REGULAR RING SPECTRA AND WEIGHT STRUCTURES

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ABSTRACT. We prove a gluing result for weight structures and adjacent t-structures, in the setting of a semi-orthogonal decomposition of stable ∞-categories. We apply this to the study of regular ring spectra, by interpreting the property of regularity in terms of the existence of a t-structure adjacent to the weight structure on the category of perfect complexes. Namely, we use this to construct many examples of regular ring spectra from existing ones, as upper triangular ring spectra.

CONTENTS

1. INTRODUCTION

Let R be a left noetherian $\mathcal{E}_1$-ring spectrum. We will say that R is *regular* if every coherent left R-module is perfect. By Serre's criterion, this agrees with the usual definition when R is discrete (an Eilenberg–MacLane spectrum).

The notion was first studied in [BL14], where it was used to construct certain localization fiber sequences for algebraic K-theory. As shown there, examples of regular $\mathcal{E}_1$-ring spectra include ko, ku, $BP\langle n \rangle$, and tmf. In all the known examples, regular ring spectra are either discrete or have infinitely many non-trivial homotopy groups. According a result of J. Lurie [SAG, Lemma 11.3.3.3], this is in fact a general phenomenon, at least if we restrict our attention to $\mathcal{E}_\infty$-ring spectra. As we note here (Theorem 2.2.1), Lurie's result holds (with the same proof) more generally for $\mathcal{E}_1$-ring spectra satisfying a much weaker commutativity condition: namely, that $\pi_0(R)$ is central in $\pi_*(R)$. However, in the noncommutative case, i.e., when $\pi_0(R)$ is not central in $\pi_*(R)$, this is no longer the case: we construct a simple example of a non-discrete ring spectrum which is regular but has only finitely many homotopy groups (Sect. 2.3).

Our example is an instance of a general construction. Given ring spectra R and S, and an R-S-bimodule M, we consider the upper triangular ring spectrum

$$T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}.$$

If R and S are regular, we prove that T is regular as long as M is connective and perfect as an S-module (Theorem 3.4.5).

To prove Theorem 3.4.5 it is useful to shift perspective. Recall that, for any connective ring spectrum R, the category of perfect complexes over R is endowed with a canonical weight structure (Example 3.1.5). The condition of regularity of R can be reformulated as the existence of a t-structure on this category which is *adjacent* to the weight structure; that is, its non-negative part coincides with the non-negative part of the weight structure. Adjacent t-structures have been studied in general contexts by Bondarko (see [Bon18]).

Our main result (Theorem 3.3.3) deals with semi-orthogonal decompositions or split short exact sequences of stable ∞-categories

$$\mathcal{C} \underset{L_{\mathcal{C}}}{\overset{i_{\mathcal{C}}}{\rightleftarrows}} \mathcal{E} \underset{i_{\mathcal{D}}}{\overset{L_{\mathcal{D}}}{\rightleftarrows}} \mathcal{D}.$$

When $\mathcal{C}$ and $\mathcal{D}$ are endowed with weight structures, we will explain how to construct a weight structure on $\mathcal{E}$ extending those on $\mathcal{C}$ and $\mathcal{D}$, if the essential images of $\mathcal{D}_{w \leqslant 0}$ and $\Sigma \mathcal{C}_{w \geqslant 0}$ in $\mathcal{E}$ are orthogonal. Moreover, if there are adjacent t-structures on $\mathcal{C}$ and $\mathcal{D}$, then under a mild (and necessary) condition, we also construct an adjacent t-structure on $\mathcal{E}$. This easily specializes to Theorem 3.4.5.

2. Regular ring spectra

2.1. **Preliminaries.** Let R be a connective $\mathcal{E}_1$-ring, and write Mod_R for the category of left R-modules. We recall a few finiteness conditions from [HA, § 7.2.4].

2.1.1. The *perfect* left R-modules are those built out of R under finite (co)limits and direct summands. These span a thick subcategory $\mathrm{Mod}_R^{\mathrm{perf}} \subset \mathrm{Mod}_R$.

2.1.2. Suppose that R is *left noetherian*, i.e. that $\pi_0(R)$ is left noetherian as an ordinary ring and the homotopy groups $\pi_i(R)$ are finitely generated as left $\pi_0(R)$-modules. A left R-module M is *almost perfect* if it is bounded below and each $\pi_i(M)$ is finitely generated as a left $\pi_0(R)$-module [HA, Prop. 7.2.4.17], and *coherent* if it is almost perfect and bounded above. We write $\mathrm{Mod}_R^{\mathrm{coh}}$ for the full subcategory of Mod_R spanned by coherent left R-modules; this is also a thick subcategory by [HA, Prop. 7.2.4.11].

Remark 2.1.3. Note that the noetherian hypothesis guarantees that R is almost perfect as a left R-module. Thus if R is bounded above, then it is moreover coherent as a left R-module. It follows then that every perfect left R-module is coherent, i.e., that there is an inclusion $\mathrm{Mod}_R^{\mathrm{perf}} \subset \mathrm{Mod}_R^{\mathrm{coh}}$ when R is bounded above.

Definition 2.1.4. Let R be a left noetherian $\mathcal{E}_1$-ring. We will say that R is *regular* if every coherent left R-module is perfect.

It suffices to require that every *discrete* coherent left R-module is perfect, in view of the exact triangles

$$\Sigma^i \pi_i(M) \to \tau_{\leqslant i}(M) \to \tau_{\leqslant i-1}(M)$$

for $M \in \mathrm{Mod}_R$ and $i \in \mathbf{Z}$. Moreover, one has the following useful criterion (see [BL14, Prop. 1.3]):

Proposition 2.1.5. *Let R be a left noetherian $\mathcal{E}_1$-ring such that $\pi_0(R)$ is a regular commutative ring. Then R is regular if and only if $\pi_0(R)$ is perfect as an R-module.*

Remark 2.1.6. Let R be a left noetherian $\mathcal{E}_1$-ring. The ∞-category Mod_R admits a canonical t-structure, where the non-negative part $(\mathrm{Mod}_R)_{t\geqslant 0}$ is spanned by the connective modules. This always restricts to the full subcategories of almost perfect and coherent modules [HA, Prop. 7.2.4.18]. One might ask whether it also restricts to the full subcategory $\mathrm{Mod}_R^{\mathrm{perf}}$ of perfect R-modules. This is the case if and only if $\pi_i(M)$ belongs to $\mathrm{Mod}_R^{\mathrm{perf}}$ for any $M \in \mathrm{Mod}_R^{\mathrm{perf}}$, i.e., precisely when R is regular.

