal{O}(-k')), \mathcal{O}(-k)) \\ &\simeq \mathrm{Maps}(\mathcal{O}(-k') \oplus \mathcal{O}(-k'+1)[1], \mathcal{O}(-k)) \end{aligned}$$

by Lemma 4.1.1 and the projection formula, and this space is contractible by Theorem 3.2.1.

4.3.5. *Proof of (iv).* Denote by  $\mathbf{D}$  the full subcategory of  $\mathrm{Qcoh}(\tilde{X})$  generated by  $\mathbf{D}(0)$ ,  $\mathbf{D}(-1)$ ,  $\dots$ ,  $\mathbf{D}(-n+1)$  under finite colimits and finite limits. The claim is that the inclusion  $\mathbf{D} \subseteq \mathrm{Qcoh}(\tilde{X})$  is an equality. Note that  $\mathcal{O}_{\tilde{X}} \in \mathbf{D}(0) \subset \mathbf{D}$  and  $(i_D)_*(\mathcal{O}_{\mathbf{D}}(-k)) \in \mathbf{D}(-k) \subset \mathbf{D}$  for  $1 \leq k \leq n-1$ . Consider the exact triangle  $\mathcal{O}_{\tilde{X}}(-D) \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow (i_D)_*(\mathcal{O}_{\mathbf{D}})$  and recall that  $\mathcal{O}_{\tilde{X}}(-D) \simeq \mathcal{O}_{\tilde{X}}(1)$ . Tensoring with  $\mathcal{O}(-k)$  and using the projection formula, we get the exact triangle

$$\mathcal{O}_{\tilde{X}}(-k+1) \rightarrow \mathcal{O}_{\tilde{X}}(-k) \rightarrow (i_D)_*(\mathcal{O}_{\mathbf{D}}(-k))$$

for each  $1 \leq k \leq n-1$ . Taking  $k=1$  we deduce that  $\mathcal{O}_{\tilde{X}}(-1) \in \mathbf{D}$ . Continuing recursively we find that  $\mathcal{O}_{\tilde{X}}(-k) \in \mathbf{D}$  for all  $1 \leq k \leq n-1$ .

Now let  $\mathcal{F} \in \mathbf{Qcoh}(\tilde{X})$ . Denote by  $\mathcal{G}_0 \in \mathbf{Qcoh}(\tilde{X})$  the cofibre of the co-unit map  $p^*p_*(\mathcal{F}) \rightarrow \mathcal{F}$ . Note that  $\mathcal{G}_0$  is right orthogonal to  $\mathbf{D}(0)$ . For  $1 \leq m \leq n-1$  define  $\mathcal{G}_m$  recursively by the exact triangles

$$(i_{\mathbf{D}})_*(q^*q_*((i_{\mathbf{D}})^!(\mathcal{G}_{m-1}) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(-m)) \xrightarrow{\text{counit}} \mathcal{G}_{m-1} \rightarrow \mathcal{G}_m.$$

Just as in the proof of Theorem 3.2.1, a simple induction argument shows that each  $\mathcal{G}_m$  is right orthogonal to all of the subcategories  $\mathbf{D}(0), \dots, \mathbf{D}(m-1)$ . We now claim that  $\mathcal{G}_{n-1}$  is zero; it will follow by recursion that  $\mathcal{F}$  belongs to  $\mathbf{D}$ , as desired.

Since the objects  $\mathcal{G}_k$  are stable under base change, we may use fpqc descent and base change to assume that  $X$  is affine. Moreover we may assume that  $i: Z \rightarrow X$  fits in a cartesian square of the form (2.1.a). By [Kha18, 3.3.6],  $p: \tilde{X} \rightarrow X$  factors through a quasi-smooth closed immersion  $i': \tilde{X} \rightarrow \mathbf{P}_X^{n-1}$ . Recall from Lemma 3.2.2 that there is a canonical isomorphism  $\varinjlim_{J \subsetneq [n-1]} \mathcal{O}(|J|) \simeq \mathcal{O}(n)$  in  $\mathbf{Qcoh}(\mathbf{P}_X^{n-1})$ . Applying  $(i')^*$ , we get  $\varinjlim_{J \subsetneq [n-1]} \mathcal{O}_{\tilde{X}}(|J|) \simeq \mathcal{O}_{\tilde{X}}(n)$  in  $\mathbf{Qcoh}(\tilde{X})$ . In particular, every  $\mathcal{O}_{\tilde{X}}(k)$  belongs to  $\mathbf{D}$  for all  $k \in \mathbf{Z}$ . Recall also that we may find a map  $\bigoplus_{\alpha} \mathcal{O}(d_{\alpha})[n_{\alpha}] \rightarrow i'_*(\mathcal{G}_{n-1})$  which is surjective on all homotopy groups (Lemma 3.2.3). By adjunction this corresponds to a map  $\bigoplus_{\alpha} \mathcal{O}(d_{\alpha})[n_{\alpha}] \rightarrow \mathcal{G}_{n-1}$  (which is also surjective on homotopy groups). But the source belongs to  $\mathbf{D}$ , and the target is right orthogonal to  $\mathbf{D}$ , so this map is null-homotopic. Thus  $\mathcal{G}_{n-1}$  is zero.

