Lecture 1

Higher categories and simplicial commutative rings

Last lecture we saw that the multiplicity of an intersection of subvarieties V and W in X can be calculated using the Tor groups $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{O}_W)$. In order to realize these groups as "coming from a geometric object", the first step is to consider the whole complex $\mathcal{O}_V \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_W$ as a single unit, before passing to homotopy groups. This is a chain complex which, by nature of derived functors, is only well-defined up to quasi-isomorphism. In this lecture we will review the theory of ∞-categories, which is an effective language for working with chain complexes up to quasi-isomorphism (or any situation of a similar "homotopical" flavour). We will also introduce the notion of *simplicial commutative ring*, which will finally allow us to view the construction $\mathcal{O}_V \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_W$ as a simplicial commutative ring (when X is affine).

Construction 1. Let R be a commutative ring, and let C(R) be the category of *chain complexes* of R-modules. Recall that an object $M_\bullet \in C(R)$ is a diagram

$$\cdots \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} M_0 \xrightarrow{d_0} \cdots$$

where the relation $d^2 = 0$ holds for all n. A *morphism* of chain complexes $\phi : M_\bullet \to N_\bullet$ is a collection of morphisms $\phi_n : M_n \to N_n$ which are compatible with the differentials ($d \circ \varphi = \varphi \circ d$).

The *homology groups* of $M_\bullet$ are the abelian groups $H_n(M_\bullet) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n+1})$. A *quasi-isomorphism* of chain complexes is a morphism that induces isomorphisms on all homology groups.

Example 2. Let M and N be R-modules. Let $\tilde{M}_\bullet \in C(R)$ be a projective resolution of M, so that we have a quasi-isomorphism $\tilde{M}_\bullet \to M$ and each $\tilde{M}_n$ is projective. Then we can perform the objectwise tensor product $\tilde{M}_\bullet \otimes_R N$ to obtain a new chain complex, which is a model for the *derived tensor product* $M \otimes^{\mathbf{L}}_R N$. In particular its homology groups $H_*(M \otimes^{\mathbf{L}}_R N)$ are the Tor groups $\mathrm{Tor}^R_*(M, N)$.

Note though that the chain complex $\tilde{M}_\bullet \otimes_R N$ depends on the choice of resolution $\tilde{M}_\bullet$: if we use another resolution, we get a new chain complex which may not be isomorphic to the first. However, it will still be *quasi-isomorphic* to the first. In other words, the derived tensor product $M \otimes^{\mathbf{L}}_R N$ is only well-defined up to quasi-isomorphism.

Thus, as far as the constructions of homological algebra go, we only care about chain complexes up to quasi-isomorphism.

Construction 3. There exists a category D(R), called the *derived category* of R, which is obtained by formally inverting quasi-isomorphisms. More precisely, it satisfies the following universal property:

(*) Let F be a functor $C(R) \to \mathbf{D}$ that sends all quasi-isomorphisms to isomorphisms. Then there exists a unique functor $\overline{F} : D(R) \to \mathbf{D}$ such that the diagram

$$\begin{array}{ccc} C(R) & \xrightarrow{F} & \mathbf{D} \\ \downarrow{\scriptstyle \gamma} & \nearrow_{\overline{F}} & \\ D(R) & & \end{array}$$

commutes.

We say that D(R) is the *(Gabriel–Zisman) localization* of C(R) at the class of quasi-isomorphisms.

Remark 4. One can describe the category D(R) explicitly using Verdier's calculus of fractions. Roughly speaking, D(R) has the same objects as C(R), and the morphisms $M_\bullet \to N_\bullet$ are (equivalence classes of) diagrams

$$\mathrm{M}_\bullet \xleftarrow{\phi} \mathrm{P}_\bullet \xrightarrow{\psi} \mathrm{N}_\bullet,$$

where ϕ and ψ are morphisms in C(R) with ϕ a quasi-isomorphism.

Example 5. Given R-modules M and N, the group $\mathrm{Hom}_{\mathrm{D(R)}}(\mathrm{M}, \mathrm{N}[i])$ can be identified with the Ext group $\mathrm{Ext}^i_{\mathrm{R}}(\mathrm{M}, \mathrm{N})$. Here $\mathrm{N}[i]$ denotes the chain complex with N concentrated in degree i.