Remark 2.1.7. For every $M \in \mathrm{Mod}_R^{\mathrm{perf}}$ and $N \in \mathrm{Mod}_R^{\mathrm{coh}}$, there exists an integer n such that the mapping space $\mathrm{Mod}_R(\Sigma^n M, N)$ is contractible. Indeed, this is true for $M = \Sigma^j R$ since N is bounded above, and therefore for an arbitary M, since it is built out of $\Sigma^j R$ using finite colimits and retracts. For regular R, this says that the space $\mathrm{Mod}_R(M, N)$ has finitely many non-zero homotopy groups, for any $M, N \in \mathrm{Mod}_R^{\mathrm{perf}}$.

Examples 2.1.8. (i) If R is discrete, then it is regular if and only if it is regular in the sense of ordinary commutative algebra. This follows from Serre's criterion.

(ii) Let k be a regular commutative ring and $R = k[T]$ with T in degree 2. Then R is regular.

(iii) The $\mathcal{E}_\infty$-ring spectra ku, ko, and tmf are regular [BL14].

2.1.9. We say that R is *quasi-commutative* if, for every $x \in \pi_0(R)$ and $y \in \pi_n(R)$, the equality $xy = yx$ holds in $\pi_n(R)$. Under this assumption, we can show that regularity is stable under localizations:

Lemma 2.1.10. *Let R be a quasi-commutative connective $\mathcal{E}_1$-ring spectrum. If R is regular, then for any set $S \subset \pi_0(R)$, the localization $S^{-1}R$ is also regular.*

Proof. It will suffice to show that, for every perfect $S^{-1}R$-module N, we have $\pi_0(N) \in \mathrm{Mod}_{S^{-1}R}^{\mathrm{perf}}$. Denote by L the extension of scalars functor $\mathrm{Mod}_R^{\mathrm{perf}} \to \mathrm{Mod}_{S^{-1}R}^{\mathrm{perf}}$. Since any perfect $S^{-1}R$-module N is a retract of an object of the form $L(M)$ for some $M \in \mathrm{Mod}_R^{\mathrm{perf}}$, we may restrict our attention to $N = L(M)$. For $M \in \mathrm{Mod}_R^{\mathrm{perf}}$, it follows from [HA, Prop. 7.2.3.20] that we have

$$\pi_n(L(M)) \simeq S^{-1}\pi_n(M) \simeq \pi_n(M) \otimes_{\pi_0(R)} S^{-1}\pi_0(R).$$

This implies in particular that $\pi_0(L(M)) \simeq L(\pi_0(M))$ belongs to $\mathrm{Mod}_{S^{-1}R}^{\mathrm{perf}}$, as claimed. $\qquad\square$

2.2. **Bounded regular ring spectra.** This subsection is dedicated to the following result, due to J. Lurie; see [SAG, Lemma 11.3.3.3], where it is proven for $\mathcal{E}_\infty$-ring spectra. Here we only note that the same proof works more generally for quasi-commutative $\mathcal{E}_1$-ring spectra. Note that this is optimal in the sense that the statement is false without the quasi-commutativity assumption (see Subsect. 2.3).

Theorem 2.2.1. *Let R be a quasi-commutative connective $\mathcal{E}_1$-ring spectrum. If R is regular and has only finitely many nontrivial homotopy groups, then R is discrete.*

Proof. We can clearly assume that R is nonzero. To show that R is discrete, it will suffice to show that its localization $R_{\mathfrak{p}}$, at any prime ideal $\mathfrak{p} \subset \pi_0(R)$, is discrete. Since $R_{\mathfrak{p}}$ is regular by Lemma 2.1.10, we may replace R by $R_{\mathfrak{p}}$ and thereby assume that $\pi_0(R)$ is local. Denote by $\mathfrak{m} \subset \pi_0(R)$ the maximal ideal, and κ the residue field. Let $n \geqslant 0$ be the largest integer such that $\pi_n(R) \neq 0$. For the sake of contradiction, we will assume that $n > 0$.

Since R is noetherian, the $\pi_0(R)$-module $\pi_n(R)$ is finitely generated. Let $\mathfrak{p} \subset \pi_0(R)$ be a minimal prime ideal such that $\mathfrak{p} \in \mathrm{Supp}_{\pi_0(R)}(\pi_n(R))$. By Lemma 2.1.10, the localization $R_{\mathfrak{p}}$ is regular. Therefore, replacing R by $R_{\mathfrak{p}}$, we may assume that $\pi_n(R)$ is supported at $\mathfrak{m}$, so that in particular $\mathfrak{m} \subset \pi_0(R)$ is an associated prime ideal of $\pi_n(R)$.

Since R is regular, the coherent left R-module κ is perfect. Therefore, the following claim will yield the desired contradiction:

($*$) Let N be a nonzero connective perfect left R-module. Let $k \geqslant 0$ be the smallest natural number such that N is of tor-amplitude $\leqslant k$. Then we have $\pi_{n+k}(N) \neq 0$.

To prove ($*$) we argue by induction. In the case $k = 0$, N is flat with $\pi_0(N) \neq 0$, so that $\pi_n(N) \simeq \pi_n(R) \otimes_{\pi_0(R)} \pi_0(N) \neq 0$. Assume therefore that $k > 0$. Choose elements of $\pi_0(N)$ that induce a basis of the κ-vector space $\pi_0(N) \otimes_{\pi_0(R)} \kappa$ and consider the surjective R-module morphism $u : R^{\oplus m} \to N$ they determine. Its fiber, which we denote N', is a connective perfect left R-module fitting in an exact triangle

$$N' \xrightarrow{v} R^{\oplus m} \xrightarrow{u} N.$$

Considering the induced exact sequence of abelian groups

$$0 = \pi_{n+k}(R^{\oplus m}) \to \pi_{n+k}(N) \to \pi_{n+k-1}(N') \xrightarrow{\varphi} \pi_{n+k-1}(R^{\oplus m}),$$

it will suffice to show that φ is not injective.

