A MODERN INTRODUCTION TO ALGEBRAIC STACKS

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0. Overview 1
1. ∞-Categories 4
   1.1. Simplicial sets 4
   1.2. Categories as simplicial sets 6
   1.3. Groupoids and Kan complexes 7
   1.4. ∞-Categories as weak Kan complexes 8
   1.5. The ∞-category of (∞-)groupoids 10
2. Sheaves and stacks 11
   2.1. Sheaves 11
   2.2. Bases of topologies 14
3. The stack of quasi-coherent sheaves 15
   3.1. Cartesian fibrations 15
   3.2. Descent for quasi-coherent sheaves 17
   3.3. Quasi-coherent sheaves on stacks 20
4. Quotient stacks 21
   4.1. Groupoid objects and effective epimorphisms 22
   4.2. Group actions 24
   4.3. Torsors 25
   4.4. Quotient stacks 27
5. Algebraic spaces and stacks 29
   5.1. Algebraic spaces 29
   5.2. Algebraic stacks 30
   5.3. Algebraicity of quotient stacks 31
References 32

0. Overview

"Definition" 0.1. Given a class of geometric objects, say "gadgets", a *moduli space* for gadgets is a space whose points correspond to gadgets, modulo some notion of equivalence between gadgets:

$$\text{moduli space of gadgets} = \{\text{gadgets}\}/\text{equivalence}.$$

The idea of moduli theory is to transform questions about gadgets into questions about the moduli space, which we may then try to tackle via, topological, geometric, or cohomological methods.

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Example 0.2. The complex projective n-space $\mathbf{P}^n(\mathbf{C})$ is the moduli space of lines in $\mathbf{C}^{n+1}$ which pass through the origin. Algebraically, these are 1-dimensional linear subspaces $L \subseteq \mathbf{C}^{n+1}$, or equivalently 1-dimensional linear quotients $\mathbf{C}^{n+1} \twoheadrightarrow L$. Alternatively, $\mathbf{P}^n(\mathbf{C})$ can be described as the moduli space of points of $\mathbf{C}^{n+1} \smallsetminus \{0\}$, where points are identified modulo scaling by $\lambda \in \mathbf{C}^*$.

Example 0.3. The Grassmannian $\mathrm{Gr}(k, V)$ is the moduli space of k-dimensional linear subspaces of a given vector space V.

One can then solve certain problems in enumerative geometry (e.g. "how many lines in $\mathbf{P}^3(\mathbf{C})$ intersect four given general lines?") by analyzing the cohomology of Grassmannians (see: Schubert calculus).

Example 0.4. Given an algebraic variety X, the moduli space of (algebraic) vector bundles on X is the set of vector bundles on X modulo isomorphism. More generally, given an algebraic group G, the moduli space of principal G-bundles is the set of principal G-bundles on X modulo isomorphism.

Example 0.5. Given an algebraic variety X, the moduli space of coherent sheaves on X is the set of coherent sheaves on X modulo isomorphism.

In order to get a useful theory of moduli spaces, we will need to refine this naive picture in two ways.

Theorem 0.6 (Grothendieck)**.** *A scheme X over a field k is completely determined by its functor of points, i.e., the functor*

$$X : \mathrm{CAlg}_k \to \mathrm{Set}$$

sending a commutative k-algebra A to the set of A-valued points $X(A)$. (Recall that an A-valued point of X is a morphism of schemes $\mathrm{Spec}(A) \to X$*.)*

Thus we may regard a scheme as a family (or fibration) of sets $X(A)$ parametrized by commutative algebras A. That is, a scheme is literally a "scheme" prescribing the A-valued points of some algebro-geometric space.

For example, in order to define complex projective n-space as a scheme, it is not sufficient to specify the set $\mathbf{P}^n(\mathbf{C})$ as above. Instead, we must specify the sets

$$\mathbf{P}^n(A) = \{A\text{-linear surjections } A^{n+1} \twoheadrightarrow L \mid L \text{ projective } A\text{-module of rank } 1\},$$

for all commutative $\mathbf{C}$-algebras A, together with the natural maps $\mathbf{P}^n(A) \to \mathbf{P}^n(A')$ for ring homomorphisms $A \to A'$.

Secondly, we will need to be smarter about quotients. Let X be a set and $R \subseteq X \times X$ an equivalence relation on X. We may depict this via the diagram

$$R \; \begin{matrix} \xrightarrow{\mathrm{pr}_1} \\ \xleftarrow{s} \\ \xrightarrow[\mathrm{pr}_2]{} \end{matrix} \; X$$

where pr_i are the projections and $s : X \to X \times X$ is the diagonal (which factors through R since the relation is reflexive). This diagram defines a *groupoid* $[X/R]$:

- The objects of $[X/R]$ are the elements of X.
- The morphisms of $[X/R]$ are the elements of R.
- The "source" and "target" maps $\mathrm{Mor}[X/R] \to \mathrm{Obj}[X/R]$ are given by the projections pr_1 and pr_2.
- The "identity" map $\mathrm{Obj}[X/R] \to \mathrm{Mor}[X/R]$ (sending an object to its identity morphism) is given by the diagonal s.
- Composition of morphisms is well-defined since the relation is transitive.
- All morphisms are invertible since the relation is symmetric.

Note that the set of connected components $\pi_0[X/R]$ (where objects of $[X/R]$ are identified if and only if they connected by some chain of morphisms) is canonically isomorphic to the usual set-theoretic quotient X/R. Unlike X/R, the quotient groupoid $[X/R]$ remembers *how* elements are identified.

Similarly, if we have a group G acting on the set X, there is a quotient groupoid $[X/G]$ defined by the diagram

$$G \times X \underset{\mathrm{act}}{\overset{\mathrm{pr}_2}{\underset{\xleftarrow{s}}{\rightrightarrows}}} X$$

where the "source" map is the projection $(g,x) \mapsto x$, the "target" map is the action map $(g,x) \mapsto g \cdot x$, and the "identity" map is $x \mapsto (e,x)$ where $e \in G$ is the neutral element. Whereas the set-theoretic quotient X/G remembers only the binary information of whether two elements $x, y \in X$ belong to the same equivalence class, the groupoid $[X/G]$ contains one isomorphism $x \simeq y$ for every $g \in G$ such that $g \cdot x = y$.

As we will see in this course, it is highly advantageous to allow moduli spaces to have *groupoids* of points rather than sets. Combining these two ideas leads one to the replace our naive definition of moduli space above by the following:

Definition 0.7. A *stack* (over a field k) is a functor

$$\mathcal{M} : \mathrm{CAlg}_k \to \mathrm{Grpd}$$

satisfying certain conditions.

One of our goals will be to prove the following theorem:

Theorem 0.8. *Let C be a smooth proper curve over k and G an algebraic group over k. Let*

$$\mathcal{M}_{\mathrm{Bun}_G(C)} : A \mapsto \mathrm{Bun}_G(C_A)^{\simeq}$$

be the stack defined by the functor sending a commutative algebra A to the groupoid of principal G-bundles on the scheme $C_A := C \otimes_k A$. Then $\mathcal{M}_{\mathrm{Bun}_G(C)}$ is a smooth *algebraic stack.*

In particular, taking G to be the general linear group GL_n, we find the same holds for the moduli stack of rank n vector bundles on C. We also have a similar result for the moduli stack $\mathcal{M}_{\mathrm{Coh}(C)}$ of coherent sheaves on C.

Remark 0.9. It is important here to work with groupoids rather than sets; the functor sending A to the set $\pi_0 \operatorname{Bun}_G(C)^{\simeq}$ of isomorphism classes of principal G-bundles on $C \otimes_k A$ is very poorly behaved.

Warning 0.10. Making the definition of "stack" above precise is more involved than for schemes, since Grpd is naturally a 2-category (where the 2-morphisms are natural transformations). In other words, in practice we only want to distinguish between groupoids up to *equivalence* rather than isomorphism.

One way to handle this subtlety is to use the language of 2-categories and pseudofunctors. In this course we will instead use the language of ∞-categories. This language is much more general than that of 2-categories, but we will see that it has some practical advantages even when all ∞-categories involved are "2-truncated". Moreover, the extra generality of ∞-categories will also be useful to us later in the course when we study concepts like the derived category of (quasi-)coherent sheaves on a stack, and the cotangent complex of a stack.

1. ∞-Categories

1.1. **Simplicial sets.** For every integer $n \geqslant 0$, let $[n]$ denote the finite set $\{0, 1, \ldots, n\}$. Let $\mathbf{\Delta}$ denote the category whose objects are the finite sets $[n]$, for all $n \geqslant 0$, and whose morphisms are order-preserving maps.

Definition 1.1. A *simplicial set* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$, i.e., a contravariant functor on $\mathbf{\Delta}$ with values in the category of sets. A morphism of simplicial sets is a natural transformation of the corresponding functors. The category of simplicial sets is the functor category $\mathrm{SSet} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$.

In other words, a simplicial set X is a sequence of sets $X_n \coloneqq X([n])$ together with a collection of maps $\alpha^* : X_n \to X_m$ for all order-preserving maps $\alpha : [m] \to [n]$, subject to the identities $\mathrm{id}^* = \mathrm{id}$ and $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$ whenever α and β are composable. Elements of the set X_n are called *n-simplices* of X.

Example 1.2. Given a set X, we let $c(X)$ denote the *constant* simplicial set on X. We have $c(X)_n = X$ for all n, and every map $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ induces the identity map $\mathrm{id} : X \to X$. The assignment $X \mapsto c(X)$ defines a canonical functor

$$c : \mathrm{Set} \to \mathrm{SSet}$$

which is fully faithful.

Notation 1.3. Given integers $n \geqslant 0$ and $0 \leqslant i \leqslant n$, we denote by

$$\delta^i_n : [n-1] \to [n]$$

the injective map that "skips" i, and by

$$\sigma^i_n : [n+1] \to [n]$$

the surjective map that "doubles" i. Given a simplicial set X, the induced maps

$$d_n^i := X(\delta_n^i) : X_n \to X_{n-1}$$

are called *face maps* and the induced maps

$$s_n^i := X(\sigma_n^i) : X_n \to X_{n+1}$$

are called *degeneracy maps*.

Remark 1.4. To specify a simplicial set X, it is enough to specify the sets X_n along with face and degeneracy maps satisfying certain relations (which one can read off the simplex category $\mathbf{\Delta}$). We will often depict X by the diagram

$$\cdots \substack{\to\\\to\\\to\\\to} X_2 \substack{\to\\\to\\\to} X_1 \rightrightarrows X_0$$

where for simplicity we only draw the face maps.

Example 1.5. For every $n \geqslant 0$, the *standard n-simplex* is a simplicial set Δ^n whose set of k-simplices ($k \geqslant 0$) is

$$\Delta^n_k = \mathrm{Hom}_{\mathbf{\Delta}}([k],[n]).$$

That is, an k-simplex of Δ^n is an increasing sequence of integers $(a_0, \dots, a_k)$ with $0 \leqslant a_i \leqslant a_j \leqslant n$ for all $i \leqslant j$. Given a morphism $\alpha : [j] \to [k]$, the induced map

$$\alpha^* : \Delta^n_k \to \Delta^n_j$$

sends $([k] \to [n])$ to the composite $([j] \to [k] \to [n])$.

Remark 1.6. Let X be a simplicial set. By the Yoneda lemma, the datum of an n-simplex $x \in X_n$ is the same as that of a morphism $x : \Delta^n \to X$.

Example 1.7. For every $n \geqslant 0$ and $0 \leqslant k \leqslant n$, let $\Delta^{n-1} \to \Delta^n$ denote the map of standard simplices induced by $\delta_n^k : [n-1] \to [n]$; on i-simplices it sends $([i] \to [n-1])$ to $([i] \to [n-1] \to [n])$. Its image is a simplicial subset

$$\partial^k \Delta^n \subseteq \Delta^n$$

called the *kth face* of the standard n-simplex. The union of $\partial^k \Delta^n$ over k is a simplicial subset

$$\partial \Delta^n \subseteq \Delta^n$$

called the *boundary* of the standard n-simplex.

