AN INTRODUCTION TO DERIVED ALGEBRAIC GEOMETRY

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0. Overview

Roughly speaking, ∞-categories are the language of "homotopical" or "derived" mathematics. This refers to the situation where we are interested in some class of objects up to some weaker notion than isomorphism: for example, this is the case in homological algebra, where we are interested in chain complexes only up to quasi-isomorphism, as well as in the theory of stacks, where we are only interested in groupoids up to equivalence. In these lectures we will see that ∞-categories are well-adapted to this situation. For example:

Theorem 0.1.

(i) *The assignment $X \mapsto \mathrm{C}^\bullet(X;\mathbf{Z})$, sending a topological space to its singular chain complex, is a sheaf with values in the derived ∞-category.*

(ii) *The assignment $X \mapsto \mathrm{D}_{\mathrm{qc}}(X)$, resp. $X \mapsto \mathrm{D}_{\mathrm{coh}}(X)$, sending a scheme to its derived ∞-category of quasi-coherent (resp. coherent) sheaves, is a sheaf of ∞-categories.*

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After giving a brief user's guide to ∞-category theory for algebraic geometers, and specifically explaining the ∞-categorical point of view on derived categories, we will then use this to introduce derived stacks and cotangent complexes, and eventually show:

Theorem 0.2. *Let X be a smooth proper scheme over a field k. Let $\mathcal{M}$ be the derived moduli stack $\mathcal{M}_{\mathrm{Vect}(X)}$ of vector bundles on X, $\mathcal{M}_{\mathrm{Coh}(X)}$ of coherent sheaves on X, or $\mathcal{M}_{\mathrm{Bun}_G(X)}$ of principal G-bundles on X (for an algebraic group G.) Then $\mathcal{M}$ is a derived algebraic stack which is "homotopically smooth" in the sense that its cotangent complex $\mathbf{L}_{\mathcal{M}}$ is perfect. In addition, if X is of dimension $\leqslant d$, then $\mathbf{L}_{\mathcal{M}}$ is of Tor-amplitude $\leqslant d-1$. In particular, it is smooth if X is a curve and quasi-smooth if X is a surface.*

1. Derived ∞-categories

1.1. **∞-Categories.** Although our main interest is in ∞-categories that arise from algebraic geometry, it will be useful to begin with the study of the "universal" homotopy theory, namely that of ∞-groupoids or homotopy types. We will see that ∞-groupoids play the role of sets in ∞-category theory.

Definition 1.1. A 1-*groupoid* is a 1-category in which all 1-morphisms are invertible.

Example 1.2. Let X be a topological space. Its *fundamental* 1-*groupoid* $\Pi_1(X)$ is defined as follows. Its objects are points of X. Morphisms $x \to y$, where $x, y \in X$, are paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, modulo (endpoint-preserving) homotopy. Composition is defined by concatenation of paths (which is associative up to homotopy).

Definition 1.3. Let us say a continuous map $f : X \to Y$ is a *homotopy* 1-*equivalence* if it induces isomorphisms

$$f_* : \pi_0(X) \to \pi_0(Y) \quad \text{and } f_* : \pi_1(X, x) \to \pi_1(Y, f(x)) \text{ for all } x \in X.$$

The *homotopy category of* 1-*types* is the localization of the category of topological spaces at the class of homotopy 1-equivalences: that is, it is the category formed by formally adjoining inverses to all homotopy 1-equivalences (see [GZ]).

Proposition 1.4.

(i) *The* 1-*groupoid only depends on the* homotopy 1-type *of X.*

(ii) *The assignment $X \mapsto \Pi_1(X)$ determines an equivalence from the homotopy category of* 1-*types to the homotopy category of groupoids (= the localization of the category of groupoids at equivalences).*

The definition of n-groupoids for higher n is much more subtle, but as a guiding principle we might similarly expect to be able to associate with any topological space X a *fundamental n-groupoid* $\Pi_n(X)$ that sees the *homotopy n-type* of X, and that this determines an equivalence from the homotopy category of n-types to the homotopy category of n-groupoids.

"Definition" 1.5. Let X be a topological space. An *n-groupoid* (which we think of as $\Pi_n(X)$ for some space X) has:

(i) Objects (corresponding to points of X).

(ii) 1-morphisms (corresponding to paths in X).

(iii) 2-morphisms (corresponding to homotopies between paths in X).

(iv) $\cdots$

A subtlety here is that the composition (concatenation of paths) should only be associative up to (coherent) homotopy. It is an essentially impossible task to write down by hand a full description of all the homotopy coherence data present in the fundamental n-groupoid as soon as n exceeds 3 or 4. To be able to make Definition 1.5 precise we would need some suitable bookkeeping device. Somewhat counterintuitively, it turns out that the theory simplifies considerably once we take the limit $n \to \infty$.

Definition 1.6. Let $\mathbf{\Delta}$ be the category whose objects are the finite sets $[n] := \{0, 1, \ldots, n\}$ for $n \geqslant 0$ and morphisms are order-preserving maps. A *simplicial set* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. Equivalently, this is a collection of sets $(X_n)_{n \geqslant 0}$ together with "face" and "degeneracy" maps

$$d_n^i : X_n \to X_{n-1}, \quad s_n^i : X_n \to X_{n+1}$$

which we often depict by the diagram

$$\cdots \substack{\to\\ \to\\ \to\\ \to} X_2 \substack{\to\\ \to\\ \to} X_1 \rightrightarrows X_0,$$

only drawing the face maps for simplicity.

Definition 1.7. An *∞-groupoid* is a simplicial set $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ satisfying the Kan condition. That is, for every solid arrow diagram as follows, there exists a lift:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Here Δ^n is the standard n-simplex and Λ^n_k is the result of removing the kth face from the boundary $\partial\Delta^n$. For example, for $n = 2$ such lifts corresponding to the existence of composites of morphisms ($k = 1$) and left and right inverses to all morphisms ($k = 0$ and $k = 2$).

Theorem 1.8 (Milnor)**.** *Given a topological space X, its fundamental groupoid $\Pi_\infty(X)$ is the simplicial set whose n-simplices are continuous map $\Delta^n_{\mathrm{top}} \to X$ from the topological standard n-simplex. Then $\Pi_\infty(X)$ is an ∞-groupoid, and the assignment $X \mapsto \Pi_\infty(X)$ determines an equivalence from the homotopy category of topological spaces to the homotopy category of ∞-groupoids.*

Exercise 1.9. Let $\mathcal{C}$ be a category. The *nerve* of $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$ whose n-simplices are n-folds composites of morphisms in $\mathcal{C}$, i.e., diagrams

$$c_0 \to c_1 \to \cdots \to c_n$$

in $\mathcal{C}$. Show that $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

"Definition" 1.10. An ∞-*category* is a category that is "homotopically enriched" in ∞-groupoids: for any two objects x, y is a mapping ∞-groupoid

$$\mathrm{Maps}(x, y) \in \mathrm{Grpd}_\infty,$$

and for x, y, z there is a composition map

$$\mathrm{Maps}(x, y) \times \mathrm{Maps}(y, z) \to \mathrm{Maps}(x, z)$$

which is associative up to coherent homotopy.

Remarkably, it turns out that all the relevant homotopy coherence data can be again encoded by the category $\mathbf{\Delta}$. In fact, it is possible to give a precise version of Definition 1.10, where an ∞-category $\mathcal{C}$ is a simplicial diagram $X_\bullet$ of ∞-groupoids satisfying certain conditions that ensure that it looks like an ∞-categorical version of the nerve, i.e., X_n is the ∞-groupoid of n-fold composites of morphisms in $\mathcal{C}$. We will not make this precise here.

Definition 1.11 (Limits and colimits)**.** Let F be a diagram in an ∞-category $\mathcal{C}$ indexed by an ∞-category I, i.e., a functor of ∞-categories $F : I \to \mathcal{C}$. Suppose given an object $X \in \mathcal{C}$ and a natural transformation $\alpha : X_{\mathrm{cst}} \to F$ where X_{cst} denotes the constant diagram $(i \in I) \mapsto (X \in \mathcal{C})$. We say that the pair (X, α) in $\mathcal{C}$ *exhibits* X *as the limit of* F if for every object $Y \in \mathcal{C}$ the induced functor of mapping ∞-groupoids

$$\mathrm{Maps}_{\mathcal{C}}(Y, X) \to \mathrm{Maps}_{\mathrm{Fun}(I,\mathcal{C})}(Y_{\mathrm{cst}}, F),$$

sending $(Y \to X) \mapsto (Y_{\mathrm{cst}} \to X_{\mathrm{cst}} \to F)$, is invertible. In that case we write

$$X \simeq \varprojlim_{i \in I} F_i.$$

Dually, we may speak of the *colimit* of F, which is the same as the limit of $F^{\mathrm{op}} : I^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

Warning 1.12. In the ∞-category of ∞-groupoids, co/limits correspond to *homotopy* co/limits of homotopy types. Thus for example, if X is a topological space with homotopy type $X^{\mathrm{ho}} \in \mathrm{Grpd}_\infty$ and $x \in X$ is a point, the limit in Grpd_∞ of the diagram

$$\mathrm{pt} \xrightarrow{x} X^{\mathrm{ho}} \xleftarrow{x} \mathrm{pt}$$

is the loop space $\Omega_x(X)$, since a commutative square in Grpd_∞ of the form

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{pt} \\ \big\downarrow & & \big\downarrow{\scriptstyle x} \\ \mathrm{pt} & \xrightarrow{x} & X^{\mathrm{ho}} \end{array}$$

encodes a self-homotopy of the constant map $x : Y \to X^{\mathrm{ho}}$. Contrast this with the fact that the limit of the diagram

$$\mathrm{pt} \xrightarrow{x} X \xleftarrow{x} \mathrm{pt}$$

in the category of topological spaces is just pt.