Since N is of tor-amplitude $\leqslant k$, with k minimal and > 0, it follows that N' is nonzero and of tor-amplitude $\leqslant k - 1$. Therefore by the inductive hypothesis, $\pi_{n+k-1}(N') \neq 0$. If $k > 1$, then $\pi_{n+k-1}(R^{\oplus m}) = 0$, so φ is not injective. It remains to consider the case $k = 1$. In this case, the perfect left R-module N' is flat, and hence free of finite rank $r \geqslant 0$, since $\pi_0(R)$ is local. Therefore the map $v : N' \to R^{\oplus m}$ is determined up to homotopy by a matrix $\{v_{i,j}\}_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant m}$ of elements $v_{i,j} \in \pi_0(R)$. Since $\mathfrak{m} \subset \pi_0(R)$ is an associated prime of $\pi_n(R)$, we may choose a nonzero element $x \in \pi_n(R)$ such that multiplication by x kills $\mathfrak{m}$. We claim that the element $(x, x, \ldots, x) \in \pi_n(R)^{\oplus r} \simeq \pi_n(N')$ is sent by φ to zero in $\pi_n(R^{\oplus m})$. For this purpose it will suffice to show that the elements $v_{i,j} \in \pi_0(R)$ all belong to $\mathfrak{m}$. Consider the exact sequence

$$\pi_0(N')/\mathfrak{m} \xrightarrow{v} \pi_0(R^{\oplus m})/\mathfrak{m} \xrightarrow{u} \pi_0(N)/\mathfrak{m} \to 0.$$

Since the map $u : \pi_0(R^{\oplus m})/\mathfrak{m} \to \pi_0(N)/\mathfrak{m}$ is injective, it follows that $v : \pi_0(N')/\mathfrak{m} \to \pi_0(R^{\oplus m})/\mathfrak{m}$ is the zero map. In other words, the image of $v : \pi_0(N') \to \pi_0(R^{\oplus m})$ lands in $\mathfrak{m}^{\oplus m} \subset \pi_0(R)^{\oplus m}$, as desired. $\square$

2.3. A noncommutative counterexample.

We now give an example of a *noncommutative* regular ring spectrum that is bounded but not discrete. This shows that the quasi-commutativity hypothesis in Theorem 2.2.1 is necessary.

Construction 2.3.1. Let k be a field. Consider the graded ring R, concentrated in degrees 0 and 1, where $R_0 = k \times k$ and $R_1 = k$ with multiplication defined by formulas

$$(a, b) \cdot (c, d) = (ac, bd)$$

$$x \cdot (a, b) = bx$$

$$(a, b) \cdot x = ax,$$

where $(a, b), (c, d) \in R_0$, $x \in R_1$.

The graded ring R is associative but not commutative, and gives rise to a connective $\mathcal{E}_1$-ring spectrum, which we denote again by R, that is not quasi-commutative. However, it has regular π_0, and is also itself regular:

Proposition 2.3.2. *Let R be the $\mathcal{E}_1$-ring spectrum defined by Construction 2.3.1. Then R is regular.*

Proof. By Proposition 2.1.5 it suffices to show that $\pi_0(R)$ is perfect.

Denote by X the direct summand of the free R-module R corresponding to the projector $(0, 1) \in \pi_0(R) = k \times k$. We first note that $X \simeq k$ is discrete. Indeed we have by definition $\pi_0(X) = k$ and $\pi_i(X) = 0$ for $i \neq 0, 1$. The only $x \in k$ satisfying $(0, 1) \cdot x = x$ is zero, so $\pi_1(X)$ is also zero.

Since $x \cdot (0, 1) = x \in \pi_1(R)$, the element $1 \in k \simeq \pi_1(R)$ gives rise to a map $X[1] \xrightarrow{1} R$. This map induces an isomorphism $\pi_0(X) \to \pi_1(R)$, and therefore exhibits $X[1]$ as the 1-connective cover $\tau_{\geqslant 1} R$. It follows that its cofiber is an object of $\mathrm{Mod}_R^{\mathrm{perf}}$ equivalent to $\tau_{\leqslant 0} R = \pi_0(R)$. $\qquad\square$

Remark 2.3.3. In fact it is easy to see using the Künneth spectral sequence that any $\mathcal{E}_1$-ring spectrum R is regular as soon as the ring $\pi_*(R) = \bigoplus_n \pi_n(R)$ is regular. In the example above, the ring $\pi_*(R)$ is isomorphic to the ring of triangular matrices, which is regular.

3. ADJACENT STRUCTURES

In this section we recall the notion of *adjacent structures* and show its relation to regularity. An adjacent structure consists of a pair of a t-structure and a weight structure compatible in a strict sense. We begin by recalling the latter notion.

3.1. Weight structures.

Definition 3.1.1. A *weight structure w* on a stable ∞-category $\mathcal{C}$ is the data of two Karoubi closed full subcategories $\mathcal{C}_{w\geqslant 0}, \mathcal{C}_{w\leqslant 0}$ satisfying the following axioms.

(i)   *Semi-invariance with respect to translations.* We have the inclusions

$$\Sigma \mathcal{C}_{w\geqslant 0} \subset \mathcal{C}_{w\geqslant 0}, \quad \Omega \mathcal{C}_{w\leqslant 0} \subset \mathcal{C}_{w\leqslant 0}.$$

(ii)   *Orthogonality.* For any $X \in \mathcal{C}_{w\leqslant 0}$, $Y \in \mathcal{C}_{w\geqslant 0}$, we have

$$\pi_0 \mathrm{Maps}_{\mathcal{C}}(X, \Sigma Y) = 0.$$

(iii) *Weight decompositions.* For any $X \in \mathcal{C}$, there exists an exact triangle

$$w_{\leqslant 0} X \to X \to w_{\geqslant 1} X,$$

where $w_{\leqslant 0} X \in \mathcal{C}_{w\leqslant 0}$ and $w_{\geqslant 1} X \in \Sigma \mathcal{C}_{\geqslant 0}$.

3.1.2.   The *heart $\underline{Hw}$* of a weight structure is the full subcategory whose objects belong to both $\mathcal{C}_{w\geqslant 0}$ and $\mathcal{C}_{w\leqslant 0}$. A weight structure is *bounded* if any object X is an object of $\Sigma^n \mathcal{C}_{w\leqslant 0}$ and of $\Omega^n \mathcal{C}_{w\geqslant 0}$ for some n. A functor between stable ∞-categories with weight structures is *weight-exact* if it preserves both classes.

The following construction gives rise to a variety of examples of weight structures:

Proposition 3.1.3. *Let $\mathcal{A}$ be an additive ∞-category. Denote by $\widehat{\mathcal{A}}$ the stable ∞-category* $\mathrm{Fun}_{\mathrm{add}}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})$ *of additive presheaves of spectra on $\mathcal{A}$. Then the full subcategory $\widehat{\mathcal{A}}^c$ of compact objects admits a bounded weight structure w with the following properties:*

(i)   *The full subcategory $(\widehat{\mathcal{A}}^c)_{w \geqslant 0}$ is spanned by presheaves with values in connective spectra.*

(ii)   *The Yoneda embedding $\mathcal{A} \to \widehat{\mathcal{A}}$ factors through a canonical functor $\mathcal{A} \to \underline{Hw}$ which exhibits the heart $\underline{Hw}$ as the idempotent completion of $\mathcal{A}$.*

In fact, this is essentially the only example, as the following proposition shows.