Example 1.8. For every $n \geqslant 0$ and $0 \leqslant k \leqslant n$, the union of the faces $\partial^j \Delta^n$ over $j \neq k$ is a simplicial subset

$$\Lambda^n_k \subseteq \Delta^n$$

called the *kth horn* of the standard n-simplex. In other words, Λ^n_k is the boundary $\partial \Delta^n$ minus the kth face $\partial^k \Delta^n$.

1.2. **Categories as simplicial sets.**

Construction 1.9. Let $\mathcal{C}$ be a category. The *nerve of* $\mathcal{C}$ is a simplicial set $N(\mathcal{C})$ defined as follows. For every $n \geqslant 0$, we set

$$N(\mathcal{C})_n := \mathrm{Fun}([n], \mathcal{C}).$$

For every $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, $\alpha^* : N(\mathcal{C})_n \to N(\mathcal{C})_m$ is given by composition: $([n] \to \mathcal{C}) \mapsto ([m] \to [n] \to \mathcal{C})$.

Remark 1.10. In other words, n-simplices of $N(\mathcal{C})$ are strings

$$c_0 \to c_1 \to \cdots \to c_n$$

of morphisms in $\mathcal{C}$ (where c_i are objects of $\mathcal{C}$). For example, 0-simplices are objects of $\mathcal{C}$, 1-simplices are morphisms of $\mathcal{C}$, 2-simplices are diagrams $c_0 \to c_1 \to c_2$ in $\mathcal{C}$, and so on. Informally speaking, the simplicial set $N(\mathcal{C})$ contains all the information about the category $\mathcal{C}$.

Exercise 1.11. The assignment $\mathcal{C} \mapsto N(\mathcal{C})$ determines a fully faithful functor $N : \mathrm{Cat} \to \mathrm{SSet}$ from the category of categories to the category of simplicial sets. Moreover, it admits a left adjoint $\tau : \mathrm{SSet} \to \mathrm{Cat}$ (hint: left Kan extension along $\Delta \to \mathrm{Cat}$).

Exercise 1.11 means that we can think of categories as "special" simplicial sets, or conversely of simplicial sets as "generalized" categories. The second point of view leads to the question: to what extent do simplicial sets admit a category theory?

Definition 1.12. Let X be a simplicial set.

(i) An *object* x of X is a 0-simplex $x \in X_0$, or by Yoneda, a morphism $x : \Delta^0 \to X$.

(ii) A *morphism* f in X is a 1-simplex $f \in X_1$, or by Yoneda, a morphism $f : \Delta^1 \to X$.

(iii) The *source* (resp. *target*) of a morphism f in X is the image of the corresponding 1-simplex $f \in X_1$ along the face map $d_1^1 : X_1 \to X_0$ (resp. $d_1^0 : X_1 \to X_0$). Equivalently, in terms of the corresponding morphism $f : \Delta^1 \to X$, these are the composites

$$\Delta^0 \to \Delta^1 \xrightarrow{f} X$$

with the inclusions of the two faces of the standard 1-simplex.

(iv) For an object x of X, the *identity morphism* $\mathrm{id}_x \in X_1$ is the image of the 0-simplex $x \in X_0$ by the degeneracy map $s_0^0 : X_0 \to X_1$. Equivalently, in terms of the corresponding morphism $x : X^0 \to X$, it is the composite

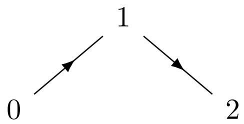
$$\Delta^1 \to \Delta^0 \xrightarrow{x} X$$

with the (unique) morphism $\Delta^1 \to \Delta^0$.

Notation 1.13. We write $s := d_1^1 : X_1 \to X_0$ and $t := d_1^0 : X_1 \to X_0$ for the source and target maps, respectively. Given two objects $x, y \in X_0$, we use the shorthand $f : x \to y$ to indicate that $f \in X_1$ is a 1-simplex with source $x = s(f)$ and target $y = t(f)$.

A defining feature of morphisms in category theory is that they are *composable*. So to justify the above definitions we need to understand how composition should work in this context.

Remark 1.14. Consider the horn Λ^2_1, which we may depict as "two Δ^1's attached at a Δ^0":



A more precise way of putting this is that there is a cartesian and cocartesian square of simplicial sets

$$\begin{array}{ccc} \Delta^0 & \hookrightarrow & \Delta^1 \\ \downarrow & & \downarrow \\ \Delta^1 & \hookrightarrow & \Lambda^2_1. \end{array}$$

Definition 1.15. Let X be a simplicial set.

(i) A *composable pair of morphisms* in X is a morphism $\Lambda^2_1 \to X$. This is the same data as that of two morphisms f and g in X such that $t(f) = s(g)$.

(ii) A *composition* of a composable pair $\sigma : \Lambda^2_1 \to X$ is a 2-simplex $\widetilde{\sigma}$ extending σ. That is, it is a morphism $\widetilde{\sigma} : \Delta^2 \to X$ such that $\widetilde{\sigma}|_{\Lambda^2_1} = \sigma$.

(iii) More generally, given a morphism $\sigma : \Lambda^n_k \to X$, where $n \geqslant 2$ and $0 < k < n$, a *composition* of σ is an extension $\widetilde{\sigma} : \Delta^n \to X$.

The question now becomes: in which simplicial sets do compositions exist (uniquely)? If we include the uniqueness requirement, it turns out that this exactly characterizes nerves of categories.

Proposition 1.16 (Grothendieck–Segal)**.** *Let X be a simplicial set. Then the following two conditions are equivalent:*

(i) *X belongs to the essential image of the fully faithful functor $N : \mathrm{Cat} \to \mathrm{SSet}$ (see Exercise 1.11).*

(ii) *For every $n \geqslant 2$ and every $0 < k < n$, the map*

$$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$$

given by restriction along the inclusion $\Lambda^n_k \subseteq \Delta^n$, is bijective.

1.3. **Groupoids and Kan complexes.** In the previous subsection we attempted to set up a "category theory" for simplicial sets, but just ended up recovering usual category theory. Things start to get more interesting if we relax the *uniqueness* condition on composition. Let's first play with this idea in the context of groupoids (categories in which all morphisms are invertible).

Remark 1.17. There is a variant of Proposition 1.16 which characterizes *groupoids* $\mathcal{C}$ by the bijectivity of the restriction map

$$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$$

for all $n \geqslant 2$ and $0 \leqslant k \leqslant n$. The edge cases $k = 0$ and $k = n$ correspond to invertibility of morphisms (rather than composition).

Definition 1.18. A simplicial set X is called a *Kan complex* if

$$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$$

is *surjective* for all $n \neq 0$ and $0 \leqslant k \leqslant n$. For example, for a category $\mathcal{C}$, $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

Kan complexes (named after Daniel Kan) are like generalized groupoids where compositions (and inverses) exist, but not uniquely. This weak composition still turns out to yield surprisingly good behaviour, at least up to homotopy:

Theorem 1.19 (Milnor)**.** *There is an equivalence between the homotopy category*[1] *of CW complexes and that of Kan complexes, given by the construction* $X \mapsto \mathrm{Sing}(X)_\bullet$ *sending a CW complex to its singular simplicial set, whose n-simplices are continuous maps* $\Delta^n_{\mathrm{top}} \to X$ *(with* Δ^n_{top} *the* topological *standard n-simplex).*

In view of Theorem 1.19 we can think of objects (0-simplices) of a Kan complex as points in a space, and of morphisms as paths between points. Through this equivalence, we see that up to homotopy, composition in a Kan complex is not that bad: for example, composites not only exist but are unique at least *up to homotopy*. This is encouraging.

1.4. **∞-Categories as weak Kan complexes.** Building on what we have seen so far, our next hope to isolate a class of simplicial sets where composition is well-behaved up to homotopy, but where not all morphisms are required to be invertible.

$$\begin{array}{ccc} \text{groupoid} & \longrightarrow & \text{category} \\ \downarrow & & \downarrow \\ \text{Kan complex} & \longrightarrow & ? \end{array}$$

Definition 1.20 (Boardman–Vogt)**.** A simplicial set X is called a *weak Kan complex* (a.k.a. *quasi-category*) if the restriction map

$$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$$

is *surjective* for all $n \geqslant 2$ and $0 < k < n$.

[1]The *homotopy category* of CW complexes is the categorical localization (in the sense of Gabriel–Zisman, see [GZ, Chap. I]) with respect to weak homotopy equivalences. This is analogous to the derived category (which is a categorical localization with respect to quasi-isomorphisms). The homotopy category of Kan complexes is defined similarly; for the definition of homotopy groups and weak homotopy equivalences of Kan complexes, see [GZ, Chap. VI, §3]. See [GZ, Chap. VII, §3] for a proof of Theorem 1.19.

Construction 1.21. Given a weak Kan complex X and two morphisms $f, g : x \to y$ in X, a *homotopy* $f \sim g$ is a 2-simplex $\sigma : \Delta^2 \to X$ of the form

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & & \overset{\mathrm{id}}{\searrow} & \\ x & & \xrightarrow{\quad g \quad} & & y. \end{array}$$

This defines an equivalence relation on the set of morphisms $x \to y$. The *homotopy category* $\mathrm{h}(X)$ is the category whose set of objects is X_0 and, for $x, y \in X_0$, the set $\mathrm{Hom}_{\mathrm{h}(X)}(x, y)$ is the set of equivalence classes of morphisms $x \to y$. Since X is a weak Kan complex, we can compose such equivalence classes and check that this gives a well-defined category $\mathrm{h}(X)$. Moreover, one can prove that $\mathrm{h}(X) \simeq \tau(X)$ (where τ is as in Exercise 1.11).

Definition 1.22. Let X be a weak Kan complex. A morphism $f : x \to y$ is an *isomorphism* if it is invertible, i.e., if there exists a morphism $g : y \to x$ and homotopies $f \circ g \sim \mathrm{id}_y$ and $g \circ f \sim \mathrm{id}_x$. One can prove that a morphism is an isomorphism if and only if it induces an isomorphism in $\mathrm{h}(X)$. We say that X is an ∞-*groupoid* if every morphism in X is an isomorphism.

Theorem 1.23 (Joyal)**.** *Let X be a weak Kan complex. Then X is an ∞-groupoid if and only if X is a Kan complex.*

Remark 1.24. We can think of weak Kan complexes as those simplicial sets which admit compositions up to coherent homotopy. Moreover, after the extensive work of André Joyal and Jacob Lurie, weak Kan complexes do admit a full "category theory":

(i) Given weak Kan complexes X and Y, the internal hom $\underline{\mathrm{Hom}}(X, Y)$ behaves like a functor category $\mathrm{Fun}(X, Y)$. Recall that the n-simplices of $\underline{\mathrm{Hom}}(X, Y)$ are maps $\Delta^n \times X \to Y$.

(ii) Given a weak Kan complex X and objects x, y of X, there is a Kan complex $\mathrm{Maps}_X(x, y)$ of maps $x \to y$, defined by the cartesian square

$$\begin{array}{ccc} \mathrm{Maps}_X(x,y) & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^1, X) \\ \downarrow & & \downarrow {\scriptstyle (s,t)} \\ \Delta^0 & \xrightarrow{(x,y)} & X \times X. \end{array}$$

This is a replacement for the Hom-set $\mathrm{Hom}(x, y)$.

This justifies the following definition:

Definition 1.25. An ∞-*category* is a weak Kan complex.