The category D(R) is a first approximation to what we want. For example, all projective resolutions of an R-module M are *isomorphic* in D(R), so the derived tensor product is actually well-defined as an object of D(R). However, it turns out to be rather poorly behaved: unlike the category C(R), D(R) is not abelian and does not admit (co)limits. It does however admit *homotopy* (co)limits:

Construction 6. Let I be a category of "diagram shapes". For example, take the category with three objects and nontrivial morphisms as follows:

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array} \tag{0.1}$$

Let $\mathrm{C^I(R)}$ denote the category of functors $\mathrm{I} \to \mathrm{C(R)}$ (these are I-shaped diagrams in C(R)). Let $\mathrm{D^I(R)}$ denote the Gabriel–Zisman localization of $\mathrm{C^I(R)}$ at the class of levelwise quasi-isomorphisms.

Fact 7. *The "constant diagram" functor* $c : \mathrm{C(R)} \to \mathrm{C^I(R)}$ *preserves quasi-isomorphisms and induces a functor*

$$c : \mathrm{D(R)} \to \mathrm{D^I(R)}.$$

It admits left and right adjoints

$$\mathbf{L}\varinjlim : \mathrm{D^I(R)} \to \mathrm{D(R)},$$
$$\mathbf{R}\varprojlim : \mathrm{D^I(R)} \to \mathrm{D(R)}.$$

These are called the "homotopy colimit" and "homotopy limit" functors, respectively.

Example 8. Take I to be the category (0.1). For any diagram $\mathrm{X} : \mathrm{I} \to \mathrm{C(R)}$, we can view X as an object of $\mathrm{D^I(R)}$ (via the canonical functor $\mathrm{C^I(R)} \to \mathrm{D^I(R)}$), and form the homotopy colimit $\mathbf{L}\varinjlim(\mathrm{X}) \in \mathrm{D(R)}$. This is called the *homotopy push-out* of X.

The theory of homotopy colimits gives us an analogue of cokernels:

Example 9. Let $\phi : \mathrm{M}_\bullet \to \mathrm{N}_\bullet$ be a morphism of chain complexes. Then we can consider the I-shaped diagram

$$\begin{array}{ccc} \mathrm{M}_\bullet & \xrightarrow{\phi} & \mathrm{N}_\bullet \\ \downarrow & & \\ 0 & & \end{array}$$

and the induced object in $\mathrm{D^I(R)}$. Its *homotopy push-out* is called the *homotopy cokernel* or *homotopy cofibre* of f.

The homotopy cofibre can be modelled by a chain complex $\mathrm{Cone}(\phi)_\bullet$:

$$\mathrm{Cone}(\phi)_n = \mathrm{N}_n \oplus \mathrm{M}_{n-1},$$

with differential given by

$$(d_{n+1} : \mathrm{N}_{n+1} \oplus \mathrm{M}_n \to \mathrm{N}_n \oplus \mathrm{M}_{n-1}) = \begin{bmatrix} d^{\mathrm{N}}_{i+1} & \phi_i \\ 0 & -d^{\mathrm{M}}_i \end{bmatrix}$$

Its image by the functor $\mathrm{C(R)} \to \mathrm{D(R)}$ is the homotopy cofibre of f.

The problem with the category $D(R)$ is that *the theory of homotopy (co)limits is not internal to it.* This is because we cannot recover the categories $D^I(R)$ from the category $D(R)$ (but only from the non-localized version $C(R)$):

Fact 10. *The canonical functor*

$$D^I(R) \to \mathrm{Funct}(I, D(R))$$

is not an equivalence.

In other words, while there does exist a good theory of homotopy (co)limits, we cannot make use of it if we only consider the $D(R)$ by itself. We will need a more refined version of $D(R)$.

Definition 11. A *dg-category* (over a commutative ring R) is a category enriched over $C(R)$. That is, a dg-category $\mathbf{C}$ has a set of objects and for each pair of objects X, Y, a *chain complex* of morphisms ("Hom-complex") $\mathrm{Hom}(X, Y)_\bullet \in C(R)$. It also has a composition law satisfying similar axioms as for ordinary categories.

Construction 12. The *homotopy category* of a dg-category $\mathbf{C}$ is an ordinary category $\mathrm{Ho}(\mathbf{C})$ with the same objects as $\mathbf{C}$, and with Hom-sets given by:

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(x, y) = H_0(\mathrm{Hom}_{\mathbf{C}}(x, y)_\bullet),$$

and with composition law induced from that of $\mathbf{C}$.

Example 13. The category $C(R)$ can be viewed as a dg-category. It is enriched over itself by the following formula:

$$\mathrm{Hom}(M_\bullet, N_\bullet)_n = \prod_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathrm{Mod}_R}(M_i, N_{i+n}).$$

The chain complex $\mathrm{Hom}(M_\bullet, N_\bullet)_\bullet$ has differentials given by:

$$d(f) = d^N \circ f - (-1)^n f \circ d^M,$$

for any $f \in \mathrm{Hom}(M_\bullet, N_\bullet)_n$.

