DESCENT BY QUASI-SMOOTH BLOW-UPS IN ALGEBRAIC K-THEORY

ADEEL A. KHAN

Abstract. We construct a semi-orthogonal decomposition on the category of perfect complexes on the blow-up of a derived Artin stack in a quasi-smooth centre. This gives a generalization of Thomason's blow-up formula in algebraic K-theory to derived stacks. We also provide a new criterion for descent in Voevodsky's cdh topology, which we use to give a direct proof of Cisinski's theorem that Weibel's homotopy invariant K-theory satisfies cdh descent.

1. Introduction

1.1. Let X be a scheme and $i : \mathrm{Z} \to \mathrm{X}$ a regular closed immersion. This means that Z is, Zariski-locally on X, the zero-locus of some regular sequence of functions $f_1, \ldots, f_n \in \Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$. Then the blow-up $\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ fits into a square

$$\begin{tikzcd} \mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \arrow[r, "i_\mathrm{D}"] \arrow[d, "q"] & \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \arrow[d, "p"] \\ \mathrm{Z} \arrow[r, "i"] & \mathrm{X}, \end{tikzcd} \tag{1.1.a}$$

where the exceptional divisor is the projective bundle associated to the conormal sheaf $\mathcal{N}_{\mathrm{Z}/\mathrm{X}}$, which under the assumptions is locally free of rank n. A result of Thomason [Tho93b] asserts that after taking algebraic K-theory, the induced square of spectra

$$\begin{tikzcd} \mathrm{K}(\mathrm{X}) \arrow[r, "i^*"] \arrow[d, "p^*"] & \mathrm{K}(\mathrm{Z}) \arrow[d] \\ \mathrm{K}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}) \arrow[r] & \mathrm{K}(\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})) \end{tikzcd}$$

is homotopy cartesian. Here K(X) denotes the Bass–Thomason–Trobaugh algebraic K-theory spectrum of perfect complexes on a scheme X. We may summarize this property by saying that algebraic K-theory satisfies *descent* with respect to blow-ups in regularly immersed centres.

Now suppose that i is more generally a *quasi-smooth* closed immersion of derived schemes. This means that Z is, Zariski-locally on X, the *derived* zero-locus of some arbitrary sequence of functions $f_1, \ldots, f_n \in \Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X})$. (When X is a classical scheme and the sequence is regular, this is the same as the classical zero-locus, and we are in the situation discussed above.) In the derived setting there is still a conormal sheaf $\mathcal{N}_{\mathrm{Z}/\mathrm{X}}$ on Z, locally free of rank n, and one may still form the blow-up square (1.1.a), see [Kha18]. Our goal in this paper is to generalize Thomason's result above to this situation. At the same time we also allow X to be a derived *Artin stack*, and consider any *additive invariant* of stable ∞-categories (see Definition 2.3.1). Examples of additive invariants include algebraic K-theory K, connective algebraic K-theory K^{cn}, topological Hochschild homology THH, and topological cyclic homology TC.

Theorem A. *Let* E *be an additive invariant of stable* ∞*-categories. Given a derived Artin stack* X *and a quasi-smooth closed immersion* $i : \mathrm{Z} \to \mathrm{X}$ *of virtual* *codimension $n \geqslant 1$, form the blow-up square (1.1.a). Then the induced commutative square*

$$\begin{array}{ccc} \mathrm{E}(\mathrm{X}) & \xrightarrow{i^*} & \mathrm{E}(\mathrm{Z}) \\ \downarrow{\scriptstyle p^*} & & \downarrow \\ \mathrm{E}(\mathrm{Bl}_{\mathrm{Z/X}}) & \longrightarrow & \mathrm{E}(\mathbf{P}(\mathcal{N}_{\mathrm{Z/X}})) \end{array}$$

is homotopy cartesian.

We deduce Theorem A from an analysis of the categories of perfect complexes on $\mathrm{Bl}_{\mathrm{Z/X}}$ and on the exceptional divisor $\mathbf{P}(\mathcal{N}_{\mathrm{Z/X}})$. The relevant notion is that of a *semi-orthogonal decomposition*, see Definition 2.2.2.

Theorem B. *Let* X *be a derived Artin stack. For any locally free $\mathcal{O}_{\mathrm{X}}$-module $\mathcal{E}$ of rank $n+1$, $n \geqslant 0$, consider the projective bundle $q : \mathbf{P}(\mathcal{E}) \to \mathrm{X}$. Then we have:*

(i) *For each $0 \leqslant k \leqslant n$, the assignment $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(-k)$ defines a fully faithful functor* $\mathrm{Perf}(\mathrm{X}) \to \mathrm{Perf}(\mathbf{P}(\mathcal{E}))$, *whose essential image we denote $\mathbf{A}(-k)$.*

(ii) *The sequence of full subcategories $(\mathbf{A}(0), \ldots, \mathbf{A}(-n))$ forms a semi-orthogonal decomposition of* $\mathrm{Perf}(\mathbf{P}(\mathcal{E}))$.

Theorem C. *Let* X *be a derived Artin stack. For any quasi-smooth closed immersion $i : \mathrm{Z} \to \mathrm{X}$ of virtual codimension $n \geqslant 1$, form the blow-up square (1.1.a). Then we have:*

(i) *The assignment $\mathcal{F} \mapsto p^*(\mathcal{F})$ defines a fully faithful functor* $\mathrm{Perf}(\mathrm{X}) \to \mathrm{Perf}(\mathrm{Bl}_{\mathrm{Z/X}})$, *whose essential image we denote $\mathbf{B}(0)$.*

(ii) *For each $1 \leqslant k \leqslant n-1$, the assignment $\mathcal{F} \mapsto (i_{\mathrm{D}})_*(q^*(\mathcal{F}) \otimes \mathcal{O}(-k))$ defines a fully faithful functor* $\mathrm{Perf}(\mathrm{Z}) \to \mathrm{Perf}(\mathrm{Bl}_{\mathrm{Z/X}})$, *whose essential image we denote $\mathbf{B}(-k)$.*

(iii) *The sequence of full subcategories $(\mathbf{B}(0), \ldots, \mathbf{B}(-n+1))$ forms a* semi-orthogonal decomposition *of* $\mathrm{Perf}(\mathrm{Bl}_{\mathrm{Z/X}})$.

We immediately deduce the projective bundle and blow-up formulas

$$\mathrm{E}(\mathbf{P}(\mathcal{E})) \simeq \bigoplus_{m=0}^{n} \mathrm{E}(\mathrm{X}), \qquad \mathrm{E}(\mathrm{Bl}_{\mathrm{Z/X}}) \simeq \mathrm{E}(\mathrm{X}) \oplus \bigoplus_{k=1}^{n-1} \mathrm{E}(\mathrm{Z}),$$

for any additive invariant E, see Corollaries 3.4.1 and 4.5.2, from which Theorem A immediately follows (see Subsect. 4.5).

1.2. The results mentioned above admit the following interesting special cases:

(a) Suppose that X is a smooth projective variety over the field of complex numbers. This case of Theorem B was proven by Orlov in [Orl92]. He also proved Theorem C for any *smooth* subvariety $\mathrm{Z} \hookrightarrow \mathrm{X}$.

(b) More generally suppose that X is a quasi-compact quasi-separated classical scheme. Then the projective bundle formula (Corollary 3.4.1) for algebraic K-theory was proven by Thomason [TT90, Tho93a]. Similarly suppose that $i : \mathrm{Z} \to \mathrm{X}$ is a quasi-smooth closed immersion of quasi-compact quasi-separated classical schemes. Then it is automatically a regular closed immersion, and in this case Thomason also proved Corollary 4.5.2 for algebraic K-theory [Tho93b]. In fact, the papers [Tho93a] and [Tho93b] essentially contain under these assumptions proofs of Theorems B and C, respectively, even if the term "semi-orthogonal decomposition" is not used explicitly. For THH and TC, these cases of Corollaries 3.4.1 and 4.5.2 were proven by Blumberg and Mandell [BM12].

(c) More generally still, let X and Z be classical Artin stacks. These cases of Theorems B and C are proven by by Bergh and Schnürer in [BS17]. However we note that Corollaries 3.4.1 and 4.5.2 were obtained earlier by Krishna and Ravi in [KR18], and their arguments in fact prove Theorems B and C for classical Artin stacks.

(d) Let X be a noetherian affine classical scheme, and let Z be the derived zero-locus of some functions $f_1, \ldots, f_n \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$. Then the canonical morphism $i : \mathrm{Z} \to \mathrm{X}$ is a quasi-smooth closed immersion. In this case, Theorem A for algebraic K-theory was proven by Kerz–Strunk–Tamme [KST18] (where the blow-up $\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ was explicitly modelled as the derived fibred product $\mathrm{X} \times_{\mathbf{A}^n} \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$), as part of their proof of Weibel's conjecture on negative K-theory.