Remark 1.26. The only difference between the two terms is that *weak Kan complex* refers to a specific model ("shadow") of the platonic notion of ∞-*category*; similarly for Kan complexes vs. ∞-groupoids. In accordance with this point of view, we will simply write $\mathcal{C}$ instead of $N(\mathcal{C})$ when we want to think of an ordinary category $\mathcal{C}$ as an ∞-category (as opposed to a weak Kan complex). Similarly, we will use letters like $\mathcal{C}$ and $\mathcal{D}$ instead of X and Y when we want to think of them as ∞-categories, and we will write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ instead of $\underline{\mathrm{Hom}}(X, Y)$.

1.5. **The ∞-category of (∞-)groupoids.** Recall that for a category $\mathcal{C}$ we have the weak Kan complex $N(\mathcal{C})$ in which an n-simplex is determined by the following data:

- objects $C_i \in \mathcal{C}$ for $0 \leqslant i \leqslant n$;
- morphisms $f_{i,j} : C_i \to C_j$ for $0 \leqslant i < j \leqslant n$;

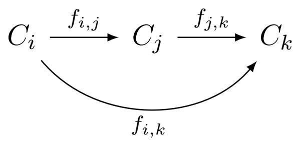
satisfying the relations $f_{j,k} \circ f_{i,j} = f_{i,k}$ for all $0 \leqslant i < j < k \leqslant n$.

Construction 1.27. Let $\mathcal{C}$ be a 2-category. The *nerve* (or *Duskin nerve*) of $\mathcal{C}$ is a simplicial set $N^D(\mathcal{C})$. An n-simplex of $N^D(\mathcal{C})$ is determined by the following data:

- objects $C_i \in \mathcal{C}$ for $0 \leqslant i \leqslant n$;
- morphisms $f_{i,j} : C_i \to C_j$ for $0 \leqslant i < j \leqslant n$;
- 2-morphisms $f_{j,k} \circ f_{i,j} \Rightarrow f_{i,k}$ for all $0 \leqslant i < j < k \leqslant n$;

This data is required to satisfy certain compatibility relations involving $\mu_{i,j,k}$, $\mu_{i,j,l}$, $\mu_{i,k,l}$, and $\mu_{j,k,l}$ (for $0 \leqslant i < j < k < l \leqslant n$).

Informally speaking, the difference between $N^D(\mathcal{C})$ and the nerve of the underlying 1-category (where we discard the 2-morphisms) is that the diagrams



only commute up to the specified natural transformation $\mu_{i,j,k}$ (which need not be invertible).

Theorem 1.28 (Duskin)**.** *Let $\mathcal{C}$ be a 2-category. The following conditions are equivalent:*

(i) *$\mathcal{C}$ is a $(2,1)$-category; i.e., the 2-morphisms of $\mathcal{C}$ are all invertible.*

(ii) *$N^D(\mathcal{C})$ is a weak Kan complex.*

Notation 1.29. Let $\mathcal{C}$ be a category, resp. $(2,1)$-category. We will write simply $\mathcal{C}$ for the ∞-category whose underlying weak Kan complex is $N(\mathcal{C})$, resp. $N^D(\mathcal{C})$.

Example 1.30. Groupoids naturally form a 2-category whose 2-morphisms are natural transformations. We denote by Grpd the $(2,1)$-category where we discard the non-invertible 2-morphisms. We also use the same notation for the associated ∞-category (whose underlying weak Kan complex is $N^D(\mathrm{Grpd})$).

Construction 1.31. Consider the (large) simplicial set $\mathrm{Kan}_\bullet$ in which an n-simplex is given by the following data:

- Kan complexes K_i for $0 \leqslant i \leqslant n$.
- Maps of Kan complexes $f_{i,j} : K_i \to K_j$ for $0 \leqslant i \leqslant j \leqslant n$.
- A system of "coherent homotopies" up to which $f_{i,j}$ are compatible under composition.

For example, a 2-simplex is a tuple $(X, Y, Z, f, g, h, \sigma)$, where X, Y and Z are Kan complexes, $f : X \to Y$, $g : Y \to Z$, $h : X \to Z$ are maps, and $\sigma \in \underline{\mathrm{Hom}}(X, Z)_1$ with $d_1^0(\sigma) = g \circ f$ and $d_1^1(\sigma) = h$. Here σ is a "homotopy" $g \circ f \simeq h$. The term "coherent" is a shorthand which indicates that not only do we have such homotopies, we are also given higher homotopies between these homotopies (starting from $n \geqslant 3$), even higher homotopies between those homotopies, and so on.

Remark 1.32. The simplicial set $\mathrm{Kan}_\bullet$ is a weak Kan complex. In fact, it is an instance of a general construction called the *homotopy coherent nerve* which takes a simplicially enriched category (in this case, that of Kan complexes) as input and yields a weak Kan complex as output. See [Lur, Tag 00KS].

Remark 1.33. There is a map of weak Kan complexes $N(\mathrm{Set}) \to \mathrm{Kan}_\bullet$ that sends $X \mapsto c(X)$ on 0-simplices. As a functor of ∞-categories, it is fully faithful with essential image spanned by Kan complexes X that are homotopy equivalent to a constant simplicial set, or equivalently, which satisfy $\pi_i(X) = 0$ for all $i > 0$.

Later on, we will see how the ∞-category corresponding to the weak Kan complex $\mathrm{Kan}_\bullet$ is a very fundamental object called the ∞-category of *anima*. In practice, we will work with this ∞-category by manipulating its universal properties and deliberately avoid any considerations involving the simplices of $\mathrm{Kan}_\bullet$.

2. Sheaves and stacks

2.1. **Sheaves.** Let $\mathcal{C}$ be a site, i.e., a category equipped with a Grothendieck topology τ. Roughly speaking, τ amounts to a notion of *covering sieves* for every object in the category $\mathcal{C}$. For simplicity, we will assume that τ arises from the following construction.

Construction 2.1. Assume that $\mathcal{C}$ admits fibred products and finite coproducts, and satisfies the following conditions:

(i) For any finite collection of objects $X_i \in \mathcal{C}$, any morphism $f : \coprod_i X_i \to Y$, and any morphism $Y' \to Y$, the canonical morphism $\coprod_i X_i \times_Y Y' \to (\coprod_i X_i) \times_Y Y'$ is invertible.

(ii) Coproducts are disjoint: for any pair of objects X and Y in $\mathcal{C}$, the fibred product $X \times_{X \coprod Y} Y$ is an initial object in $\mathcal{C}$.

Let S be a collection of morphisms in $\mathcal{C}$, which contains all isomorphisms and is stable under composition, and satisfies the following conditions:

(i) For every morphism $f : X \to Y$ in S, the base change $f' : X \times_Y Y' \to Y'$ along any morphism $Y' \to Y$ in $\mathcal{C}$ belongs to S.

(ii) For every finite collection of morphisms $(f_i : X_i \to Y_i)_i$ in S, the induced morphism $\coprod_i X_i \to \coprod_i Y_i$ belongs to S.

Then there is a Grothendieck topology τ on $\mathcal{C}$ where a sieve on an object $X \in \mathcal{C}$ is covering if and only if it contains a finite collection of morphisms $\{X_i \to X\}_i$ such that the induced morphism $\coprod_i X_i \to X$ belongs to S. We refer to this as the Grothendieck topology *generated by* the collection S. See [SAG, Prop. A.3.2.1].

Example 2.2. Let X be a topological space. Then the category $\mathcal{U}(X)$ of opens $U \subseteq X$ (where there is a morphism $U \to V$ if and only if $U \subseteq V$) admits a Grothendieck topology generated by surjections. In particular, a sieve on $U \subseteq X$ is covering if and only if it contains a finite collection of morphisms $(U_i \hookrightarrow U)_i$ such that $U = \bigcup_i U_i$.

Example 2.3. Let X be a scheme. The *small étale site* $X_{\text{ét}}$ is the category of étale morphisms $U \to X$ (where U is a scheme), with the Grothendieck topology generated by surjections (equivalently, faithfully flat morphisms).

Example 2.4. Let X be a scheme. The *big étale site* is the category $\text{Sch}_{/X}$ of (arbitrary) morphisms $Y \to X$ (where Y is a scheme), with the Grothendieck topology generated by étale surjections (equivalently, faithfully flat and étale morphisms). (We also have the étale topology on the category Sch, which is the special case where $X = \text{Spec}(\mathbf{Z})$.)

Definition 2.5. Let $\mathcal{V}$ be an ∞-category. Let F be a diagram in $\mathcal{V}$ indexed by an ∞-category I, i.e., a functor of ∞-categories $F : I \to \mathcal{C}$. Suppose given an object $V \in \mathcal{V}$ and a natural transformation $\alpha : V_{\text{cst}} \to F$ where V_{cst} denotes the constant diagram $(i \in I) \mapsto (V \in \mathcal{V})$. We say that the pair (V, α) in $\mathcal{V}$ *exhibits V as the limit of F* if for every object $V' \in \mathcal{V}$ the induced functor of mapping ∞-groupoids

$$\text{Maps}_{\mathcal{V}}(V', V) \to \text{Maps}_{\text{Fun}(I,\mathcal{V})}(V'_{\text{cst}}, F),$$

sending $(V' \to V) \mapsto (V'_{\text{cst}} \to V_{\text{cst}} \to F)$, is invertible. In this case, we will write

$$V \simeq \varprojlim_{i \in I} F_i := \varprojlim(F).$$

Definition 2.6. Let $\mathcal{V}$ be an ∞-category. A *presheaf* on $\mathcal{C}$ with values in $\mathcal{V}$ is a functor $F : \mathcal{C}^{\text{op}} \to \mathcal{V}$. A presheaf F is a *sheaf* if it satisfies the following conditions:

(i) F sends finite coproducts in $\mathcal{C}$ to products in $\mathcal{V}$. In other words, for every finite collection $(X_i)_i$ of objects of $\mathcal{C}$, the canonical morphism

$$F(\coprod_i X_i) \to \prod_i F(X_i)$$

is invertible.

(ii) For every morphism $f : U \to X$ in S, let $U_\bullet$ denote the Čech nerve of f, i.e., the simplicial object

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} U \times_U U \times_X U \substack{\rightarrow\\\rightarrow\\\rightarrow} U \times_X U \rightrightarrows U$$

whose nth term is the $(n+1)$-fold fibre power $U \times_X \cdots \times_X U$. Then the canonical map

$$F(X) \to \varprojlim_{[n] \in \mathbf{\Delta}} F(U_n)$$

is invertible. In other words, the diagram

$$F(X) \to F(U) \rightrightarrows F(U \underset{X}{\times} U) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} F(U \underset{U}{\times} U \underset{X}{\times} U) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

exhibits $F(X)$ as the limit of $F(U_\bullet)$.

Notation 2.7. We denote by $\mathrm{Shv}(\mathcal{C}; \mathcal{V})$ the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})$ spanned by sheaves.

When $\mathcal{V}$ is a 1-category (resp. $(2,1)$-category), this is equivalent to the usual sheaf condition:

Proposition 2.8. *Let $V^\bullet : \mathbf{\Delta} \to \mathcal{V}$ be a cosimplicial diagram in an ∞-category $\mathcal{V}$. If $\mathcal{V}$ is equivalent to a 1-category, then the limit of $V^\bullet$ is identified with the equalizer of $V^0 \rightrightarrows V^1$:*

$$\varprojlim_{[n] \in \mathbf{\Delta}} V^n \simeq \varprojlim (V^0 \rightrightarrows V^1).$$

Similarly, if $\mathcal{V}$ is equivalent to a $(2,1)$-category, then it is identified with the 2-limit:

$$\varprojlim_{[n] \in \mathbf{\Delta}} V^n \simeq 2\text{-}\varprojlim (V^0 \rightrightarrows V^1 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} V^2).$$

Remark 2.9. More generally, say $\mathcal{V}$ is an $(n,1)$-*category* if all mapping ∞-groupoids $\mathrm{Maps}_{\mathcal{V}}(V, V')$ are $(n-1)$-truncated (have trivial higher homotopy groups $\pi_i \mathrm{Maps}_{\mathcal{V}}(V, V') = 0$ for $i \geqslant n$). In this case the limit of $V^\bullet$ is isomorphic to the limit of the restriction $V|_{\mathbf{\Delta}_{\leqslant n}}$ to the full subcategory of $\mathbf{\Delta}$ spanned by the objects $[0], [1], \ldots, [n]$. (This follows from a variant of Quillen's Theorem A, because the inclusion $\mathbf{\Delta}_{\leqslant n} \hookrightarrow \mathbf{\Delta}$ is an n-*final* functor, i.e. the category $\mathbf{\Delta}_{\leqslant n} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[m]}$ has n-connected nerve for every $[m] \in \mathbf{\Delta}$.) Note also that a limit over $\mathbf{\Delta}_{\leqslant 1}$ is (by an easy finality argument) isomorphic to the limit over the subcategory where the morphism $[1] \to [0]$ is discarded; i.e., it is the equalizer of the two parallel arrows $V^0 \twoheadrightarrow V^1$.