Proposition 3.1.4. *Let $\mathcal{C}$ be a idempotent complete stable ∞-category endowed with a bounded weight structure. Then there is an equivalence of ∞-categories $\mathcal{C} \to \widehat{\underline{Hw}}^c$, where $\widehat{\underline{Hw}}$ denotes the construction of Proposition 3.1.3.*

Both Propositions 3.1.3 and 3.1.4 follow from [Sos17, Cor. 3.4]. The following example will be of special interest for us:

Example 3.1.5. Let R be a connective $\mathcal{E}_1$-ring spectrum and let $\mathcal{A}$ be the additive ∞-category $\mathrm{Mod}_R^{\mathrm{free}}$ of finitely generated free R-modules. Then the construction $\widehat{\mathcal{A}}^c$ of Proposition 3.1.3 is canonically equivalent to the stable ∞-category $\mathrm{Mod}_R^{\mathrm{perf}}$. It follows that $\mathrm{Mod}_R^{\mathrm{perf}}$ admits a canonical bounded weight structure w such that the non-negative part $(\mathrm{Mod}_R^{\mathrm{perf}})_{w \geqslant 0}$ is spanned by the connective perfect R-modules.

3.2. **Adjacent weight and t-structures.** Weight structures look quite similar to t-structures. However, the orthogonality conditions and the directions of the maps in the weight and t-decompositions are opposite to each other. A less direct relation between the two notions is given by the following result of Bondarko [Bon18, Thm. 0.1].

Theorem 3.2.1. *Let $\mathcal{C}$ be a stable ∞-category having all coproducts and satisfying the Brown representability theorem (e.g., compactly generated). Suppose $(\mathcal{C}_{w \geqslant 0}, \mathcal{C}_{w \leqslant 0})$ is a weight structure such that $\mathcal{C}_{w \geqslant 0}$ is closed under arbitrary coproducts. Then there exists a t-structure on $\mathcal{C}$ such that $\mathcal{C}_{t \geqslant 0} = \mathcal{C}_{w \geqslant 0}$.*

When $\mathcal{C}$ does not have all coproducts, such a t-structure may not exist. When it does, it is called an adjacent t-structure:

Definition 3.2.2. Let $\mathcal{C}$ be a stable ∞-category. Given a weight structure $(\mathcal{C}_{w \geqslant 0}, \mathcal{C}_{w \leqslant 0})$ and a t-structure $(\mathcal{C}_{t \geqslant 0}, \mathcal{C}_{t \leqslant 0})$, we say that they are *adjacent* if the equality $\mathcal{C}_{t \geqslant 0} = \mathcal{C}_{w \geqslant 0}$ holds.

Any non-regular ring spectrum provides an example of a weight structure with no adjacent t-structure:

Example 3.2.3. Let R be a left noetherian $\mathcal{E}_1$-ring spectrum and consider the canonical weight structure w on $\mathrm{Mod}_R^{\mathrm{perf}}$ (Example 3.1.5). Note that there a t-structure t on $\mathrm{Mod}_R^{\mathrm{perf}}$ is adjacent to w if and only if it is the restriction of the homotopy t-structure on Mod_R. In particular, such an adjacent t-structure exists if and only if R is regular (see Remark 2.1.6). If this t-structure is *bounded*, then R has finitely many homotopy groups and hence, by Theorem 2.2.1, is discrete.

3.3. **Gluing weight structures.** Recall the following definition from [BGT13]:

Definition 3.3.1. A *split short exact sequence* (or semi-orthogonal decomposition) of stable ∞-categories is a diagram

$$\mathcal{C} \xrightarrow[\;L_{\mathcal{C}}\;]{\;i_{\mathcal{C}}\;} \mathcal{E} \xrightarrow[\;i_{\mathcal{D}}\;]{\;L_{\mathcal{D}}\;} \mathcal{D}$$

satisfying the following conditions:

(i)   The functors $i_{\mathcal{C}}$ and $i_{\mathcal{D}}$ are fully faithful.

(ii)   The functor $L_{\mathcal{C}}$ is right adjoint to $i_{\mathcal{C}}$, and $i_{\mathcal{D}}$ is right adjoint to $L_{\mathcal{D}}$.

(iii)   The essential image of $i_{\mathcal{D}}$ is the right orthogonal to the essential image of $i_{\mathcal{C}}$.

Lemma 3.3.2. *Given a split short exact sequence of stable ∞-categories as above, the unit and counit maps of the adjunctions form an exact triangle*

$$i_{\mathcal{C}} L_{\mathcal{C}}(X) \to X \to i_{\mathcal{D}} L_{\mathcal{D}}(X).$$

Proof. Condition (iii) of the definition implies that the fiber of the map $X \to i_{\mathcal{D}} L_{\mathcal{D}}$ has the form $i_{\mathcal{C}} Y$ for some Y. Now $Y \simeq L_{\mathcal{C}} i_{\mathcal{C}} Y \to L_{\mathcal{C}} X$ is an equivalence. $\qquad\square$

Our main result is as follows:

Theorem 3.3.3. *Suppose given a split short exact sequence of stable ∞-categories:*

$$\mathcal{C} \underset{L_{\mathcal{C}}}{\overset{i_{\mathcal{C}}}{\rightleftarrows}} \mathcal{E} \underset{i_{\mathcal{D}}}{\overset{L_{\mathcal{D}}}{\rightleftarrows}} \mathcal{D}.$$

Assume that $\mathcal{C}$ and $\mathcal{D}$ admit weight structures such that $\pi_0 \mathcal{E}(i_{\mathcal{D}} X, i_{\mathcal{C}} Y) = 0$ for any $X \in \mathcal{D}_{w \leqslant 0}$ and $Y \in \Sigma \mathcal{C}_{w \geqslant 0}$. Then there exists a unique weight structure on $\mathcal{E}$ such that all the functors in the diagram are weight-exact. Moreover, if $i_{\mathcal{D}}$ has a right adjoint $R_{\mathcal{D}}$ and the weight structures on $\mathcal{C}$ and $\mathcal{D}$ admit adjacent t-structures, then so does the weight structure on $\mathcal{E}$.