1.2. **Animation.**

Definition 1.13. A category $\mathcal{C}$ is *algebraic* if it is equivalent to

$$\mathrm{Fun}_{\Pi}(\mathcal{F}^{\mathrm{op}}, \mathrm{Set})$$

for some category $\mathcal{F}$ admitting finite coproducts, where the subscript indicates functors sending finite coproducts in $\mathcal{F}$ to finite products.

Remark 1.14. If $\mathcal{F}$ is generated under finite coproducts by an object $1 \in \mathcal{F}$, so that every object of $\mathcal{F}$ is $1^{\oplus n}$ for some $n \geqslant 0$, then we may think of $\mathrm{Fun}_{\Pi}(\mathcal{F}^{\mathrm{op}}, \mathrm{Set})$ as the category of sets X equipped with some algebraic operations

$$X^{\times m} \to X^{\times n}$$

encoded by elements of $\mathrm{Hom}_{\mathcal{F}}(1^{\oplus n}, 1^{\oplus m})$.

Example 1.15.

(i) The category of sets is algebraic with

$$\mathrm{Set} \simeq \mathrm{Fun}_{\Pi}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Set})$$

where Fin is the category of finite sets; there are no nontrivial operations in this case.

(ii) For a commutative ring R the category Mod_R of R-modules is algebraic with

$$\mathrm{Mod}_R \simeq \mathrm{Fun}_{\Pi}(\mathrm{FFree}_R^{\mathrm{op}}, \mathrm{Set})$$

where FFree_R is the category of finitely generated free R-modules. Operations are encoded by elements of $\mathrm{Hom}(R^{\oplus m}, R^{\oplus n})$, which are $(n \times m)$-matrices with values in R.

(iii) The category CAlg_R of commutative R-algebras is algebraic with

$$\mathrm{CAlg}_R \simeq \mathrm{Fun}_{\Pi}(\mathrm{Poly}_R^{\mathrm{op}}, \mathrm{Set})$$

where Poly_R is the category of finitely generated polynomial R-algebras $R[t_1, \ldots, t_n]$, $n \geqslant 0$. Operations are encoded by elements of $\mathrm{Hom}(R[t_1, \ldots, t_m], R[t_1, \ldots, t_n])$, which are collections of polynomials with coefficients in R.

Definition 1.16. Let $\mathcal{C}$ be an algebraic category. The *animation* of $\mathcal{C}$ is the ∞-category

$$\mathrm{Anim}(\mathcal{C}) \coloneqq \mathrm{Fun}_{\Pi}(\mathcal{F}^{\mathrm{op}}, \mathrm{Grpd}_{\infty})$$

of functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Grpd}_{\infty}$ that send finite coproducts in $\mathcal{F}$ to finite products.

Remark 1.17. Historically, animation was called the *nonabelian derived category.* Since we also want to apply it to abelian contexts like Mod_R, and since we want the terminology to differentiate between the connective (bounded on the right) and nonconnective (unbounded) derived categories, we use this newer terminology from [CS].

Example 1.18. The ∞-category Grpd_{∞} is the animation of the category of sets. Indeed, a functor $F \in \mathrm{Fun}_{\Pi}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Grpd}_{\infty})$ is completely determined by the ∞-groupoid $F(\{*\})$ (because of the subscript $_{\Pi}$).

Example 1.19. For R a commutative ring, we denote the animation $\mathrm{Anim}(\mathrm{Mod}_R)$ by $\mathrm{D}(R)_{\geqslant 0}$ (or $\mathrm{D}(R)^{\leqslant 0}$). This is an ∞-category version of the right-bounded derived category: that is, it is equivalent to the ∞-categorical localization at quasi-isomorphisms of chain complexes with $H_i = H^{-i} = 0$ for $i < 0$.

Example 1.20. We denote the animation dCAlg_R by $\mathrm{Anim}(\mathrm{CAlg}_R)$, and $\mathrm{dCRing} = \mathrm{Anim}(\mathrm{CRing}) \simeq \mathrm{dCAlg}_{\mathbf{Z}}$. We call their objects *derived commutative R-algebras* and derived commutative rings, respectively. There are also strict models in this case: dCAlg_R is equivalent to the ∞-categorical localization at weak homotopy equivalences of simplicial commutative R-algebras; if moreover R is a $\mathbf{Q}$-algebra, it is equivalent to the ∞-categorical localization at quasi-isomorphisms of commutative dg-R-algebras (with $H_i = H^{-i} = 0$ for $i < 0$).

Definition 1.21. An animated object $X \in \mathrm{Anim}(\mathcal{C})$ is *discrete* if the functor $X : \mathcal{F}^{\mathrm{op}} \to \mathrm{Grpd}_\infty$ takes values in sets. The inclusion $\mathrm{Set} \hookrightarrow \mathrm{Grpd}_\infty$, regarding sets as discrete ∞-groupoids, induces a fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$ which identifies $\mathcal{C}$ with the discrete objects of $\mathrm{Anim}(\mathcal{C})$. There is a left adjoint $\pi_0 : \mathrm{Anim}(\mathcal{C}) \to \mathcal{C}$ sending

$$(\mathcal{F}^{\mathrm{op}} \to \mathrm{Grpd}_\infty) \mapsto (\mathcal{F}^{\mathrm{op}} \to \mathrm{Grpd}_\infty \xrightarrow{\pi_0} \mathrm{Set})$$

where $\pi_0 : \mathrm{Grpd}_\infty \to \mathrm{Set}$ sends a ∞-groupoid to its set of isomorphism classes of objects (or connected components).

Remark 1.22. An animated R-module $M \in \mathrm{D}(R)_{\geqslant 0}$ amounts to the following data:

(i) For every integer $n \geqslant 0$, an ∞-groupoid $M_n \in \mathrm{Grpd}_\infty$.

(ii) For every R-linear map $\phi : R^{\oplus n} \to R^{\oplus m}$ (or $\phi \in \mathrm{Mat}_{m\times n}(R)$), a map $M_\phi : M_m \to M_n$.

(iii) For every two R-linear maps $\phi : R^{\oplus n} \to R^{\oplus m}$ and $\psi : R^{\oplus m} \to R^{\oplus l}$, a homotopy $h_{\phi,\psi} : M_\phi \circ M_\psi \simeq M_{\psi\circ\phi}$ of maps $M_l \to M_n$.

(iv) For every three R-linear maps ϕ, ψ, ω, a tetrahedron-shaped diagram expressing a "higher" homotopy between the homotopies $h_{\phi,\psi}$, $h_{\psi,\omega}$, $h_{\phi,\omega\circ\psi}$, and $h_{\psi\circ\phi,\omega}$.

(v) . . .

This data is subject to the condition that the canonical map $M_n \to (M_1)^{\times n}$ is invertible for every $n \geqslant 0$ (in particular, $M_0 \simeq \mathrm{pt}$). We summarize (iii) and (iv) by saying that the maps M_ϕ are functorial *up to coherent homotopy*.

In particular, this data encodes:

(i) The underlying ∞-groupoid $M^\circ := M_1 \in \mathrm{Grpd}_\infty$.

(ii) Operations $(M^\circ)^{\times n} \to M^\circ$ on M°, for every $\phi \in \mathrm{Mat}_{n\times 1}(R)$. In particular, an addition operation $\mathrm{add} : M^\circ \times M^\circ \to M^\circ$.

(iii) An action of R on M°, i.e., a map $R \to \mathrm{End}(M^\circ)$ given by

$$R \simeq \mathrm{Mat}_{1\times 1}(R) = \mathrm{Hom}_{\mathcal{F}_R}(1,1) \xrightarrow{M} \mathrm{Maps}_{\mathrm{Grpd}_\infty}(M_1, M_1) = \mathrm{End}(M^\circ).$$

The endomorphism induced by $a \in R$ is the operation encoded by the matrix $a \in \mathrm{Mat}_{1\times 1}(R)$.

(iv) Associativity up to coherent homotopy. For example, given three points $x, y, z \in M$ we have a homotopy

$$\mathrm{add}(\mathrm{add}(x,y),z) \simeq \mathrm{add}(x, \mathrm{add}(y,z)).$$

Diagrammatically,

$$\begin{array}{ccc} M^\circ \times M^\circ \times M^\circ & \xrightarrow{\mathrm{add}\times\mathrm{id}} & M^\circ \times M^\circ \\ \downarrow{\scriptstyle \mathrm{id}\times\mathrm{add}} & & \downarrow{\scriptstyle \mathrm{add}} \\ M^\circ \times M^\circ & \xrightarrow{\mathrm{add}} & M^\circ. \end{array}$$

Informally speaking, we can think of an animated R-module as an ∞-groupoid equipped with a homotopy coherent R-module structure.

