ORLOV'S THEOREM VIA NONCOMMUTATIVE MOTIVES

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Abstract

In [12] Orlov showed that the derived category of a smooth projective variety determines its rational Chow motive up to Tate twists. In this note we show how this result follows formally from the theory of noncommutative motives developed by Tabuada.

§ 1. INTRODUCTION

An important invariant of a scheme X is its bounded derived category $\mathbf{D}(X)$ of coherent sheaves. It has been studied for example by Beilinson [2], Mukai [10], Bondal-Orlov [4], and Bridgeland [5]. Lunts-Orlov [9] showed that for a quasi-projective scheme X over a commutative ring k, there exists a unique dg-category $\underline{\mathbf{D}}(X)$ whose homotopy category $H^0(\underline{\mathbf{D}}(X))$ is equivalent to $\mathbf{D}(X)$ (we say that $\underline{\mathbf{D}}(X)$ is the unique dg-enhancement of $\mathbf{D}(X)$). Further, results of Orlov [11] and Toën [17] imply that, for smooth proper schemes X over a field k, any triangulated equivalence $\mathbf{D}(X) \xrightarrow{\sim} \mathbf{D}(Y)$ lifts to an equivalence on the dg-enhancements; this will be recalled in section §2. In section §3 we briefly review the theory of noncommutative motives as developed by Tabuada [15], and in the final section we apply it to recover a theorem of Orlov [12] stating that the derived category determines the rational Chow motive up to Tate twists.

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Throughout the note, we will fix a field k and work in the category **SmProj** of smooth projective schemes over k. For $X, Y \in \mathbf{SmProj}$, we will write $X \times Y$ instead of $X \times_k Y$ for the fibred product over k.

§ 2. FUNCTORIAL ENHANCEMENT

2.1. Recall that a *dg-category* (*differential graded category*) over k is a category enriched over the symmetric monoidal category of complexes of k-modules. We will write **DGCat** for the category of small dg-categories over k. Given a dg-category $\mathscr{A}$, we will write $H^0(\mathscr{A})$ for its *homotopy category*, given by taking the zeroth cohomologies of all the mapping complexes. By a *dg-enhancement* of a triangulated category $\mathscr{A}$ we mean a dg-category whose homotopy category is equivalent to $\mathscr{A}$. We refer the reader to the introductions of Toën [18] or Keller [7].

Recall that there is a natural notion of equivalence of dg-categories, called *quasi-equivalence*. There is a model structure on **DGCat** where the weak equivalences are the quasi-equivalences (Tabuada [14]), and we let $\mathbf{DGCat}[\mathscr{W}_{\mathrm{dg}}^{-1}]$ denote the associated homotopy category, that is to say, the localization of **DGCat** at the class of quasi-equivalences.

2.2. For two schemes $\mathrm{X}, \mathrm{Y} \in \mathbf{SmProj}$, a *derived correspondence* between X and Y is an object of the derived category $\mathbf{D}(\mathrm{X} \times \mathrm{Y})$. Given derived correspondences $\mathscr{E}^\bullet \in \mathbf{D}(\mathrm{X} \times \mathrm{Y})$ and $\mathscr{E}'^\bullet \in \mathbf{D}(\mathrm{Y} \times \mathrm{Z})$, one defines their composite as the complex

$$\mathscr{E}'^\bullet \circ \mathscr{E}^\bullet = \mathbf{R}(p_{\mathrm{XZ}})_*(\mathbf{L}(p_{\mathrm{XY}})^*(\mathscr{E}^\bullet) \otimes^{\mathbf{L}} \mathbf{L}(p_{\mathrm{YZ}})^*(\mathscr{E}'^\bullet))$$

in $\mathbf{D}(\mathrm{X} \times \mathrm{Z})$, where p_{XY}, p_{XZ} and p_{YZ} are the projections from $\mathrm{X} \times \mathrm{Y} \times \mathrm{Z}$. This defines a category $\mathbf{SmProj}^{\mathrm{cor}}$ where morphisms are isomorphism classes of derived correspondences.

