Fourier analysis in derived algebraic geometry

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TMS 2026

$$f : \mathbb{R}^n \longrightarrow \mathbb{C}$$

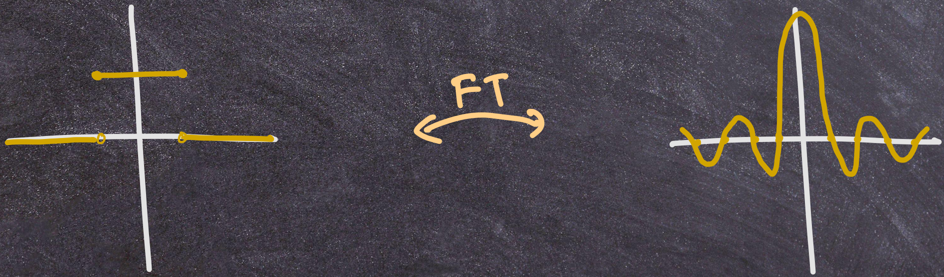
$$\Big\updownarrow \; \text{FT}$$

$$\hat{f} : \mathbb{R}^n \longrightarrow \mathbb{C}$$

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \cdot \xi \cdot x} \, dx$$

$f(x) = \mathbb{1}_{[-1,1]}(x)$

$\hat{f}(\xi) = \frac{\sin(2\pi\xi)}{\pi\xi}$



local

spread out

$$\mathrm{FT}\left(\frac{\partial}{\partial x_k}(f)\right) = (2\pi i \cdot \xi_k) \cdot \hat{f}(\xi)$$

differentiation $\rightsquigarrow$ multiplication

T : translation-invariant linear operator

$$T_v T(f) = T T_v(f)$$

where $T_v f(x) := f(x-v)$

translation by $v \in \mathbb{R}^n$

FT **diagonalizes** translation-invariant linear operators:

$$FT(T_v f)(\xi) = e^{-2\pi i \cdot \xi v} \cdot \hat{f}(\xi)$$

eigenvalue

diagonalization

→

nonabelian

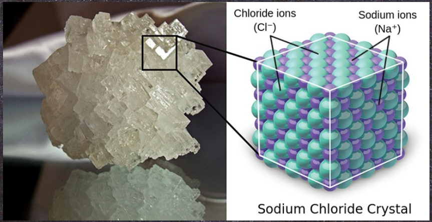
matrix multiplication

abelian

scalar multiplication

The whole story arises naturally by considering the translational symmetry of Euclidean space

M.C. Escher's "Pegasus"



$(\mathbb{R}^n, +)$ $\rightsquigarrow$ $T_v f(x) = f(x-v)$

$(V, +)$

translation operators

eigenbasis expansion

using T_v we decompose $f: V \to \mathbb{C}$

over characters $\chi \in \hat{V} = \mathrm{Hom}_{\mathrm{cont}}(V, S^1)$

$$f(x) = \int_{\chi \in \hat{V}} \hat{f}(\chi) \cdot \chi(x)\, d\chi$$

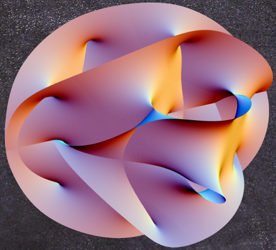
every character $\chi : V \to S^1$ is of the form

$$\chi(x) = e^{2\pi i \cdot \langle \xi, x \rangle}$$

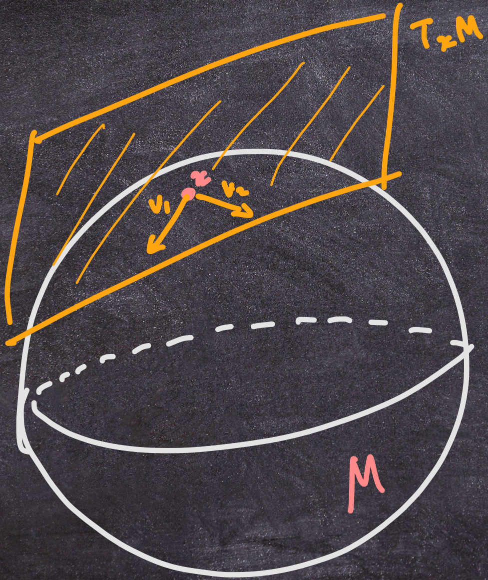
for some $\xi \in V^*$, so $\hat{V} \simeq V^*$

$$f(x) = \int_{V^*} \hat{f}(\xi) \cdot e^{2\pi i \cdot \langle \xi, x \rangle}\, d\xi$$

geometry: M smooth manifold



Can we adapt Fourier duality to help study M?