1.3. **Derived functors.**

Proposition 1.23 (Universal property)**.** *Let $\mathcal{C} \simeq \mathrm{Fun}_\Pi(\mathcal{F}^{\mathrm{op}}, \mathrm{Set})$ be an algebraic category.*

(i) *The category $\mathcal{C}$ is freely generated by $\mathcal{F}$ under filtered colimits ("unions") and reflexive coequalizers ("quotients by equivalence relations").*

(ii) *The ∞-category $\mathrm{Anim}(\mathcal{C})$ is freely generated by $\mathcal{F}$ under filtered colimits and geometric realizations ("derived quotients by equivalence relations"). More precisely, for every ∞-category $\mathcal{D}$ admitting filtered colimits and geometric realizations, the canonical functor*

$$\mathrm{Fun}_{\mathrm{filt},\mathbf{\Delta}}(\mathrm{Anim}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{F}, \mathcal{D})$$

is an equivalence, where the source is the ∞-category of functors that preserve filtered colimits and geometric realizations.

Definition 1.24. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between algebraic categories. Write $\mathcal{C} \simeq \mathrm{Fun}_\Pi(\mathcal{F}^{\mathrm{op}}, \mathrm{Set})$ and consider the restriction $F|_{\mathcal{F}} : \mathcal{F} \to \mathcal{D} \hookrightarrow \mathrm{Anim}(\mathcal{D})$. Applying the universal property of animation we obtain a unique functor

$$F^{\mathrm{anim}} : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}(\mathcal{D})$$

which preserves filtered colimits and geometric realizations, and sends $F^{\mathrm{anim}}(X) = F(X)$ if $X \in \mathcal{F}$. Moreover, it satisfies $\pi_0(F^{\mathrm{anim}}(X)) \simeq F(\pi_0(X))$ for any $X \in \mathrm{Anim}(\mathcal{C})$. We call F^{anim} the *animation* of F, and think of this as a left-derived functor of F.

Example 1.25. If $R \to S$ is a ring homomorphism, extension of scalars defines a functor $(-) \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$, whose animation we denote

$$(-) \otimes_R^{\mathbf{L}} S : \mathrm{D}(R)_{\geqslant 0} \to \mathrm{D}(S)_{\geqslant 0}.$$

Similarly, restriction of scalars $\mathrm{Mod}_S \to \mathrm{Mod}_S$ animates to a functor $\mathrm{D}(S)_{\geqslant 0} \to \mathrm{D}(R)_{\geqslant 0}$, right adjoint to $(-) \otimes^{\mathbf{L}}_R S$. If $M \in \mathrm{Mod}_R$ is a flat R-module, then we have

$$M \otimes^{\mathbf{L}}_R S \simeq M \otimes_R S$$

since by Lazard's theorem M can be written as a filtered colimit of finitely generated free modules, and $(-) \otimes^{\mathbf{L}}_R S$ commutes with filtered colimits.

1.4. **Cotangent complex.**

Remark 1.26. Let R be a commutative ring. Relative algebraic Kähler differentials define a canonical functor

$$\Omega_{-/R} : A \in \mathrm{CAlg}_R \quad \mapsto \quad (A, \Omega_{A/R}) \in \mathrm{CAlgMod}_R \tag{1.27}$$

where the target $\mathrm{CAlgMod}_R$ is the category of pairs (A, M) with $A \in \mathrm{CAlg}_R$ and $M \in \mathrm{Mod}_A$; a morphism $(A, M) \to (A', M')$ is an R-algebra homomorphism $A \to A'$ together with an A'-module homomorphism $M \otimes_A A' \to M'$. This functor is a section of the projection $\pi : \mathrm{CAlgMod}_R \to \mathrm{CAlg}_R$, $(A, M) \mapsto A$.

Construction 1.28. The animation of (1.27) is a functor

$$\Omega^{\mathrm{anim}}_{-/R} : \mathrm{dCAlg}_R \to \mathrm{Anim}(\mathrm{CAlgMod}_R)$$

which is a section of the projection $\pi^{\mathrm{anim}} : \mathrm{Anim}(\mathrm{CAlgMod}_R) \to \mathrm{Anim}(\mathrm{CAlg}_R)$. Thus the image of $A \in \mathrm{dCAlg}_R$ may be written as a pair $(A, \mathbf{L}_{A/R})$ with $\mathbf{L}_{A/R} \in \mathrm{D}(A)_{\geqslant 0}$, where $\mathrm{D}(A)_{\geqslant 0}$ is by definition the fibre of π^{anim} over A. (If A is discrete, this is consistent with our previous definition of $\mathrm{D}(A)_{\geqslant 0}$.) We call $\mathbf{L}_{A/R}$ *the (relative) cotangent complex of* A.

Remark 1.29. In Construction 1.28 we used the fact that $\mathrm{CAlgMod}_R$ is algebraic: $\mathcal{F}$ in this case is the full subcategory of pairs (A, M) where $A \in \mathrm{Poly}_R$ and $M \in \mathrm{FFree}_R$.

Proposition 1.30.

(i) *For every* $A \in \mathrm{dCAlg}_R$ *there is a canonical isomorphism* $\pi_0 \mathbf{L}_{A/R} \simeq \Omega_{\pi_0(A)/R}$.

(ii) *Let* $A \to B$ *be a morphism in* $\mathrm{Anim}(\mathrm{CAlg}_R)$. *Then there is an exact triangle*

$$\mathbf{L}_{A/R} \otimes^{\mathbf{L}}_A B \to \mathbf{L}_{B/R} \to \mathbf{L}_{B/A}$$

in $\mathrm{D}(B)$.

(iii) *Given morphisms* $A \to B$ *and* $A \to A'$ *in* dCAlg_R, *there is a canonical isomorphism*

$$\mathbf{L}_{B/A} \otimes^{\mathbf{L}}_B B' \simeq \mathbf{L}_{A' \otimes^{\mathbf{L}}_A B/A'}.$$

The cotangent complex corepresents derived derivations:

Proposition 1.31 (Universal property)**.** *For every* $A \in \mathrm{dCAlg}_R$ *there are canonical isomorphisms, functorial in* $M \in \mathrm{D}(R)_{\geqslant 0}$,

$$\mathrm{Maps}_{\mathrm{D}(A)_{\geqslant 0}}(\mathbf{L}_{A/R}, M) \simeq \mathrm{Fib}(\mathrm{Maps}_{\mathrm{dCAlg}_R}(A, A \oplus M) \to \mathrm{Maps}_{\mathrm{dCAlg}_R}(A, A))$$

where the fibre is taken over the identity map id_A.

2. Derived stacks

2.1. Sheaves.

Definition/Proposition 2.1. *Let $A \to B$ be a morphism of derived commutative rings. We say that it is* flat *if it satisfies the following equivalent conditions:*

(i) *The functor $(-) \otimes^{\mathbf{L}}_A B : \mathrm{D}(A)_{\geqslant 0} \to \mathrm{D}(B)_{\geqslant 0}$ is left-exact, i.e., it preserves discrete objects.*

(ii) *The induced ring homomorphism $\pi_0(A) \to \pi_0(B)$ is flat, and the canonical homomorphisms*

$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B)$$

are bijective for all $i \geqslant 0$.

We say that $A \to B$ is faithfully flat *if it is flat and $\pi_0(A) \to \pi_0(B)$ is faithfully flat.*

Definition 2.2. We extend the flat topology on $\mathrm{CRing}^{\mathrm{op}}$ to $\mathrm{dCRing}^{\mathrm{op}}$ as follows: we say that a morphism $A \to B$ in dCRing is *faithfully flat* if it is flat in the sense of Definition 2.1, and the induced ring homomorphism $\pi_0(A) \to \pi_0(B)$ is faithfully flat.

Definition 2.3. We say that $A \to B$ is *étale*, resp. *smooth*, if it is flat and the induced ring homomorphism $\pi_0(A) \to \pi_0(B)$ is étale, resp. smooth (in the sense of ordinary commutative algebra). One can show that this is equivalent to the following condition: $\pi_0(A) \to \pi_0(B)$ is of finite presentation, and the relative cotangent complex $\mathbf{L}_{B/A}$ is zero, resp. of Tor-amplitude $[0, 0]$.

Definition 2.4. Let $\mathcal{V}$ be an ∞-category. A functor $F : \mathrm{dCRing} \to \mathcal{V}$ is a *sheaf* (for the flat, resp. étale, topology) if it satisfies the following conditions:

(i) F preserves finite products. That is, for every finite collection $(A_i)_i$ of derived commutative rings, the canonical morphism

$$F(\prod_i A_i) \to \prod_i F(A_i)$$

is invertible.