1.3. Let KH denote homotopy invariant K-theory. Recall that this is the $\mathbf{A}^1$-localization of the presheaf $\mathrm{X} \mapsto \mathrm{K}(\mathrm{X})$. That is, it is obtained by forcing the property of $\mathbf{A}^1$-homotopy invariance: for every quasi-compact quasi-separated algebraic space X, the map

$$\mathrm{KH}(\mathrm{X}) \to \mathrm{KH}(\mathrm{X} \times \mathbf{A}^1)$$

is invertible (see [Wei89, Cis13]). As an application of Theorem A, we give a new proof of the following theorem of Cisinski [Cis13]:

Theorem D. *The presheaf of spectra* $\mathrm{S} \mapsto \mathrm{KH}(\mathrm{S})$ *satisfies cdh descent on the site of quasi-compact quasi-separated algebraic spaces.*

This was first proven by Haesemeyer [Hae04] for schemes over a field of characteristic zero, using resolution of singularities. Cisinski's proof over general bases (noetherian schemes of finite dimension) relies on Ayoub's proper base change theorem in motivic homotopy theory. Another proof of Theorem D (also in the noetherian setting) was given recently by Kerz–Strunk–Tamme [KST18, Thm. C], as an application of pro-cdh descent and their resolution of Weibel's conjecture on negative K-theory. The proof we give here is much more direct and uses a new criterion for cdh descent (Theorem 5.1.2). This criterion says that cdh descent is equivalent to Nisnevich descent, descent by quasi-smooth blow-ups, and closed descent. Since algebraic K-theory satisfies the first two properties, the only obstruction is the latter property, which vanishes after passing from K to KH. A similar cdh descent criterion has been noticed independently by Markus Land and Georg Tamme, see [LT18, Thm. A.2]. We do not know if it can be applied here since we do not know that KH can be extended to an invariant of stable ∞-categories which is truncating in the sense of *op. cit.* In fact the main new input here is the result of [CK17] which asserts that its extension to *derived schemes* (or algebraic spaces) does satisfy the property that $\mathrm{KH}(\mathrm{X}) \to \mathrm{KH}(\mathrm{X}_{\mathrm{cl}})$ is invertible for all X.

Theorem D was extended to certain nice Artin stacks recently by Hoyois and Hoyois–Krishna [Hoy16, HK17]. We do not know whether our proof can be extended to stacks, since we do not know whether, given a closed substack Z of a stack X, there always exists a quasi-smooth closed immersion $\tilde{\mathrm{Z}} \to \mathrm{X}$ such that $\tilde{\mathrm{Z}}_{\mathrm{cl}} = \mathrm{Z}$ (at least Nisnevich-locally on X). This also seems to be an obstruction to extending the pro-cdh descent results of [KST18] to stacks.

1.4. The organization of this paper is as follows. We begin in Sect. 2 with some background on derived algebraic geometry and on semi-orthogonal decompositions of stable ∞-categories.

Sect. 3 is dedicated to the proof of Theorem B. We first show that the semi-orthogonal decomposition exists on the larger stable ∞-category $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ (Theorem 3.2.1). Then we show that it restricts to $\mathrm{Perf}(\mathbf{P}(\mathcal{E}))$ (Subsect. 3.3), and deduce the projective bundle formula (Corollary 3.4.1) for any additive invariant. We follow a similar pattern in Sect. 4 to prove Theorem C. There is a semi-orthogonal decomposition on $\mathrm{Qcoh}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}})$ (Theorem 4.3.1) which then restricts to $\mathrm{Perf}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}})$ (Subsect. 4.4). This gives both the blow-up formula (Corollary 4.5.2) as well as Theorem A (4.5.3) for additive invariants. As input we prove a Grothendieck duality statement for virtual Cartier divisors (Proposition 4.2.1) that should be of independent interest.

Sect. 5 contains our results on cdh descent and KH. We first give the general cdh descent criterion (Theorems 5.1.2 and 5.2.4). We apply this criterion to KH to give our proof of Theorem D (5.3.3).

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2. Preliminaries

Throughout the paper we work with the language of ∞-categories as in [Lur09, Lur12].

2.1. *Derived algebraic geometry.* This paper is set in the world of derived algebraic geometry, as in [TV08, Lur16, GR17].

2.1.1. Let SCRing denote the ∞-category of simplicial commutative rings. A *derived stack* is an étale sheaf of spaces $\mathrm{X} : \mathrm{SCRing} \to \mathrm{Spc}$. If X is corepresentable by a simplicial commutative ring A, we write $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ and call X an *affine derived scheme*. A *derived scheme* is a derived stack X that admits a Zariski atlas by affine derived schemes, i.e., a jointly surjective family $(\mathrm{U}_i \to \mathrm{X})_i$ of Zariski open immersions with each U_i an affine derived scheme. Allowing Nisnevich, étale or smooth atlases, respectively, gives rise to the notions of *derived algebraic spaces*[1], *derived Deligne–Mumford stacks*, and *derived Artin stacks*. The precise definition is slightly more involved, see e.g. [GR17, Vol. I, Sect. 4.1].

Any derived stack X admits an underlying classical stack which we denote X_{cl}. If X is a derived scheme, algebraic space, Deligne–Mumford or Artin stack, then X_{cl} is a classical such.

2.1.2. Let X be a derived scheme and let $f_1, \ldots, f_n \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ be functions classifying a morphism $f : \mathrm{X} \to \mathbf{A}^n$ to affine space. The *derived zero-locus* of these functions is given by the derived fibred product

$$\begin{tikzcd} \mathrm{Z} \arrow[r] \arrow[d] & \mathrm{X} \arrow[d, "f"] \\ \{0\} \arrow[r] & \mathbf{A}^n. \end{tikzcd} \tag{2.1.a}$$

If X is classical, then Z is classical if and only if the sequence $(f_1, \ldots, f_n)$ is regular in the sense of [BGI71], in which case Z is regularly immersed. A closed immersion of derived schemes $i : \mathrm{Z} \to \mathrm{X}$ is called *quasi-smooth* (of virtual codimension n) if it is cut out Zariski-locally as the derived zero-locus of n functions on X. Equivalently, this means that i is locally finitely presented and its shifted cotangent complex $\mathcal{N}_{\mathrm{Z}/\mathrm{X}} := \mathcal{L}_{\mathrm{Z}/\mathrm{X}}[-1]$ is locally free (of rank n). A closed immersion of derived Artin stacks is quasi-smooth if it satisfies this condition smooth-locally.

A morphism of derived schemes $f : \mathrm{Y} \to \mathrm{X}$ is quasi-smooth if it can be factored, Zariski-locally on Y, through a quasi-smooth closed immersion $i : \mathrm{Y} \to \mathrm{X}'$ and a smooth morphism $\mathrm{X}' \to \mathrm{X}$. A morphism of derived Artin stacks is quasi-smooth if it satisfies this condition smooth-locally on Y. We refer to [Kha18] for more details on quasi-smoothness.

[1]That this agrees with the classical notion of algebraic space (at least under quasi-compactness and quasi-separatedness hypotheses) follows from [RG71, Prop. 5.7.6]. That it agrees with Lurie's definition follows from [Lur16, Ex. 3.7.1.5].

2.1.3. Important in this paper is the following construction from [Kha18]. Given any quasi-smooth closed immersion $i : \mathrm{Z} \to \mathrm{X}$ of derived Artin stacks, there is an associated *quasi-smooth blow-up square*:

$$
\begin{tikzcd}
\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \arrow[r, "i_{\mathrm{D}}"] \arrow[d, "q"] & \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \arrow[d, "p"] \\
\mathrm{Z} \arrow[r, "i"] & \mathrm{X}.
\end{tikzcd}
\tag{2.1.b}
$$

Here $\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ is the blow-up of X in Z, which is a quasi-smooth proper derived Artin stack over X, and $\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$ is the projectivized normal bundle, which is a smooth proper derived Artin stack over X. This square is universal with the following properties: (a) the morphism i_{D} is a quasi-smooth closed immersion of virtual codimension 1, i.e., a virtual effective Cartier divisor; (b) the underlying square of classical Artin stacks is cartesian; and (c) the canonical map $q^*\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \to \mathcal{N}_{\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})/\,\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}}$ is surjective on π_0. When X is a derived Deligne–Mumford stack (resp. derived algebraic space, derived scheme), then so is $\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$.

2.1.4. Given a derived stack X, the stable ∞-category of quasi-coherent sheaves Qcoh(X) is the limit

$$\mathrm{Qcoh}(\mathrm{X}) = \varprojlim_{\mathrm{Spec}(\mathrm{A})\to\mathrm{X}} \mathrm{Qcoh}(\mathrm{Spec}(\mathrm{A}))$$

taken over all morphisms $\mathrm{Spec}(\mathrm{A}) \to \mathrm{X}$ with $\mathrm{A} \in \mathrm{SCRing}$. Here Qcoh(Spec(A)) is the stable ∞-category $\mathrm{Mod}_{\mathrm{A}}$ of A-modules[2] in the sense of Lurie. Informally speaking, a quasi-coherent sheaf $\mathcal{F}$ on X is thus a collection of quasi-coherent sheaves $x^*(\mathcal{F}) \in \mathrm{Qcoh}(\mathrm{Spec}(\mathrm{A}))$, for every simplicial commutative ring A and every A-point $x : \mathrm{Spec}(\mathrm{A}) \to \mathrm{X}$, together with a homotopy coherent system of compatibilities.