**4.4. Proof of Theorem C.** We now deduce Theorem C from Theorem 4.3.1. First note that the fully faithful functor  $\mathcal{F} \mapsto p^*(\mathcal{F})$  of Theorem 4.3.1(i) preserves perfect complexes and therefore restricts to a fully faithful functor  $\text{Perf}(X) \rightarrow \text{Perf}(\text{Bl}_{Z/X})$ . This shows Theorem C(i).

Similarly, part (ii) follows from the fact that the functors  $q^*$  and  $(i_{\mathbf{D}})_*$  preserve perfect complexes. For the latter, this is because  $i_{\mathbf{D}}$  is quasi-smooth (and hence locally of finite presentation and of finite tor-amplitude).

For part (iii) we argue again as in the proof of Theorem 4.3.1(iv). The point is that if  $\mathcal{F} \in \mathbf{Qcoh}(\text{Bl}_{Z/X})$  is perfect, then so is each  $\mathcal{G}_m \in \mathbf{Qcoh}(\mathbf{P}(\mathcal{E}))$ , since  $q^*$ ,  $q_*$ ,  $(i_{\mathbf{D}})_*$  and  $(i_{\mathbf{D}})^!$  all preserve perfect complexes. For the latter this follows from Proposition 4.2.1.

#### 4.5. Blow-up formula.

4.5.1. By Theorem C and Lemma 2.3.3 we get:

**Corollary 4.5.2.** *Let  $X$  be a derived Artin stack and  $i: Z \rightarrow X$  a quasi-smooth closed immersion of virtual codimension  $n \geq 1$ . Then for any additive invariant  $E$ , there is a canonical isomorphism*

$$E(\text{Bl}_{Z/X}) \simeq E(X) \oplus \bigoplus_{k=1}^{n-1} E(Z).$$

4.5.3. *Proof of Theorem A.* Combine Corollaries 4.5.2 and 3.4.1 (with  $\mathcal{E} = \mathcal{N}_{Z/X}$ ).

## 5. THE CDH TOPOLOGY

**5.1. The cdh topology.** The following notion was introduced by Voevodsky [Voe10b] for noetherian schemes:

**Definition 5.1.1.** Suppose given a cartesian square  $Q$  of algebraic spaces

$$(5.1.a) \quad \begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow{e} & X. \end{array}$$

- (i) We say that  $Q$  is a *Nisnevich square* if  $e$  is an open immersion, and  $p$  is an étale morphism inducing an isomorphism  $(Y \setminus B)_{\text{red}} \simeq (X \setminus A)_{\text{red}}$ .
- (ii) We say that  $Q$  is a *proper cdh square*, or *abstract blow-up square*, if  $e$  is a closed immersion that is locally of finite presentation, and  $p$  is a proper morphism inducing an isomorphism  $(Y \setminus B)_{\text{red}} \simeq (X \setminus A)_{\text{red}}$ .
- (iii) We say that  $Q$  is a *cdh square* if it is either a Nisnevich square or a proper cdh square.

5.1.2. Given any class of commutative squares of algebraic spaces, we say that a presheaf satisfies *descent* for this class if it sends all such squares to homotopy cartesian squares, and the empty scheme to a terminal object. In case of the three classes considered in Definition 5.1.1, it follows from a theorem of Voevodsky [Voe10a, Cor. 5.10] that descent in this sense is equivalent to Čech descent with respect to the associated Grothendieck topology.