a manifold doesn't have translational symmetry globally, but it does *infinitesimally*

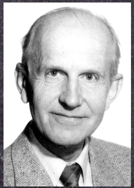
$$\begin{array}{l} TM = \bigsqcup_x T_x M \ni (x, v) \\ \ \downarrow \\ M \end{array}$$

fibrewise Fourier transform

$TM \longleftrightarrow T^*M$

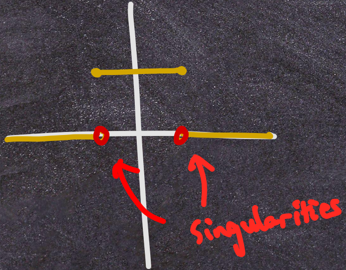
this line of thought leads to the field of microlocal analysis

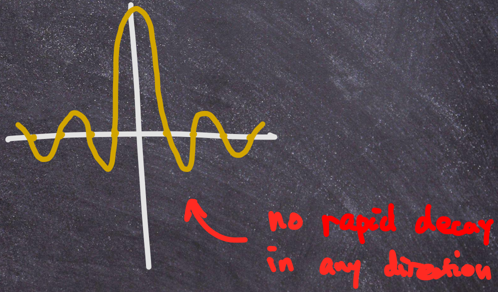
Mikio Sato



Lars Hörmander

$$f(x) = \mathbb{1}_{[-1,1]}(x) \quad \overset{FT}{\longleftrightarrow} \quad \hat{f}(\xi) = \frac{\sin(2\pi\xi)}{\pi\xi}$$



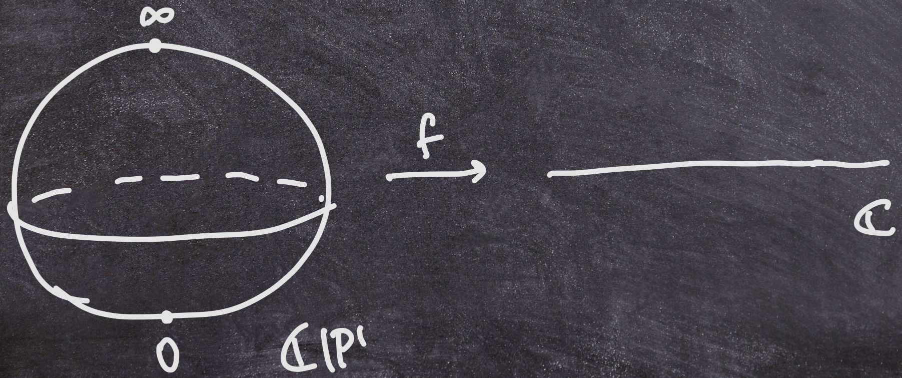


in algebraic geometry, we study **algebraic varieties** and regular maps between them, defined by **polynomial equations**:

$$X = \{y = x^2\} \subseteq \mathbb{C}^2$$

$$Y = \{yz = x^2\} \subseteq \mathbb{CP}^2$$

algebraic varieties often do not have many **global** regular functions



however, they do have an ample supply of sheaves

$\mathrm{Shv}(X) = \{ \text{sheaves on } X \}$ ← a category

$\oplus, \otimes, \ldots$

Shv(X) remembers much of the topology of X, e.g. $H_*(X)$, $H^*(X)$,

and there is a very close analogy between functions and sheaves

Grothendieck

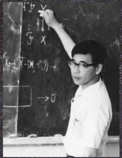
we can define a Fourier transform

$$\mathrm{FT} : \mathrm{Shv}(V) \longrightarrow \mathrm{Shv}(V^\vee)$$

V
$\downarrow$ vector bundle $\leftrightsquigarrow$ $V^\vee$ fibrewise dual
X $\qquad\qquad\qquad\qquad$ $\downarrow$
$\qquad\qquad\qquad\qquad\qquad\quad$ X

$$\begin{array}{c} V \\ \downarrow \\ X \end{array} \text{ vector bundle } \leftrightsquigarrow \begin{array}{c} V^\vee \\ \downarrow \\ X \end{array} \text{ fibrewise dual}$$

$$\mathrm{FT}(\mathcal{F}) := pr_{1!}\left(pr_2^{*}(\mathcal{F}) \otimes ev^{*}(\mathcal{L}^{\exp})\right)$$



