Exercise sheet 3

1. Let $\mathbf{C}$ be a stable ∞-category. Show that $\mathbf{C}$ is additive, i.e., that the canonical morphisms $x \sqcup y \to x \times y$ are invertible for all objects $x, y \in \mathbf{C}$.

2. Let R be a simplicial commutative ring. Show that the assignment $\mathrm{M} \mapsto \pi_0(\mathrm{M})$, viewed as a functor $\mathrm{Mod}_{\mathrm{R}} \to (\mathrm{Mod}_{\pi_0(\mathrm{R})})^{\heartsuit}$, induces an equivalence

$$(\mathrm{Mod}_{\mathrm{R}})^{\heartsuit} \simeq (\mathrm{Mod}_{\pi_0 \mathrm{R}})^{\heartsuit}.$$

3. Let R be a simplicial commutative ring and $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}$. Show that M is finitely generated projective iff it is locally free of finite rank; that is, if there exists a Zariski covering $(\mathrm{R} \to \mathrm{R}_\alpha)_\alpha$ such that each $\mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}_\alpha$ is isomorphic to $\mathrm{R}^{\oplus n_\alpha}$ for some n_α.

3. Let R be a simplicial commutative ring and $\mathrm{M} \in \mathrm{Mod}_{\mathrm{R}}$. Suppose that M is flat. Then show that the following conditions are equivalent:

(i) M is *projective* in the sense that for any map of connective R-modules $\mathrm{N}_1 \to \mathrm{N}_2$ that is surjective on π_0, any map $\mathrm{M} \to \mathrm{N}_2$ lifts to N_1 (up to homotopy).

(ii) $\pi_0(\mathrm{M})$ is projective as a $\pi_0(\mathrm{R})$-module.

4. Let R be a simplicial commutative ring.

(a) Show that the condition "of finite tor-amplitude" is stable under finite colimits and direct summands in $\mathrm{Mod}_{\mathrm{R}}$.

(b) Deduce that any perfect R-module is of finite tor-amplitude.

5. Show that for any R-module M, we have functorial isomorphisms

$$\mathrm{M} \xrightarrow{\sim} \varprojlim_n \tau_{\leqslant n}(\mathrm{M}),$$

$$\varinjlim_n \tau_{\geqslant n}(\mathrm{M}) \xrightarrow{\sim} \mathrm{M}.$$

6. Show that the subcategory $(\mathrm{Mod}_{\mathrm{R}})_{\geqslant 0}$ is generated under colimits by the object R.

7. Let R be an ordinary commutative ring. Show that a discrete R-module $\mathrm{M} \in (\mathrm{Mod}_{\mathrm{R}})^{\heartsuit}$ is compact iff it is of finite presentation.

8. Let R be a simplicial commutative ring. Suppose that $\pi_0(\mathrm{R})$ is noetherian and $\pi_i(\mathrm{R})$ are finitely generated $\pi_0(\mathrm{R})$-modules. Say that an R-module M is *coherent* if its homotopy groups $\pi_i(\mathrm{M})$ are bounded above and below, and are all finitely generated $\pi_0(\mathrm{R})$-modules.

(a) Show that the coherent R-modules generate a *thick* subcategory $\mathrm{Mod}^{\mathrm{coh}}_{\mathrm{R}} \subset \mathrm{Mod}_{\mathrm{R}}$, i.e., they are closed under finite (co)limits and direct summands.

(b) Show that there is an inclusion $\mathrm{Mod}^{\mathrm{perf}}_{\mathrm{R}} \subset \mathrm{Mod}^{\mathrm{coh}}_{\mathrm{R}}$ iff R has only finitely many homotopy groups.

(c) Suppose that $\pi_0(\mathrm{R})$ is regular. Then show that there is an inclusion $\mathrm{Mod}^{\mathrm{coh}}_{\mathrm{R}} \subset \mathrm{Mod}^{\mathrm{perf}}_{\mathrm{R}}$ iff $\pi_0(\mathrm{R})$ is perfect as an R-module.