We postpone the proof to Subsect. 3.5.

3.4. **Upper triangular matrix rings.** In this subsection we explain how Theorem 3.3.3 can be applied to produce new examples of regular ring spectra.

3.4.1.   Any split short exact sequence as in Definition 3.3.1 is determined by $\mathcal{C}$, $\mathcal{D}$, and the $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{E}(i_{\mathcal{C}}(-), i_{\mathcal{D}}(-))$, in the following sense. Given stable ∞-categories $\mathcal{C}$, $\mathcal{D}$, and an exact $\mathcal{C}$-$\mathcal{D}$-bimodule M, one can define the upper triangular ∞-category

$$\begin{pmatrix} \mathcal{C} & M \\ 0 & \mathcal{D} \end{pmatrix}$$

as the comma category $j_{\mathcal{D}} \downarrow M(-, \mathcal{C})$, where $j_{\mathcal{D}}$ is the Yoneda embedding $\mathcal{D} \to \mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Spt})$ and $M(-, \mathcal{C})$ is the functor $\mathcal{C} \to \mathrm{Fun}_{\mathrm{ex}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Spt})$ sending $X \in \mathcal{C}$ to $M(-, X)$. This ∞-category is always stable and is equipped with fully faithful functors from $\mathcal{C}$ and $\mathcal{D}$, admitting right and left adjoints, respectively, which fit into a split short exact sequence.

Conversely, given a split short exact sequence

$$\mathcal{C} \underset{L_{\mathcal{C}}}{\overset{i_{\mathcal{C}}}{\rightleftarrows}} \mathcal{E} \underset{i_{\mathcal{D}}}{\overset{L_{\mathcal{D}}}{\rightleftarrows}} \mathcal{D},$$

the ∞-category $\mathcal{E}$ is canonically equivalent to the upper triangular category

$$\begin{pmatrix} \mathcal{C} & \mathcal{E}(i_{\mathcal{C}}(-), i_{\mathcal{D}}(-)) \\ 0 & \mathcal{D} \end{pmatrix}.$$

3.4.2. Let R and S be $\mathcal{E}_1$-rings. Then any R-S-bimodule M gives rise to a split short exact sequence

$$\mathcal{C} \xrightarrow[L_{\mathcal{C}}]{i_{\mathcal{C}}} \mathcal{E} \xrightarrow[i_{\mathcal{D}}]{L_{\mathcal{D}}} \mathcal{D},$$

where $\mathcal{C} = \mathrm{Mod}_R^{\mathrm{perf}}$, $\mathcal{D} = \mathrm{Mod}_S^{\mathrm{perf}}$, and $\mathcal{E}$ is the upper triangular ∞-category

$$\begin{pmatrix} \mathcal{C} & - \otimes_R M \otimes_S - \\ 0 & \mathcal{D} \end{pmatrix}.$$

The latter is compactly generated by the object $G = i_{\mathcal{C}}(R) \oplus i_{\mathcal{D}}(S) \in \mathcal{E}$. In particular, there is an equivalence $\mathrm{Mod}_{\mathrm{End}_{\mathcal{E}}(G)}^{\mathrm{perf}} \simeq \mathcal{E}$ ([SS03]; see also Proposition 3.1.4).

Definition 3.4.3. For R, S, M as above, we call the endomorphism spectrum $\mathrm{End}_{\mathcal{E}}(G)$ the *upper triangular matrix ring* and denote it by

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}.$$

Remark 3.4.4. When R and S are discrete, the above definition is compatible with the classical one. Moreover, we have an isomorphism

$$\pi_* \begin{pmatrix} R & \mathcal{E}(i_{\mathcal{D}}(S), i_{\mathcal{C}}(R)) \\ 0 & S \end{pmatrix} \simeq \begin{pmatrix} \pi_* R & \pi_* \mathcal{E}(i_{\mathcal{D}}(S), i_{\mathcal{C}}(R)) \\ 0 & \pi_* S \end{pmatrix}.$$

In particular, if R and S are left noetherian and M is connective and almost perfect, then the upper triangular matrix ring is left noetherian.

In this setting, Theorem 3.3.3 yields:

Theorem 3.4.5. *Let R and S be regular connective $\mathcal{E}_1$-rings, and M a connective R-S-bimodule which is perfect as an S-module. Then the $\mathcal{E}_1$-ring spectrum*

$$T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

is regular.

Proof. Let $\mathcal{C} = \mathrm{Mod}_R^{\mathrm{perf}}$, $\mathcal{D} = \mathrm{Mod}_S^{\mathrm{perf}}$, and

$$\mathcal{E} = \begin{pmatrix} \mathcal{C} & - \otimes_R M \otimes_S - \\ 0 & \mathcal{D} \end{pmatrix}.$$

Recall that $\mathcal{C}$ and $\mathcal{D}$ are equipped with canonical weight structures (Example 3.1.5). To show that T is regular, it will suffice to show that $\mathrm{Mod}_T^{\mathrm{perf}} \simeq \mathcal{E}$ admits a weight structure with an adjacent t-structure (Example 3.2.3). In order to apply Theorem 3.3.3, note that the first condition translates as connectivity of the spectrum $M = \mathcal{E}(i_{\mathcal{D}}(S), i_{\mathcal{C}}(R))$, while the second condition translates as perfectness of M as a right S-module. $\square$

3.5. **Proof of Theorem 3.3.3.**

3.5.1. *Weight structure.* We set $\mathcal{E}_{w\geqslant 0}$ and $\mathcal{E}_{w\leqslant 0}$ to be the extension-closures of the unions $i_{\mathcal{C}}\mathcal{C}_{w\geqslant 0} \cup i_{\mathcal{D}}\mathcal{D}_{w\geqslant 0}$ and $i_{\mathcal{C}}\mathcal{C}_{w\leqslant 0} \cup i_{\mathcal{D}}\mathcal{D}_{w\leqslant 0}$, respectively. These classes are semi-invariant with respect to translations, and the orthogonality axiom follows from fully faithfulness of $i_{\mathcal{C}}$ and $i_{\mathcal{D}}$ and from the extra condition in the statement. It suffices to construct a weight decomposition for an object $X \in \mathcal{E}$.