2.3. Let **TriCat** denote the category of small triangulated categories and isomorphism classes of triangulated functors. There is a canonical functor

$$\mathbf{D} : \mathbf{SmProj}^{\mathrm{cor}} \to \mathbf{TriCat} \tag{2.3.1}$$

which maps a scheme $\mathrm{X} \in \mathbf{SmProj}$ to $\mathbf{D}(\mathrm{X})$ and a derived correspondence $\mathscr{E}^\bullet \in \mathbf{D}(\mathrm{X} \times \mathrm{Y})$ to the triangulated functor $\mathbf{D}(\mathrm{X}) \to \mathbf{D}(\mathrm{Y})$ defined by

$$\mathbf{D}(\mathscr{E}^\bullet) := \mathbf{R}(p_{\mathrm{Y}})_*(\mathscr{E}^\bullet \otimes^{\mathbf{L}} \mathbf{L}(p_{\mathrm{X}})^*(-)),$$

where p_{X} and p_{Y} are the respective projections from $\mathrm{X} \times \mathrm{Y}$. The functoriality of this construction was shown by Mukai [10]. $\mathbf{D}(\mathscr{E}^\bullet)$ is called the functor *represented* by $\mathscr{E}^\bullet$, or the *Fourier-Mukai functor* associated to $\mathscr{E}^\bullet$.

2.4. By Toën's representability theorem ([17], Theorem 8.15), there are canonical bifunctorial isomorphisms of sets

$$\mathrm{Iso}(\mathbf{D}(\mathrm{X} \times \mathrm{Y})) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{DGCat}[\mathscr{W}_{\mathrm{dg}}^{-1}]}(\underline{\mathbf{D}}(\mathrm{X}), \underline{\mathbf{D}}(\mathrm{Y})).$$

Writing $\mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}]$ for the full subcategory of $\mathbf{DGCat}[\mathscr{W}_{\mathrm{dg}}^{-1}]$ spanned by the dg-categories $\underline{\mathbf{D}}(\mathrm{X})$ for $\mathrm{X} \in \mathbf{SmProj}$, one gets a canonical equivalence of categories

$$\mathbf{SmProj}^{\mathrm{cor}} \xrightarrow{\sim} \mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}] \tag{2.4.1}$$

which is given on objects by $\mathrm{X} \rightsquigarrow \underline{\mathbf{D}}(\mathrm{X})$.

2.5. By Orlov's representability theorem ([11], Theorem 2.2), every fully faithful triangulated functor $\mathbf{D}(\mathrm{X}) \to \mathbf{D}(\mathrm{Y})$ is represented by some derived correspondence $\mathscr{E}^\bullet \in \mathbf{D}(\mathrm{X} \times \mathrm{Y})$ which is unique up to isomorphism.

By abuse of notation let $\mathbf{SmProj}[\mathscr{W}_{\mathrm{tri}}^{-1}] \subset \mathbf{TriCat}$ denote the full subcategory spanned by triangulated categories of the form $\mathbf{D}(\mathrm{X})$ for $\mathrm{X} \in \mathbf{SmProj}$. Let $\mathbf{SmProj}[\mathscr{W}_{\mathrm{tri}}^{-1}]_0 \subset \mathbf{SmProj}[\mathscr{W}_{\mathrm{tri}}^{-1}]$ denote the (nonfull) subcategory where the morphisms are only the fully faithful functors. We have a canonical functor

$$\mathbf{SmProj}[\mathscr{W}_{\mathrm{tri}}^{-1}]_0 \longrightarrow \mathbf{SmProj}^{\mathrm{cor}} \tag{2.5.1}$$

which is *faithful* (but not full).