(ii) For every faithfully flat (resp. étale and faithfully flat) morphism $A \to B$, the diagram

$$F(A) \to F(B) \rightrightarrows F(B \otimes^{\mathbf{L}}_A B) \substack{\to\\\to\\\to} F(B \otimes^{\mathbf{L}}_A B \otimes^{\mathbf{L}}_A B) \substack{\to\\\to\\\to\\\to} \cdots$$

is a limit diagram. In other words, $F(X)$ is isomorphic to the totalization $\mathrm{Tot}(F(B^\bullet))$ of the cosimplicial diagram $B^\bullet = B^{\otimes \bullet+1}$ (tensor product over A).

Remark 2.5. The above definition makes sense more generally for a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ where $\mathcal{C}$ is any ∞-category with a Grothendieck topology. We write $\mathrm{Shv}(\mathcal{C}; \mathcal{V})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})$ spanned by sheaves.

2.2. **Derived stacks.**

Definition 2.6. Let R be a commutative ring. A *derived stack* over R is a functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ satisfying étale descent. We denote the ∞-category of derived stacks by $\mathrm{dStk} = \mathrm{Shv}(\mathrm{dCAlg}_R^{\mathrm{op}}; \mathrm{Grpd}_\infty)$.

Example 2.7. An *affine derived scheme* over R is the functor $\mathrm{Spec}(A) : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$, $B \mapsto \mathrm{Maps}_{\mathrm{dCAlg}_R}(A, B)$, corepresented by an animated algebra $A \in \mathrm{dCAlg}_R$.

Definition 2.8. Given a derived stack X, the restriction of the functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ along $\mathrm{CAlg}_R \hookrightarrow \mathrm{dCAlg}_R$ is called the *classical truncation* of X and is denoted X_{cl}. For example, $\mathrm{Spec}(A)_{\mathrm{cl}} \simeq \mathrm{Spec}(\pi_0(A))$ for any $A \in \mathrm{dCAlg}_R$. In general, X is a *higher stack* in the sense of [HS]. If the ∞-groupoid $X(A)$ is 1-truncated for every $A \in \mathrm{CAlg}_R$, then $X_{\mathrm{cl}} : \mathrm{CAlg}_R \to \mathrm{Grpd}$ is a 1-stack. For example, this will be the case if X is 1-Artin.

Remark 2.9. Let $X \to \mathrm{Spec}(A)$ be a morphism of derived stacks with affine target. Then X may be regarded equivalently as a functor

$$X : \mathrm{dCAlg}_A \to \mathrm{Grpd}_\infty,$$

where $\mathrm{dCAlg}_A = \mathrm{dCRing}_{A\backslash -}$ is the ∞-category of A-algebras, via a canonical equivalence

$$\mathrm{Shv}(\mathrm{dCRing}^{\mathrm{op}}; \mathrm{Grpd}_\infty)_{/\mathrm{Spec}(A)} \simeq \mathrm{Shv}(\mathrm{dCAlg}_A^{\mathrm{op}}; \mathrm{Grpd}_\infty).$$

Definition 2.10. Let $j : U \to X$ be a morphism of derived stacks.

(i) If X and U are affine, we say j is an open immersion if it is étale ($\mathcal{O}_X \to \mathcal{O}_U$ is an étale morphism of derived commutative rings) and $U_{\mathrm{cl}} \to X_{\mathrm{cl}}$ is an open immersion of classical affines.

(ii) If X is affine, we say j is an open immersion if it is a monomorphism (the diagonal $U \to U \times_X U$ is an isomorphism) and there exists a collection of affines $(U_\alpha)_\alpha$ and a surjection $\coprod_\alpha U_\alpha \twoheadrightarrow U$ such that each composite $U_\alpha \to X$ is an open immersion of affines.

(iii) In general, j is an open immersion if for every affine S and every morphism $S \to X$, the base change $U \times_X S \to S$ is an open immersion to an affine.

Definition 2.11 (Derived schemes)**.** A derived stack X is a *derived scheme* if there exists a collection $(U_\alpha \hookrightarrow X)_\alpha$ of open immersions where U_α are affine derived schemes, and a surjection $\coprod_\alpha U_\alpha \twoheadrightarrow X$. A morphism $f : X \to Y$ is *schematic* if for every affine V and every morphism $V \to Y$, the fibre $X \times^{\mathbf{R}}_Y V$ is a derived scheme. A schematic morphism $f : X \to Y$ of derived stacks is *smooth*, resp. *étale*, if for every affine V and every morphism $V \to Y$, there exists a collection of open immersions $(U_\alpha \hookrightarrow X \times_Y V)_\alpha$ where each U_α is affine and each composite $U_\alpha \to X \times_Y V \to V$ is a smooth, resp. étale, morphism of affines.

Remark 2.12. A derived scheme X is 0-truncated, in the sense that the functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ takes values in sets (= 0-truncated or discrete ∞-groupoids).

2.3. **Derived ∞-categories of quasi-coherent sheaves.**

Definition 2.13. Let A be a derived commutative ring. In the ∞-category $\mathrm{D}(A)_{\geqslant 0}$, or more generally any ∞-category with a zero object and finite co/limits, we define the *suspension* and *loop space* functors Σ and Ω by the cocartesian and cartesian squares

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma(X), \end{array} \qquad \begin{array}{ccc} \Omega(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X, \end{array}$$

respectively. These form an adjoint pair (Σ, Ω). We define the ∞-category $\mathrm{D}(A)$, the (unbounded) derived ∞-category of A, by forcing these functors to become mutually inverse; that is, we take the limit

$$\mathrm{D}(A) \to \cdots \xrightarrow{\Omega} \mathrm{D}(A)_{\geqslant 0} \xrightarrow{\Omega} \mathrm{D}(A)_{\geqslant 0}.$$

In the resulting ∞-category, we have $M \simeq \Omega\Sigma(M) \simeq \Sigma\Omega(M)$ for all $M \in \mathrm{D}(A)$. This property means that $\mathrm{D}(A)$ is a *stable ∞-category*. Moreover, it has a t-structure $(\mathrm{D}(A)_{\geqslant 0}, \mathrm{D}(A)_{<0})$. We set $M[1] \coloneqq \Sigma(M)$ and $M[-1] \coloneqq M(\mathcal{F})$ for any $M \in \mathrm{D}(A)$.

Definition 2.14. A derived A-module $M \in \mathrm{D}(A)$ is *perfect* if it belongs to the thick subcategory generated by A; i.e., if it can be built out of A using finite limits, finite colimits, and direct summands. We write $\mathrm{Perf}(A) \subseteq \mathrm{D}(A)$ for the full subcategory of perfect A-modules.

Theorem 2.15 (Lurie, Toën)**.** *The functor*

$$\mathrm{dCAlg}_R \to \mathrm{Cat}_\infty, \quad A \mapsto \mathrm{D}(A)$$

is a sheaf for the flat topology. The same holds for $A \mapsto \mathrm{D}(A)_{\geqslant 0}$ and $A \mapsto \mathrm{Perf}(A)$.

Definition 2.16. Let X be a derived stack. The derived ∞-category of quasi-coherent sheaves $\mathrm{D}_{\mathrm{qc}}(X)$ is the limit

$$\mathrm{D}_{\mathrm{qc}}(X) \coloneqq \varprojlim_{(A,x)} \mathrm{D}(A)$$

over the category of pairs (A, x) where $A \in \mathrm{dCAlg}_R$ and $x \in X(A)$. In other words,

$$\mathrm{D}_{\mathrm{qc}} : \mathrm{dStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

is the right Kan extension of the presheaf $\mathrm{Spec}(A) \mapsto \mathrm{D}(A)$ along the inclusion $\mathrm{Aff} \hookrightarrow \mathrm{dStk}$. We refer to objects of $\mathrm{D}_{\mathrm{qc}}(X)$ as *quasi-coherent complexes* on X. Note that $\mathrm{D}_{\mathrm{qc}}(X)$ is stable, since this property is preserved under formation of limits.

Remark 2.17. Recall that, by one characterization of right Kan extensions, the presheaf $\mathrm{D}_{\mathrm{qc}} : \mathrm{dStk}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is the unique limit-preserving functor extending $\mathrm{D}_{\mathrm{qc}} : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, $\mathrm{Spec}(A) \mapsto \mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(A)) \simeq \mathrm{D}(A)$. In particular, it sends colimits of derived stacks to limits of ∞-categories.

Example 2.18. Let G be a group scheme over R, acting on a derived stack X. The *quotient stack* $[X/G]$ is the colimit of the action groupoid

$$X_\bullet = [\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} G \times X \rightrightarrows X].$$

Then there is a limit diagram of ∞-categories

$$\mathrm{QCoh}([X/G]) \to \mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(G\times X) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{QCoh}(G\times G\times X) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots.$$

In other words, the canonical functor $\mathrm{QCoh}([X/G]) \to \mathrm{Tot}(\mathrm{QCoh}(X_\bullet))$ is an equivalence, where $X_\bullet = [\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} G \times X \rightrightarrows X]$ is the action groupoid (whose colimit is the quotient stack $[X/G]$). We call

$$\mathrm{QCoh}^G(X) \coloneqq \mathrm{Tot}(\mathrm{QCoh}(X_\bullet))$$

the *G-equivariant derived ∞-category of quasi-coherent sheaves on X*; its objects, which we call *G-equivariant quasi-coherent sheaves* on X, are quasi-coherent sheaves $\mathcal{F}$ on X together with a (specified) isomorphism $\mathrm{act}^*\mathcal{F} \simeq \mathrm{pr}^*\mathcal{F}$ on $G\times X$, as well as a homotopy coherent system of isomorphisms on the higher terms $G^{\times n}\times X$. Thus Example 2.18 says that quasi-coherent sheaves on $[X/G]$ are quasi-coherent sheaves on X that are G-equivariant in the homotopy coherent sense.