**Example 5.1.3.** Every localizing invariant  $E$  satisfies Nisnevich descent when regarded as a presheaf on quasi-compact quasi-separated algebraic spaces with  $E(X) = E(\text{Perf}(X))$ . This is essentially due to Thomason [TT90] and in the asserted generality is a consequence of the study of compact generation properties of the  $\infty$ -categories  $\text{Qcoh}(X)$  carried out by Bondal–Van den Bergh [BVdB03].

**Example 5.1.4.** Any quasi-smooth blow-up square (2.1.b) induces a proper cdh square

$$\begin{array}{ccc} \mathbf{P}(\mathcal{N}_{Z/X}|_{Z_{\text{cl}}}) & \longrightarrow & (\text{Bl}_{Z/X})_{\text{cl}} \\ \downarrow & & \downarrow \\ Z_{\text{cl}} & \longrightarrow & X_{\text{cl}} \end{array}$$

on underlying classical algebraic spaces.

**Example 5.1.5.** Consider the class of proper cdh squares (5.1.a) where the proper morphism  $p$  is a closed immersion (with quasi-compact open complement). The associated Grothendieck topology is the same as the one generated by *closed squares*, i.e. cartesian squares as in (5.1.a) such that  $i$  and  $p$  are closed immersions,  $i$  is locally of finite presentation and  $p$  has quasi-compact open complement, and  $A \sqcup Y \rightarrow X$  is surjective on underlying topological spaces.

**Example 5.1.6.** Note that for any algebraic space  $X$ , the square

$$\begin{array}{ccc} \emptyset & \longrightarrow & X_{\text{red}} \\ \downarrow & & \downarrow \\ \emptyset & \longrightarrow & X \end{array}$$

is a closed square as in Example 5.1.5.

## 5.2. A cdh descent criterion.

**Theorem 5.2.1.** *Let  $\mathcal{F}$  be a presheaf on the category  $\mathbf{C}$  of algebraic spaces, with values in a stable  $\infty$ -category. Then  $\mathcal{F}$  satisfies cdh descent if and only if it satisfies the following conditions:*

- (i) *It sends the empty scheme to a terminal object.*
- (ii) *It sends Nisnevich squares to cartesian squares.*
- (iii) *It sends closed squares to cartesian squares.*

- (iv) For every  $X \in \mathbf{C}$  and every quasi-smooth closed immersion  $Z \rightarrow X$ , it sends the square (Example 5.1.4)

$$\begin{array}{ccc} \mathbf{P}(\mathcal{N}_{Z/X}|_{Z_{\text{cl}}}) & \longrightarrow & (\text{Bl}_{Z/X})_{\text{cl}} \\ \downarrow & & \downarrow \\ Z_{\text{cl}} & \longrightarrow & X \end{array}$$

to a cartesian square.

Moreover, the same holds if  $\mathbf{C}$  is replaced by the full subcategory of (a) quasi-compact quasi-separated (qcqs) algebraic spaces, (b) schemes, (c) or qcqs schemes.

**Remark 5.2.2.** Any presheaf  $\mathcal{F}$  on algebraic spaces can be trivially extended to derived algebraic spaces, by setting  $\Gamma(X, \mathcal{F}) = \Gamma(X_{\text{cl}}, \mathcal{F})$  for every derived algebraic space  $X$ . The condition (iv) in Theorem 5.2.1 is equivalent to requiring this extension to satisfy descent for quasi-smooth blow-up squares (2.1.b).

**Example 5.2.3.** Let  $E$  be a localizing invariant of stable  $\infty$ -categories. Then it satisfies Nisnevich descent on qcqs algebraic spaces (Example 5.1.3) and quasi-smooth blow-up descent (Theorem A). Assume that  $E$  also satisfies *derived nilpotent invariance*, i.e., that the canonical map  $E(X) \rightarrow E(X_{\text{cl}})$  is invertible for every derived algebraic space  $X$ . Then the condition (iv) in Theorem 5.2.1 holds. Therefore,  $E$  satisfies cdh descent if and only if it satisfies closed descent. Moreover, by Nisnevich descent it suffices to consider closed squares of affine schemes.