The full subcategory $\mathrm{Perf}(\mathrm{X}) \subset \mathrm{Qcoh}(\mathrm{X})$ is similarly the limit

$$\mathrm{Perf}(\mathrm{X}) = \varprojlim_{\mathrm{Spec}(\mathrm{A})\to\mathrm{X}} \mathrm{Perf}(\mathrm{Spec}(\mathrm{A})),$$

where Perf(Spec(A)) is the stable ∞-category $\mathrm{Mod}_{\mathrm{A}}^{\mathrm{perf}}$ of *perfect* A-modules. In other words, $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{X})$ belongs to Perf(X) if and only if $x^*(\mathcal{F})$ is perfect for every simplicial commutative ring A and every morphism $x : \mathrm{Spec}(\mathrm{A}) \to \mathrm{X}$.

We will use repeatedly the fact that the assignments $\mathrm{X} \mapsto \mathrm{Qcoh}(\mathrm{X})$ and $\mathrm{X} \mapsto \mathrm{Perf}(\mathrm{X})$, as presheaves of ∞-categories, satisfy *descent* for the fpqc topology ([Lur16, Cor. D.6.3.3], [GR17, Thm. 1.3.4]). This means in particular that given any fpqc covering family $(f_\alpha : \mathrm{X}_\alpha \to \mathrm{X})_\alpha$, the family of inverse image functors $f_\alpha^* : \mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\mathrm{X}_\alpha)$ is jointly conservative.

2.2. *Semi-orthogonal decompositions.* The following definitions were originally formulated by [BK89] in the language of triangulated categories and are standard.

Definition 2.2.1. Let $\mathbf{C}$ be a stable ∞-category and $\mathbf{D}$ a stable full subcategory. An object $x \in \mathbf{C}$ is *left orthogonal*, resp. *right orthogonal*, to $\mathbf{D}$ if the mapping space $\mathrm{Maps}_{\mathbf{C}}(x, d)$, resp. $\mathrm{Maps}_{\mathbf{C}}(d, x)$, is contractible for all objects $d \in \mathbf{D}$. We let ${}^\perp\mathbf{D} \subseteq \mathbf{C}$ and $\mathbf{D}^\perp \subseteq \mathbf{C}$ denote the full subcategories of left orthogonal and right orthogonal objects, respectively.

Definition 2.2.2. Let $\mathbf{C}$ be a stable ∞-category and let $\mathbf{C}(0), \ldots, \mathbf{C}(-n)$ be full stable subcategories. Suppose that the following conditions hold:

(i) For all integers $i > j$, there is an inclusion $\mathbf{C}(i) \subseteq {}^\perp\mathbf{C}(j)$.

(ii) The ∞-category $\mathbf{C}$ is generated by the subcategories $\mathbf{C}(0), \ldots, \mathbf{C}(-n)$, under finite limits and finite colimits.

[2]Note that if A is discrete (an ordinary commutative ring), then this is not the abelian category of discrete A-modules, but rather the derived ∞-category of this abelian category as in [Lur12, Chap. 1].

Then we say that the sequence $(\mathbf{C}(0), \ldots, \mathbf{C}(-n))$ forms a *semi-orthogonal decomposition* of $\mathbf{C}$.

Semi-orthogonal decompositions of length 2 come from *split short exact sequences* of stable ∞-categories, as in [BGT13].

Definition 2.2.3.

(i) A *short exact sequence* of small stable ∞-categories is a diagram

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'',$$

where i and p are exact, the composite $p \circ i$ is null-homotopic, i is fully faithful, and p induces an equivalence $(\mathbf{C}/\mathbf{C}')^{\mathrm{idem}} \simeq (\mathbf{C}'')^{\mathrm{idem}}$ (where $(-)^{\mathrm{idem}}$ denotes idempotent completion).

(ii) A short exact sequence of small stable ∞-categories

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}''$$

is *split* if there exist functors $q : \mathbf{C} \to \mathbf{C}'$ and $j : \mathbf{C}'' \to \mathbf{C}$, right adjoint to i and p, respectively, such that the unit $\mathrm{id} \to q \circ i$ and co-unit $p \circ j \to \mathrm{id}$ are invertible.

Remark 2.2.4. Let $\mathbf{C}$ be a small stable ∞-category, and let $(\mathbf{C}(0), \mathbf{C}(-1))$ be a semi-orthogonal decomposition. Then for any object $x \in \mathbf{C}$, there exists an exact triangle

$$x(0) \to x \to x(-1),$$

where $x(0) \in \mathbf{C}(0)$ and $x(-1) \in \mathbf{C}(-1)$. To see this, simply observe that the full subcategory spanned by objects x for which such a triangle exists, is closed under finite limits and colimits, and contains $\mathbf{C}(0)$ and $\mathbf{C}(-1)$. Moreover, the assignments $x \mapsto x(0)$ and $x \mapsto x(-1)$ determine well-defined functors $q : \mathbf{C} \to \mathbf{C}(0)$ and $p : \mathbf{C} \to \mathbf{C}(-1)$, respectively, which are right and left adjoint, respectively, to the inclusions (see e.g. [Lur16, Rem. 7.2.0.2]). It follows from this that any semi-orthogonal decomposition $(\mathbf{C}(0), \mathbf{C}(-1))$ induces a split short exact sequence

$$\mathbf{C}(0) \to \mathbf{C} \xrightarrow{p} \mathbf{C}(-1).$$

Lemma 2.2.5. *Let $\mathbf{C}$ be a stable ∞-category, and let $(\mathbf{C}(0), \ldots, \mathbf{C}(-n))$ be a sequence of full stable subcategories forming a semi-orthogonal decomposition of $\mathbf{C}$. For each $0 \leqslant m \leqslant n$, let $\mathbf{C}_{\leqslant -m} \subseteq \mathbf{C}$ denote the union $\mathbf{C}(-m) \cup \cdots \cup \mathbf{C}(-n)$, and let $\mathbf{C}_{\leqslant -n-1} \subseteq \mathbf{C}$ denote the full subcategory spanned by the zero object. Then there are split short exact sequences*

$$\mathbf{C}_{\leqslant -m-1} \hookrightarrow \mathbf{C}_{\leqslant -m} \to \mathbf{C}(-m)$$

for each $0 \leqslant m \leqslant n$.

Proof. It follows from the definitions that for each $0 \leqslant m \leqslant n$, the sequence $(\mathbf{C}(-m), \mathbf{C}_{\leqslant -m-1})$ forms a semi-orthogonal decomposition of $\mathbf{C}$. Therefore the claim follows from Remark 2.2.4. □

2.3. *Additive and localizing invariants.* The following definition is from [BGT13], except that we do not require commutativity with filtered colimits.

Definition 2.3.1. Let $\mathbf{A}$ be a stable presentable ∞-category. Let E be an $\mathbf{A}$-valued functor E from the ∞-category of small stable ∞-categories and exact functors.

(i) We say that E is an *additive invariant* if for any split short exact sequence

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'',$$

the induced map

$$\mathrm{E}(\mathbf{C}') \oplus \mathrm{E}(\mathbf{C}'') \xrightarrow{(i,j)} \mathrm{E}(\mathbf{C})$$

is invertible, where j is a right adjoint to p.

(ii) We say that E is a *localizing invariant* if for any short exact sequence

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'',$$

the induced diagram

$$\mathrm{E}(\mathbf{C}') \to \mathrm{E}(\mathbf{C}) \to \mathrm{E}(\mathbf{C}'')$$

is an exact triangle.

Remark 2.3.2. Any localizing invariant is also additive.

Lemma 2.3.3. *Let* $\mathbf{C}$ *be a stable* ∞*-category, and let* $(\mathbf{C}(0), \ldots, \mathbf{C}(-n))$ *be a sequence of full stable subcategories forming a semi-orthogonal decomposition of* $\mathbf{C}$. *Then for any additive invariant* E *there is a canonical isomorphism*

$$\mathrm{E}(\mathbf{C}) \simeq \bigoplus_{m=0}^{n} \mathrm{E}(\mathbf{C}(-m)).$$

Proof. Follows immediately from Lemma 2.2.5. □

We now recall that every localizing invariant satisfies Nisnevich descent, see e.g. [CMNN, Prop. A.13].

Definition 2.3.4. A *Nisnevich square* of derived algebraic spaces is a commutative square

$$\begin{tikzcd} \mathrm{W} \arrow[r] \arrow[d] & \mathrm{V} \arrow[d, "p"] \\ \mathrm{U} \arrow[r, "j"] & \mathrm{X} \end{tikzcd}$$

where j is an open immersion, and p is étale and induces an isomorphism $\mathrm{V} - \mathrm{W} \simeq \mathrm{X} - \mathrm{U}$. We say that a presheaf $\mathcal{F}$ satisfies *Nisnevich descent* if it sends the empty scheme to a terminal object, and Nisnevich squares to homotopy cartesian squares.