Mikio Sato

Kashiwara

$$V^{\vee} \oplus V$$

$$pr_1 : V^{\vee} \oplus V \to V^{\vee}, \quad ev : V^{\vee} \oplus V \to \mathbb{C}, \quad pr_2 : V^{\vee} \oplus V \to V$$

Deligne

Laumon

$$FT(f)(\xi) := \int_{\xi \in V^\vee} f(x) \cdot e^{-2\pi i \cdot \langle \xi, x \rangle} \, d\xi$$

$$FT(\mathcal{F}) := pr_{1!}\left(pr_2^*(\mathcal{F}) \otimes ev^*(\mathcal{L}^{exp}) \right)$$

just like FT on functions,
this diagonalizes translation:

$$FT(T_v(\mathcal{F})) \simeq FT(\mathcal{F}) \otimes \mathcal{L}_v$$

translation operator

character sheaves

the sheaf-theoretic Fourier transform was applied with great success in

geometric representation theory,

arithmetic geometry,

microlocal sheaf theory,

symplectic geometry, ...

when X happens to be smooth (nonsingular)

we may apply this to

$$FT : Shv(TX) \to Shv(T^*X)$$

$\rightsquigarrow$ microlocal analysis on smooth algebraic varieties in the XXI century, we are often dealing with moduli spaces which are inherently singular

typical local structure:

zero loci

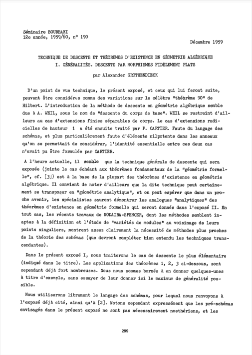
$$X = \{ f_1 = \cdots = f_m = 0 \} \subseteq \mathbb{C}^n$$

$f_i : \mathbb{C}^n \to \mathbb{C}$ regular fn's

quotients

$$X = V/G$$

$V \simeq \mathbb{C}^n$ a G-representation both types of singularities can be better understood by incorporating ideas of homotopy theory


Séminaire BOURBAKI
12e année, 1959/60, n° 190

Décembre 1959

TECHNIQUE DE DESCENTE ET THÉORÈMES D'EXISTENCE EN GÉOMETRIE ALGÉBRIQUE
I. GÉNÉRALITÉS. DESCENTE PAR MORPHISMES FIDÈLEMENT PLATS

par Alexander GROTHENDIECK

D'un point de vue technique, le présent exposé, et ceux qui lui feront suite, peuvent être considérés comme des variations sur le célèbre "théorème 90" de Hilbert. L'introduction de la méthode de descente en géométrie algébrique semble due à A. WEIL, sous le nom de "descente du corps de base". WEIL se restreint d'ailleurs au cas d'extensions finies séparables de corps. Le cas d'extensions radicielles de hauteur 1 a été ensuite traité par P. CARTIER. Faute du langage des schémas, et plus particulièrement faute d'éléments nilpotents dans les anneaux qu'on se permettait de considérer, l'identité essentielle entre ces deux cas n'avait pu être formulée par CARTIER.

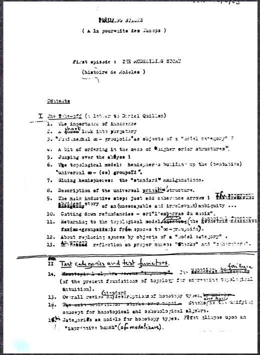
A l'heure actuelle, il semble que la technique générale de descente qui sera exposée (jointe le cas échéant aux théorèmes fondamentaux de la "géométrie formelle", cf. [3]) est à la base de la plupart des théorèmes d'existence en géométrie algébrique. Il convient de noter d'ailleurs que la dite technique peut certainement se transposer en "géométrie analytique", et on peut espérer que dans un proche avenir, les spécialistes sauront démontrer les analogues "analytiques" des théorèmes d'existence en géométrie formelle qui seront donnés dans l'exposé II. En tout cas, les récents travaux de KODAIRA-SPENCER, dont les méthodes semblent inaptes à la définition et l'étude de "variétés de modules" au voisinage de leurs points singuliers, montrent assez clairement la nécessité de méthodes plus proches de la théorie des schémas (que devront compléter bien entendu les techniques transcendantes).

Dans le présent exposé I, nous traiterons le cas de descente le plus élémentaire (indiqué dans le titre). Les applications des théorèmes 1, 2, 3 ci-dessous, sont cependant déjà fort nombreuses. Nous nous sommes bornés à en donner quelques-unes à titre d'exemple, sans essayer de leur donner ici le maximum de généralité possible.