2.6. Define the *enhancement functor* $\varepsilon : \mathbf{SmProj}[\mathscr{W}_{\mathrm{tri}}^{-1}]_0 \to \mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}]$ as the composite of the functor $\mathbf{SmProj}[\mathscr{W}_{\mathrm{tri}}^{-1}]_0 \to \mathbf{SmProj}^{\mathrm{cor}}$ (2.5.1) with the equivalence $\mathbf{SmProj}^{\mathrm{cor}} \xrightarrow{\sim} \mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}]$ (2.4.1).

$$\begin{array}{ccc} & \mathbf{SmProj}^{\mathrm{cor}} & \\ \nearrow & & \overset{\sim}{\searrow} \\ \mathbf{SmProj}[\mathscr{W}_{\mathrm{tri}}^{-1}]_0 & \xrightarrow{\quad\varepsilon\quad} & \mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}] \end{array}$$

This associates to the triangulated category $\mathbf{D}(\mathrm{X})$ its dg-enhancement $\underline{\mathbf{D}}(\mathrm{X})$. Though it is not fully faithful, note that it is conservative (i.e. reflects isomorphisms).

2.7. ***Theorem*** *(Orlov, Toën).* — *Let* $\mathrm{X}, \mathrm{Y} \in \mathbf{SmProj}$ *be smooth projective schemes over a field* k*. The three conditions*

- *(i)* X *and* Y *are isomorphic in* $\mathbf{SmProj}^{\mathrm{cor}}$*.*
- *(ii) The triangulated categories* $\mathbf{D}(\mathrm{X})$ *and* $\mathbf{D}(\mathrm{Y})$ *are equivalent.*
- *(iii) The dg-categories* $\underline{\mathbf{D}}(\mathrm{X})$ *and* $\underline{\mathbf{D}}(\mathrm{Y})$ *are quasi-equivalent.*

are equivalent.

This is an immediate consequence of the existence of the above functors.

§ 3. NONCOMMUTATIVE MOTIVES

3.1. Let **NCSpc** be the category of noncommutative spaces, i.e. the full subcategory of $\mathbf{DGCat}[\mathscr{W}_{\mathrm{dg}}^{-1}]$ spanned by smooth proper pretriangulated dg-categories. The category **NCMot** of *noncommutative motives* is the karoubian envelope of the category with the same objects as **NCSpc** and where morphisms are Grothendieck groups of internal homs; see Tabuada [15]. The canonical functor

$$\mathrm{U} : \mathbf{NCSpc} \longrightarrow \mathbf{NCMot} \tag{3.1.1}$$

is the *universal additive invariant*, i.e. the universal functor sending semi-orthogonal decompositions (of the homotopy category) to direct sums (in some additive category). See (*loc. cit.*, Theorem 4.2).

3.2. For $X \in \mathbf{SmProj}$, the dg-category $\underline{\mathbf{D}}(X)$ is smooth and proper, and hence is a noncommutative space. In particular $\mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}]$ is a full subcategory of $\mathbf{NCSpc}$. The noncommutative motive of X, which we will denote NM(X), is defined as the noncommutative motive of its associated noncommutative space $\underline{\mathbf{D}}(X)$.

3.3. ***Corollary.*** — *Let* $X, Y \in \mathbf{SmProj}$ *be smooth projective schemes over a field* k*. If the triangulated categories* $\mathbf{D}(X)$ *and* $\mathbf{D}(Y)$ *are equivalent, then the noncommutative motives* NM(X) *and* NM(Y) *are isomorphic.*

Proof. — The enhancement functor $\varepsilon : \mathbf{SmProj}[\mathscr{W}_{\mathrm{tri}}^{-1}]_0 \to \mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}]$ lifts a triangulated equivalence $\mathbf{D}(X) \xrightarrow{\sim} \mathbf{D}(Y)$ to an isomorphism $\underline{\mathbf{D}}(X) \xrightarrow{\sim} \underline{\mathbf{D}}(Y)$ in $\mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}]$, and therefore in $\mathbf{NCSpc}$. Hence $U : \mathbf{NCSpc} \to \mathbf{NCMot}$ gives an isomorphism $\mathrm{NM}(X) \xrightarrow{\sim} \mathrm{NM}(Y)$.