By Lemma 3.3.2 we have an exact triangle $i_{\mathcal{C}}L_{\mathcal{C}}X \to X \to i_{\mathcal{D}}L_{\mathcal{D}}$. Take any weight decompositions of $L_{\mathcal{C}}X$ and $L_{\mathcal{D}}X$. Since there are no non-zero maps $\Omega i_{\mathcal{C}}w_{\leqslant 0}L_{\mathcal{D}}X \to i_{\mathcal{C}}w_{\geqslant 1}L_{\mathcal{C}}X$ we obtain a homotopy commutative square

$$
\begin{array}{ccc}
i_{\mathcal{C}}w_{\leqslant 0}L_{\mathcal{D}}\Omega X & \longrightarrow & i_{\mathcal{C}}w_{\leqslant 0}L_{\mathcal{C}}X \\
\downarrow & & \downarrow \\
i_{\mathcal{C}}L_{\mathcal{D}}\Omega X & \longrightarrow & i_{\mathcal{C}}Y
\end{array}
$$

Using Lemma 1.1.11 of [BBD82] we get a commutative diagram

$$
\begin{array}{ccccc}
i_{\mathcal{C}}w_{\leqslant 0}L_{\mathcal{C}}X & \longrightarrow & X' & \longrightarrow & i_{\mathcal{D}}w_{\leqslant 0}L_{\mathcal{D}}X \\
\downarrow & & \downarrow & & \downarrow \\
i_{\mathcal{C}}Y & \longrightarrow & X & \longrightarrow & i_{\mathcal{D}}L_{\mathcal{D}}X \\
\downarrow & & \downarrow & & \downarrow \\
i_{\mathcal{C}}w_{\geqslant 1}L_{\mathcal{C}}X & \longrightarrow & X'' & \longrightarrow & i_{\mathcal{D}}w_{\geqslant 1}L_{\mathcal{D}}X
\end{array}
$$

in the homotopy category of $\mathcal{E}$, whose rows and columns come from exact triangles. In particular, $X' \in \mathcal{E}_{w\leqslant 0}$ and $X'' \in \Sigma\mathcal{E}_{w\geqslant 0}$, so the exact triangle $X' \to X \to X''$ is a weight decomposition.

3.5.2. *Adjacent t-structure.* We now assume that the weight structures on $\mathcal{C}$ and $\mathcal{D}$ admit adjacent t-structures, and that i_D admits a right adjoint R_D. In this case we will show that the weight structure on $\mathcal{E}$ constructed above also admits an adjacent t-structure.

It suffices to construct maps $\tau\colon \tau_{\geqslant 0}X \to X$ for all X where $\tau_{\geqslant 0}X \in \mathcal{E}_{w\geqslant 0}$ and $\pi_0\mathcal{E}(Y, \mathrm{Cofib}(\tau)) = 0$ for any $Y \in \mathcal{E}_{w\geqslant 0}$. First consider the case $X = i_{\mathcal{D}}Y$. Then the map $i_{\mathcal{D}}\tau\colon i_{\mathcal{D}}\tau_{\geqslant 0}Y \to i_{\mathcal{D}}Y$ satisfies the conditions. Indeed,

$$
\pi_0\mathcal{E}(i_{\mathcal{D}}Z, \mathrm{Cofib}(i_{\mathcal{D}}\tau)) = \pi_0\mathcal{E}(i_{\mathcal{D}}Z, i_{\mathcal{D}}\mathrm{Cofib}(\tau)) = \pi_0\mathcal{D}(Z, \mathrm{Cofib}(\tau)) = 0
$$

for any $Z \in \mathcal{D}_{w\geqslant 0}$ and

$$
\pi_0\mathcal{E}(i_{\mathcal{C}}Z, \mathrm{Cofib}(i_{\mathcal{D}}\tau)) = \pi_0\mathcal{E}(i_{\mathcal{C}}Z, i_{\mathcal{D}}\mathrm{Cofib}(\tau)) = \pi_0\mathcal{C}(Z, L_{\mathcal{C}}i_{\mathcal{D}}\mathrm{Cofib}(\tau)) = 0
$$

for any $Z \in \mathcal{C}$. So, $\pi_0\mathcal{E}(Z, \mathrm{Cofib}(i_{\mathcal{D}}\tau)) = 0$ for any Z from the extension-closure of $i_{\mathcal{C}}\mathcal{C}_{w\geqslant 0} \cup i_{\mathcal{D}}\mathcal{D}_{w\geqslant 0}$.

Next consider the case $X = i_{\mathcal{C}}Y$. The cofiber Y' of the map $i_{\mathcal{C}}\tau\colon i_{\mathcal{C}}\tau_{\geqslant 0}Y \to i_{\mathcal{C}}Y$ satisfies

$$
\pi_0\mathcal{E}(i_{\mathcal{C}}Z, Y') = \pi_0\mathcal{C}(Z, \mathrm{Cofib}(\tau)) = 0.
$$

for any $Z \in \mathcal{C}_{w\geqslant 0}$. This orthogonality property is also satisfied by the cofiber Y'' of the map $i_{\mathcal{D}}\tau_{\geqslant 0}R_{\mathcal{D}}Y' \to i_{\mathcal{D}}R_{\mathcal{D}}Y' \to Y'$ since

$$
\pi_0\mathcal{E}(i_{\mathcal{C}}Z, i_{\mathcal{D}}\tau_{\geqslant 0}R_{\mathcal{D}}Y') = \pi_0\mathcal{E}(Z, L_{\mathcal{C}}i_{\mathcal{D}}\tau_{\geqslant 0}R_{\mathcal{D}}Y') = 0
$$

for any $Z \in \mathcal{C}$. The map

$$
\pi_0\mathcal{E}(i_{\mathcal{D}}Z, i_{\mathcal{D}}\tau_{\geqslant 0}R_{\mathcal{D}}Y') \simeq \pi_0\mathcal{D}(Z, i_{\mathcal{D}}\tau_{\geqslant 0}R_{\mathcal{D}}Y') \to \pi_0\mathcal{D}(Z, R_{\mathcal{D}}Y') \simeq \pi_0\mathcal{E}(i_{\mathcal{D}}Z, Y')
$$

is an isomorphism for $Z \in \mathcal{D}_{w\geqslant 0}$ and an injection for $Z \in \Omega\mathcal{D}_{w\geqslant 0}$ by orthogonality properties of the t-structure on $\mathcal{D}$. Therefore from the long exact sequence associated to an exact triangle we see that $\pi_0\mathcal{E}(i_{\mathcal{D}}Z, Y'') = 0$ for any $Z \in \mathcal{D}_{w\geqslant 0}$. The fiber F of the map $X \to Y''$ is an extension of

$i_{\mathcal{D}}\tau_{\geqslant 0}R_{\mathcal{D}}Y' = \mathrm{Cofib}(Y' \to Y'')$ by $i_{\mathcal{C}}\tau_{\geqslant 0}Y = \mathrm{Cofib}(X \to Y')$, so it belongs to $\mathcal{E}_{w\geqslant 0}$ and the map $F \to X$ satisfies the desired conditions.