Definition 2.10. A *stack* is a sheaf of groupoids on the category of schemes (with the étale topology).

In other words, a stack $\mathcal{X}$ is a functor

$$\mathcal{X} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpd}$$

to the $(2,1)$-category of groupoids such that for every finite collection of étale morphisms $(U_i \to U)_i$ which is jointly surjective, the diagram

$$\mathcal{X}(U) \to \prod_i \mathcal{X}(U_i) \rightrightarrows \prod_{i,j} \mathcal{X}(U_i \underset{U}{\times} U_j) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \prod_{i,j,k} \mathcal{X}(U_i \underset{U}{\times} U_j \underset{U}{\times} U_k)$$

is a limit diagram in the $(2,1)$-category of groupoids.

2.2. **Bases of topologies.** Denote by Aff ⊆ Sch the full subcategory spanned by affine schemes. Note that the étale topology on Sch restricts to Aff, and is still generated in the sense of Construction 2.1 by étale surjections.

Theorem 2.11. *Let $\mathcal{V}$ be an ∞-category admitting limits. The canonical functor*

$$\mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathcal{V}) \to \mathrm{Fun}(\mathrm{Aff}^{\mathrm{op}}, \mathcal{V}),$$

given by restriction along the inclusion Aff ⊆ Sch*, restricts to an equivalence of ∞-categories*

$$\mathrm{Shv}(\mathrm{Sch}; \mathcal{V}) \to \mathrm{Shv}(\mathrm{Aff}; \mathcal{V}).$$

Example 2.12. Stacks can be defined as sheaves of groupoids on Aff.

Theorem 2.11 is a special case of a following more general result.

Definition 2.13. Let $\mathcal{C}$ be a site with Grothendieck topology generated by a class of morphisms S as in Construction 2.1. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under fibred products and finite coproducts, and regard it with the Grothendieck topology generated by $S \cap \mathcal{C}_0$ (the subclass of morphisms in S whose source and target belong to $\mathcal{C}_0$). We say that $\mathcal{C}_0$ is a *basis* for $\mathcal{C}$ if for every object $X \in \mathcal{C}$ there exists a collection of morphisms $(Y_i \to X)_i$ such that $Y_i \in \mathcal{C}_0$, the coproduct $\coprod_i Y_i$ exists in $\mathcal{C}$, and $\coprod_i Y_i \to X$ belongs to S.

Theorem 2.14. *In the situation of Definition 2.13, the functor*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{V})$$

restricts to an equivalence

$$\mathrm{Shv}(\mathcal{C}; \mathcal{V}) \to \mathrm{Shv}(\mathcal{C}_0; \mathcal{V})$$

for all ∞-categories $\mathcal{V}$ admitting limits.

We can moreover give a more precise version of Theorem 2.14.

Definition 2.15. Let $i : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ be a fully faithful functor of categories. Let $F_0 : \mathcal{C}_0^{\mathrm{op}} \to \mathcal{V}$ be a presheaf with values in an ∞-category $\mathcal{V}$ admitting limits. The *right Kan extension* of F_0, denoted[2]

$$F \coloneqq \mathrm{RKE}_{\mathcal{C}_0 \hookrightarrow \mathcal{C}}(F_0) \coloneqq i_*(F_0)$$

is the unique limit-preserving functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ which restricts to F_0. Explicitly, it is given by the formula

$$F(X) \simeq \varprojlim_{(Y,f)} F(Y)$$

where the limit is taken over the category of pairs (Y, f) where $Y \in \mathcal{C}_0$ and $f : i(Y) \to X$ is a morphism in $\mathcal{C}$ (and morphisms $(Y', f') \to (Y, f)$ are morphisms $Y' \to Y$ in $\mathcal{C}_0$ which are compatible with f and f').

Theorem 2.16. *In the situation of Definition 2.13, let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ be a $\mathcal{V}$-valued presheaf on $\mathcal{C}$ where $\mathcal{V}$ is an ∞-category with limits. Then F is a sheaf if and only if the following conditions hold:*

[2] The explanation for the notation $i_*(F_0)$ is that the assignment $F_0 \mapsto F$ actually determines a right adjoint i_* to the restriction functor $i^* : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{V})$.

(i) $F_0 := F|_{\mathcal{C}_0}$ *is a sheaf on* $\mathcal{C}_0$.

(ii) F *is the right Kan extension of* F_0 *along* $\mathcal{C}_0 \to \mathcal{C}$.

See [Aok, Cor. A.8] for a proof.

3. The stack of quasi-coherent sheaves

3.1. **Cartesian fibrations.** Given a scheme X, quasi-coherent sheaves on X form a category $\mathrm{QCoh}(X)$. For any morphism $f : X \to Y$, we have the adjoint pair of functors

$$f^* : \mathrm{QCoh}(Y) \rightleftarrows \mathrm{QCoh}(X) : f_*$$

where the left adjoint f^* is inverse image and the right adjoint f_* is direct image. Note that for a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$, the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Z) & \xrightarrow{\quad g^* \quad} & \mathrm{QCoh}(Y) \\ & {\scriptstyle (g\circ f)^*} \searrow \quad \swarrow {\scriptstyle f^*} & \\ & \mathrm{QCoh}(X) & \end{array}$$

does not commute in the 1-category of categories. Instead, there is an invertible natural transformation

$$(g \circ f)^* \to f^* \circ g^*$$

up to which it commutes. Together with this extra piece of data, the diagram above does determine a commutative diagram in the ∞-category Grpd.

In particular, the assignment $X \mapsto \mathrm{QCoh}(X)$ cannot be assembled into a functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}'$ into the 1-category Cat' of categories, but only into a functor

$$\mathrm{QCoh} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}$$

into the $(2,1)$-category[3] (or equivalently, ∞-category) of categories. Still, a precise construction of this functor requires more than just the data of the invertible natural transformation above for all pairs of morphisms f and g; for example, we need to require compatibilities between this data whenever we have three composable morhpisms. This is somewhat messy, so we will prefer to take the following alternative perspective.

Definition 3.1. Let $\pi : \mathcal{E} \to \mathcal{C}$ be a functor of categories. Let $f : C \to D$ be a morphism in $\mathcal{C}$ and $\widetilde{D} \in \mathcal{E}$ a lift of D (so that $\pi(\widetilde{D}) = D$). Let $\widetilde{f} : \widetilde{C} \to \widetilde{D}$ be a lift of f, i.e. $\pi(\widetilde{C}) = C$ and $\pi(\widetilde{f}) = f$. We say that $\widetilde{f}$ is π-*cartesian* if for every $E \in \mathcal{E}$ we require that the canonical map

$$\mathrm{Hom}_{\mathcal{E}}(E, \widetilde{C}) \to \mathrm{Hom}_{\mathcal{E}}(E, \widetilde{D}) \underset{\mathrm{Hom}_{\mathcal{C}}(\pi(E), D)}{\times} \mathrm{Hom}_{\mathcal{C}}(\pi(E), \pi(\widetilde{C}))$$

is bijective. Informally speaking, $\widetilde{f}$ is terminal among all lifts of f with target $\widetilde{D}$.

[3] Recall that by convention, we identify $(2,1)$-categories $\mathcal{C}$ with the corresponding ∞-category (whose underlying weak Kan complex is the Duskin nerve $N^D(\mathcal{C})$).

Definition 3.2. Let $\pi : \mathcal{E} \to \mathcal{C}$ be a functor of categories. We say that π is a *cartesian fibration* if for every $E \in \mathcal{E}$, every $C \in \mathcal{C}$, and every morphism $f : C \to \pi(E)$ in $\mathcal{C}$, there exists a lift $\widetilde{C} \in \mathcal{E}$ of C and a π-cartesian morphism $\widetilde{f} : \widetilde{C} \to E$ lifting f.

Example 3.3. Let $\mathrm{QCoh}_{\mathrm{Sch}}$ denote the category of pairs $(X, \mathcal{F})$ where $X \in \mathrm{Sch}$ and $\mathcal{F} \in \mathrm{QCoh}(X)$. A morphism $(X', \mathcal{F}') \to (X, \mathcal{F})$ is a morphism $f : X' \to X$ together with a morphism $\phi : f^*\mathcal{F} \to \mathcal{F}'$ in $\mathrm{QCoh}(X')$. Given morphisms $(f, \phi) : (X', \mathcal{F}') \to (X, \mathcal{F})$ and $(g, \psi) : (X'', \mathcal{F}'') \to (X', \mathcal{F}')$, the composite is defined by

$$f \circ g : X'' \to X \to X$$

and

$$g^* f^* \mathcal{F} \xrightarrow{g^*\phi} g^*\mathcal{F}' \xrightarrow{\psi} \mathcal{F}''.$$

Then the projection $(X, \mathcal{F}) \mapsto X$ determines a functor

$$\mathrm{QCoh}_{\mathrm{Sch}} \to \mathrm{Sch}$$

which is a cartesian fibration.

Construction 3.4. Let $\mathcal{C}$ be a fixed category. We denote by $\mathrm{Cart}(\mathcal{C})$ the $(2,1)$-category whose objects are cartesian fibrations $\pi : \mathcal{E} \to \mathcal{C}$, whose 1-morphisms $(\pi' : \mathcal{E}' \to \mathcal{C}) \to (\pi : \mathcal{E} \to \mathcal{C})$ are morphisms $f : \mathcal{E}' \to \mathcal{E}$ that are compatible with π and π', and whose 2-morphisms $f \Rightarrow g$ (where f, g are morphisms $\pi' \to \pi$) are invertible natural transformations $\theta : f \Rightarrow g$ such that for all $E' \in \mathcal{E}'$, $\pi : \mathcal{E} \to \mathcal{C}$ sends

$$\theta_{E'} : f(E') \to g(E')$$

to the identity of $\pi(f(E')) = \pi(g(E'))$.

Theorem 3.5 (Grothendieck)**.** *For every category $\mathcal{C}$, there is an equivalence of ∞-categories*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Cart}(C).$$

Definition 3.6. Given $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$, the corresponding cartesian fibration is called the *unstraightening* of F. Given a cartesian fibration $\pi : \mathcal{E} \to \mathcal{C}$, the *straightening* of π is the (essentially unique) presheaf of categories whose unstraightening is π.

Remark 3.7. Lurie proved a generalization of Theorem 3.5, where $\mathcal{C}$ is allowed to be an ∞-category and Cat is replaced by the ∞-category of ∞-categories.

Definition 3.8. We let $\mathrm{QCoh} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}$ denote the unstraightening of the cartesian fibration $\mathrm{QCoh}_{\mathrm{Sch}} \to \mathrm{Sch}$. We let $\mathrm{QCoh}^{\simeq} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpd}$ denote the presheaf of groupoids given by sending

$$X \mapsto \mathrm{QCoh}(X)^{\simeq}$$

where $(-)^{\simeq}$ indicates that we discard all non-invertible morphisms.