Nous utiliserons librement le langage des schémas, pour lequel nous renvoyons à l'exposé déjà cité, ainsi qu'à [2]. Notons cependant expressément que les pré-schémas envisagés dans le présent exposé ne sont pas nécessairement noethériens, et les

299

1959


PURSUING STACKS

(A la poursuite des Champs)

First episode : THE MODELLING STORY

(histoire de modeles)

Contents

I The take-off (a letter to Daniel Quillen)
1. The importance of innocence
2. A short look into purgatory
3. "Fundamental ∞-groupoids" as objects of a "model category" ?
4. A bit of ordering in the mess of "higher order structures".
5. Jumping over the abyss !
6. The topological model: hemispheres building up the (tentative) "universal ∞- (co) groupoid".
7. Gluing hemispheres: the "standard" amalgamations.
8. Description of the universal primitive structure.
9. The main inductive step: just add coherence arrows ! ... story of an (unescapable and irrelevant) ambiguity ...
10. Cutting down redundancies – or "l'embarras du choix".
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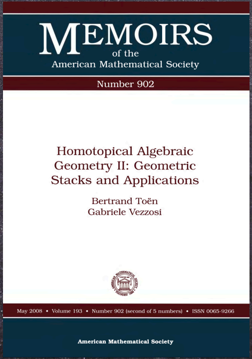
12. About replacing spaces by objects of a "model category".
13. ... reflection on proper names: "Stacks" and "...".

II Test categories and test functors.
14. ... (of the present foundations of topology for expressing topological intuition).
15. Overall review of standard descriptions of homotopy types ...
16. ... Stacks ... concept for homotopical and cohomological algebra.
17. Categories as models for homotopy types. First glimpse upon an "impressive bunch" (of modelizers).

1983


MEMOIRS
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Number 902

Homotopical Algebraic Geometry II: Geometric Stacks and Applications

Bertrand Toën
Gabriele Vezzosi

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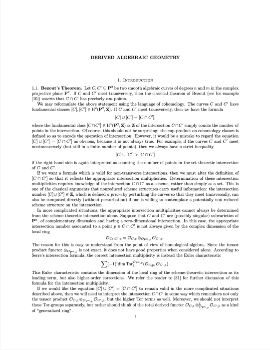
American Mathematical Society

2004


DERIVED ALGEBRAIC GEOMETRY

1. Introduction

1.1. **Bezout's Theorem.** Let $C, C' \subseteq \mathbf{P}^2$ be two smooth algebraic curves of degrees n and m in the complex projective plane $\mathbf{P}^2$. If C and C' meet transversely, then the classical theorem of Bezout (see for example [10]) asserts that $C \cap C'$ has precisely nm points.

We may reformulate the above statement using the language of cohomology. The curves C and C' have fundamental classes $[C], [C'] \in \mathrm{H}^2(\mathbf{P}^2, \mathbf{Z})$. If C and C' meet transversely, then we have the formula

$$[C] \cup [C'] = [C \cap C'],$$

where the fundamental class $[C \cap C'] \in \mathrm{H}^4(\mathbf{P}^2, \mathbf{Z}) \simeq \mathbf{Z}$ of the intersection $C \cap C'$ simply counts the number of points in the intersection. Of course, this should not be surprising: the cup-product on cohomology classes is defined so as to encode the operation of intersection. However, it would be a mistake to regard the equation $[C] \cup [C'] = [C \cap C']$ as obvious, because it is not always true. For example, if the curves C and C' meet nontransversely (but still in a finite number of points), then we always have a strict inequality

$$[C] \cup [C'] > [C \cap C']$$

if the right hand side is again interpreted as counting the number of points in the set-theoretic intersection of C and C'.

If we want a formula which is valid for non-transverse intersections, then we must alter the definition of $[C \cap C']$ so that it reflects the appropriate intersection multiplicities. Determination of these intersection multiplicities requires knowledge of the intersection $C \cap C'$ as a scheme, rather than simply as a set. This is one of the classical arguments that nonreduced scheme structures carry useful information: the intersection number $[C] \cup [C'] \in \mathbf{Z}$, which is defined *a priori* by perturbing the curves so that they meet transversally, can also be computed directly (without perturbation) if one is willing to contemplate a potentially non-reduced scheme structure on the intersection.