**Example 5.2.4.** In the presence of  $\mathbf{A}^1$ -homotopy invariance, the Morel–Voevodsky localization theorem [MV99, Theorem 3.2.21] provides the following sufficient condition for closed descent. Let  $\mathcal{F}$  be an  $\mathbf{A}^1$ -invariant Nisnevich sheaf on the category of algebraic spaces. Suppose that, for every algebraic space  $S$ , its restriction  $\mathcal{F}_S$  to the site of *smooth* algebraic spaces over  $S$  is stable under arbitrary base change. That is, for every morphism of algebraic spaces  $f : T \rightarrow S$ , the canonical map  $f^*(\mathcal{F}_S) \rightarrow \mathcal{F}_T$  is invertible. Then  $\mathcal{F}$  satisfies closed descent. This follows immediately from the closed base change formula (cf. [Kha16, Prop. 3.3.2]).

**Remark 5.2.5.** Let  $E$  be a localizing invariant and suppose that it is moreover *truncating* in the sense of [LT18]. That is, if  $R$  is a connective  $\mathcal{E}_1$ -ring spectrum and  $\text{Mod}_R^{\text{perf}}$  denotes the stable  $\infty$ -category of left  $R$ -modules, then the canonical map  $E(\text{Mod}_R^{\text{perf}}) \rightarrow E(\text{Mod}_{\tau_0(R)}^{\text{perf}})$  is invertible. Then Land–Tamme have recently proven that (see Step 1 in the proof of [LT18, Thm. A.2]) that  $E$  has closed descent, at least if we restrict to *noetherian* algebraic spaces.

**Remark 5.2.6.** There are a few variants of Theorem 5.2.1 with the same proof. For example:

- (i) On the category of (qcqs) schemes, descent with respect to the *rh* topology (generated by Zariski squares and proper cdh squares) can be checked with the same criteria, except that Nisnevich squares are replaced by Zariski squares in condition (ii).
- (ii) If we do not assume either Nisnevich or Zariski descent, descent for the proper cdh topology is still equivalent to conditions (i), (iii), and (iv), as long as we restrict to a full subcategory of algebraic spaces or schemes which satisfy Thomason’s *resolution property*. For example, this holds on the category of quasi-projective schemes.
- (iii) One can extend the criterion to qcqs Artin stacks as follows. The definition of Nisnevich square extends without modification (cf. [HK17, Subsect. 2.3]). In the definition of proper cdh square, we add the requirement that the proper morphism  $p$  is representable (cf. *op. cit.*). Then the criterion of Theorem 5.2.1 holds for stacks *which admit the resolution property Nisnevich-locally*, see (5.3.3). This condition is relatively mild. For example, many quotient stacks have the resolution property ([Tho87, Lem. 2.4], [HR17, Exam. 7.5]). By the Nisnevich-local structure

theorem of Alper–Hall–Rydh [HK17, Thm. 2.9], any stack with linearly reductive and almost multiplicative stabilizers satisfies the resolution property Nisnevich-locally.

**5.3.** *Proof of Theorem 5.2.1.* Since Nisnevich squares, closed squares, and quasi-smooth blow-up squares are all cdh squares, the conditions are clearly necessary. Conversely suppose that  $\mathcal{F}$  is a presheaf satisfying the conditions and let  $Q$  be a proper cdh square of algebraic spaces of the form

$$(5.3.a) \quad \begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ Z & \xrightarrow{i} & X. \end{array}$$

It will suffice to show that the induced square  $\Gamma(Q, \mathcal{F})$  is homotopy cartesian.

5.3.1. Assume first that  $Q$  is a blow-up square, i.e., that  $Y = \mathrm{Bl}_{Z/X}$  is the blow-up of  $X$  centred in  $Z$  (and  $E = \mathbf{P}(\mathcal{C}_{Z/X})$  is the projectivized normal cone). By Nisnevich descent we may assume that  $X$  satisfies the resolution property (e.g.  $X$  is affine) and that  $i$  is of finite presentation. Then the ideal of definition  $J \subset \mathcal{O}_X$  is of finite type, so that by the resolution property there exists a surjection  $u : \mathcal{E} \rightarrow J$  with  $\mathcal{E}$  a locally free  $\mathcal{O}_X$ -module of finite rank. Denote by  $V = \mathbf{V}_X(\mathcal{E}) = \mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}))$  the associated vector bundle and  $s : X \rightarrow V$  the zero section. The  $\mathcal{O}_X$ -module homomorphism  $u : \mathcal{E} \rightarrow J \subset \mathcal{O}_X$  induces a section of  $V$  which we denote again by  $u$ . We define  $\tilde{Z}$  as the derived zero-locus of  $u$ , so that it fits into a homotopy cartesian square