3.2. **Descent for quasi-coherent sheaves.**

Theorem 3.9. *The presheaf of categories* $\mathrm{QCoh} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}$ *satisfies étale descent.*

In fact, we will see that it even satisfies descent for the *fpqc*[4] topology on Sch, which is the Grothendieck topology generated by faithfully flat quasi-compact morphisms.

Corollary 3.10. *The presheaf of groupoids* $\mathrm{QCoh}^{\simeq} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpd}$ *satisfies descent. In particular, it is a stack.*

Proof. The functor $\mathrm{Cat} \to \mathrm{Grpd}$ sending $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$ preserves limits. In fact, it is right adjoint to the inclusion $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$. Thus the sheaf condition for $\mathrm{QCoh}^{\simeq}$ follows from that of QCoh. □

To prove Theorem 3.9 we will apply a general descent criterion.

Definition 3.11. Given a cosimplicial diagram $X^\bullet : \mathbf{\Delta} \to \mathcal{C}$ in an ∞-category $\mathcal{C}$, we refer to its limit as the *totalization* of $X^\bullet$, and write

$$\mathrm{Tot}(X^\bullet) := \varprojlim_{\mathbf{\Delta}} X^\bullet.$$

Definition 3.12. Let $\mathbf{\Delta}_+$ denote the category whose objects are the finite sets $[n] = \{0, 1, \dots, n\}$ for all $n \geqslant -1$, where $[-1] = \varnothing$ by convention, and whose morphisms are order-preserving maps. An *augmented cosimplicial diagram* in any ∞-category $\mathcal{C}$ is a functor $X^\bullet : \mathbf{\Delta}_+ \to \mathcal{C}$. We will depict $X^\bullet$ by the diagram

$$X^{-1} \to X^0 \rightrightarrows X^1 \substack{\to\\\to\\\to} X^2 \substack{\to\\\to\\\to\\\to} \cdots.$$

We will say this is a *limit diagram* if the induced morphism of cosimplicial diagrams

$$X^{-1}_{\mathrm{cst}} \to X^\bullet|_{\mathbf{\Delta}}$$

exhibits X^{-1} as the limit (totalization) of $X^\bullet|_{\mathbf{\Delta}}$.

Definition 3.13. Let $\mathbf{\Delta}_{-\infty}$ denote the category whose objects are the finite sets $[n]$ for all $n \geqslant -1$, and whose morphisms $[m] \to [n]$ are order-preserving maps $[m] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ which preserve $-\infty$. A *splitting* of a cosimplicial diagram $X^\bullet : \mathbf{\Delta} \to \mathcal{C}$ is an extension to a functor $X^\bullet : \mathbf{\Delta}_{-\infty} \to \mathcal{C}$. A simplicial diagram $X^\bullet$ is *split* if it admits a splitting. In this case, $X^\bullet|_{\mathbf{\Delta}_+}$ is a limit diagram, i.e. we have $\mathrm{Tot}(X^\bullet|_{\mathbf{\Delta}}) \simeq X^{-1}$.

Theorem 3.14 (Descent criterion)**.** *Let* $\mathcal{C}^\bullet : \mathbf{\Delta}_+ \to \mathrm{Cat}_\infty$ [5] *be an augmented cosimplicial diagram of ∞-categories, which we depict as follows:*

$$\mathcal{C}^{-1} \xrightarrow{F} \mathcal{C}^0 \rightrightarrows \mathcal{C}^1 \substack{\to\\\to\\\to} \mathcal{C}^2 \substack{\to\\\to\\\to\\\to} \cdots. \tag{3.15}$$

Suppose the following conditions hold:

(i) *The functor* $F : \mathcal{C}^{-1} \to \mathcal{C}^0$ *is conservative.*

[4] This stands for "fidèlement plat et quasi-compact" in French.

[5] Here the ∞-category Cat_∞ of ∞-categories may be defined just as we defined the ∞-category of ∞-groupoids, replacing Kan complexes with weak Kan complexes.

(ii) *For every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_+$, let $\beta : [m+1] \to [n+1]$ denote the unique morphism which commutes with $\delta^0 : [m] \to [m+1]$ and $\delta^0 : [n] \to [n+1]$, and consider the commutative square*

$$\begin{array}{ccc} \mathcal{C}^m & \xrightarrow{d^0} & \mathcal{C}^{m+1} \\ \downarrow \alpha & & \downarrow \beta \\ \mathcal{C}^n & \xrightarrow{d^0} & \mathcal{C}^{n+1}. \end{array}$$

Then the horizontal arrows admit right adjoints $d^{0,R}$ which also commute with the vertical arrows; more precisely, the natural transformation

$$\alpha \circ d^{0,R} \xrightarrow{\text{unit}} d^{0,R} \circ d^0 \circ \alpha \circ d^{0,R} \simeq d^{0,R} \circ \beta \circ d^0 \circ d^{0,R} \xrightarrow{\text{counit}} d^{0,R} \circ \beta$$

is invertible.

(iii) *The functor $F : \mathcal{C}^{-1} \to \mathcal{C}^0$ preserves totalizations of F-split simplicial diagrams in $\mathcal{C}^{-1}$. That is, for every cosimplicial diagram $X^\bullet$ in $\mathcal{C}^{-1}$ whose image $F(X^\bullet)$ is split, the canonical map $F(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}(F(X^\bullet))$ is invertible.*

Then the induced functor

$$\mathcal{C}^{-1} \to \mathrm{Tot}(\mathcal{C}^\bullet|_{\mathbf{\Delta}})$$

is an equivalence of ∞-categories. That is, (3.15) is a limit diagram.

This result is a corollary of the monadicity theorem of Barr–Beck, generalized to ∞-categories by Lurie. See [SAG, Cor. 4.7.5.3].

Consider the functor

$$\mathrm{CRing} \to \mathrm{Cat}, \quad R \mapsto \mathrm{Mod}_R$$

sending a commutative ring R to the category Mod_R of R-modules, and a ring homomorphism $\phi : R \to S$ to the extension of scalars functor

$$\phi^* := (-) \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S.$$

It can be defined using Theorem 3.5, or alternatively is the restriction of $\mathrm{QCoh} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}$ to affine schemes (under the equivalence $\mathrm{Aff}^{\mathrm{op}} \simeq \mathrm{CRing}$).

Theorem 3.16. *The functor $R \mapsto \mathrm{Mod}_R$ satisfies descent for the flat topology, i.e. the Grothendieck topology on* CRing *generated by faithfully flat ring homomorphisms.*

Proof. Let $\phi : R \to S$ be a faithfully flat ring homomorphism. We need to show that the augmented cosimplicial diagram

$$\mathrm{Mod}_R \xrightarrow{\phi^*} \mathrm{Mod}_S \rightrightarrows \mathrm{Mod}_{S \otimes_R S} \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \cdots$$

is a limit diagram, to which end we apply the criterion of Theorem 3.14:

(i) Conservativity of the functor ϕ^* is a consequence of the assumption that ϕ is faithfully flat.

(ii) For every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_+$, consider the commutative square

$$\begin{array}{ccc} \mathrm{Mod}_{S^{\otimes m+1}} & \xrightarrow{d^{0,*}} & \mathrm{Mod}_{S^{\otimes m+2}} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \mathrm{Mod}_{S^{\otimes n+1}} & \xrightarrow{d^{0,*}} & \mathrm{Mod}_{S^{\otimes n+2}}, \end{array}$$

where the tensor powers are taken over R and the horizontal arrows are extension of scalars along $d^0 : S^{\otimes m+1} \to S^{\otimes m+2}$. These functors are left adjoint to the restriction of scalars functors d^0_*. The vertical arrows are flat (as base changes of flat homomorphisms), and hence commute with d^0_* by the flat base change formula.

(iii) Let $M^\bullet$ be a cosimplicial diagram in Mod_R such that the cosimplicial diagram $\phi^*(M^\bullet)$ in Mod_S is split. The claim is that the S-module homomorphism

$$\phi^*(\mathrm{Tot}(M^\bullet)) \to \mathrm{Tot}(\phi^*(M^\bullet))$$

is invertible. Since Mod_R and Mod_S are 1-categories, these totalizations can be computed as equalizers (Proposition 2.8). Since ϕ^* is an exact functor, it preserves equalizers.

We also need to show that the functor $R \mapsto \mathrm{Mod}_R$ preserves finite products:

- The category of modules over the zero ring is equivalent to the trivial category.
- For any pair of commutative rings R_1 and R_2, the category of modules over $R_1 \times R_2$ is equivalent to the product of categories $\mathrm{Mod}_{R_1} \times \mathrm{Mod}_{R_2}$.

It follows that $R \mapsto \mathrm{Mod}_R$ is a sheaf for the flat topology. □

The proof of Theorem 3.9 is almost the same, using the following lemma:

Lemma 3.17. *Let $f : X \to Y$ be a morphism of schemes. If f is faithfully flat and quasi-compact, then the inverse image functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is conservative.*

Proof. Let $(V_j \to X)_j$ be a (possibly infinite) family of affine opens $V_j \subseteq Y$ covering Y. For each j, let $V'_j := V_j \times_Y X = f^{-1}(V_j) \subseteq X$. Since f is quasi-compact, V'_j is quasi-compact. Thus there exists a finite family $(U_{i,j} \to V'_j)_i$ where $U_{i,j} \subseteq V'_j$ are affine opens covering V'_j. We have the commutative square

$$\begin{array}{ccc} \coprod_i U_{i,j} & \xrightarrow{f_j} & V_j \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $f_j : \coprod_i U_{i,j} \to V'_j \to V_j$ is a faithfully flat of affine schemes, since faithfully flat morphisms are stable under composition and base change.

Now let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of quasi-coherent sheaves on Y, whose inverse image $f^*(\phi) : f^*(\mathcal{F}) \to f^*(\mathcal{G})$ is invertible. We need to show that ϕ is itself invertible. It will clearly suffice to show that each restriction $\phi|_{V_j}$ is invertible. Since f_j is a faithfully flat morphism between affines, we know (by definition) that f_j^* is conservative (since it corresponds to extension of scalars of a module along a faithfully flat ring homomorphism), so it will moreover suffice to show that ϕ is invertible after inverse image along $\coprod_i U_{i,j} \to V_j \subseteq Y$. But the latter factors through f, hence is invertible by assumption. □

Proof of Theorem 3.9. Same as the proof of Theorem 3.16, except Lemma 3.17 is used to check the first condition of Theorem 3.14. □

3.3. **Quasi-coherent sheaves on stacks.** Since the full subcategory $\mathrm{Aff} \subseteq \mathrm{Sch}$ is a basis with respect to the Zariski topology, we have by Theorem 2.16 the following further consequence of Theorem 3.9:

Corollary 3.18. *The presheaf of categories* $\mathrm{QCoh} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}$ *is right Kan extended from its restriction to* Aff*.*

Under the equivalence $\mathrm{Aff}^{\mathrm{op}} \simeq \mathrm{CRing}$, $\mathrm{QCoh}\,|_{\mathrm{Aff}}$ is identified with the functor $\mathrm{CRing} \to \mathrm{Cat}$ sending $R \mapsto \mathrm{Mod}_R$. Thus we have:

Corollary 3.19. *For every scheme X, there is a canonical equivalence of categories*

$$\mathrm{QCoh}(X) \simeq \varprojlim_{(R,x)} \mathrm{Mod}_R$$

where the limit is taken over the category of pairs (R,x) where $R \in \mathrm{CRing}$ and $x \in X(R)$ is an R-point, where morphisms $(R,x) \to (R',x')$ are ring homomorphisms $R \to R'$ such that $X(R) \to X(R')$ sends x to x'.