2.4. **Cotangent complexes.**

Definition 2.19. Let $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ be a derived stack. We say that X *admits a cotangent complex* $\mathbf{L}_{X/R}$ if and only if the following conditions hold:

(i) For every $A \in \mathrm{dCAlg}_R$ and every $x \in X(A)$, denote by $F_x(N)$ the fibre at x of the map

$$X(A\oplus N) \to X(A)$$

for every $N \in \mathrm{D}(A)_{\geqslant 0}$. Then the functor $F_x(-)$ is corepresented by a derived A-module M_x which is eventually coconnective, i.e., $M_x[n] \in \mathrm{D}(A)_{\geqslant 0}$ for some n.

(ii) For every morphism $A \to B$ in dCAlg_R and every $N \in \mathrm{D}(B)_{\geqslant 0}$, the commutative square

$$\begin{array}{ccc} X(A\oplus N) & \longrightarrow & X(B\oplus N) \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array}$$

is cartesian.

Note that under these conditions, there exists an object $\mathbf{L}_{X/R} \in \mathrm{D}_{\mathrm{qc}}(X)$ such that $x^*\mathbf{L}_{X/R} \simeq M_x$ for every $A \in \mathrm{dCAlg}_R$ and $x \in X(A)$ (modulo the equivalence $\mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(A)) \simeq \mathrm{D}(A)$). We will often write $\mathbf{L}_X \coloneqq \mathbf{L}_{X/R}$.

Remark 2.20. As in the affine case (the case of derived commutative rings), it is possible to talk about *relative* cotangent complexes $\mathbf{L}_{X/Y}$ for morphisms $X \to Y$. See [Kha, Def. 8.31], [Toë, §5.3], or [TV2, §1.4.1]. We have $\mathbf{L}_{X/\mathrm{Spec}(R)} \simeq \mathbf{L}_{X/R}$, and by Theorem 2.21 below, there is an exact triangle

$$f^*\mathbf{L}_Y \to \mathbf{L}_X \to \mathbf{L}_{X/Y}$$

in the stable ∞-category $\mathrm{D}_{\mathrm{qc}}(X)$.

Theorem 1.30 immediately globalizes as follows:

Theorem 2.21.

(i) *Let S be a derived stack and $f : X \to Y$ a morphism over S. If X and Y admit cotangent complexes over S, then f admits a relative cotangent complex such that there is an exact triangle*

$$f^*\mathbf{L}_{Y/S} \to \mathbf{L}_{X/S} \to \mathbf{L}_{X/Y}$$

in $\mathrm{D}_{\mathrm{qc}}(X)$.

(ii) *Let $f : X \to Y$ be a morphism of derived stacks. If f admits a relative cotangent complex, then for every morphism $Y' \to Y$ the derived base change $X \times^{\mathbf{R}}_Y Y' \to Y'$ admits a relative cotangent complex, and moreover there is a canonical isomorphism*

$$p^*\mathbf{L}_{X/Y} \simeq \mathbf{L}_{X \times^{\mathbf{R}}_Y Y'/Y'}$$

in $\mathrm{D}_{\mathrm{qc}}(X \times^{\mathbf{R}}_Y Y')$, where $p : X \times^{\mathbf{R}}_Y Y' \to X$ is the projection.

Example 2.22. Let G be a smooth group scheme over R, and let U be a derived stack over R with a G-action. Assuming that U admits a cotangent complex $\mathbf{L}_U$, we can compute the cotangent complex of the quotient stack $[U/G]$ as follows. Consider the cartesian square:

$$\begin{array}{ccc} G \times U & \xrightarrow{\mathrm{pr}} & U \\ \big\downarrow{\scriptstyle \mathrm{act}} & & \big\downarrow{\scriptstyle p} \\ U & \xrightarrow{p} & [U/G]. \end{array}$$

We have

$$\mathbf{L}_{U/[U/G]} \simeq d^*\mathrm{act}^*\mathbf{L}_{U/[U/G]} \simeq d^*\mathbf{L}_{G\times U/U} \simeq d^*\mathrm{pr}_1^*\mathbf{L}_G$$

where $d = (e, \mathrm{id}) : U \to G \times U$ and $\mathrm{pr}_1 : G \times U \to G$, using Theorem 2.21(ii) twice. Since $\mathrm{pr}_1 \circ d$ factors as the projection $f : U \to \mathrm{Spec}(R)$ followed by the identity section $e : \mathrm{Spec}(R) \to G$, we get

$$\mathbf{L}_{U/[U/G]} \simeq f^*e^*\mathbf{L}_G \simeq f^*\mathfrak{g}^\vee$$

where $\mathfrak{g}^\vee = e^*\mathbf{L}_G \simeq e^*\Omega_G$ is the dual Lie algebra of G (recall that G is smooth over R). Finally, we have by Theorem 2.21(i) an exact triangle

$$p^*\mathbf{L}_{[U/G]} \to \mathbf{L}_U \to \mathbf{L}_{U/[U/G]}$$

where $p : U \twoheadrightarrow [U/G]$ is the quotient morphism. Under the equivalence $\mathrm{D}_{\mathrm{qc}}([U/G]) \simeq \mathrm{D}^G_{\mathrm{qc}}(U)$, $\mathbf{L}_{[U/G]}$ may be regarded as the quasi-coherent complex

$$\mathrm{Fib}(\mathbf{L}_U \to f^*\mathfrak{g}^\vee) \in \mathrm{D}_{\mathrm{qc}}(X)$$

with a natural G-action (induced naturally by the action on U). For example, if U is a smooth scheme, then this is a 2-term complex with Ω_U in degree

0 and $f^*\mathfrak{g}^\vee$ in (homological) degree -1. Note that if G is finite (and hence étale), we have $\mathfrak{g}^\vee \simeq 0$.

2.5. Moduli of sheaves.

Definition 2.23. [TV1] Let $\mathcal{M}_{\mathrm{Perf}}$ denote the functor

$$\mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty, \quad A \mapsto \mathrm{Perf}(A)^{\simeq}$$

sending A to the ∞-groupoid of perfect derived A-modules. The superscript $^{\simeq}$ indicates that we take the underlying ∞-groupoid, obtained by discarding all non-invertible morphisms. By Theorem 2.15, this satisfies étale descent and hence defines a derived stack over R. By Yoneda, there is a *universal perfect complex*

$$\mathcal{E}^{\mathrm{univ}} \in \mathrm{Perf}(\mathcal{M}_{\mathrm{Perf}})$$

such that for every derived stack X and every perfect complex $\mathcal{E} \in \mathrm{Perf}(X)$, there is a unique morphism $f : X \to \mathcal{M}_{\mathrm{Perf}}$ together with an isomorphism $f^*(\mathcal{E}^{\mathrm{univ}}) \simeq \mathcal{E}$.

Remark 2.24. We can similarly consider the larger stacks $\mathcal{M}_{\mathrm{D_{coh}}}$ and $\mathcal{M}_{\mathrm{D_{pscoh}}}$ sending $A \in \mathrm{dCRing}_R$ to $\mathrm{D_{coh}}(A)^{\simeq}$ or $\mathrm{D_{pscoh}}(A)^{\simeq}$, respectively. Here $\mathrm{D_{pscoh}}(A) \subseteq \mathrm{D_{qc}}(A)$ is the full subcategory of *pseudocoherent* derived A-modules (sometimes called *almost perfect* A-modules) and $\mathrm{D_{coh}}(A) \subseteq \mathrm{D_{pscoh}}(A)$ is the full subcategory of *coherent* derived A-modules.[1] There are open immersions of derived stacks

$$\mathcal{M}_{\mathrm{D_{perf}}} \hookrightarrow \mathcal{M}_{\mathrm{D_{coh}}} \hookrightarrow \mathcal{M}_{\mathrm{D_{pscoh}}},$$

but these larger stacks do not admit a cotangent complex (because the perfect complexes are precisely the dualizable objects in $\mathrm{D_{pscoh}}(X)$).

Theorem 2.25. *The perfect complex*

$$\mathbf{L}_{\mathcal{M}_{\mathrm{Perf}}} \coloneqq \mathcal{E}^{\mathrm{univ}} \otimes^{\mathbf{L}} \mathcal{E}^{\mathrm{univ},\vee}[-1]$$

is a cotangent complex for the derived stack $\mathcal{M}_{\mathrm{Perf}}$.