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{i}} & X \\ \downarrow & & \downarrow u \\ X & \xrightarrow{s} & V. \end{array}$$

Now by construction, the morphism  $\tilde{i}$  is a quasi-smooth closed immersion (as the derived base change of a quasi-smooth closed immersion), and there is a canonical morphism  $Z \rightarrow \tilde{Z}$  which exhibits  $Z$  as the underlying classical scheme of  $\tilde{Z}$ . Regarding  $\mathcal{F}$  as a presheaf on derived algebraic spaces as in Remark 5.2.2, the square  $\Gamma(Q, \mathcal{F})$  now factors as follows:

$$\begin{array}{ccccc} \Gamma(X, \mathcal{F}) & \longrightarrow & \Gamma(\tilde{Z}, \mathcal{F}) & \xlongequal{\quad} & \Gamma(Z, \mathcal{F}) \\ \downarrow & & \downarrow & & \searrow \\ \Gamma(\mathrm{Bl}_{\tilde{Z}/X}, \mathcal{F}) & \longrightarrow & \Gamma(\mathbf{P}(\mathcal{N}_{\tilde{Z}/X}), \mathcal{F}) & & \\ \downarrow & & \downarrow & \swarrow & \\ \Gamma(\mathrm{Bl}_{Z/X}, \mathcal{F}) & \longrightarrow & \Gamma(\mathbf{P}(\mathcal{C}_{Z/X}), \mathcal{F}) & & \end{array}$$

The upper square is induced by a quasi-smooth blow-up square, hence is cartesian. The lower square is induced by a closed square, hence is also cartesian. Therefore it follows that the outer composite square is also cartesian. This shows that  $\mathcal{F}$  satisfies descent for blow-up squares.

5.3.2. Now consider an arbitrary proper cdh square  $Q$  of the form (5.3.a). Since  $\mathcal{F}$  is a Nisnevich sheaf, we may assume that  $X$  is quasi-compact and quasi-separated and that  $i : Z \rightarrow X$  is of finite presentation (so in particular has quasi-compact open complement). Then one reduces to the case considered above using Raynaud–Gruson’s technique of *platification par éclatements* [RG71, I, Cor. 5.7.9], just as in [KST18, Subsect. 5.2]. Instead of pro-descent for finite cdh squares (Lemma 5.1 in *op. cit.*), we use descent for closed squares, which holds by assumption. Note also that Raynaud–Gruson do not impose any noetherian hypotheses and that the statement cited applies to qcqs algebraic spaces. We remark that the assumption that  $\mathcal{F}$  takes values in a stable  $\infty$ -category only becomes relevant at the end of the

argument, in order to check that a square is homotopy cartesian by showing that it induces an isomorphism on homotopy fibres.

5.3.3. We now discuss the extension to stacks mentioned in Remark 5.2.6(iii). The precise statement is as follows. Let  $\mathbf{C}$  be a category of qcqs Artin stacks such that (a) every stack  $X \in \mathbf{C}$  admits a Nisnevich atlas by stacks with the resolution property; (b) for every stack  $X \in \mathbf{C}$  and every blow-up  $Y \rightarrow X$ , the qcqs Artin stack  $Y$  also belongs to  $\mathbf{C}$ . Then the statement of Theorem 5.2.1 holds for presheaves on  $\mathbf{C}$ .

The proof for the case of a blow-up square (5.3.1) has been presented in such a way that it holds *mutatis mutandis* under the above assumptions. The argument of [KST18, Claim 5.3] also goes through, using descent for closed squares and blow-up squares, to deal with the slightly more general case where  $Y = \mathrm{Bl}_{Z'/X}$  is a blow-up centred in some closed immersion  $Z' \rightarrow X$  that factors through  $Z$ . To reduce a general proper cdh square to that case, we use Rydh's extension of Raynaud–Gruson [HK17, Thm. 2.2]. First, closed descent allows us to assume that  $X \setminus Z$  is dense in  $X$ . Then we apply Rydh–Raynaud–Gruson just as in the proof of [HK17, Cor. 2.4]. The only difference with the case of schemes or algebraic spaces is that in general we get a *sequence* of  $(X \setminus Z)$ -admissible blow-ups  $\tilde{X} \rightarrow X$  which factors through  $p : Y \rightarrow X$ . The addition of a simple induction is then the only modification required to run the same argument.