Definition 3.20. Let $\mathcal{X}$ be a stack. We define the category $\mathrm{QCoh}(\mathcal{X})$ of quasi-coherent sheaves on $\mathcal{X}$ as the limit

$$\mathrm{QCoh}(\mathcal{X}) := \varprojlim_{(R,x)} \mathrm{Mod}_R$$

over the category of pairs (R,x) where $R \in \mathrm{CRing}$ and $x \in \mathcal{X}(R)$ is an R-point. More precisely, we define

$$\mathrm{QCoh} : \mathrm{Stk}^{\mathrm{op}} \to \mathrm{Cat}$$

as the right Kan extension of the presheaf $\mathrm{Spec}(R) \mapsto \mathrm{Mod}_R$ along the inclusion $\mathrm{Aff} \hookrightarrow \mathrm{Stk}$, where Stk is the ∞-category of stacks.

Remark 3.21. By definition, a quasi-coherent sheaf $\mathcal{F}$ on a stack $\mathcal{X}$ amounts to the following data:

(i) For every commutative ring R and every R-point $x \in \mathcal{X}(R)$, an R-module $\mathcal{F}(x)$.

(ii) For every ring homomorphism $R \to R'$, R-point $x \in \mathcal{X}(R)$, and R'-point $x' \in \mathcal{X}(R')$ such that $\mathcal{X}(R) \to \mathcal{X}(R')$ sends $x \mapsto x'$, an R'-module isomorphism

$$\alpha_{x,x'} : \mathcal{F}(x) \otimes_R R' \simeq \mathcal{F}(x').$$

This data is subject to the following cocycle condition: for every pair of ring homomorphisms $R \to R'$ and $R' \to R''$, R-point $x \in \mathcal{X}(R)$, R'-point $x' \in \mathcal{X}(R')$, and R''-point $x'' \in \mathcal{X}(R'')$ such that $\mathcal{X}(R) \to \mathcal{X}(R')$ sends $x \mapsto x'$ and $\mathcal{X}(R') \to \mathcal{X}(R'')$ sends $x' \mapsto x''$, there is a commutative diagram

$$\begin{array}{ccccc} (\mathcal{F}(x) \otimes_R R') \otimes_{R'} R'' & \xrightarrow{\alpha_{x,x'}} & \mathcal{F}(x') \otimes_{R'} R'' & \xrightarrow{\alpha_{x',x''}} & \mathcal{F}(x'') \\ \| & & & & \| \\ \mathcal{F}(x) \otimes_R R'' & & \xrightarrow{\alpha_{x,x''}} & & \mathcal{F}(x''). \end{array}$$

4. Quotient stacks

Let G be a group. As we discussed in Sect. 0, a G-action on a set X may be encoded by the diagram

$$G \times X \rightleftarrows X$$

where the two rightwards arrows are the action map $(g,x) \mapsto g \cdot x$ and the projection $(g,x) \mapsto x$, and the leftward arrow is the section $x \mapsto (e,x)$ where $e \in G$ is the neutral element. The colimit of this diagram is the (set-theoretic) quotient X/G. If we instead take the colimit in the ∞-category Grpd, call it the *quotient groupoid* $[X/G]$, then we may recover the above diagram by taking the fibred product of the canonical functor $X \to [X/G]$ along itself: there is a canonical isomorphism in Grpd

$$X \underset{[X/G]}{\times} X \simeq G \times X$$

under which the two projections of $X \times_{[X/G]} X$ are identified with the action and projection maps $G \times X \to X$, and the diagonal $X \to X \times_{[X/G]} X$ is identified with the section $(e, \mathrm{id}) : X \to G \times X$. Informally speaking, the quotient groupoid $[X/G]$ remembers everything about the G-action on X.

This story may be generalized to the case where X is a groupoid or even ∞-groupoid. In that setting, the diagram above must be replaced by a simplicial diagram

$$\cdots \overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}} G \times G \times X \overset{\rightarrow}{\rightrightarrows} G \times X \rightrightarrows X.$$

We let $\mathcal{X} = [X/G]$ denote the the geometric realization, i.e. the colimit taken in ∞-groupoids. Then we can recover the above diagram as the Čech nerve of the quotient map $X \twoheadrightarrow \mathcal{X}$. This leads to a bijective correspondence between *groupoid objects* acting on X (certain simplicial diagrams) and maps $X \twoheadrightarrow \mathcal{X}$ which are surjective on π_0, where one direction is formation of the geometric realization, and the other is formation of the Čech nerve.

We begin by briefly recalling how this works in the greater generality where ∞-groupoids are replaced by objects of any "∞-topos", such as the ∞-topos of ∞-stacks, i.e., étale sheaves of ∞-groupoids on Sch.

4.1. **Groupoid objects and effective epimorphisms.**

Definition 4.1. A *groupoid object* in an ∞-category $\mathcal{C}$ is a simplicial diagram $U_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that for all $[n] \in \mathbf{\Delta}$ and all partitions $[n] = S \cup S'$ into subsets S and S' such that $S \cap S'$ consists of a single element $m \in [n]$, the square

$$\begin{array}{ccc} U_n = U_\bullet([n]) & \longrightarrow & U_\bullet(S) \\ \downarrow & & \downarrow \\ U_\bullet(S') & \longrightarrow & U_\bullet(\{m\}) = U_0 \end{array}$$

is cartesian. A *morphism* of groupoid objects is a morphism of simplicial diagrams (i.e., a natural transformation).

Remark 4.2. Given a groupoid object $U_\bullet$, we think of $U_0 \in \mathcal{C}$ as the object of "objects" of $U_\bullet$ and $U_1 \in \mathcal{C}$ as the object of "morphisms" of $U_\bullet$. The above condition for "ordered"[6] partitions such as $[2] = \{0,1\} \cup \{1,2\}$ means, informally speaking, that we may fill all inner horns: for example, the morphism $(d_2^0, d_2^2) : U_2 \to U_1 \times_{U_0} U_1$ is invertible, where the two morphisms $U_1 \to U_0$ are "source" and "target", so that there is a "composition" morphism

$$U_1 \underset{U_0}{\times} U_1 \simeq U_2 \xrightarrow{d_2^1} U_1.$$

Similarly, for "unordered" partitions such as $[2] = \{0,1\} \cup \{0,2\}$ or $[2] = \{0,2\} \cup \{1,2\}$, the condition allows us to fill outer horns (and thus choose inverses to "morphisms" in $U_\bullet$).

Example 4.3. Given a 1-groupoid $\mathcal{G}$, its nerve $\mathcal{G}_\bullet := N(\mathcal{G}) \in \mathrm{SSet}$ defines a groupoid object in the 1-category Set. In fact, the assignment $\mathcal{G} \mapsto \mathcal{G}_\bullet$ determines an equivalence from the 1-category of 1-groupoids to the ∞-category of groupoid objects in Set. We may also regard $\mathcal{G}_\bullet$ as a groupoid object in Grpd_∞ (via the fully faithful functor $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}) \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Grpd}_\infty)$ induced by the embedding $\mathrm{Set} \hookrightarrow \mathrm{Grpd}_\infty$ of sets as discrete ∞-groupoids).

Example 4.4. Given a group G, we may consider the 1-groupoid with a single object $*$ and endomorphism group $\mathrm{End}(*) = G$, with composition law defined by the multiplication law of G. The corresponding groupoid object $G_\bullet$ is a simplicial diagram of the form

$$\cdots \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} G \times G \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} G \rightrightarrows \mathrm{pt},$$

where the face maps are induced by the group multiplication $m : G \times G \to G$ and the degeneracy maps are induced by the neutral element $e : \mathrm{pt} \to G$.

Example 4.5. Let $f : U \to X$ be a morphism in $\mathcal{C}$ such that the iterated fibred products $U \times_X \cdots \times_X U$ all exist in $\mathcal{C}$. The *Čech nerve* $U_\bullet$ of f is the simplicial diagram

$$\cdots \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} U \underset{U}{\times} U \underset{X}{\times} U \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} U \underset{X}{\times} U \rightrightarrows U.$$

More precisely, let $\mathbf{\Delta}_{\leqslant 0,+} \subseteq \mathbf{\Delta}_+$ denote the full subcategory spanned by the objects $[0]$ and $[-1]$. Then the morphism f determines a diagram $\mathbf{\Delta}_{\leqslant 0,+}^{\mathrm{op}} \to \mathcal{C}$, whose right Kan extension is an augmented simplicial diagram $U_\bullet^+ : \mathbf{\Delta}_+^{\mathrm{op}} \to \mathcal{C}$.

[6] i.e., partitions $[n] = S \cup S'$ such that $s \leqslant s'$ for all $s \in S$ and $s' \in S'$

The *Čech nerve* of f is its restriction $U_\bullet := U_\bullet^+|_{\mathbf{\Delta}}$. This is always a groupoid object in $\mathcal{C}$ (see [HTT, Prop. 6.1.2.11]).

Notation 4.6. Given a simplicial diagram $U_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, we will also refer to its colimit as its *geometric realization*, and denote it by

$$|U_\bullet| := \varinjlim_{[n]\in\mathbf{\Delta}} U_n \in \mathcal{C}$$

when it exists.

Definition 4.7. A morphism $f : U \to X$ in $\mathcal{C}$ is an *effective epimorphism* if the iterated fibred products $U \times_X \cdots \times_X U$ all exist in $\mathcal{C}$, and the augmented simplicial diagram

$$\cdots \substack{\to\\\to\\\to\\\to} U \underset{U}{\times} U \underset{X}{\times} U \substack{\to\\\to\\\to} U \underset{X}{\times} U \rightrightarrows U \to X$$

is a colimit diagram, i.e., exhibits X as the geometric realization of the Čech nerve $U_\bullet$.

We now specialize to the ∞-category of ∞-stacks.

Notation 4.8. Let Stk_∞ denote the ∞-category of ∞-*stacks*, i.e., étale sheaves of ∞-groupoids $\mathcal{X} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpd}_\infty$. Any stack can be regarded as an ∞-stack, via the fully faithful functor $\mathrm{Stk} \hookrightarrow \mathrm{Stk}_\infty$ induced by the inclusion $\mathrm{Grpd} \hookrightarrow \mathrm{Grpd}_\infty$.

The following two results hold more generally for sheaves of ∞-groupoids on any site (see [HTT, Thm. 6.1.0.6, Cor. 6.2.3.5]):

Theorem 4.9 (Lurie)**.**

(i) *Let $U_\bullet$ be a groupoid object in* Stk_∞*. Then the canonical morphism $U_0 \to |U_\bullet|$ is an effective epimorphism in* Stk_∞*.*

(ii) *The assignment sending $U_\bullet$ to the morphism $U_0 \twoheadrightarrow |U_\bullet|$ determines an equivalence from the ∞-category of groupoid objects in* Stk_∞ *to the ∞-category of effective epimorphisms in* Stk_∞*. Its inverse is the functor forming the Čech nerve.*

It will also be useful to have the following more explicit description of effective epimorphisms:

Proposition 4.10. *A morphism $f : U \to X$ is an effective epimorphism in* Stk_∞ *if and only if it is* étale-locally surjective*, i.e., for every commutative ring R and every R-point $x \in X(R)$, there exists a faithfully flat étale homomorphism $R \to R'$ such that the image of x in $X(R')$ belongs to the essential image of the functor $f(R') : U(R') \to X(R')$.*

Example 4.11. A smooth morphism of schemes $f : X \to Y$ is surjective if and only if it is an effective epimorphism.

4.2. **Group actions.**

Definition 4.12. A *group object* in an ∞-category $\mathcal{C}$ is a groupoid object $G_\bullet$ such that $G_0 \in \mathcal{C}$ is a terminal object. Given a group object $G_\bullet$, we will often abuse notation by saying that $G := G_1 \in \mathcal{C}$ is a group object.

Example 4.13. For any group G, the corresponding groupoid object $G_\bullet$ in Set (Example 4.4) is a group object. It may also be regarded as a group object in Grpd, which we still denote by $G_\bullet$. Similarly, if $G : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grp}$ is an (étale) sheaf of groups (such as represented by a group scheme), then it determines a group object in $\mathrm{Shv}(\mathrm{Sch}; \mathrm{Set})$ and hence also in Stk (or Stk_∞).