In more complicated situations, the appropriate intersection multiplicities cannot always be determined from the scheme-theoretic intersection alone. Suppose that C and C' are (possibly singular) subvarieties of $\mathbf{P}^n$, of complementary dimension and having a zero-dimensional intersection. In this case, the appropriate intersection number associated to a point $p \in C \cap C'$ is not always given by the complex dimension of the local ring

$$\mathcal{O}_{C \cap C', p} = \mathcal{O}_{C,p} \otimes_{\mathcal{O}_{\mathbf{P}^n,p}} \mathcal{O}_{C',p}.$$

The reason for this is easy to understand from the point of view of homological algebra. Since the tensor product functor $\otimes_{\mathcal{O}_{\mathbf{P}^n,p}}$ is not exact, it does not have good properties when considered alone. According to Serre's intersection formula, the correct intersection multiplicity is instead the Euler characteristic

$$\sum (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}^n,p}}(\mathcal{O}_{C,p}, \mathcal{O}_{C',p}).$$

This Euler characteristic contains the dimension of the local ring of the scheme-theoretic intersection as its leading term, but also higher-order corrections. We refer the reader to [31] for further discussion of this formula for the intersection multiplicity.

If we would like the equation $[C] \cup [C'] = [C \cap C']$ to remain valid in the more complicated situations described above, then we will need to interpret the intersection $C \cap C'$ in some way which remembers not only the tensor product $\mathcal{O}_{C,p} \otimes_{\mathcal{O}_{\mathbf{P}^n,p}} \mathcal{O}_{C',p}$, but the higher Tor terms as well. Moreover, we should not interpret these Tor-groups separately, but rather should think of the total derived functor $\mathcal{O}_{C,p} \otimes^{L}_{\mathcal{O}_{\mathbf{P}^n,p}} \mathcal{O}_{C',p}$ as a kind of "generalized ring".

1

2004

$G \circlearrowright Y$ group action on a set Y

in the quotient Y/G, we collapse any two y_1, y_2 in the same orbit

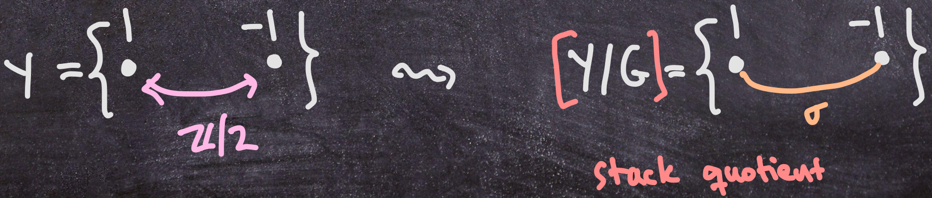
$$y_1 = g \cdot y_2$$

$$Y = \{ \overset{1}{\bullet} \quad \overset{-1}{\bullet} \} \rightsquigarrow Y/G = \{ \bullet \}$$

$\mathbb{Z}/2$

in homotopy theory, one remembers **why** $y_1 = y_2$, by attaching a path

whenever $y_1 = g \cdot y_2$.



tangent complex:

$$T[Y/G] = [\underset{\text{deg } 1}{\mathfrak{g}_Y} \longrightarrow \underset{\text{deg } 0}{TY}]$$

↖ fibres = Lie algebra $\mathfrak{g}$

cotangent complex:

$$T^*[Y/G] = [\underset{\text{deg } 0}{T^*Y} \xrightarrow{d\mu} \underset{\text{deg } -1}{\mathfrak{g}_Y^*}]$$

Similarly, if $X = \{ f_1 = \cdots = f_m = 0 \} \subseteq Y$,

regular functions on X are given by

$$\mathbb{C}[X] \simeq \mathbb{C}[Y]/(f_1, \ldots, f_m).$$

i.e. $f_1 = \cdots = f_m = 0$ in $\mathbb{C}[X]$.