#### 5.4. Homotopy invariant K-theory.

5.4.1. For any qcqs algebraic space  $X$ , its homotopy invariant K-theory spectrum is given by the formula

$$(5.4.a) \quad \Gamma(X, \mathrm{KH}) = \varinjlim_{[n] \in \Delta^{\mathrm{op}}} \mathrm{K}(X \times \mathbf{A}^n).$$

That is,  $\Gamma(X, \mathrm{KH})$  is the geometric realization of the simplicial diagram  $\mathrm{K}(X \times \mathbf{A}^\bullet)$ , where  $\mathbf{A}^\bullet$  is regarded as a cosimplicial scheme in the usual way (see e.g. [MV99, p. 45]). This extends the usual definition [Wei89, TT90], and is a way to formally impose the property of  $\mathbf{A}^1$ -homotopy invariance: for any qcqs algebraic space  $X$ , the projection  $p : X \times \mathbf{A}^1 \rightarrow X$  induces an isomorphism of spectra

$$p^* : \Gamma(X, \mathrm{KH}) \rightarrow \Gamma(X \times \mathbf{A}^1, \mathrm{KH}).$$

5.4.2. *Proof of Theorem D.* We use the criterion of Theorem 5.2.1. Condition (i) is obvious.

For Nisnevich descent, suppose given a Nisnevich square as in (5.1.a). Then for every  $[n] \in \Delta^{\mathrm{op}}$  we have, by Nisnevich descent for  $\mathrm{K}$  (Example 5.1.3), a homotopy cartesian square

$$\begin{array}{ccc} \mathrm{K}(X \times \mathbf{A}^n) & \longrightarrow & \mathrm{K}(A \times \mathbf{A}^n) \\ \downarrow & & \downarrow \\ \mathrm{K}(Y \times \mathbf{A}^n) & \longrightarrow & \mathrm{K}(B \times \mathbf{A}^n) \end{array}$$

and we conclude by passing to the colimit over  $n$  (as colimits of spectra commute with finite limits). This shows condition (ii).

For condition (iv), we use the extension of  $\mathrm{KH}$  to derived algebraic spaces<sup>3</sup> studied in [CK17]. It was proven in *loc. cit.* that  $\mathrm{KH}$  satisfies derived nilpotent invariance, i.e., that the canonical map  $\mathrm{KH}(X) \rightarrow \mathrm{KH}(X_{\mathrm{cl}})$  is invertible for any derived algebraic space  $X$ . Therefore, by Remark 5.2.2, it will suffice to show that  $\mathrm{KH}$  sends quasi-smooth blow-up squares of derived algebraic spaces to homotopy

<sup>3</sup>In *op. cit.* the authors work in the more exotic setting of *spectral* algebraic geometry, but the proofs apply *mutatis mutandis* also in the derived setting.



cartesian squares. This property holds for  $K$  by Theorem A, so just as for Nisnevich squares, we conclude using the formula (5.4.a) (which is still valid when  $X$  is derived).

The remaining condition (iii) is closed descent. By Nisnevich descent, we may restrict our attention to closed squares of *affine* schemes. This is classical, see [TT90, Exer. 9.11(f)] or [Wei89, Cor. 4.10].

**Remark 5.4.3.** By continuity for KH (e.g. [HK17, Thm. 4.9(5)]), once we have descent for proper cdh squares as in Definition 5.1.1(ii), we can immediately drop the finite presentation hypothesis on  $e$ .