Definition 4.14. Let $G_\bullet$ be a group object in an ∞-category $\mathcal{C}$. An *action* of $G_\bullet$ on an object $U \in \mathcal{C}$ is a groupoid object $U_\bullet$ with an isomorphism $U_0 \simeq U$ and a morphism of simplicial diagrams $U_\bullet \to G_\bullet$ such that for every morphism $[m] \to [n]$ in $\mathbf{\Delta}$ sending $0 \mapsto 0$, the square

$$\begin{array}{ccc} U_n & \longrightarrow & U_m \\ \downarrow & & \downarrow \\ G_n & \longrightarrow & G_m \end{array}$$

is cartesian.

Remark 4.15. In the definition above, it is equivalent to require that for every $[n] \in \mathbf{\Delta}$, the square

$$\begin{array}{ccc} U_n & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ G_n & \longrightarrow & G_0, \end{array}$$

where the horizontal arrows are induced by the morphism $[0] \to [n]$ in $\mathbf{\Delta}$ sending $0 \mapsto 0$, is cartesian. In particular, there is for every $[n] \in \mathbf{\Delta}$ a canonical isomorphism $U_n \simeq G_n \times_{G_0} U_0 \simeq G^{\times n} \times U$. Thus the simplicial diagram $U_\bullet$ is of the form

$$\cdots \substack{\to\\\to\\\to\\\to} G \times G \times U \substack{\to\\\to\\\to} G \times U \rightrightarrows U.$$

Moreover, the face map $d^1 : G \times U \to U$ is the projection $(g, u) \mapsto u$, since the square

$$\begin{array}{ccc} U_1 & \xrightarrow{d^1} & U_0 \\ \downarrow & & \downarrow \\ G_1 & \xrightarrow{d^1} & G_0 \end{array}$$

is cartesian by assumption. Similarly, the degeneracy map $s^0 : U \to G \times U$ is the section (e', id) where $e' : U \to \mathrm{pt} \to G$ with pt the terminal object and $e = s^0 : \mathrm{pt} \to G$ the "neutral element" (part of the data of the group object $G_\bullet$).

Notation 4.16. Let $U_\bullet$ be an action of a group object $G_\bullet$ on $U \in \mathcal{C}$. By Remark 4.15, the face map $d^0 : U_1 \to U_0$ determines an isomorphism

$$\mathrm{act} : G \times U \to U$$

which we call the *action morphism*. Informally speaking, we may summarize the data of the action $U_\bullet$ as an action morphism $G \times U \to U$ which satisfies the analogues of the usual axioms of a group action *up to coherent homotopy*.

Definition 4.17. We say that an action $U_\bullet$ is *trivial* if for *every* morphism $[m] \to [n]$ the square

$$\begin{array}{ccc} U_n & \xrightarrow{d^0} & U_m \\ \downarrow & & \downarrow \\ G_n & \xrightarrow{d^0} & G_m \end{array}$$

is cartesian. In particular, the action morphism $G \times U \to U$ is in this case also the projection. For any object U, the trivial action on U is defined by the groupoid $U_\bullet^{\mathrm{triv}} := G_\bullet \times_{G_0} U$.

Notation 4.18. We will often abuse language by saying "let G be a group object acting on U" instead of "let $U_\bullet$ be an action of a group object $G_\bullet$ on U". If we need to speak of $U_\bullet$, we will refer to it as the *action groupoid* (or *quotient groupoid*). If G acts on U and V (meaning that there are actions $U_\bullet$ and $V_\bullet$ with $U_0 \simeq U$ and $V_0 \simeq V$), a *G-equivariant morphism* $f : U \to V$ is a morphism of simplicial diagrams $f_\bullet : U_\bullet \to V_\bullet$ fitting in a commutative diagram as follows.

$$\begin{array}{ccccc} U_\bullet & & \xrightarrow{f_\bullet} & & V_\bullet \\ & \searrow & & \swarrow & \\ & & G_\bullet & & \end{array}$$

4.3. **Torsors.**

Definition 4.19. Let X be a stack and let G be a group object in Stk. A *G-torsor* over X is a G-equivariant morphism $\pi : U \to X$ where U is a stack with an action of G and X is regarded with trivial G-action, satisfying the following conditions:

- (i) $\pi : U \to X$ is an effective epimorphism;
- (ii) the canonical morphism of simplicial diagrams $U_\bullet \to C_\bullet$ is invertible, where $U_\bullet$ is the action groupoid and $C_\bullet$ denotes the Čech nerve of $\pi : U \to X$.

Remark 4.20. It is immediate from the definition that if $\pi : U \to X$ is a G-torsor, then we may identify X as the geometric realization of the action groupoid $U_\bullet$. That is, there is a colimit diagram

$$\cdots \substack{\to\\\to\\\to\\\to} G \times G \times U \substack{\to\\\to\\\to} G \times U \rightrightarrows U \to X.$$

Remark 4.21. Condition (ii) in Definition 4.19 implies that the square

$$\begin{array}{ccc} U_1 & \xrightarrow{d^0} & U_0 \\ \downarrow{\scriptstyle d^1} & & \downarrow \\ U_0 & \longrightarrow & X \end{array}$$

is cartesian, i.e., that the canonical morphism

$$(\mathrm{act}, \mathrm{pr}) : G \times U \to U \underset{X}{\times} U$$

is invertible. In fact, one can show that this is equivalent to (ii).

Remark 4.22. We will also consider the variant of Definition 4.19 over a fixed base scheme S. This amounts to looking at presheaves on Sch_S instead of Sch; for example, a stack $X : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpd}$ over S is the same data as a sheaf $X : \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Grpd}$. Below, we will leave S implicit and simply denote it by "pt".

Example 4.23. The trivial G-torsor is the projection $\pi : G \times X \to X$. More precisely:

(i) The G-action on $G \times X$ is given by the groupoid object $U_\bullet$:

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} G \times G \times G \times X \substack{\rightarrow\\\rightarrow\\\rightarrow} G \times G \times X \rightrightarrows G \times X$$

where the face maps are $d^0 = m \times \mathrm{id}$ (where $m : G \times G \to G$ is the multiplication of the group) and $d^1 = \mathrm{pr}_1$. In other words, G acts by multiplication on the first component and trivially on the second.

(ii) The G-equivariant map $\pi : G \times X \to X$ is given by the morphism of simplicial diagrams $U_\bullet \to X^{\mathrm{triv}} = G_\bullet \times X$:

$$\begin{array}{ccccc} \cdots & \substack{\rightarrow\\\rightarrow\\\rightarrow} & G \times G \times X & \rightrightarrows & G \times X \\ & & \downarrow & & \downarrow \\ \cdots & \substack{\rightarrow\\\rightarrow\\\rightarrow} & G \times X & \rightrightarrows & X \end{array}$$

which on level n is given by projecting onto the last $n+1$ components of $G^{n+1} \times X$.

(iii) Note that $G \to \mathrm{pt}$ is an effective epimorphism, since it admits a section $e : \mathrm{pt} \to G$. Hence its base change $\pi : G \times X \to X$ is an effective epimorphism.

(iv) The square

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \mathrm{id}} & G \times X \\ \downarrow{\scriptstyle \mathrm{pr}_{2,3}} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ G \times X & \xrightarrow{\mathrm{pr}_2} & X \end{array}$$

is cartesian, hence $U_\bullet$ is identified with the Čech nerve of π by Remark 4.21. Indeed, this square is obtained from the cartesian square

$$\begin{array}{ccc} G \times G & \xrightarrow{\mathrm{pr}_1} & G \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow \\ G & \longrightarrow & \mathrm{pt} \end{array}$$

by precomposing with the inverse of the morphism $(m, \mathrm{pr}_2) : G \times G \to G \times G$ (which is invertible since G is a group).

Definition 4.24. We say that a G-torsor $\pi : U \to X$ is *trivial* if U is G-equivariantly isomorphic to $G \times X$ over X. Note that this is equivalent to the existence of a section $s : X \to U$ (so that $\pi \circ s \simeq \mathrm{id}$), since such s gives rise to an isomorphism

$$G \times X \xrightarrow{\mathrm{id} \times s} G \times U \xrightarrow{\mathrm{act}} U.$$

4.4. **Quotient stacks.**

Definition 4.25. Let G a group stack over a base scheme S (i.e., a group object in Stk). Let U be a stack over S with G-action. The *quotient stack* $[U/G]$ is defined as the geometric realization of the action groupoid $U_\bullet$, i.e.:

$$[U/G] := |U_\bullet| = \varinjlim_{[n] \in \mathbf{\Delta}} U_n,$$

where the colimit is taken in Stk. In particular, there is a colimit diagram

$$\cdots \overset{\to}{\underset{\to}{\rightrightarrows}} G \times G \times U \overset{\to}{\underset{\to}{\to}} G \times U \rightrightarrows U \twoheadrightarrow [U/G]$$

in Stk.

Remark 4.26. Let $|U_\bullet|^{\mathrm{PreStk}}$ denote the geometric realization taken in the category of prestacks (i.e., in the category $\mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{Grpd})$). Then $|U_\bullet|^{\mathrm{PreStk}}$ is given by the functor of points sending $T \in \mathrm{Sch}_S$ to the geometric realization $|U_\bullet(T)|$ of the simplicial diagram of groupoids $U_\bullet(T)$. This is rarely a stack, and there is a canonical morphism of prestacks

$$|U_\bullet|^{\mathrm{PreStk}} \to |U_\bullet| = [U/G]$$

which exhibits $[U/G]$ as the sheafification.

Remark 4.27. By construction, the quotient morphism $p : U \twoheadrightarrow [U/G]$ is a G-torsor.

We can compute the functor of points of $[U/G]$ as follows:

Theorem 4.28. *For every S-scheme T, there is a (functorial) equivalence of groupoids between $[U/G](T)$ and the groupoid of diagrams*

$$T \xleftarrow{\pi} Y \xrightarrow{f} U$$

where Y is an S-scheme with G-action, $f : Y \to U$ is a G-equivariant S-morphism, and π is a G-torsor over Y. An isomorphism $(Y, f, \pi) \to (Y', f', \pi')$ is a G-equivariant isomorphism $\phi : Y \to Y'$ and commutative squares expressing the compatibility of ϕ with f and f', and with π and π'.

Proof. By the Yoneda lemma, $[U/G](T)$ is the groupoid of morphisms $T \to [U/G]$. Such a morphism gives rise by base change to a cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{f} & U \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ T & \longrightarrow & [U/G] \end{array}$$

where the vertical arrows are G-torsors. The morphism f is G-equivariant, since it lifts to a morphism of simplicial diagrams

$$U_\bullet \underset{[U/G]}{\times} T \to U_\bullet$$

by base change, where $U_\bullet$ is the action groupoid of the G-action on U.

Conversely, given a G-torsor $\pi : Y \to T$, with G-action on Y given by an action groupoid $Y_\bullet$, and a G-equivariant morphism $f : Y \to U$ given by a morphism of simplicial diagrams $Y_\bullet \to U_\bullet$, we may recover $T \to [U/G]$ as the geometric realization

$$T \simeq [Y/G] = |Y_\bullet| \to |U_\bullet| = [U/G]$$

by Theorem 4.9. □

Example 4.29. The *classifying stack* of G is defined as the quotient stack

$$BG := B_S(G) := [S/G]$$

with respect to the trivial G-action on S. By Theorem 4.28, every G-torsor $\pi : Y \to T$ for an S-scheme T is classified by a morphism $T \to BG$ fitting into a cartesian square

$$\begin{array}{ccc} Y & \longrightarrow & S \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ T & \longrightarrow & BG. \end{array}$$

Informally speaking, the quotient morphism $p : S \twoheadrightarrow BG$ is the universal G-torsor.