Theorem 2.3.5. *Let* E *be a localizing invariant. Regard* E *as a presheaf on the* ∞*-category of quasi-compact quasi-separated derived algebraic spaces by setting* $\mathrm{E}(\mathrm{X}) = \mathrm{E}(\mathrm{Perf}(\mathrm{X}))$. *Then* E *satisfies Nisnevich descent.*

Remark 2.3.6. It follows from [Kha16, Thm. 2.2.7] that a presheaf satisfies Nisnevich descent in the above sense if and only if it satisfies Čech descent with respect to the Grothendieck topology generated by Nisnevich squares.

3. The projective bundle formula

3.1. *Projective bundles.* Let X be a derived stack and $\mathcal{E}$ a locally free $\mathcal{O}_{\mathrm{X}}$-module of finite rank. Recall that the *projective bundle* associated to $\mathcal{E}$ is a derived stack $\mathbf{P}(\mathcal{E})$ over X equipped with an invertible sheaf $\mathcal{O}(1)$ together with a surjection $\mathcal{E} \to \mathcal{O}(1)$. More precisely, for any derived scheme S over X, with structural morphism morphism $x : \mathrm{S} \to \mathrm{X}$, the space of S-points of $\mathbf{P}(\mathcal{E})$ is the space of pairs $(\mathcal{L}, u)$, where $\mathcal{L}$ is a locally free $\mathcal{O}_{\mathrm{S}}$-module of rank 1, and $u : x^*(\mathcal{E}) \to \mathcal{L}$ is surjective on π_0. We recall the standard properties of this construction:

Proposition 3.1.1.

(i) *If* $f : \mathrm{X}' \to \mathrm{X}$ *is a morphism of derived stacks, then there is a canonical isomorphism* $\mathbf{P}(f^*(\mathcal{E})) \to \mathbf{P}(\mathcal{E}) \times_{\mathrm{X}} \mathrm{X}'$ *of derived stacks over* X'.

(ii) *If* X *is a derived scheme (resp. derived algebraic space,* k*-geometric derived Deligne–Mumford stack,* k*-geometric derived Artin stack), then the same holds for the derived stack* $\mathbf{P}(\mathcal{E})$.

(iii) *The projection* $\mathbf{P}(\mathcal{E}) \to \mathrm{X}$ *is proper.*

(iv) *The relative cotangent complex* $\mathcal{L}_{\mathbf{P}(\mathcal{E})/\mathrm{X}}$ *is canonically isomorphic to* $\mathcal{F} \otimes \mathcal{O}(-1)$, *where the locally free sheaf* $\mathcal{F}$ *is the fibre of the canonical map* $\mathcal{E} \to \mathcal{O}(1)$. *In particular, the morphism* $\mathbf{P}(\mathcal{E}) \to \mathrm{X}$ *is smooth of relative dimension equal to* $\mathrm{rk}(\mathcal{E}) - 1$.

Proposition 3.1.2 (Serre). *Let* X *be a derived Artin stack, and* $\mathcal{E}$ *a locally free sheaf of rank* $n+1$, $n \geqslant 0$. *If* $q : \mathbf{P}(\mathcal{E}) \to \mathrm{X}$ *denotes the associated projective bundle, then we have canonical isomorphisms*

$$q_*\mathcal{O}(0) \simeq \mathcal{O}_{\mathrm{X}}, \qquad q_*\mathcal{O}(-m) = 0 \quad (1 \leqslant m \leqslant n)$$

in $\mathrm{Qcoh(X)}$.

Proof. Since $\mathrm{Qcoh}(-)$ satisfies fpqc descent and base change for p_*, we may reduce to the case where X is affine and $\mathcal{E}$ is free. In this case the result follows immediately from Serre's computation (which holds in the derived setting by [Lur16, Thm. 5.4.2.6]). □

3.2. *Semi-orthogonal decomposition on* $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$. In this subsection we will show that the stable ∞-category $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ admits a canonical semi-orthogonal decomposition.

Theorem 3.2.1. *Let* X *be a derived Artin stack. Let* $\mathcal{E}$ *be a locally free* $\mathcal{O}_{\mathrm{X}}$*-module of rank* $n+1$, $n \geqslant 0$, *and* $q : \mathbf{P}(\mathcal{E}) \to \mathrm{X}$ *the associated projective bundle. Then we have:*

(i) *For every integer* $m \in \mathbf{Z}$, *the assignment* $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(m)$ *defines a fully faithful functor* $\mathrm{Qcoh(X)} \to \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$.

(ii) *For every integer* $m \in \mathbf{Z}$, *let* $\mathbf{C}(m) \subset \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ *denote the essential image of the functor in (i). Then the subcategories* $\mathbf{C}(m), \ldots, \mathbf{C}(m-n)$ *form a semi-orthogonal decomposition of* $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$.

We will need the following facts (see Lemmas 7.2.2.2 and 5.6.2.2 in [Lur16]):

Lemma 3.2.2. *Let* R *be a simplicial commutative ring and* $\mathrm{X} = \mathrm{Spec(R)}$. *Denote by* $\mathbf{P}^n_{\mathrm{R}} = \mathbf{P}(\mathcal{O}_{\mathrm{X}}^{n+1})$ *the* n*-dimensional projective space over* R. *Then for every integer* $m \in \mathbf{Z}$, *there is a canonical isomorphism*

$$\varinjlim_{\mathrm{J} \subsetneq [n]} \mathcal{O}(m + |\mathrm{J}|) \xrightarrow{\sim} \mathcal{O}(m+n+1)$$

in $\mathrm{Qcoh}(\mathbf{P}^n_{\mathrm{R}})$, *where the colimit is taken over the proper subsets* J *of the set* $[n] = \{0, 1, \ldots, n\}$, *and* $0 \leqslant |\mathrm{J}| \leqslant n$ *denotes the cardinality of such a subset.*

Lemma 3.2.3. *Let* R *be a simplicial commutative ring and* $\mathrm{X} = \mathrm{Spec(R)}$. *Denote by* $\mathbf{P}^n_{\mathrm{R}} = \mathbf{P}(\mathcal{O}_{\mathrm{X}}^{n+1})$ *the* n*-dimensional projective space over* R. *Then for any connective quasi-coherent sheaf* $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}^n_{\mathrm{R}})$, *there exists a map*

$$\bigoplus_{\alpha} \mathcal{O}(d_\alpha) \to \mathcal{F},$$

with $d_\alpha \in \mathbf{Z}$, *which is surjective on* π_0.

Proof of Theorem 3.2.1. Since the functors $-\otimes\mathcal{O}(k)$ are equivalences, it will suffice to take $k = 0$ in both claims. For claim (i) we want to show that the unit map $\mathcal{F} \to q_*q^*(\mathcal{F})$ is invertible for all $\mathcal{F} \in \mathrm{Qcoh(X)}$. Since the presheaf $\mathrm{Qcoh}(-)$ satisfies fpqc descent and base change for p_*, we may reduce recursively to the case where $\mathrm{X} = \mathrm{Spec(R)}$ is affine and $\mathcal{E} = \mathcal{O}_{\mathrm{S}}^{n+1}$ is free. Now both functors q^* and q_* are exact and moreover commute with arbitrary colimits (the latter since q is quasi-compact), and $\mathrm{Qcoh(X)} \simeq \mathrm{Mod_R}$ is generated by $\mathcal{O}_{\mathrm{X}}$ under colimits and finite limits. Therefore we may assume $\mathcal{F} = \mathcal{O}_{\mathrm{X}}$, in which case the claim holds by Proposition 3.1.2.

For claim (ii), let us first check the orthogonality condition in Definition 2.2.2. Thus take $\mathcal{F}, \mathcal{G} \in \mathrm{Qcoh(X)}$ and consider the mapping space

$$\mathrm{Maps}(q^*(\mathcal{F}), q^*(\mathcal{G}) \otimes \mathcal{O}(-m)) \simeq \mathrm{Maps}(\mathcal{F}, q_*(\mathcal{O}(-m)) \otimes \mathcal{G})$$

for $1 \leqslant m \leqslant n$, where the identification results from the projection formula. Since $q_*(\mathcal{O}(-m)) \simeq 0$ by Proposition 3.1.2, this space is contractible.