3.4. Let **Spt** denote the category of spectra. By work of Blumberg-Mandell [3], Keller [6], Schlichting [13] and Thomason-Trobaugh [16] (cf. Tabuada [15]), each of the following can be defined as functors $\mathbf{NCSpc} \to \mathbf{Spt}$ and are additive invariants in the above sense:

(i) algebraic K-theory,
(ii) cyclic homology,
(iii) topological cyclic homology,
(iv) Hochschild homology,
(v) topological Hochschild homology.

Universality of the functor $U : \mathbf{NCSpc} \to \mathbf{NCMot}$ (3.1.1) gives immediately the following corollary.

Corollary. — *Let* $X, Y \in \mathbf{SmProj}$ *be smooth projective schemes over a field* k *and let* H_* *denote one of the above functors. If the triangulated categories* $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ *are equivalent, then* $H_*(X)$ *and* $H_*(Y)$ *are isomorphic.*

§ 4. CHOW MOTIVES

4.1. Let $\mathscr{A}$ be an additive category and let $T : \mathscr{A} \xrightarrow{\sim} \mathscr{A}$ be an auto-equivalence. The *orbit category of* $\mathscr{A}$ *with respect to* T is the category $\mathscr{A}/T$ whose objects are the same as those of $\mathscr{A}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathscr{A}/T}(X, Y) = \bigoplus_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathscr{A}}(X, T^i(Y))$$

for all $\mathrm{X}, \mathrm{Y} \in \mathscr{A}$. The law of composition is defined as follows: for two morphisms $f = (f^i)_i : \mathrm{X} \to \mathrm{Y}$ and $g = (g^j)_j : \mathrm{Y} \to \mathrm{Z}$ in $\mathscr{A}/\mathrm{T}$, the composite $g \circ f$ is defined as the morphism whose k-th component is the sum

$$(g \circ f)^k = \sum_{i+j=k} \mathrm{T}^i(g^j) \circ f^i.$$

Let $\pi_{\mathscr{A}/\mathrm{T}} : \mathscr{A} \to \mathscr{A}/\mathrm{T}$ denote the canonical functor which maps a morphism f to the morphism which is f in the zeroth component and 0 everywhere else.

The following lemma is an easy exercise.

Lemma. — *Let $\mathscr{A}$ be an additive category and $\mathrm{T} : \mathscr{A} \xrightarrow{\sim} \mathscr{A}$ an additive auto-equivalence. Suppose that $\mathscr{A}$ admits arbitrary direct sums. Then the projection functor $\pi = \pi_{\mathscr{A}/\mathrm{T}} : \mathscr{A} \to \mathscr{A}/\mathrm{T}$ admits a right adjoint*

$$\tau = \tau_{\mathscr{A}/\mathrm{T}} : \mathscr{A}/\mathrm{T} \longrightarrow \mathscr{A}$$

which maps an object X *to the direct sum of all the objects* $\mathrm{T}^i(\mathrm{X})$ $(i \in \mathbf{Z})$.

4.2. Let $\mathbf{ChMot}(\mathbf{Q})$ be the category of Chow motives with rational coefficients, and $\mathrm{M} : \mathbf{SmProj}^{\mathrm{op}} \to \mathbf{ChMot}(\mathbf{Q})$ the canonical functor (see André [1]). Let $\mathbf{Q}(1) \in \mathbf{ChMot}(\mathbf{Q})$ denote the Tate motive, and recall that the functor $\mathbf{Q}(1) \otimes - : \mathbf{ChMot}(\mathbf{Q}) \to \mathbf{ChMot}(\mathbf{Q})$ is an auto-equivalence. Let $\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1)$ denote the associated orbit category; we call this the *category of Chow motives modulo Tate twists*. We will write $\mathrm{M}(i) = \mathrm{M} \otimes \mathbf{Q}(i) = \mathrm{M} \otimes \mathbf{Q}(1)^{\otimes i}$ for a motive $\mathrm{M} \in \mathbf{ChMot}(\mathbf{Q})$.