Lemma 2.26. *Let $A \in \mathrm{dCAlg}_R$ and $M \in \mathrm{Perf}(A)$. For every $N \in \mathrm{Perf}(A)_{\geqslant 0}$ denote by $F_M(N)$ the fibre at M of the map of anima*

$$\mathrm{Perf}(A \oplus N)^{\simeq} \to \mathrm{Perf}(A)^{\simeq}$$

given by extending scalars along the canonical homomorphism $A \oplus N \to A$. Then we have canonical isomorphisms

$$F_M(N) \simeq \mathrm{Maps}_{\mathrm{D}(A)}(M \otimes^{\mathbf{L}}_A M^\vee[-1], N),$$

natural in N.

[1] When A is noetherian these are defined as follows: $M \in \mathrm{D}(A)$ is pseudocohererent if it is homologically bounded above ($\pi_i(M) = 0$ for $i \gg 0$) and its homotopy groups $\pi_i(M)$ are finitely generated $\pi_0(A)$-modules (see [HA, Def. 7.2.4.10]); it is *coherent* if it is additionally bounded below ($\pi_i(M) = 0$ for $i \ll 0$).

Proof. By definition, $F_M(N)$ is the ∞-groupoid of deformations of M along $A \oplus N \to A$, i.e., of pairs $(\widetilde{M}, \theta)$ where $\widetilde{M}$ is an $A \oplus N$-module and θ is an A-linear isomorphism $\widetilde{M} \otimes^{\mathbf{L}}_{A\oplus N} A \simeq M$. Since the square

$$\begin{array}{ccc} A \oplus N & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus N[1] \end{array}$$

is cartesian, this is equivalent to the ∞-groupoid of deformations of $M \otimes^{\mathbf{L}}_A (A \oplus N[1]) \simeq M \oplus (M \otimes^{\mathbf{L}}_A N[1])$ along the trivial derivation $A \to A \oplus N[1]$. Equivalently, this is the ∞-groupoid of automorphisms of $M \oplus (M \otimes^{\mathbf{L}} N[1])$ over $A \oplus N[1]$ which extend to the identity $\mathrm{id}_M : M = M$ along $A \oplus N[1] \to A$. That is,

$$F_M(N) \simeq \mathrm{End}_{A\oplus N[1]}(M \oplus (M \otimes^{\mathbf{L}} N[1])) \underset{\mathrm{End}_{\mathrm{D}(A)}(M)}{\times} \{\mathrm{id}_M\}$$

where we can write End instead of Aut since every such endomorphism is necessarily invertible. Thus we have

$$\begin{aligned} F_M(N) &\simeq \mathrm{Maps}_{\mathrm{D}(A)}(M, (M \oplus (M \otimes^{\mathbf{L}} N[1])) \underset{M}{\times} 0) \\ &\simeq \mathrm{Maps}_{\mathrm{D}(A)}(M, M \otimes^{\mathbf{L}} N[1]) \simeq \mathrm{Maps}_{\mathrm{D}(A)}(M \otimes^{\mathbf{L}} M^\vee[-1], N) \end{aligned}$$

where the last isomorphism follows from the fact that M is perfect, hence dualizable. □

Proof of Theorem 2.25. Let $A \in \mathrm{dCAlg}_R$ and $x : \mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Perf}}$ an A-point classifying a perfect derived A-module $M \in \mathrm{Perf}(A)$. By Lemma 2.26, the ∞-groupoid of R-linear derivations with values in M is corepresented by $M \otimes^{\mathbf{L}} M^\vee[-1]$. As (A, x) varies, the perfect complexes $M \otimes^{\mathbf{L}} M^\vee[-1]$ assemble into the perfect complex $\mathcal{E}^{\mathrm{univ}} \otimes^{\mathbf{L}} \mathcal{E}^{\mathrm{univ},\vee}[-1] \in \mathrm{Perf}(\mathcal{M}_{\mathrm{Perf}})$, which is therefore a cotangent complex for $\mathcal{M}_{\mathrm{Perf}}$. □

Construction 2.27. Let $R \in \mathrm{dCRing}$ and let X and Y be derived stacks over R. The *derived mapping stack* $\underline{\mathrm{Maps}}(X, Y)$ is the functor

$$\underline{\mathrm{Maps}}(X, Y) : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty, \quad A \mapsto \mathrm{Maps}_{\mathrm{Spec}(R)}(X \times \mathrm{Spec}(A), Y)$$

sending A to the ∞-groupoid of R-morphisms $X \times \mathrm{Spec}(A) \to Y$. More generally, we may form the derived mapping stack

$$\underline{\mathrm{Maps}}_S(X, Y) : \mathrm{dAff}^{\mathrm{op}}_{/S} \to \mathrm{Grpd}_\infty, \quad (\mathrm{Spec}(A) \to S) \mapsto \mathrm{Maps}_S(X \underset{S}{\times} \mathrm{Spec}(A), Y)$$

over a derived stack S, whenever X and Y are defined over S. There is an *evaluation morphism*

$$\mathrm{ev} : \underline{\mathrm{Maps}}_S(X, Y) \underset{S}{\times} X \to Y,$$

classified by the identity $\mathrm{id} : \underline{\mathrm{Maps}}_S(X, Y) \to \underline{\mathrm{Maps}}_S(X, Y)$.

Warning 2.28. By abuse of notation, all products are implicitly fibred over the $\mathrm{Spec}(R)$. For example, $X \times \mathrm{Spec}(A)$ really means the derived fibred product $X \times^{\mathbf{R}}_{\mathrm{Spec}(R)} \mathrm{Spec}(A)$, which need not agree with the classical fibred product if X is not flat over R. In particular, there is an identification

$$\underline{\mathrm{Maps}}(X,Y)_{\mathrm{cl}} \simeq \underline{\mathrm{Hom}}(X_{\mathrm{cl}}, Y_{\mathrm{cl}})$$

of the classical truncation of $\underline{\mathrm{Maps}}_R(X,Y)$ with the classical Hom stack when X is flat over R, but not in general.

Theorem 2.29. *Suppose X and Y are derived stacks over R. Set $H := \underline{\mathrm{Maps}}(X,Y)$ and consider the diagram*

$$H \xleftarrow{\pi} X \times H \xrightarrow{\mathrm{ev}} Y$$

where π is the projection. If X is a derived scheme, proper and of finite Tor-amplitude over R, and Y admits a cotangent complex $\mathbf{L}_Y$, then the perfect complex

$$\mathbf{L}_H \simeq \pi_*(\mathrm{ev}^*(\mathbf{L}_Y) \otimes (K_X \boxtimes \mathcal{O}_H))$$

is a cotangent complex for H. Here $K_X = p^!(\mathcal{O}_{\mathrm{Spec}(R)})$ denotes the dualizing complex, where $p : X \to \mathrm{Spec}(R)$ is the projection, and $K_X \boxtimes \mathcal{O}_H = \mathrm{pr}_1^*(K_X)$ where $\mathrm{pr}_1 : X \times H \to X$ is the projection.

Proof. Given $A \in \mathrm{dCAlg}_R$, an A-point $h \in H(A)$ classifying a morphism $f : X_R \to Y$, and $M \in \mathrm{D}(A)_{\geqslant 0}$, derivations of H at h are extensions of the morphism $f : X_A := X \times \mathrm{Spec}(A) \to Y$ along $X_A \hookrightarrow X_{A\oplus M}$. Since the latter can be regarded as the trivial square-zero extension of X_A by $p_A^*(M)$, where $p_A : X_A \to \mathrm{Spec}(A)$ is the projection, these are classified by the cotangent complex of Y, i.e.,

$$\mathrm{Der}_h(H, M) \simeq \mathrm{Maps}_{\mathrm{D}(X_A)}(f^*\mathbf{L}_Y, p_A^*(M)).$$

The assumptions on X imply that p_A^* admits a left adjoint $p_{A,\sharp} := p_*(- \otimes K_{X_A/A})$, hence $\mathrm{Der}_h(H, -)$ is corepresented by $p_{A,\sharp} f^* \mathbf{L}_Y$. Now

$$\mathbf{L}_H := \pi_\sharp \mathrm{ev}^*(\mathbf{L}_Y) := \pi_*(\mathrm{ev}^*(\mathbf{L}_Y) \otimes K_{X\times H/H}) \simeq \pi_*(\mathrm{ev}^*(\mathbf{L}_Y) \otimes (K_X \boxtimes \mathcal{O}_H))$$

is the unique perfect complex on H such that $h^*(\mathbf{L}_H) \simeq p_{A,\sharp} f^*(\mathbf{L}_Y)$, in view of the commutative diagram

$$\begin{array}{ccccc} H & \xleftarrow{\pi} & X \times H & \xrightarrow{\mathrm{ev}} & Y \\ {\scriptstyle h}\uparrow & & \uparrow & & \uparrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xleftarrow{p_A} & X_A & = & X_A \end{array}$$

where the left-hand square is cartesian. It follows that $\mathbf{L}_H$ is a cotangent complex for H. □

Definition 2.30. Let X be a smooth proper scheme over R.