It now remains to show that every $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ belongs to the full subcategory $\langle \mathbf{C}(0), \ldots, \mathbf{C}(-n) \rangle \subseteq \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ generated under finite colimits and limits by the subcategories $\mathbf{C}(0), \ldots, \mathbf{C}(-n)$. Set $\mathcal{G}_{-1} = \mathcal{F} \otimes \mathcal{O}(-1)$ and define $\mathcal{G}_m$, for $m \geqslant 0$, so that we have exact triangles

$$q^* q_*(\mathcal{G}_{m-1} \otimes \mathcal{O}(1)) \xrightarrow{\text{counit}} \mathcal{G}_{m-1} \otimes \mathcal{O}(1) \to \mathcal{G}_m. \tag{3.2.a}$$

For each $m \geqslant -1$, we claim that $\mathcal{G}_m$ is right orthogonal to each of the subcategories $\mathbf{C}(0), \ldots, \mathbf{C}(m)$. For $m = -1$ the claim is vacuous, so take $m \geqslant 0$ and assume by induction that it holds for $m-1$. Since $p^* p_*(\mathcal{G}_{m-1} \otimes \mathcal{O}(1))$ is contained in $\mathbf{C}(0)$, it follows that $\mathcal{G}_m$ is right orthogonal to $\mathbf{C}(0)$. To show that $\mathcal{G}_m$ is right orthogonal to $\mathbf{C}(i)$, for $1 \leqslant i \leqslant m$, it will suffice to show that the left-hand and middle terms of the exact triangle (3.2.a) are both right orthogonal to $\mathbf{C}(i)$. For the left-hand term this follows from the inclusion $\mathbf{C}(0) \subset \mathbf{C}(i)^{\perp}$, demonstrated above. For the middle term $\mathcal{G}_{m-1} \otimes \mathcal{O}(1)$, the claim follows by the induction hypothesis.

Now we claim that $\mathcal{G}_n$ is zero. Using fpqc descent again, we may assume that $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ and $\mathcal{E} = \mathcal{O}_{\mathrm{X}}^{\oplus n+1}$ is free (since the sequence $(\mathcal{G}_{-1}, \mathcal{G}_0, \ldots, \mathcal{G}_n)$ is stable under base change). Using Lemma 3.2.3 we can build a map

$$\varphi : \bigoplus_{\alpha} \mathcal{O}(m_\alpha)[k_\alpha] \to \mathcal{G}_n$$

which is surjective on all homotopy groups. From Lemma 3.2.2 it follows that $\mathcal{G}_n$ is right orthogonal to all $\mathbf{C}(i)$, $i \in \mathbf{Z}$. Thus φ must be null-homotopic, so $\mathcal{G}_n \simeq 0$ as claimed. Working backwards, we deduce that $\mathcal{G}_{n-1} \in \mathbf{C}(-1)$, ..., $\mathcal{G}_0 \in \langle \mathbf{C}(-1), \ldots, \mathbf{C}(-n) \rangle$, and then finally that $\mathcal{F} \in \langle \mathbf{C}(0), \mathbf{C}(-1), \ldots, \mathbf{C}(-n) \rangle$ as claimed. □

3.3. *Proof of Theorem B.* We now deduce Theorem B from Theorem 3.2.1. First note that the fully faithful functor $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(m)$ of Theorem 3.2.1(i) restricts to a fully faithful functor $\mathrm{Perf}(\mathrm{X}) \to \mathrm{Perf}(\mathbf{P}(\mathcal{E}))$, since q^* preserves perfect complexes. This shows Theorem B(i).

For part (ii) we argue again as in the proof of Theorem 3.2.1. The point is that if $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ is perfect, then so is each $\mathcal{G}_m \in \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$, since q^* and q_* preserve perfect complexes [Lur16, Thm. 6.1.3.2].

3.4. *Projective bundle formula.* From Theorem B and Lemma 2.3.3 we deduce:

Corollary 3.4.1. *Let* X *be a derived Artin stack,* $\mathcal{E}$ *a locally free* $\mathcal{O}_{\mathrm{X}}$*-module of rank* $n+1$, $n \geqslant 0$, *and* $q : \mathbf{P}(\mathcal{E}) \to \mathrm{X}$ *the associated projective bundle. Then for any additive invariant* E, *there is a canonical isomorphism*

$$\mathrm{E}(\mathbf{P}(\mathcal{E})) \simeq \bigoplus_{m=0}^{n} \mathrm{E}(\mathrm{X})$$

induced by the functors $q^*(-) \otimes \mathcal{O}(-m) : \mathrm{Perf}(\mathrm{X}) \to \mathrm{Perf}(\mathbf{P}(\mathcal{E}))$.

4. The blow-up formula

4.1. *Virtual Cartier divisors.* Recall from [Kha18] that a *virtual (effective) Cartier divisor* on a derived Artin stack X is a quasi-smooth closed immersion $i : \mathrm{D} \to \mathrm{X}$ of virtual codimension 1. For any such $i : \mathrm{D} \to \mathrm{X}$, there is the canonical exact triangle

$$\mathcal{O}_{\mathrm{X}}(-\mathrm{D}) \to \mathcal{O}_{\mathrm{X}} \to i_*(\mathcal{O}_{\mathrm{D}}),$$

where $\mathcal{O}_{\mathrm{X}}(-\mathrm{D})$ is a locally free sheaf of rank 1, equipped with a canonical isomorphism $i^*(\mathcal{O}_{\mathrm{X}}(-\mathrm{D})) \simeq \mathcal{N}_{\mathrm{D}/\mathrm{X}}$.

Lemma 4.1.1. *Let* X *be a derived Artin stack and* $i : \mathrm{D} \to \mathrm{X}$ *a virtual Cartier divisor. Then there is a canonical isomorphism*

$$i^* i_*(\mathcal{O}_{\mathrm{D}}) \simeq \mathcal{O}_{\mathrm{D}} \oplus \mathcal{N}_{\mathrm{D}/\mathrm{X}}[1].$$

Proof. Applying i^* to the exact triangle $\mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*(\mathcal{O}_D)$ (and rotating it), we get the exact triangle

$$\mathcal{O}_D \to i^*i_*(\mathcal{O}_D) \to \mathcal{N}_{D/X}[1].$$

The map $\mathcal{O}_D \to i^*i_*(\mathcal{O}_D)$ is induced by the natural transformation $i^*(\eta) : i^* \to i^*i_*i^*$ (where η is the adjunction unit), so by the triangle identities it has a retraction given by the co-unit map $i^*i_*(\mathcal{O}_D) \to \mathcal{O}_D$. In other words, the triangle splits. $\square$

4.2. *Grothendieck duality.* Let $i : Z \to X$ be a quasi-smooth closed immersion of derived Artin stacks. The functor i_* admits a right adjoint $i^!$, which for formal reasons can be computed by the formula

$$i^!(-) \simeq i^*(-) \otimes \omega_{D/X},$$

where $\omega_{D/X} := i^!(\mathcal{O}_X)$ is called the *relative dualizing sheaf.* See [Lur16, Cor. 6.4.2.7]. When i is a virtual Cartier divisor, $\omega_{D/X}$ can be computed as follows:

Proposition 4.2.1 (Grothendieck duality)**.** *Let* X *be a derived Artin stack. Then for any virtual Cartier divisor* $i : D \to X$*, there is a canonical isomorphism*

$$\mathcal{N}^\vee_{D/X}[-1] \xrightarrow{\sim} \omega_{D/X}$$

of perfect complexes on D*. In particular, there is a canonical identification* $i^! \simeq i^*(-) \otimes \mathcal{N}^\vee_{D/X}[-1]$*.*

Proof. Write $\mathcal{L} := \mathcal{O}_X(-D)$ and consider again the exact triangle $\mathcal{L} \to \mathcal{O}_X \to i_*(\mathcal{O}_D)$. By the projection formula, this can be refined to an exact triangle of natural transformations $\mathrm{id} \otimes \mathcal{L} \to \mathrm{id} \to i_*i^*$, or, passing to right adjoints, an exact triangle $i_*i^! \to \mathrm{id} \to \mathrm{id} \otimes \mathcal{L}^\vee$. In particular we get the exact triangle

$$i_*i^!(\mathcal{O}_X) \to \mathcal{O}_X \to \mathcal{L}^\vee. \tag{4.2.a}$$

The associated map $\mathcal{L}^\vee[-1] \to i_*i^!(\mathcal{O}_X)$ gives by adjunction a canonical morphism

$$\mathcal{N}^\vee_{D/X}[-1] \simeq i^*(\mathcal{L}^\vee)[-1] \to i^!(\mathcal{O}_X),$$

which we claim is invertible. By fpqc descent and the fact that $i^!$ commutes with the operation f^*, for any morphism f [Lur16, Prop. 6.4.2.1], we may assume that X is affine. In this case the functor i_* is conservative, so it will suffice to show that the canonical map

$$i_*(\mathcal{N}^\vee_{D/X})[-1] \to i_*i^!(\mathcal{O}_X)$$

is invertible. Considering again the triangle $\mathcal{F} \otimes \mathcal{L} \to \mathcal{F} \to i_*i^*(\mathcal{F})$ above and taking $\mathcal{F} = \mathcal{L}^\vee$, we get the exact triangle

$$\mathcal{O}_X \to \mathcal{L}^\vee \to i_*i^*(\mathcal{L}^\vee) \simeq i_*(\mathcal{N}^\vee_{D/X}),$$

since $\mathcal{L}$ is invertible. Comparing with (4.2.a) yields the claim. $\square$