Remark 4.30. In particular, the G-torsor $p : U \twoheadrightarrow [U/G]$ is classified by the morphism

$$[U/G] \to BG$$

induced by the G-equivariant morphism $U \to S$. That is, there is a cartesian square

$$\begin{array}{ccc} U & \longrightarrow & S \\ \downarrow & & \downarrow \\ [U/G] & \longrightarrow & BG. \end{array}$$

Example 4.31. Let $G = \mathrm{GL}_n$ be the general linear group scheme over S. Let X be an S-scheme. For every locally free sheaf $\mathcal{E}$ on X of rank n, there exists a GL_n-torsor

$$\pi : \mathrm{Isom}_X(\mathcal{O}_X^n, \mathcal{E}) \to X$$

whose total space represents the sheaf $\mathrm{Sch}_{/X}^{\mathrm{op}} \to \mathrm{Set}$ which sends $t : T \to X$ to the set of isomorphisms $\mathcal{O}_T^n \simeq t^*\mathcal{E}$. The assignment $\mathcal{E} \mapsto \mathrm{Isom}_X(\mathcal{O}_X^n, \mathcal{E})$ determines an equivalence between the groupoid of locally free sheaves on X of rank n and the groupoid of GL_n-torsors over X. In particular, morphisms $X \to \mathrm{BGL}_n$ may be identified with vector bundles over X of rank n.

Example 4.32. Let $\Theta = [\mathbf{A}^1/\mathbf{G}_m]$ denote the quotient stack with respect to the weight 1 scaling action of the multiplicative group scheme $\mathbf{G}_m = \mathrm{GL}_1$ on the affine line $\mathbf{A}^1$. Then the groupoid of morphisms $T \to \Theta$ is equivalent to the groupoid of *generalized divisors* on T, i.e., pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is a locally free sheaf of rank 1 on T and $s : \mathcal{O}_T \to \mathcal{L}$ is a section. Indeed, there is a cartesian square in Stk of the form

$$\begin{array}{ccc} \mathbf{A}^1 & \longrightarrow & \mathrm{GDiv} \\ \downarrow & & \downarrow \\ S & \twoheadrightarrow & \mathrm{B}\mathbf{G}_m \end{array}$$

where the right-hand vertical arrow is the forgetful map $(\mathcal{L}, s) \mapsto \mathcal{L}$ (or rather, the $\mathbf{G}_m$-torsor corresponding to $\mathcal{L}$); the lower horizontal arrow is the quotient map (which classifies the trivial line bundle); and the upper horizontal arrow sends a T-point of $\mathbf{A}^1$ given by $f \in \Gamma(T, \mathcal{O}_T)$ to the generalized divisor $(\mathcal{O}_T, f : \mathcal{O}_T \to \mathcal{O}_T) \in \mathrm{GDiv}(T)$. Since the lower horizontal arrow is a $\mathbf{G}_m$-torsor, it follows that the upper horizontal arrow is as well. In particular, it exhibits GDiv as the quotient stack $[\mathbf{A}^1/\mathbf{G}_m]$ as claimed.

5. Algebraic spaces and stacks

5.1. **Algebraic spaces.**

Definition 5.1. A morphism $f : X \to Y$ in an ∞-category $\mathcal{C}$ is a *monomorphism* if the square

$$\begin{array}{ccc} X & = & X \\ \| & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian. In other words, the fibred product $X \times_Y X$ exists and the diagonal $\Delta_f : X \to X \times_Y X$ is invertible.

Example 5.2. A functor of groupoids $F : \mathcal{C} \to \mathcal{D}$ is a monomorphism in the ∞-category Grpd if it is fully faithful. Equivalently, the induced map on sets of connected components $\pi_0(F) : \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ is injective, and for every object $X \in \mathcal{C}$ the map of automorphism groups $\mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{D}}(Y)$ is bijective. More generally, a functor $F : \mathcal{C} \to \mathcal{D}$ of ∞-groupoids is a monomorphism in the ∞-category Grpd_∞ if and only if it fully faithful, or equivalently injective on π_0 and bijective on $\pi_i(\mathcal{C}, X) \to \pi_i(\mathcal{D}, F(X))$ for all $i > 0$ and objects $X \in \mathcal{C}$.

Remark 5.3. Let X be an ∞-groupoid. If its diagonal $\Delta : X \to X \times X$ is a monomorphism, then X is discrete (i.e., $X \simeq \pi_0(X)$). Indeed, the ∞-groupoid of isomorphisms between two objects $x, y \in X$ is given by the fibred product

$$\begin{array}{ccc} \mathrm{Maps}_X(x, y) & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ \mathrm{pt} & \xrightarrow{(x,y)} & X \times X \end{array}$$

Hence $\mathrm{Maps}_X(x, y)$ is either empty or contractible (isomorphic in Grpd_∞ to pt) for all x, y. The converse also holds: for X discrete, the diagonal is always a monomorphism.

Example 5.4. A morphism of stacks $f : X \to Y$ is a monomorphism if and only if $f(T) : X(T) \to Y(T)$ is a monomorphism of groupoids for all schemes T. Thus if X is a stack with monomorphic diagonal, X takes values in sets (by Remark 5.3).

Definition 5.5. A morphism $f : X \to Y$ of stacks is *schematic* (or *representable by schemes*) if for every scheme V and every morphism $v : V \to Y$, the fibred product $X \times_Y V$ is a scheme.

Lemma 5.6. *Let X be a stack. The following conditions are equivalent:*

(i) *The diagonal $\Delta : X \to X \times X$ is schematic.*

(ii) *For every pair of morphisms $u : U \to X$ and $v : V \to X$, where U and V are schemes, the fibred product $U \times_X V$ is a scheme.*

(iii) *Every morphism $f : U \to X$, where U is a scheme, is schematic.*

Proof. We have (i) $\Rightarrow$ (ii) by the cartesian square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ U \times V & \xrightarrow{u \times v} & X \times X, \end{array}$$

and (ii) $=$ (iii) by definition. Suppose condition (iii) holds and let $f = (u, v) : U \to X \times X$ be a morphism. Then we have the cartesian square

$$\begin{array}{ccccc} U \times_{X \times X} X & \longrightarrow & U \times_X U & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta} \\ U & \xrightarrow{\Delta_U} & U \times U & \xrightarrow{u \times v} & X \times X, \end{array}$$

where the lower horizontal composite arrow is f. Now $U \times_X U$ is a scheme by (iii), so $U \times_{X \times X} X$ is a scheme since Δ_U is schematic (as a morphism between schemes). As U and f vary, this shows (i). □

Definition 5.7. Let X be a stack. We say that X is an *algebraic space* if it satisfies the following conditions:

(i) The diagonal $\Delta : X \to X \times X$ is schematic and a monomorphism.

(ii) There exists a scheme U and a morphism $U \twoheadrightarrow X$ which is étale and surjective, i.e. for every scheme V and every morphism $V \to X$ the base change $U \times_X V \to V$ is an étale surjection (where $U \times_X V$ is a scheme by Lemma 5.6).

Note that X takes values in sets (Example 5.4), i.e., an algebraic space is equivalently a sheaf of sets $X : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ with schematic diagonal and an étale atlas as in (ii).

5.2. **Algebraic stacks.**

Definition 5.8. A morphism $f : X \to Y$ of stacks is *representable* if for every scheme[7] V and every morphism $v : V \to Y$, the fibred product $X \times_Y V$ is an algebraic space.

Definition 5.9. A stack X is *algebraic* if it satisfies the following conditions:

(i) The diagonal $\Delta : X \to X \times X$ is representable.

(ii) There exists a scheme U and a morphism $U \twoheadrightarrow X$ which is smooth and surjective, i.e. for every scheme V and every morphism $V \to X$ the base change $U \times_X V \to V$ is a smooth surjection (where $U \times_X V$ is an algebraic space by the analogue of Lemma 5.6).

We say that X is *Deligne–Mumford* if in (ii), *smooth* is replaced by *étale.*

Remark 5.10. We call the smooth surjection $U \twoheadrightarrow X$ in (ii) an *atlas* for X.

5.3. Algebraicity of quotient stacks.

Theorem 5.11. *Let G be a smooth group algebraic space over a scheme S. Let U be a stack with G-action. If U is algebraic, then so is the quotient stack $[U/G]$.*

Proof. Let us show that the diagonal of $[U/G]$ is representable. Given a scheme T and a morphism $(t,t') : T \to [U/G] \times [U/G]$, the claim is that the fibred product $T \times_{[U/G]} T \simeq T \times_{[U/G]\times[U/G]} [U/G]$ is an algebraic space. Since the quotient morphism $p : U \twoheadrightarrow [U/G]$ is an effective epimorphism, there exists by Proposition 4.10 a scheme T' and an étale surjection $T' \twoheadrightarrow T$ such that the composite $T' \twoheadrightarrow T \to [U/G]$ factors through $p : U \twoheadrightarrow [U/G]$. Thus the composite of the cartesian squares

$$\begin{array}{ccccc} T' \times_{[U/G]} T & \longrightarrow & T \times_{[U/G]} T & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle t'} \\ T' & \twoheadrightarrow & T & \xrightarrow{t} & [U/G]. \end{array}$$

is identified with the composite of the cartesian squares

$$\begin{array}{ccccc} T' \times_{[U/G]} T & \longrightarrow & Y & \twoheadrightarrow & T \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle t'} \\ T' & \longrightarrow & U & \overset{p}{\twoheadrightarrow} & [U/G] \end{array}$$

where $Y \twoheadrightarrow T$ is the G-torsor classified by t'. Since U is algebraic (hence has representable diagonal) and T' and Y are algebraic spaces (the latter because G is), it follows by the analogue of Lemma 5.6 that $T' \times_{[U/G]} T$ is an algebraic space. It follows then that $T \times_{[U/G]} T$ is also an algebraic space.

It remains to show the existence of an atlas for $[U/G]$. Since $G \to S$ is a smooth surjection, it follows that the quotient morphism $p : U \twoheadrightarrow [U/G]$ is a smooth surjection. (Indeed, given a scheme V and a morphism $v : V \to [V/G]$,

[7] equivalently, algebraic space

the base change of p along v is the G-torsor classified by v, which is smooth and surjective because $G \to S$ is.) Since U is algebraic, there exists a scheme U' and a smooth surjection $U' \twoheadrightarrow U$. Then the composite $U' \twoheadrightarrow U \twoheadrightarrow [U/G]$ is smooth and surjective. ☐

Proposition 5.12. *Let G be a smooth group algebraic space over a scheme S. Let U be an algebraic space with G-action. Then the quotient stack $[U/G]$ is an algebraic space if and only if the action of G on U is* free*, i.e., the morphism*

$$(\mathrm{act}, \mathrm{pr}) : G \times U \to U \times U$$

is a monomorphism.

Proof. This follows from the cartesian square

$$\begin{array}{ccc} G \times U & \xrightarrow{(\mathrm{act},\mathrm{pr})} & U \times U \\ \downarrow & & \downarrow{\scriptstyle p\times p} \\ [U/G] & \xrightarrow{\Delta} & [U/G] \times [U/G]. \end{array}$$

Indeed, one can show that a morphism of stacks is a monomorphism if and only if it is such after base change along a smooth surjection. ☐

Warning 5.13. The analogue of Proposition 5.12 for schemes is false: there exist schemes X of finite type over a field (which may be taken separated or even proper), and an action of a group G (which may be taken to be a finite discrete group), such that the quotient stack $[X/G]$ is not representable by a scheme.

References

[Aok] K. Aoki, *Tensor triangular geometry of filtered objects and sheaves*. arXiv:2001.00319 (2020).

[GZ] P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **35** (1967). Springer–Verlag.

[HTT] J. Lurie, *Higher Topos Theory*.

[Lur] J. Lurie, *Kerodon*, `https://kerodon.net`.

[SAG] J. Lurie, *Spectral algebraic geometry*, version of 2018-02-03. `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`.

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