(i) The *moduli stack of perfect complexes on X* is the derived mapping stack

$$\mathcal{M}_{\mathrm{Perf}(X)} = \underline{\mathrm{Maps}}(X, \mathcal{M}_{\mathrm{Perf}}).$$

For $A \in \mathrm{dCAlg}_R$, its A-points are morphisms $X_A := X \times \mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Perf}}$ over $\mathrm{Spec}(A)$, i.e., perfect complexes on X_A.

(ii) Given a group scheme G over R, the *moduli stack of G-torsors on X* (a.k.a. *principal G-bundles* on X) is the derived mapping stack

$$\mathcal{M}_{\mathrm{Bun}_G(X)} = \underline{\mathrm{Maps}}(X, BG).$$

For $A \in \mathrm{dCAlg}_R$, its A-points are morphisms $X_A \to BG$ over $\mathrm{Spec}(A)$, i.e., G-torsors on X_A.

(iii) The *moduli stack of vector bundles on X* is the substack $\mathcal{M}_{\mathrm{Vect}(X)} \subseteq \mathcal{M}_{\mathrm{Perf}(X)}$ defined as follows: for $A \in \mathrm{dCAlg}_R$, an A-point of $\mathcal{M}_{\mathrm{Perf}(X)}$ belongs to $\mathcal{M}_{\mathrm{Vect}(X)}$ if and only if the corresponding perfect complex $\mathcal{F} \in \mathrm{D}_{\mathrm{perf}}(X_A)$ is of Tor-amplitude $[0,0]$, i.e., it is connective and flat over X_A.

(iv) The *moduli stack of coherent sheaves on X $\mathcal{M}_{\mathrm{Coh}(X)}$* is the substack of the derived mapping stack

$$\mathcal{M}_{\mathrm{D}_{\mathrm{coh}}(X)} = \underline{\mathrm{Maps}}(X, \mathrm{D}_{\mathrm{coh}})$$

defined as follows: for $A \in \mathrm{dCAlg}_R$, an A-point of $\mathcal{M}_{\mathrm{D}_{\mathrm{coh}}(X)}$ belongs to $\mathcal{M}_{\mathrm{Coh}(X)}$ if and only if the corresponding coherent complex $\mathcal{F} \in \mathrm{D}_{\mathrm{perf}}(X_A)$ is connective and flat over $\mathrm{Spec}(A)$.

Remark 2.31. Since vector bundles are locally trivial, there is a canonical isomorphism of derived stacks

$$\mathcal{M}_{\mathrm{Vect}(X)} \simeq \coprod_{n \geqslant 0} \mathcal{M}_{\mathrm{Bun}_{\mathrm{GL}_n}(X)}.$$

Remark 2.32. If A is an ordinary R-algebra, the ∞-groupoid of A-points of $\mathcal{M}_{\mathrm{Coh}(X)}$ is equivalent to the 1-groupoid of coherent sheaves on X_A which are flat over $\mathrm{Spec}(A)$.

Combining Theorems 2.29 and 2.25, we have:

Corollary 2.33. *Let X be a smooth proper scheme over R. Then the derived stack $\mathcal{M}_{\mathrm{Perf}(X)}$ admits a relative cotangent complex*

$$\mathbf{L}_{\mathcal{M}_{\mathrm{Perf}(X)}} = \mathrm{pr}_{2,*}(\mathcal{E}_X \otimes^{\mathbf{L}} \mathcal{E}_X^{\vee}[1] \otimes^{\mathbf{L}} \mathrm{pr}_1^*(K_X))$$

where pr_i are the two projections from $X \times \mathcal{M}_{\mathrm{Perf}(X)}$, and $\mathcal{E}_X := \mathrm{ev}^*(\mathcal{E}^{\mathrm{univ}})$ is the inverse image of the universal perfect complex along the evaluation morphism

$$\mathrm{ev} : X \times \mathcal{M}_{\mathrm{Perf}(X)} \to \mathcal{M}_{\mathrm{Perf}}.$$

Remark 2.34. The inclusions $\mathcal{M}_{\mathrm{Vect}(X)} \hookrightarrow \mathcal{M}_{\mathrm{Coh}(X)}$ and $\mathcal{M}_{\mathrm{Vect}(X)} \hookrightarrow \mathcal{M}_{\mathrm{Perf}(X)}$ are open immersions. That is, for any $A \in \mathrm{dCAlg}_R$ and any morphism $\mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Coh}(X)}$, resp. $\mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Perf}(X)}$, the base changes

$$\mathcal{M}_{\mathrm{Vect}(X)} \overset{\mathbf{R}}{\underset{\mathcal{M}_{\mathrm{Coh}(X)}}{\times}} \mathrm{Spec}(R) \to \mathrm{Spec}(R),$$

$$\text{resp.}\quad \mathcal{M}_{\mathrm{Vect}(X)} \overset{\mathbf{R}}{\underset{\mathcal{M}_{\mathrm{Perf}(X)}}{\times}} \mathrm{Spec}(R) \to \mathrm{Spec}(R)$$

are open immersions of derived schemes. In particular, $\mathcal{M}_{\mathrm{Vect}(X)}$ admits a cotangent complex given by the same formula as in Corollary 2.33.

Similarly, combining Theorem 2.29 with our computation $\mathbf{L}_{BG} \simeq \mathfrak{g}^\vee[-1]$ (Example 2.22) yields:

Corollary 2.35. *Let X be a smooth proper scheme over R. Then the derived stack $\mathcal{M}_{\mathrm{Bun}_G(X)}$ admits a relative cotangent complex*

$$\mathbf{L}_{\mathcal{M}_{\mathrm{Bun}_G(X)}} = \mathrm{pr}_{2,*}(\mathrm{ev}^*\mathfrak{g}^\vee[-1] \otimes^{\mathbf{L}} \mathrm{pr}_1^*(K_X))$$

where pr_i are the two projections from $X \times \mathcal{M}_{\mathrm{Bun}_G(X)}$, and

$$\mathrm{ev} : X \times \mathcal{M}_{\mathrm{Bun}_G(X)} \to BG$$

is the evaluation morphism.

2.6. Algebraicity.

Definition 2.36 (Derived algebraic spaces)**.** A derived stack X is 0-*Artin*, or a *derived algebraic space* if its diagonal $X \to X \times X$ is schematic and a monomorphism, and there exists an étale surjection $U \twoheadrightarrow X$ where U is a derived scheme. A morphism $f : X \to Y$ is 0-*Artin*, or *representable*, if for every affine V and every morphism $V \to Y$, the fibre $X \times^{\mathbf{R}}_Y V$ is a derived algebraic space. A 0-Artin morphism $f : X \to Y$ is *flat*, *smooth*, or *surjective* if for every affine V and every morphism $V \to Y$, there exists a derived scheme U and an étale surjection $U \twoheadrightarrow X \times_Y V$ such that the composite $U \twoheadrightarrow X \times_Y V \to V$ has the respective property.

Definition 2.37 (Derived Artin stacks)**.** We define, by induction:

(i) For $n > 0$, a morphism of derived stacks $f : X \to Y$ is $(n-1)$-*Artin* if for every affine V and every morphism $V \to Y$, $X \times^{\mathbf{R}}_Y V$ is $(n-1)$-Artin. A derived stack X is n-*Artin* if its diagonal is $(n-1)$-Artin and there exists a smooth surjection $U \twoheadrightarrow X$ where U is a derived scheme. An $(n-1)$-Artin morphism $f : X \to Y$ is *flat*, *smooth*, or *surjective*, if for any affine V and any morphism $V \to Y$, there exists a derived scheme U and a smooth surjection $U \twoheadrightarrow X \times_Y V$ such that the composite $U \twoheadrightarrow X \times_Y V \to V$ has the respective property.

(ii) A derived stack is *Artin* if it is n-Artin for some n. A morphism of derived stacks is *Artin* if it is n-Artin for some n. A morphism of derived stacks is *flat*, *smooth*, or *surjective*, if it is n-Artin with the respective property for some n.

Definition 2.38 (Derived Deligne–Mumford stacks)**.** A derived 1-Artin stack is *Deligne–Mumford* if it admits an *étale* surjection from a derived scheme. Equivalently, its classical truncation is Deligne–Mumford.

Artin stacks always admit a cotangent complex:

Theorem 2.39. *Let $f : X \to Y$ be an n-Artin morphism of derived stacks.*

(i) *There exists a relative cotangent complex $\mathbf{L}_{X/Y}$ for f.*

(ii) *The cotangent complex $\mathbf{L}_{X/Y}$ is $(-n)$-connective. That is, for every derived scheme U and every smooth morphism $p : U \to X$, the inverse* *image $p^*\mathbf{L}_{X/Y}$ is $(-n)$-connective, i.e.*

$$\mathrm{H}^{-i}(U, \mathbf{L}_{X/Y}) = \pi_i \mathbf{R}\Gamma(U, \mathbf{L}_{X/Y}) = \pi_i \operatorname{Maps}_{\mathrm{D}(U)}(\mathcal{O}_U, p^*\mathbf{L}_{X/Y}) = 0$$

for all $i < -1$. If f is representable by derived algebraic spaces (or derived Deligne–Mumford stacks), then $\mathbf{L}_{X/Y}$ is in fact connective.