4.3. *Semi-orthogonal decomposition on* $\mathrm{Qcoh}(\mathrm{Bl}_{Z/X})$. In this subsection we prove:

Theorem 4.3.1. *Let* $i : Z \to X$ *be a quasi-smooth closed immersion of virtual codimension* n*. Consider the blow-up square*

$$\begin{tikzcd} D \arrow[r, "i_D"] \arrow[d, "q"] & \tilde{X} \arrow[d, "p"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

(i) *The functor* $p^* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(\tilde{X})$ *is fully faithful. We denote its essential image by* $\mathbf{D}(0) \subset \mathrm{Qcoh}(\tilde{X})$*.*

(ii) *The functor* $(i_D)_*(q^*(-) \otimes \mathcal{O}(-k)) : \mathrm{Qcoh}(Z) \to \mathrm{Qcoh}(\tilde{X})$ *is fully faithful, for each* $1 \leqslant k \leqslant n-1$*. We denote its essential image by* $\mathbf{D}(-k) \subset \mathrm{Qcoh}(\tilde{X})$*.*

(iii) *For each* $1 \leqslant k \leqslant n-1$*, the full stable subcategory* $\mathbf{D}(-k) \subset \mathrm{Qcoh}(\tilde{X})$ *is right orthogonal to each of* $\mathbf{D}(0), \ldots, \mathbf{D}(-k+1)$*.*

(iv) *The stable ∞-category* $\mathrm{Qcoh}(\tilde{\mathrm{X}})$ *is generated by the full subcategories* $\mathbf{D}(0)$, $\mathbf{D}(-1)$, ..., $\mathbf{D}(-n+1)$ *under finite colimits and finite limits. In particular, the sequence* $(\mathbf{D}(0), \mathbf{D}(-1), \dots, \mathbf{D}(-n+1))$ *forms a semi-orthogonal decomposition of* $\mathrm{Qcoh}(\tilde{\mathrm{X}})$.

4.3.2. *Proof of* (i). The claim is that for any $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{X})$, the unit map $\mathcal{F} \to p_*p^*(\mathcal{F})$ is invertible. By Zariski descent we may reduce to the case where X is affine and i fits in a cartesian square of the form (2.1.a). Since $\mathrm{Qcoh}(\mathrm{X})$ is then generated under colimits and finite limits by $\mathcal{O}_\mathrm{X}$, we may assume that $\mathcal{F} = \mathcal{O}_\mathrm{X}$. In other words, it suffices to show that the canonical map $\mathcal{O}_\mathrm{X} \to p_*(\mathcal{O}_{\tilde{\mathrm{X}}})$ is invertible.

$$\begin{tikzcd} \mathrm{D} \arrow[r, "i_\mathrm{D}"] \arrow[d, "q"] & \tilde{\mathrm{X}} \arrow[d, "p"] \\ \mathrm{Z} \arrow[r, "i"] & \mathrm{X} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{P}^{n-1} \arrow[r] \arrow[d] & \mathrm{Bl}_{\{0\}/\mathbf{A}^n} \arrow[d, "p_0"] \\ \{0\} \arrow[r, "i_0"] & \mathbf{A}^n, \end{tikzcd}$$

Since the left-hand square is the (derived) base change of the right-hand square along the morphism $f : \mathrm{X} \to \mathbf{A}^n$, it follows that the map $\mathcal{O}_\mathrm{X} \to p_*(\mathcal{O}_{\tilde{\mathrm{X}}})$ is the inverse image of the canonical map $\mathcal{O}_{\mathbf{A}^n} \to (p_0)_*(\mathcal{O}_{\mathrm{Bl}_{\{0\}/\mathbf{A}^n}})$. Thus we reduce to the case where i is the immersion $\{0\} \hookrightarrow \mathbf{A}^n$, [BGI71, Exp. VII].

4.3.3. *Proof of* (ii). It suffices to show the unit map $\mathcal{F} \to q_*(i_\mathrm{D})^!(i_\mathrm{D})_*q^*(\mathcal{F})$ is invertible for all $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{Z})$. As in the previous claim we may assume X is affine and that $\mathcal{F} = \mathcal{O}_\mathrm{Z}$. Using Proposition 4.2.1 and the canonical identification $\mathcal{N}_{\mathrm{D}/\tilde{\mathrm{X}}} \simeq \mathcal{O}_\mathrm{D}(1)$, the unit map is identified with

$$\mathcal{O}_\mathrm{Z} \to q_*((i_\mathrm{D})^*(i_\mathrm{D})_*(\mathcal{O}_\mathrm{D}) \otimes \mathcal{O}_\mathrm{D}(-1))[-1] \simeq q_*(\mathcal{O}_\mathrm{D}(-1)) \oplus q_*(\mathcal{O}_\mathrm{D}),$$

where the second identification comes from Lemma 4.1.1. Since $q : \mathrm{D} \to \mathrm{Z}$ is the projection of the projective bundle $\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$, it follows from Proposition 3.1.2 that we have identifications $q_*(\mathcal{O}_\mathrm{D}(-1)) \simeq 0$ and $q_*(\mathcal{O}_\mathrm{D}) \simeq \mathcal{O}_\mathrm{Z}$, under which the map in question is the identity.

4.3.4. *Proof of* (iii). To see that $\mathbf{D}(-k)$ is right orthogonal to $\mathbf{D}(0)$, observe that by Theorem 3.2.1, the mapping space

$$\mathrm{Maps}(p^*(\mathcal{F}_\mathrm{X}), (i_\mathrm{D})_*(q^*(\mathcal{F}_\mathrm{Z}) \otimes \mathcal{O}(-k))) \simeq \mathrm{Maps}(q^*i^*(\mathcal{F}_\mathrm{X}), q^*(\mathcal{F}_\mathrm{Z}) \otimes \mathcal{O}(-k))$$

is contractible for every $\mathcal{F}_\mathrm{X} \in \mathrm{Qcoh}(\mathrm{X})$ and $\mathcal{F}_\mathrm{Z} \in \mathrm{Qcoh}(\mathrm{Z})$.

To see that $\mathbf{D}(-k)$ is right orthogonal to $\mathbf{D}(-k')$, for $1 \leqslant k' < k$, consider the mapping space

$$\mathrm{Maps}((i_\mathrm{D})_*(q^*(\mathcal{F}_\mathrm{Z}) \otimes \mathcal{O}(-k')), (i_\mathrm{D})_*(q^*(\mathcal{F}'_\mathrm{Z}) \otimes \mathcal{O}(-k))),$$

for $\mathcal{F}_\mathrm{Z}, \mathcal{F}'_\mathrm{Z} \in \mathrm{Qcoh}(\mathrm{Z})$. Using fpqc descent and base change for $(i_\mathrm{D})_*$ against f^* for any morphism $f : \mathrm{U} \to \tilde{\mathrm{X}}$, we may reduce to the case where X is affine. Since $\mathrm{Qcoh}(\mathrm{Z})$ is then generated under colimits and finite limits by $\mathcal{O}_\mathrm{Z}$, we may assume that $\mathcal{F}_\mathrm{Z} = \mathcal{F}'_\mathrm{Z} = \mathcal{O}_\mathrm{Z}$. Then we have

$$\begin{aligned} \mathrm{Maps}((i_\mathrm{D})_*(\mathcal{O}(-k')), (i_\mathrm{D})_*(\mathcal{O}(-k))) &\simeq \mathrm{Maps}((i_\mathrm{D})^*(i_\mathrm{D})_*(\mathcal{O}(-k')), \mathcal{O}(-k)) \\ &\simeq \mathrm{Maps}(\mathcal{O}(-k') \oplus \mathcal{O}(-k'+1)[1], \mathcal{O}(-k)) \end{aligned}$$

by Lemma 4.1.1 and the projection formula, and this space is contractible by Theorem 3.2.1.

4.3.5. *Proof of* (iv). Denote by $\mathbf{D}$ the full subcategory of $\mathrm{Qcoh}(\tilde{\mathrm{X}})$ generated by $\mathbf{D}(0)$, $\mathbf{D}(-1)$, ..., $\mathbf{D}(-n+1)$ under finite colimits and finite limits. The claim is that the inclusion $\mathbf{D} \subseteq \mathrm{Qcoh}(\tilde{\mathrm{X}})$ is an equality. Note that $\mathcal{O}_{\tilde{\mathrm{X}}} \in \mathbf{D}(0) \subset \mathbf{D}$ and $(i_\mathrm{D})_*(\mathcal{O}_\mathrm{D}(-k)) \in \mathbf{D}(-k) \subset \mathbf{D}$ for $1 \leqslant k \leqslant n-1$. Consider the exact triangle $\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{D}) \to \mathcal{O}_{\tilde{\mathrm{X}}} \to (i_\mathrm{D})_*(\mathcal{O}_\mathrm{D})$ and recall that $\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{D}) \simeq \mathcal{O}_{\tilde{\mathrm{X}}}(1)$. Tensoring with $\mathcal{O}(-k)$ and using the projection formula, we get the exact triangle

$$\mathcal{O}_{\tilde{\mathrm{X}}}(-k+1) \to \mathcal{O}_{\tilde{\mathrm{X}}}(-k) \to (i_\mathrm{D})_*(\mathcal{O}_\mathrm{D}(-k))$$

for each $1 \leqslant k \leqslant n-1$. Taking $k = 1$ we deduce that $\mathcal{O}_{\tilde{X}}(-1) \in \mathbf{D}$. Continuing recursively we find that $\mathcal{O}_{\tilde{X}}(-k) \in \mathbf{D}$ for all $1 \leqslant k \leqslant n-1$.