instead, we may attach paths multiplicatively to $\mathbb{C}[Y]$

to get

$$\mathbb{C}[\tilde{X}] \longleftrightarrow \tilde{X}$$

algebra geometry

$Y = \mathbb{C}$, $f(y) = y^2$, $X = \{ y^2 = 0 \}$

$$\mathbb{C}[X] = \mathbb{C}[y]/(y^2)$$

$$\mathbb{C}[\tilde{X}] = \mathbb{C}[y, \alpha]$$

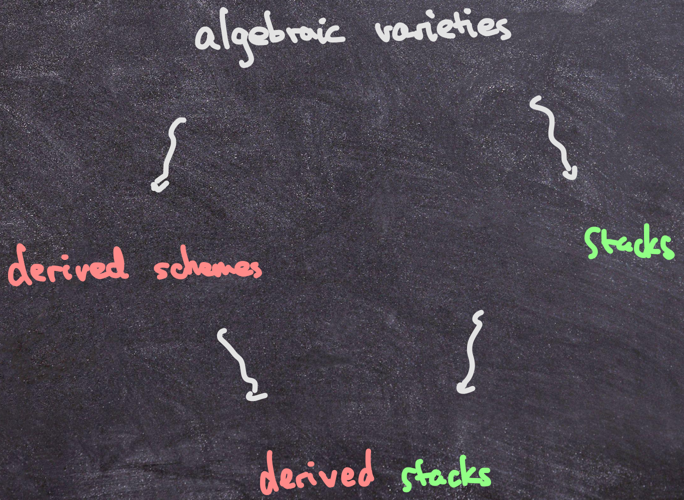
where α is a degree 1 element such that $d\alpha = y^2$

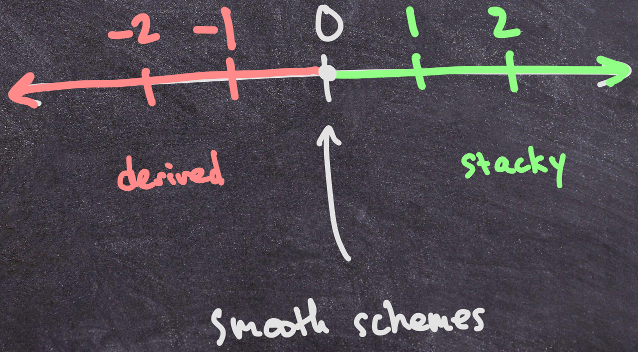
tangent complex:

$$T\tilde{X} = \left[\underset{\text{deg } 0}{TY|_x} \xrightarrow{df} \underset{\text{deg } -1}{\mathbb{C}^n_x} \right]$$

cotangent complex:

$$T^*\tilde{X} = \left[\underset{\text{deg } 1}{\mathbb{C}^n_x} \longrightarrow \underset{\text{deg } 0}{T^*Y|_x} \right]$$


algebraic varieties
derived schemes
stacks
derived stacks



-2
-1
0
1
2
derived
stacky
smooth schemes

Example: if X is a smooth projective variety of dim. d, the moduli space $\mathcal{M}(X) = \{$ principal G-bundles on $X \}$ is a derived stack which is 1-stacky, $(d-1)$-derived

while the classical version of $\mathcal{M}(X)$ is singular it turns out that we can extend

$$FT : Shv(V) \longrightarrow Shv(V^\vee)$$

to derived vector bundles $V \to X$, e.g.

$$FT : Shv(TX) \longrightarrow Shv(T^*X)$$

for any derived stack X

So we can try to apply methods of Fourier analysis in the world of derived algebraic geometry!

This is the beginning of the emerging story of derived microlocal sheaf theory...

so far, applying these ideas to various moduli spaces of bundles (and sheaves) has resulted in applications in various directions...

enumerative geometry

KK ↓

Donaldson-Thomas theory

representation theory

KK, KKPS ↓

cohomological Hall algebras

KKPS ↓

relative Langlands duality

arithmetic geometry

FK ↓

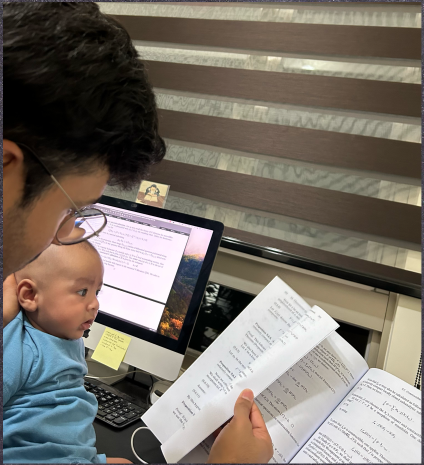
arithmetic theta series

[KK] —, T. Kinjo. 3d CoHAs for local surfaces. (2023)

[FK] T. Feng, —. Modularity of higher theta series II. (2024)

[KKPS] —, T. Kinjo, H. Park, P. Safronov. Perverse pullbacks. (2025),
Period sheaves... (2025),
Lagrangian classes. (soon)

still much more to explore!



it's never too early to start
learning microlocal sheaf theory