The Artin–Lurie representability theorem is a sort of converse:

Theorem 2.40 (Artin–Lurie representability)**.** *Let R be a commutative ring, which we assume is of finite type over a field (or more generally is a G-ring), and X a derived stack over R. Then X is 1-Artin if and only if the following conditions hold:*

(i) *X admits a cotangent complex $\mathbf{L}_X$.*

(ii) *The restriction of X to ordinary R-algebras takes values in 1-groupoids.*

(iii) Almost of finite presentation. *For any $n \geqslant 0$, the functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ preserves filtered colimits when restricted to n-truncated algebras.*

(iv) Integrability. *For every complete local noetherian k-algebra R, the canonical map $X(R) \to \varprojlim_n X(R/\mathfrak{m}^n)$ is invertible, where $\mathfrak{m} \subseteq R$ is the maximal ideal.*

(v) Nil-completeness. *For every $R \in \mathrm{dCAlg}_k$, the canonical map $X(R) \to \varprojlim_n X(\tau_{\leqslant n}(R))$ is invertible.*

(vi) Infinitesimal cohesion. *For every cartesian square in dCAlg_k*

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \twoheadrightarrow & B \end{array}$$

such that $A \to B$ and $B' \to B$ are surjective on π_0 with nilpotent kernel, X sends the square to a cartesian square.

Example 2.41. Let G be a smooth group scheme over a scheme R and U a derived stack over R with G-action. If U is n-Artin, then so is the quotient stack $[U/G]$. See e.g. [Kha, Thm. 5.11].

Theorem 2.42. *Let R be a G-ring and X a smooth proper scheme over R. Then the following derived stacks are 1-Artin:*

(i) *The moduli stack $\mathcal{M}_{\mathrm{Vect}(X)}$ of vector bundles over X.*

(ii) *The moduli stack $\mathcal{M}_{\mathrm{Bun}_G(X)}$ of G-bundles over X, for every smooth group scheme G over R.*

Theorem 2.42 can be proven by appealing to Theorem 2.40 and applying Corollaries 2.33 and 2.35 to verify condition (i) of Theorem 2.40.

Theorem 2.43. *If R is of finite type over a field (or more generally, is a G-ring admitting a dualizing complex), the moduli stack $\mathcal{M}_{\mathrm{Coh}(X)}$ is also 1-Artin.*

See [HP, Thm. 5.2.2].

2.7. Smoothness.

Example 2.44. If X is a smooth 1-Artin stack over R, then by definition there exists a smooth scheme U and a smooth representable surjection $p : U \twoheadrightarrow X$. We have the exact triangle

$$p^*\mathbf{L}_X \to \mathbf{L}_U \to \mathbf{L}_{U/X}.$$

Since U is smooth, $\mathbf{L}_U \simeq \Omega_U$ is locally free and has Tor-amplitude in $[0,0]$. Since $p : U \twoheadrightarrow X$ is smooth and representable, $\mathbf{L}_{U/X}$ also has Tor-amplitude in $[0,0]$. It follows that the fibre $p^*\mathbf{L}_X$ has Tor-amplitude in $[-1,0]$. Since p is smooth surjective, it follows that $\mathbf{L}_X$ has Tor-amplitude in $[-1,0]$.

More generally we have:

Proposition 2.45. *Let $f : X \to Y$ be a morphism of derived Artin stacks. Then f is smooth if and only if $f_{\mathrm{cl}} : X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$ is locally of finite presentation and $\mathbf{L}_{X/Y}$ is perfect of Tor-amplitude $\leqslant 0$, i.e., if and only if $\mathbf{L}_{X/Y}$ is a perfect complex such that*

$$\pi_i(\mathbf{L}_{X/Y} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$$

for all discrete $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X)^{\heartsuit}$ and all $i > 0$.

This suggests the following generalization of smoothness:

Definition 2.46. Let $f : X \to Y$ be a morphism of derived Artin stacks. We say that f is *homotopically smooth* if f_{cl} is locally of finite presentation and $\mathbf{L}_{X/Y}$ is a perfect complex. We say that f is *homotopically n-smooth*, $n \geqslant 0$, if moreover $\mathbf{L}_{X/Y}$ is of Tor-amplitude $\leqslant n$.

Example 2.47. We say that $f : X \to Y$ is *quasi-smooth* if it is homotopically 1-smooth. This admits the following more geometric characterization: there exists a smooth surjection $U \twoheadrightarrow X$ such that $f|_U$ factors via a smooth morphism $Y' \to Y$ and a closed immersion $U \to Y'$ which exhibits U as the derived zero locus of a section s of a vector bundle E over Y'.

$$\begin{array}{ccccc} U & \longrightarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle s} & & \\ Y' & \xrightarrow{0} & E & & \end{array}$$

See [KR, Prop. 2.3.14]. In fact, it is possible to generalize this to characterize homotopical n-smoothness by taking into account "shifted" vector bundles $E[-i]$, $0 \leqslant i < n$.

Corollary 2.48. *Let X be a smooth proper scheme over R. The derived stacks $\mathcal{M}_{\mathrm{Vect}(X)}$ and $\mathcal{M}_{\mathrm{Bun}_G(X)}$ (for any smooth group scheme G over R) are homotopically smooth. More precisely, they are homotopically $(n-1)$-smooth if X is of dimension $\leqslant n$.*

Proof. Follows from corollaries 2.33 and 2.35. □

For example, when X is a curve these moduli stacks are smooth (a fortiori flat, hence classical). When X is a surface, they are quasi-smooth, even though their *classical truncations* are generally very singular (not even homotopically smooth). This is a general phenomenon in derived algebraic geometry: derived moduli problems tend to be homotopically smooth. The following fact may be regarded as a conceptual explanation for this phenomenon: it turns out that homotopical smoothness is just the derived analogue of being locally finitely presented.

Theorem 2.49 (Lurie). *A derived stack X over R is homotopically smooth if and only if it is* locally homotopically of finite presentation*.*

Recall that if X is classical, it is locally of finite presentation if and only if $X : \mathrm{CAlg}_R \to \mathrm{Grpd}$ preserves filtered colimits.

Definition 2.50. A derived stack X is *locally homotopically of finite presentation* (or *locally hfp*) if $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ preserves filtered colimits. (In particular, Theorem 2.40(iii) is automatic in this case.)

More generally, a morphism $X \to Y$ is locally hfp if for every affine $V = \mathrm{Spec}(A)$ and every morphism $\mathrm{Spec}(A) \to Y$, the derived fibre $X \times^{\mathbf{R}}_Y \mathrm{Spec}(A)$ is locally hfp.

Warning 2.51. Sometimes (e.g. in [SAG]), the term *(locally) of finite presentation* is used instead of *(locally) homotopically of finite presentation*. We warn the reader however that the homotopical condition is much stronger than being locally of finite presentation in the sense of classical algebraic geometry. For example, if X and Y are classical noetherian schemes and $i : X \hookrightarrow Y$ is a closed immersion of finite Tor-amplitude (this is automatic say if Y is regular, e.g. smooth over a field), then the following conditions are equivalent (see [Avr, Thm. 1.3]):

(a) i is homotopically of finite presentation, or equivalently homotopically smooth: the relative cotangent complex $\mathbf{L}_{X/Y}$ is a perfect complex;

(b) i is homotopically 1-smooth: the relative cotangent complex $\mathbf{L}_{X/Y}$ is perfect of Tor-amplitude $[0, 1]$;

(c) i is a regular (or *lci*) closed immersion.

References

[Avr] L. Avramov, *Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology.* Ann. Math. **150** (1999), no. 2, 455–487.

[CS] K. Česnavicius, P. Scholze, *Purity for flat cohomology.* arXiv:arXiv:1912.10932 (2019).

[GZ] P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **35** (1967). Springer–Verlag.

[HP] D. Halpern-Leistner, A. Preygel, *Mapping stacks and categorical notions of properness.* arXiv:1402.3204 (2019).

[HS] A. Hirschowitz, C. Simpson, *Descente pour les n-champs.* math.AG/9807049 (2001).

[HTT] J. Lurie, *Higher Topos Theory.*

[HA] J. Lurie, *Higher Algebra.*

[KR] A. A. Khan, D. Rydh, *Virtual Cartier divisors and blow-ups.* arXiv:1802.05702 (2018).

[Kha] A. A. Khan, *A modern introduction to algebraic stacks.* Lecture notes (2022), https://www.preschema.com/lecture-notes/2022-stacks/stacksncts.pdf.

[SAG] J. Lurie, *Spectral algebraic geometry*, version of 2018-02-03. https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[TV1] B. Toën, M. Vaquié, *Moduli of objects in dg-categories.* Ann. Sci. Éc. Norm. Supér. **40** (2007), no. 3, 387–444.

[TV2] B. Toën, G. Vezzosi, *Homotopical algebraic geometry. II: Geometric stacks and applications.* Mem. Am. Math. Soc. **902** (2008).

[Toë] B. Toën, *Simplicial presheaves and derived algebraic geometry*, in: Simplicial methods for operads and algebraic geometry, 119–186, Adv. Courses Math. CRM Barcelona (2010).

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