## REFERENCES

- [SGA 6] Pierre Berthelot, Alexandre Grothendieck, and Luc Illusie. *Séminaire de Géométrie Algébrique du Bois Marie - 1966-67 - Théorie des intersections et théorème de Riemann-Roch - (SGA 6)*, volume 225. Springer-Verlag, 1971.
- [BGT13] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geometry & Topology*, 17(2):733–838, 2013.
- [BK89] Aleksei Igorevich Bondal and Mikhail Mikhailovich Kapranov. Representable functors, Serre functors, and mutations. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 53(6):1183–1205, 1989.
- [BM12] Andrew J. Blumberg and Michael A. Mandell. Localization theorems in topological Hochschild homology and topological cyclic homology. *Geom. Topol.*, 16(2):1053–1120, 2012.
- [BS17] Daniel Bergh and Olaf M Schnürer. Conservative descent for semi-orthogonal decompositions. *arXiv preprint arXiv:1712.06845*, 2017.
- [BVdB03] Aleksei Igorevich Bondal and Michel Van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Moscow Mathematical Journal*, 3(1):1–36, 2003.
- [Cis13] Denis-Charles Cisinski. Descente par éclatements en K-théorie invariante par homotopie. *Annals of Mathematics*, pages 425–448, 2013.
- [CK17] Denis-Charles Cisinski and Adeel A. Khan.  $\mathbf{A}^1$ -homotopy invariance in spectral algebraic geometry. *arXiv preprint arXiv:1705.03340*, 2017.
- [GR17] Dennis Gaitsgory and Nick Rozenblyum. *A Study in Derived Algebraic Geometry: Volumes I and II*. American Mathematical Soc., 2017.
- [Hae04] Christian Haesemeyer. Descent properties of homotopy K-theory. *Duke Mathematical Journal*, 125(3):589–619, 2004.
- [HK17] Marc Hoyois and Amalendu Krishna. Vanishing theorems for the negative K-theory of stacks. *arXiv preprint arXiv:1705.02295*, 2017.
- [Hoy16] Marc Hoyois. Cdh descent in equivariant homotopy K-theory. *arXiv preprint arXiv:1604.06410*, 2016.
- [Hoy17] Marc Hoyois. The six operations in equivariant motivic homotopy theory. *Advances in Mathematics*, 305:197–279, 2017.
- [HR17] Jack Hall and David Rydh. Perfect complexes on algebraic stacks. *Compositio Mathematica*, 153(11):2318–2367, 2017.
- [Kha16] Adeel A. Khan. The Morel–Voevodsky localization theorem in spectral algebraic geometry. *arXiv preprint arXiv:1610.06871*, 2016.
- [Kha18] Adeel A Khan. Virtual Cartier divisors and blow-ups. *arXiv preprint arXiv:1802.05702*, 2018.
- [KR18] Amalendu Krishna and Charanya Ravi. Algebraic K-theory of quotient stacks. *Annals of K-Theory*, 3(2):207–233, 2018.
- [KST18] Moritz Kerz, Florian Strunk, and Georg Tamme. Algebraic K-theory and descent for blow-ups. *Inventiones mathematicae*, 211(2):523–577, 2018.
- [LT18] Markus Land and Georg Tamme. On the K-theory of pullbacks. *arXiv preprint arXiv:1808.05559*, 2018.
- [HTT] Jacob Lurie. *Higher topos theory*. Number 170. Princeton University Press, 2009.
- [HA] Jacob Lurie. Higher algebra. *Preprint, available at www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf*, 2012. Version of 2017-09-18.
- [SAG] Jacob Lurie. Spectral algebraic geometry. *Preprint, available at www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf*, 2016. Version of 2018-02-03.
- [MV99] Fabien Morel and Vladimir Voevodsky.  $\mathbf{A}^1$ -homotopy theory of schemes. *Publications Mathématiques de l’IHES*, 90(1):45–143, 1999.
- [Orl92] Dmitri Olegovich Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 56(4):852–862, 1992.
- [RG71] Michel Raynaud and Laurent Gruson. Critères de platitude et de projectivité: Techniques de “platification” d’un module. *Inventiones mathematicae*, 13(1-2):1–89, 1971.

- [Tho87] Robert W Thomason. Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes. *Advances in Mathematics*, 65(1):16–34, 1987.
- [Tho93a] R. W. Thomason. Les K-groupes d'un fibré projectif. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 243–248. Kluwer Acad. Publ., Dordrecht, 1993.
- [Tho93b] R. W. Thomason. Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire. *Invent. Math.*, 112(1):195–215, 1993.
- [TT90] Robert W Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift*, pages 247–435. Springer, 1990.
- [TV08] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry II: Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.
- [Voe10a] Vladimir Voevodsky. Homotopy theory of simplicial sheaves in completely decomposable topologies. *Journal of pure and applied algebra*, 214(8):1384–1398, 2010.
- [Voe10b] Vladimir Voevodsky. Unstable motivic homotopy categories in Nisnevich and cdh-topologies. *Journal of Pure and applied Algebra*, 214(8):1399–1406, 2010.
- [Wei89] Charles A Weibel. Homotopy algebraic K-theory. *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, 83:461–488, 1989.

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