Now let $\mathcal{F} \in \mathrm{Qcoh}(\tilde{X})$. Denote by $\mathcal{G}_0 \in \mathrm{Qcoh}(\tilde{X})$ the cofibre of the co-unit $p^*p_*(\mathcal{F}) \to \mathcal{F} \to \mathcal{G}_0$; note that $\mathcal{G}_0$ is right orthogonal to $\mathbf{D}(0)$. For $1 \leqslant k \leqslant n-1$ define $\mathcal{G}_k$ recursively by the exact triangles

$$(i_D)_*(q^*q_*((i_D)^!(\mathcal{G}_{k-1}) \otimes \mathcal{O}(k)) \otimes \mathcal{O}(-k)) \xrightarrow{\text{counit}} \mathcal{G}_{k-1} \to \mathcal{G}_k.$$

Just as in the proof of Theorem 3.2.1, a simple induction argument shows that each $\mathcal{G}_k$ is right orthogonal to all of the subcategories $\mathbf{D}(0), \ldots, \mathbf{D}(k-1)$. We now claim that $\mathcal{G}_{n-1}$ is zero; it will follow by recursion that $\mathcal{F}$ belongs to $\mathbf{D}$, as desired.

Since the objects $\mathcal{G}_k$ are stable under base change, we may use fpqc descent and base change to assume that X is affine. Moreover we may assume that $i : Z \hookrightarrow X$ fits in a cartesian square of the form (2.1.a). By [Kha18, 3.3.6], $p : \tilde{X} \to X$ factors through a quasi-smooth closed immersion $i' : \tilde{X} \hookrightarrow \mathbf{P}^{n-1}_X$. Recall from Lecture 7 that there is a canonical isomorphism $\varinjlim_{J \subsetneq [n-1]} \mathcal{O}(\#J) \simeq \mathcal{O}(n)$ in $\mathrm{Qcoh}(\mathbf{P}^{n-1}_X)$. Applying $(i')^*$, we get $\varinjlim_{J \subsetneq [n-1]} \mathcal{O}_{\tilde{X}}(\#J) \simeq \mathcal{O}_{\tilde{X}}(n)$ in $\mathrm{Qcoh}(\tilde{X})$. In particular, every $\mathcal{O}_{\tilde{X}}(k)$ belongs to $\mathbf{D}$ for all $k \in \mathbf{Z}$. Recall also that we may find a map $\bigoplus_\alpha \mathcal{O}(d_\alpha)[n_\alpha] \to i'_*(\mathcal{G}_{n-1})$ which is surjective on all homotopy groups. By adjunction this corresponds to a map $\bigoplus_\alpha \mathcal{O}(d_\alpha)[n_\alpha] \to \mathcal{G}_{n-1}$ (which is also surjective on homotopy groups). But the source belongs to $\mathbf{D}$, and the target is right orthogonal to $\mathbf{D}$, so this map is null-homotopic. Thus $\mathcal{G}_{n-1}$ is zero.

4.4. *Proof of Theorem C.* We now deduce Theorem C from Theorem 4.3.1. First note that the fully faithful functor $\mathcal{F} \mapsto p^*(\mathcal{F})$ of Theorem 4.3.1(i) preserves perfect complexes and therefore restricts to a fully faithful functor $\mathrm{Perf}(X) \to \mathrm{Perf}(\mathrm{Bl}_{Z/X})$. This shows Theorem C(i).

Similarly, part (ii) follows from the fact that the functors q^* and $(i_D)_*$ preserve perfect complexes. For the latter, this is because i_D is of finite tor-amplitude [Lur16, Thm. 6.1.3.2].

For part (iii) we argue again as in the proof of Theorem 4.3.1(iv). The point is that if $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{Bl}_{Z/X})$ is perfect, then so is each $\mathcal{G}_m \in \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$, since q^*, q_*, $(i_D)_*$ and $(i_D)^!$ all preserve perfect complexes. For the latter this follows from Proposition 4.2.1.

4.5. *Blow-up formula.*

4.5.1. By Theorem C and Lemma 2.3.3 we get:

Corollary 4.5.2. *Let* E *be an additive invariant. Then there is a canonical isomorphism*

$$\mathrm{E}(\mathrm{Bl}_{Z/X}) \simeq \mathrm{E}(X) \oplus \bigoplus_{k=1}^{n-1} \mathrm{E}(Z),$$

4.5.3. *Proof of Theorem A.* Combine Corollaries 4.5.2 and 3.4.1 (with $\mathcal{E} = \mathcal{N}_{Z/X}$).

5. The cdh topology

5.1. *A cdh descent criterion.* Recall the following notion due to Voevodsky [Voe10]:

Definition 5.1.1. Suppose given a commutative square of quasi-compact quasi-separated algebraic spaces

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ Z & \xrightarrow{i} & X \end{array} \qquad (5.1.a)$$

We say that (5.1.a) is a *proper cdh square*, or *abstract blow-up square*, if i is a closed immersion, p is proper and induces an isomorphism $p^{-1}(\mathrm{X}-\mathrm{Z})_{\mathrm{red}} \simeq (\mathrm{X}-\mathrm{Z})_{\mathrm{red}}$, and the square is cartesian.

Voevodsky's *cdh topology* is generated by Nisnevich squares and proper cdh squares. In this section we will prove Theorem D, which asserts that homotopy invariant K-theory satisfies cdh descent. To prove it we will begin by showing that the class of proper cdh squares is generated by quasi-smooth blow-up squares (2.1.b) and *closed squares*, i.e., cartesian squares of the form

$$\begin{tikzcd} \mathrm{Z}\cap\mathrm{Z}' \arrow[r] \arrow[d] & \mathrm{Z}' \arrow[d, "i'"] \\ \mathrm{Z} \arrow[r, "i"] & \mathrm{X} \end{tikzcd}$$

where i and i' are closed immersions, $\mathrm{Z}\cap\mathrm{Z}'$ denotes the (classical) fibred product $\mathrm{Z}\times_{\mathrm{X}}\mathrm{Z}'$, and the morphism $\mathrm{Z}\sqcup\mathrm{Z}' \to \mathrm{X}$ is surjective on underlying topological spaces.

Theorem 5.1.2. *Let* Alg *be the category of qcqs algebraic spaces. Then a presheaf* $\mathcal{F}$ *on* Alg *satisfies cdh descent if and only if it satisfies Nisnevich descent, closed descent, and quasi-smooth blow-up descent.*

Remark 5.1.3. Given any class of commutative squares of algebraic spaces, we say that a presheaf satisfies *descent* for this class if it sends all such squares to homotopy cartesian squares, and the empty scheme to a terminal object.

Remark 5.1.4. In order to make sense of the quasi-smooth blow-up descent condition for a presheaf $\mathcal{F}$ on Alg, we tacitly extend $\mathcal{F}$ to the ∞-category of derived algebraic spaces by simply setting $\Gamma(\mathrm{X},\mathcal{F}) = \Gamma(\mathrm{X}_{\mathrm{cl}},\mathcal{F})$ for every derived algebraic space X. In other words, the condition is that for every quasi-smooth blow-up square as in (2.1.b), the induced square

$$\begin{tikzcd} \Gamma(\mathrm{X}_{\mathrm{cl}},\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(\mathrm{Z}_{\mathrm{cl}},\mathcal{F}) \arrow[d] \\ \Gamma((\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}})_{\mathrm{cl}},\mathcal{F}) \arrow[r] & \Gamma(\mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}|_{\mathrm{Z}_{\mathrm{cl}}})) \end{tikzcd}$$

is homotopy cartesian. Note that it suffices to consider only classical X, as $\mathcal{F}$ is invariant under passing to the derived base change of the square along $\mathrm{X}_{\mathrm{cl}} \to \mathrm{X}$.

Remark 5.1.5. There are a few variants of Theorem 5.1.2 with the same proof. For example, let $\mathcal{F}$ be a presheaf defined on the site of classical qcqs *schemes*. Then $\mathcal{F}$ satisfies proper cdh descent if and only if it satisfies Zariski descent, closed descent, and quasi-smooth blow-up descent.

5.2. *Proof of Theorem 5.1.2.* It is slightly more natural (though not strictly necessary) to formulate Theorem 5.1.2 on the bigger site of qcqs derived algebraic spaces. In fact, as far as the cdh topology is concerned, one can freely pass between the two sites, for reasons we explain presently.