The dg-category $C(R)$ does not take the homotopy theory of chain complexes into account; in particular, quasi-isomorphic complexes do not become isomorphic in $\mathrm{Ho}(C(R))$. Instead, we would like to construct a dg-category $\underline{D}(R)$ whose homotopy category is equivalent to $D(R)$. According to Example 5, we need the homology of the Hom-complexes in $\underline{D}(R)$ to compute the Ext groups $\mathrm{Ext}^n_R(M, N)$ (where M and N are R-modules). In fact, the naively defined Hom-complexes of $C(R)$ do have the right homology, when we restrict to nice enough objects:

Definition 14. A chain complex $M_\bullet \in C(R)$ is *K-projective* if for every quasi-isomorphism $N_\bullet \to N'_\bullet$ that is degree-wise surjective, any morphism $M_\bullet \to N'_\bullet$ lifts to $N_\bullet$.

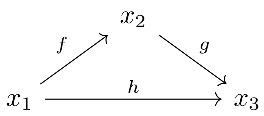
Construction 15. Let $\underline{D}(R)$ denote the full sub-dg-category of $C(R)$ whose objects are K-projective complexes. Then we have an equivalence $\mathrm{Ho}(\underline{D}(R)) \approx D(R)$.

There is an alternative way of encoding the data of the dg-category $\underline{D}(R)$, which will be more convenient for our purposes (see [2, Constr. 1.3.1.6] for details):

Construction 16 (Dg-nerve)**.** Let $\mathbf{C}$ be a dg-category. We define a sequence of sets S_n as follows:

- Let S_0 be the set of objects of $\mathbf{C}$.
- Let S_1 be the set of morphisms $f : x_1 \to x_2$ in $\mathbf{C}$, i.e. triples (x_1, x_2, f) with x_1 and x_2 objects and $f \in \mathrm{Hom}_{\mathbf{C}}(x_1, x_2)_0$ with $df = 0$.

- Let S_2 be the set of diagrams



  that commute *up to a specified chain homotopy* $h \Rightarrow g \circ f$, i.e. an element $\varphi \in \mathrm{Hom}_{\mathbf{C}}(x_1, x_3)_1$ with $d\varphi = (g \circ f) - h$.
- $\cdots$
- Let S_n be the set of n-tuples of objects $(x_1, \ldots, x_n)$ and, for each $0 \leqslant m \leqslant 0$ and all indices $0 \leqslant i_0 < \cdots < i_{m+1} \leqslant n$, a morphism

$$f_{i_0,\ldots,i_{m+1}} \in \mathrm{Hom}_{\mathbf{C}}(x_{i_0}, x_{i_{m+1}})_m$$

  satisfying

$$d(f_{i_0,\ldots,i_{m+1}}) = \sum_{1 \leqslant j \leqslant m} (-1)^j (f_{i_0,\ldots,\hat{i_j},\ldots,i_{m+1}} - f_{i_j,\ldots,i_{m+1}} \circ f_{i_0,\ldots,i_j}).$$

Moreover, these sets are connected by canonical maps. For example, there are two maps $\mathrm{S}_1 \to \mathrm{S}_0$, which pick out the source and target, respectively. In the other direction there is a map $\mathrm{S}_0 \to \mathrm{S}_1$ which picks out the identity morphism of an object. Similarly there are three maps $\mathrm{S}_2 \to \mathrm{S}_1$ picking out the morphisms on the boundary of the triangle, and two maps $\mathrm{S}_1 \to \mathrm{S}_2$ picking out, for any morphism f, the two triangles encoding homotopies $f \Rightarrow f \circ \mathrm{id}$ and $f \Rightarrow \mathrm{id} \circ f$. Generally, for any order-preserving map $\alpha : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$, there is a canonical map $\alpha^* : \mathrm{S}_n \to \mathrm{S}_m$. The data of these sets $(\mathrm{S}_n)_n$ together with these maps forms an example of a simplicial set:

Definition 17. Let $\mathbf{\Delta}$ denote the category of finite sets $[n] := \{0, 1, \ldots, n\}$ and order-preserving maps between them. A *simplicial set* X is a contravariant functor from $\mathbf{\Delta}$ to the category of sets. That is, it is a sequence of sets $\mathrm{X}_n := \mathrm{X}([n])$, $n \geqslant 0$, together with maps $\alpha^* : \mathrm{X}_n \to \mathrm{X}_m$ for all morphisms $[m] \to [n]$ in $\mathbf{\Delta}$, such that $\mathrm{id}^* = \mathrm{id}$ and $(\beta \circ \alpha)^* = \alpha^* \beta^*$ for all composable morphisms α and β in $\mathbf{\Delta}$.