Finally let X be arbitrary. By Lemma 3.3.2 there is an exact triangle $\Omega i_{\mathcal{D}}L_{\mathcal{D}}X \xrightarrow{p} i_{\mathcal{C}}L_{\mathcal{C}}X \to X$. The already constructed maps $\tau_{\geqslant 0}\Omega i_{\mathcal{D}}L_{\mathcal{D}} \to \Omega i_{\mathcal{D}}L_{\mathcal{D}}X$ and $\tau_{\geqslant 0}i_{\mathcal{C}}L_{\mathcal{C}}X \to i_{\mathcal{C}}L_{\mathcal{C}}X$ fit into a homotopy commutative diagram

$$
\begin{array}{ccc}
\tau_{\geqslant 0}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \xdashrightarrow{\tau_{\geqslant 0}p} & \tau_{\geqslant 0}i_{\mathcal{C}}L_{\mathcal{C}}X \\
\downarrow{\scriptstyle\tau_{\mathcal{D}}} & & \downarrow{\scriptstyle\tau_{\mathcal{C}}} \\
\Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \xrightarrow{\ \ p\ \ } & i_{\mathcal{C}}L_{\mathcal{C}}X.
\end{array}
$$

Applying Lemma 1.1.11 of [BBD82] we get a homotopy commutative diagram

$$
\begin{array}{ccccc}
\tau_{\geqslant 0}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \xrightarrow{\tau_{\geqslant 0}p} & \tau_{\geqslant 0}i_{\mathcal{C}}L_{\mathcal{C}}X & \longrightarrow & P \\
\downarrow{\scriptstyle\tau_{\mathcal{D}}} & & \downarrow{\scriptstyle\tau_{\mathcal{C}}} & & \downarrow \\
\Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \xrightarrow{\ p\ } & i_{\mathcal{C}}L_{\mathcal{C}}X & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \longrightarrow & \tau_{\leqslant -1}i_{\mathcal{C}}L_{\mathcal{C}}X & \longrightarrow & N
\end{array}
$$

whose rows and columns are exact triangles. We note that P belongs to $\mathcal{E}_{w\geqslant 0}$ and $\pi_0\mathcal{E}(Y,N) = 0$ for any $Y \in \mathcal{E}_{w\geqslant 1}$. Moreover, $\tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X = i_{\mathcal{D}}\tau_{\leqslant -1}\Omega L_{\mathcal{D}}X$ by construction of $\tau_{\mathcal{D}}$, so $\pi_0\mathcal{E}(Y,N) = 0$ for any $Y \in i_{\mathcal{C}}\mathcal{C}_{w\geqslant 0}$. However, the group $\pi_0\mathcal{E}(i_{\mathcal{D}}Y,N)$ might be non-zero for $Y \in \underline{Hw}_{\mathcal{D}}$. The issue is fixed using the following.

Claim. There exists an object $I \in \mathrm{Ht}_{\mathcal{D}}$ and a map $\tau_0 \colon i_{\mathcal{D}}I \to N$ such that the map $\pi_0\mathcal{E}(i_{\mathcal{D}}Y,\tau_0)$ is an isomorphism for any $Y \in \underline{Hw}_{\mathcal{D}}$.

Indeed, assume the claim is proven. Then it implies the vanishing of $\pi_0\mathcal{E}(i_{\mathcal{D}}Y,\mathrm{Cofib}(\tau_0))$ for any $Y \in \mathcal{D}_{w\geqslant 0}$. Moreover, $\pi_0\mathcal{E}(i_{\mathcal{C}}Y,\mathrm{Cofib}(\tau_0))$ is just isomorphic to $\pi_0\mathcal{E}(i_{\mathcal{C}}Y,N)$ for $Y \in \mathcal{C}$, so $\pi_0(Y,\mathrm{Cofib}(\tau_0)) = 0$ for any $Y \in \mathcal{E}_{w\geqslant 0}$. The fiber $\tau_{\geqslant 0}X$ of the map $X \to \mathrm{Cofib}(\tau_0)$ belongs to $\mathcal{E}_{w\geqslant 0}$ as it is an extension of $i_{\mathcal{D}}I$ by P. So $\tau_{\geqslant 0}X \to X$ gives us the desired map. It suffices to prove the claim.

From the bottom and the middle exact triangles in the diagram above we see that for $Y \in \underline{Hw}_{\mathcal{D}}$

$$
\pi_0\mathcal{E}(i_{\mathcal{D}}Y,N) \simeq im(\pi_0\mathcal{E}(i_{\mathcal{D}}Y,X) \xrightarrow{u} \pi_0\mathcal{E}(i_{\mathcal{D}}Y,\Sigma\tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X))
$$

and the isomorphism is induced by the map $N \to \Sigma\tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X$.

Since $i_{\mathcal{D}}$ is fully faithful, the composition of the unit and the counit $i_{\mathcal{D}}R_{\mathcal{D}}X \to X \to i_{\mathcal{D}}L_{\mathcal{D}}X$ defines a map $\gamma\colon R_{\mathcal{D}}X \to L_{\mathcal{D}}X$. We define I to be the image in the abelian category $\mathrm{Ht}_{\mathcal{D}}$ of the map $\pi_0\gamma\colon \pi_0R_{\mathcal{D}}X \to \pi_0L_{\mathcal{D}}X$. There is an obvious map f from $i_{\mathcal{D}}I$ to $\Sigma\tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X = i_{\mathcal{D}}\tau_{\leqslant 0}L_{\mathcal{D}}X$. The following commutative diagram shows that f maps $\pi_0\mathcal{E}(i_{\mathcal{D}}Y,i_{\mathcal{D}}I)$ isomorphically to the subset of $\pi_0\mathcal{E}(i_{\mathcal{D}}Y,i_{\mathcal{D}}\tau_{\leqslant 0}L_{\mathcal{D}}X)$ isomorphic to $\pi_0\mathcal{E}(i_{\mathcal{D}}Y,N)$.