5.2.1. Let DAlg denote the ∞-category of qcqs derived algebraic spaces. Recall that Alg embeds into DAlg as a full subcategory, and the inclusion admits a right adjoint v : DAlg $\to$ Alg given by the assignment $\mathrm{X} \mapsto \mathrm{X}_{\mathrm{cl}}$. It follows tautologically that we may regard the ∞-category of presheaves on Alg as a left Bousfield localization of the ∞-category of presheaves on DAlg:

Proposition 5.2.2. *The functor of restriction along v induces a fully faithful embedding of the ∞-category of presheaves on* Alg *into the ∞-category of presheaves on* DAlg*. Its essential image is the full subcategory spanned by presheaves $\mathcal{F}$ satisfying the property that for every derived algebraic spaces* X*, the canonical map*

$$\Gamma(\mathrm{X},\mathcal{F}) \to \Gamma(\mathrm{X}_{\mathrm{cl}},\mathcal{F})$$

is invertible. Moreover, this embedding admits a left adjoint, given by left Kan extension along v.

5.2.3. On the site DAlg, we define a *proper cdh square*, resp. *closed square*, to be a commutative square that induces a proper cdh square, resp. closed square, on underlying classical algebraic spaces. Every quasi-smooth blow-up square (2.1.b) is a proper cdh square in this sense. The class of *cdh squares* is as usual the union of the classes of Nisnevich squares (Definition 2.3.4) and proper cdh squares. Note that the square

$$\begin{tikzcd} \varnothing \arrow[r, equal] \arrow[d] & \varnothing \arrow[d] \\ \mathrm{X}_{\mathrm{cl}} \arrow[r] & \mathrm{X} \end{tikzcd}$$

is a closed square for any qcqs derived algebraic space X, so the canonical map $\Gamma(\mathrm{X}, \mathcal{F}) \to \Gamma(\mathrm{X}_{\mathrm{cl}}, \mathcal{F})$ is invertible as soon as $\mathcal{F}$ satisfies closed descent. For that reason, the discussion in this section takes place entirely in the essential image of the functor described in Proposition 5.2.2. All in all we see that Theorem 5.1.2 is equivalent to the following statement:

Theorem 5.2.4. *Let* DAlg *be the ∞-category of qcqs derived algebraic spaces. Then a presheaf $\mathcal{F}$ on* DAlg *satisfies cdh descent if and only if it satisfies Nisnevich descent, closed descent, and descent by quasi-smooth blow-ups.*

Proof. Since Nisnevich squares, closed squares, and quasi-smooth blow-up squares are all cdh squares, the conditions are clearly necessary. Conversely suppose that $\mathcal{F}$ satisfies the conditions and consider a proper cdh square Q of qcqs derived algebraic spaces

$$\begin{tikzcd} \mathrm{E} \arrow[r] \arrow[d] & \mathrm{Y} \arrow[d, "p"] \\ \mathrm{Z} \arrow[r, "i"] & \mathrm{X}. \end{tikzcd}$$

It will suffice to show that the induced square $\Gamma(\mathrm{Q}, \mathcal{F})$ is homotopy cartesian. By closed descent, we may as well replace Q by the underlying square of classical algebraic spaces. A well-known argument using Raynaud–Gruson's technique of *platification par éclatements* [RG71, I, Cor. 5.7.12] allows one to reduce to the case where $\mathrm{Y} = \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ is the blow-up of X centred in Z (and $\mathrm{E} = \mathbf{P}(\mathcal{C}_{\mathrm{Z}/\mathrm{X}})$ is the projectivized normal cone). If Z is *regularly* immersed then the square is a quasi-smooth blow-up square and so we are already done. In general we argue as follows. By Nisnevich descent we may assume that X is affine, say $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$. Any choice of generators $f_1, \ldots, f_n \in \mathrm{A}$ for the ideal defining Z gives rise to a quasi-smooth derived scheme $\tilde{\mathrm{Z}}$, defined as the derived zero-locus of the f_i, such that $\tilde{\mathrm{Z}}_{\mathrm{cl}} = \mathrm{Z}$. Now the square Q factors as follows:

$$\begin{tikzcd} \Gamma(\mathrm{X}, \mathcal{F}) \arrow[r] \arrow[d] & \Gamma(\tilde{\mathrm{Z}}, \mathcal{F}) \arrow[r, "\sim"] \arrow[d] & \Gamma(\mathrm{Z}, \mathcal{F}) \arrow[ddl] \\ \Gamma(\mathrm{Bl}_{\tilde{\mathrm{Z}}/\mathrm{X}}, \mathcal{F}) \arrow[r] \arrow[d] & \Gamma(\mathbf{P}(\mathcal{N}_{\tilde{\mathrm{Z}}/\mathrm{X}}), \mathcal{F}) \arrow[d] & \\ \Gamma(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}, \mathcal{F}) \arrow[r] & \Gamma(\mathbf{P}(\mathcal{C}_{\mathrm{Z}/\mathrm{X}}), \mathcal{F}) & \end{tikzcd}$$

The upper square is induced by a quasi-smooth blow-up square, hence is cartesian. The lower square is induced by a closed square, hence is also cartesian. The map $\Gamma(\tilde{\mathrm{Z}}, \mathcal{F}) \to \Gamma(\mathrm{Z}, \mathcal{F})$ is also invertible because of closed descent. Therefore it follows that the outer composite square is also cartesian, as claimed. □

5.3. *Homotopy invariant K-theory.*

5.3.1. For any qcqs algebraic space X, its homotopy invariant K-theory spectrum is given by the formula

$$\Gamma(\mathrm{X}, \mathrm{KH}) = \varinjlim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{K}(\mathrm{X} \times \mathbf{A}^n).$$

That is, $\Gamma(\mathrm{X}, \mathrm{KH})$ is the geometric realization of the simplicial diagram $\mathrm{K}(\mathrm{X}\times\mathbf{A}^\bullet)$, where $\mathbf{A}^\bullet$ is regarded as a cosimplicial scheme in the usual way (see e.g. [MV99, p. 45]). This extends the usual definition [Wei89, TT90], and is a way to formally impose the property of $\mathbf{A}^1$-homotopy invariance: for any qcqs algebraic space X, the projection $p : \mathrm{X} \times \mathbf{A}^1 \to \mathrm{X}$ induces an isomorphism of spectra

$$p^* : \Gamma(\mathrm{X}, \mathrm{KH}) \to \Gamma(\mathrm{X} \times \mathbf{A}^1, \mathrm{KH}).$$

5.3.2. The above definition is equally sensible when X is a qcqs *derived* algebraic space, and defines a presheaf of spectra $\mathrm{KH} : (\mathrm{DAlg})^{\mathrm{op}} \to \mathrm{Spt}$. This construction was studied in [CK17] in the more exotic setting of *spectral* algebraic geometry, but the proofs apply *mutatis mutandis* also in the derived setting. We now prove Theorem D:

5.3.3. *Proof of Theorem D.* We apply the criterion of Theorem 5.2.4. Since the presheaf $\mathrm{K} : (\mathrm{DAlg})^{\mathrm{op}} \to \mathrm{Spt}$ already satisfies Nisnevich descent and quasi-smooth blow-up descent, the same holds for KH in view of the formula (5.3.1). In more detail, given a quasi-smooth closed immersion $\mathrm{Z} \to \mathrm{X}$, denote by $\tilde{\mathrm{X}} = \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ its blow-up and by $\mathrm{D} = \mathbf{P}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$ the exceptional divisor. Then for every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ we have a quasi-smooth blow-up square

$$\begin{tikzcd} \mathrm{D} \times \mathbf{A}^n \arrow[r] \arrow[d] & \tilde{\mathrm{X}} \times \mathbf{A}^n \arrow[d] \\ \mathrm{Z} \times \mathbf{A}^n \arrow[r] & \mathrm{X} \times \mathbf{A}^n, \end{tikzcd}$$

and by Theorem A we have cartesian squares

$$\begin{tikzcd} \mathrm{K}(\mathrm{X} \times \mathbf{A}^n) \arrow[r] \arrow[d] & \mathrm{K}(\mathrm{Z} \times \mathbf{A}^n) \arrow[d] \\ \mathrm{K}(\tilde{\mathrm{X}} \times \mathbf{A}^n) \arrow[r] & \mathrm{K}(\mathrm{D} \times \mathbf{A}^n). \end{tikzcd}$$

We conclude by passing to the colimit over n. Exactly the same argument works for Nisnevich squares.

It remains only to show that KH satisfies *closed* descent. It was proven in [CK17] that for any derived algebraic space X, the canonical map $\mathrm{KH}(\mathrm{X}) \to \mathrm{KH}(\mathrm{X}_{\mathrm{cl}})$ is invertible. Therefore it will suffice to show that KH sends closed squares of *classical* algebraic spaces to homotopy cartesian squares. By Nisnevich descent, we may also restrict our attention to closed squares of *affine* classical schemes. Now this is easy, see [TT90, Exer. 9.11(f)] or [Wei89, Cor. 4.10].

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Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

E-mail address: adeel.khan@mathematik.uni-regensburg.de