Let $\mathbf{DM}_{\mathrm{gm}}(\mathbf{Q})$ be the triangulated category of geometric motives of Voevodsky and recall that there is a canonical functor

$$\mathbf{ChMot}(\mathbf{Q}) \longrightarrow \mathbf{DM}_{\mathrm{gm}}(\mathbf{Q}) \tag{4.2.1}$$

which sends the Tate motive $\mathbf{Q}(1)$ to $\mathbf{Q}(-1)[-2]$ (where $[n]$ denotes the n-fold composition of the translation functor); see (*loc. cit.*, Théorème 18.3.1.1). Hence one gets an induced functor $\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \longrightarrow \mathbf{DM}_{\mathrm{gm}}(\mathbf{Q})/\mathbf{Q}(-1)[-2]$ on the orbit categories and by the above lemma (4.1), as the triangulated category $\mathbf{DM}_{\mathrm{gm}}(\mathbf{Q})$ admits arbitrary direct sums, one gets a canonical functor

$$\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \longrightarrow \mathbf{DM}_{\mathrm{gm}}(\mathbf{Q})/\mathbf{Q}(-1)[-2] \xrightarrow{\tau} \mathbf{DM}_{\mathrm{gm}}(\mathbf{Q}) \tag{4.2.2}$$

which maps a motive to the direct sum of all its Tate twists:

$$\mathrm{M} \rightsquigarrow \bigoplus_{i \in \mathbf{Z}} \mathrm{M}(i)[2i].$$

4.3. Kontsevich [8] noted that there is a canonical fully faithful functor

$$\nu : \mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \hookrightarrow \mathbf{NCMot}(\mathbf{Q})$$

which is given on morphisms by

$$\bigoplus_i \mathrm{CH}^i(\mathrm{X} \times \mathrm{Y}, \mathbf{Q}) \xrightarrow{\sim} \mathrm{K}_0(\mathrm{X} \times \mathrm{Y}) \otimes \mathbf{Q},$$

the inverses of the Grothendieck-Riemann-Roch isomorphisms $\mathrm{ch}(-) \cdot \sqrt{\mathrm{td}_{\mathrm{X}\times\mathrm{Y}}}$. Note that this fits into a commutative diagram

$$\begin{array}{ccc} \mathbf{SmProj}[\mathscr{W}_{\mathrm{dg}}^{-1}] & \hookrightarrow & \mathbf{NCSpc} \\ \downarrow{\scriptstyle \mathrm{M}} & & \downarrow{\scriptstyle \mathrm{U}} \\ \mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) & \xhookrightarrow{\nu} & \mathbf{NCMot}(\mathbf{Q}). \end{array}$$

4.4. ***Theorem*** *(Orlov [12]).* — *Let* $\mathrm{X}, \mathrm{Y} \in \mathbf{SmProj}$ *be smooth projective schemes over a field* k*. If the triangulated categories* $\mathbf{D}(\mathrm{X}) \simeq \mathbf{D}(\mathrm{Y})$ *are equivalent, then there is an isomorphism*

$$\bigoplus_{i\in\mathbf{Z}} \mathrm{M}(\mathrm{X})(i)[2i] \xrightarrow{\sim} \bigoplus_{j\in\mathbf{Z}} \mathrm{M}(\mathrm{Y})(j)[2j]$$

in the triangulated category $\mathbf{DM}_{\mathrm{gm}}(\mathbf{Q})$ *of geometric motives.*

Proof. — Since $\mathrm{NM}(\mathrm{X}) \simeq \mathrm{NM}(\mathrm{Y})$ in $\mathbf{NCMot}(\mathbf{Q})$ by (3.3), and the functor $\nu : \mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \hookrightarrow \mathbf{NCMot}(\mathbf{Q})$ is fully faithful, one has $\mathrm{M}(\mathrm{X}) \simeq \mathrm{M}(\mathrm{Y})$ in $\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1)$. Then the functor $\mathbf{ChMot}(\mathbf{Q})/\mathbf{Q}(1) \to \mathbf{DM}_{\mathrm{gm}}(\mathbf{Q})$ (4.2.2) gives the desired isomorphism.

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