$$\begin{array}{ccccc}
\pi_0\mathcal{E}(i_\mathcal{D}Y, i_\mathcal{D}\pi_0 R_\mathcal{D}X) & \longrightarrow & \pi_0\mathcal{E}(i_\mathcal{D}Y, i_\mathcal{D}I) & \hookrightarrow & \pi_0\mathcal{E}(i_\mathcal{D}Y, i_\mathcal{D}\pi_0 L_\mathcal{D}X) \\
\downarrow{\scriptstyle(1)} & & {\searrow}^{f} & & \downarrow{\scriptstyle(2)} \\
\pi_0\mathcal{E}(i_\mathcal{D}Y, i_\mathcal{D}\tau_{\leqslant 0}R_\mathcal{D}X) & \longrightarrow & & & \pi_0\mathcal{E}(i_\mathcal{D}Y, i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X) \\
\uparrow{\scriptstyle(3)} & & {\nearrow}^{u} & & \uparrow{\scriptstyle=} \\
\pi_0\mathcal{E}(i_\mathcal{D}Y, i_\mathcal{D}R_\mathcal{D}X) & \xrightarrow{(4)} & \pi_0\mathcal{E}(i_\mathcal{D}Y, X) & \longrightarrow & \pi_0\mathcal{E}(i_\mathcal{D}Y, i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X)
\end{array}$$

Indeed, considering the long exact sequences the maps (1) and (2) fit into, we see that they are isomorphisms by the orthogonality axiom for weight structures. The map (3) is a surjection for the same reason. The map (4) becomes the map $\pi_0\mathcal{D}(Y, R_\mathcal{D}i_\mathcal{D}R_\mathcal{D}X) \to \pi_0\mathcal{D}(Y, R_\mathcal{D}X)$ under the adjunction isomorphism for $i_\mathcal{D}$ and $R_\mathcal{D}$. The latter map is an isomorphism because $\mathrm{id}_\mathcal{D} \to R_\mathcal{D}i_\mathcal{D}$ is an equivalence. Therefore we see that the images of $\pi_0\mathcal{E}(i_\mathcal{D}Y, i_\mathcal{D}I)$ and of $\pi_0\mathcal{E}(i_\mathcal{D}Y, X)$ in $\pi_0\mathcal{E}(i_\mathcal{D}Y, i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X)$ coincide and the former maps injectively onto the image.

Now it suffices to construct a map $i_\mathcal{D}I \to N$ that makes the triangle

$$\begin{array}{ccc}
N & \longrightarrow & i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X \\
{\scriptstyle\wedge} & {\nearrow}^{f} & \\
{\vdots} & & \\
i_\mathcal{D}I & &
\end{array}$$

commute. Since N is a fiber of the map $i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X \to \Sigma\tau_{\leqslant -1}i_\mathcal{C}L_\mathcal{C}X$, it suffices to prove that the composite map $i_\mathcal{D}I \to \Sigma\tau_{\leqslant -1}i_\mathcal{C}L_\mathcal{C}X$ is null-homotopic. We will show that the element f in the upper left corner of the following commutative diagram becomes trivial after applying the horizontal map.

$$\begin{array}{ccc}
\pi_0\mathcal{E}(i_\mathcal{D}I, i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X) & \longrightarrow & \pi_0\mathcal{E}(i_\mathcal{D}I, \Sigma\tau_{\leqslant -1}i_\mathcal{C}L_\mathcal{C}X) \\
\downarrow{\scriptstyle(1)} & & \downarrow{\scriptstyle(2)} \\
\pi_0\mathcal{E}(i_\mathcal{D}\pi_0 R_\mathcal{D}X, i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X) & \longrightarrow & \pi_0\mathcal{E}(i_\mathcal{D}\pi_0 R_\mathcal{D}X, \Sigma\tau_{\leqslant -1}i_\mathcal{C}L_\mathcal{C}X) \\
\uparrow{\scriptstyle(3)} & & \uparrow \\
\pi_0\mathcal{E}(i_\mathcal{D}\tau_{\leqslant 0}R_\mathcal{D}X, i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X) & \longrightarrow & \pi_0\mathcal{E}(i_\mathcal{D}\tau_{\leqslant 0}R_\mathcal{D}X, \Sigma\tau_{\leqslant -1}i_\mathcal{C}L_\mathcal{C}X) \\
\downarrow{\scriptstyle(4)} & & \downarrow{\scriptstyle(5)} \\
\pi_0\mathcal{E}(i_\mathcal{D}R_\mathcal{D}X, i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X) & \longrightarrow & \pi_0\mathcal{E}(i_\mathcal{D}R_\mathcal{D}X, \Sigma\tau_{\leqslant -1}i_\mathcal{C}L_\mathcal{C}X)
\end{array}$$

The long exact sequence associated to the exact triangle

$$i_\mathcal{D}K \to i_\mathcal{D}\pi_0 R_\mathcal{D}X \to i_\mathcal{D}I$$

where K denotes the kernel of the surjection $\pi_0 R_\mathcal{D}X \to I$ in the abelian category $\mathrm{Ht}_\mathcal{D}$, together with the vanishing property in the definition of $\tau_\mathcal{C}$ and $\tau_\mathcal{D}$, yields injectivity of the maps (1) and (2). So it suffices to show that the image of f in $\pi_0\mathcal{E}(i_\mathcal{D}\pi_0 R_\mathcal{D}X, i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X)$ is mapped to zero via the horizontal map. By definition of f the image lifts via (3) to a map $\tilde{f} = i_\mathcal{D}\tau_{\leqslant 0}(\gamma)$. The long exact sequence associated with the exact triangle

$$i_\mathcal{D}\tau_{\geqslant 1}R_\mathcal{D}X \to i_\mathcal{D}R_\mathcal{D}X \to i_\mathcal{D}\tau_{\leqslant 0}R_\mathcal{D}X$$

together with the vanishing property in the definition of $\tau_\mathcal{C}$ and $\tau_\mathcal{D}$ implies that the maps (4) and (5) are isomorphisms. So it suffices to show that the image of $\tilde{f}$ via (4) is mapped to zero via the horizontal map. The image of $\tilde{f}$ is the composite $i_\mathcal{D}R_\mathcal{D}X \to i_\mathcal{D}\tau_{\leqslant 0}R_\mathcal{D}X \to i_\mathcal{D}\tau_{\leqslant 0}L_\mathcal{D}X$. This

factorizes as $i_{\mathcal{D}} R_{\mathcal{D}} X \to i_{\mathcal{D}} L_{\mathcal{D}} X \to i_{\mathcal{D}} \tau_{\leqslant 0} L_{\mathcal{D}} X$. Therefore, the result follows from the fact that the composite of the two maps in a exact triangle is null-homotopic.

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