For each n and $0 \leqslant i \leqslant n$ we write $d^i_n : \mathrm{X}_n \to \mathrm{X}_{n-1}$ for the *face maps*, induced by the canonical maps $\delta^i_n : [n-1] \to [n]$ (where δ^i_n is the injective map that "skips" i). We write $s^i_n : \mathrm{X}_n \to \mathrm{X}_{n+1}$ for the *degeneracy maps*, induced by the canonical maps $\sigma^i_n : [n+1] \to [n]$ (where σ^i_n is the surjective map that "doubles" i).

Example 18. Any set X can be viewed as a constant simplicial set $c(\mathrm{X})$, with $c(\mathrm{X})_n = \mathrm{X}$ (and $\alpha^* = \mathrm{id}$ for all $\alpha : [m] \to [n]$). The assignment $\mathrm{X} \mapsto c(\mathrm{X})$ defines a fully faithful embedding of the category of sets into that of simplicial sets.

Example 19. There is a simplicial set Δ^n, called the *standard n-simplex*, whose k-simplices are given by:

$$\Delta^n_k = \mathrm{Hom}_{\mathbf{\Delta}}([k], [n]).$$

Note that such a map $[k] \to [n]$ corresponds to a sequence of integers $(a_0, \ldots, a_k)$ with $0 \leqslant a_i \leqslant a_j \leqslant n$ for all $i \leqslant j$.

Example 20. There is a simplicial set $\partial^i \Delta^n$, defined as the image of the canonical map $\Delta^{n-1} \to \Delta^n$ induced by δ^i_n. This is the *i-th face* of the standard n-simplex. The union of these is a simplicial set $\partial \Delta^n$, called the *boundary* of the standard n-simplex.

The *i-th horn* Λ^n_i is the union of the faces $\partial^j \Delta^n$ with $j \neq i$.

We can view $\partial \Delta^n$ as the result of removing the "interior" of Δ^n. Similarly, Λ^n_i is the result of removing the interior as well as the i-th face.

Example 21. For any (ordinary) category $\mathbf{C}$, there is a simplicial set $\mathrm{N}(\mathbf{C})$, called the *nerve* of $\mathbf{C}$, whose n-simplices are composable strings of morphisms of length n

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} f_n$$

in $\mathbf{C}$.

Proposition 22. *The assignment $\mathbf{C} \mapsto \mathrm{N}(\mathbf{C})$ defines a fully faithful embedding of the category of (small) categories into the category of simplicial sets. Moreover, a simplicial set X lies in its essential image if and only if it satisfies the Grothendieck–Segal condition:*

($*$) *Any map $\Lambda^n_i \to \mathrm{X}$, with $n \geqslant 0$ and $0 < i < n$, lifts uniquely to a map $\Delta^n \to \mathrm{X}$.*

One can actually take "simplicial set satisfying the Grothendieck–Segal condition" as a (very inefficient) *definition* of the term "category", where we view the 0-simplices of X as objects, and 1-simplices as morphisms. To understand the nature of the Grothendieck–Segal condition, take $n = 2$ and $i = 1$. Then a map $\Lambda^2_1 \to \mathrm{X}$ corresponds to a diagram

$$\begin{array}{ccccc} & & x_2 & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ x_1 & & & & x_3 \end{array}$$

in X. The condition that this lifts uniquely to a map $\Delta^2 \to \mathrm{X}$ means that there exists a unique map $x_1 \to x_3$, which we may suggestively denote $g \circ f$, making the diagram commute.

Remark 23. If we add the edge cases $i = 0$ and $i = n$ to the Grothendieck–Segal condition, then we get a characterization of the nerves of *groupoids*.

The dg-nerve (Construction 16) does not satisfy the Grothendieck–Segal condition, since there is not a unique way to compose morphisms. Instead, it satisfies the following weaker variant:

Definition 24. A *weak Kan complex* is a simplicial set X satisfying the following condition (the *inner horn lifting condition*):

($*$) Any map $\Lambda^n_i \to \mathrm{X}$, with $n \geqslant 0$ and $0 < i < n$, lifts to a map $\Delta^n \to \mathrm{X}$.

We will take weak Kan complexes as a definition of the term "∞-category":

Definition 25. An *∞-category* $\mathbf{C}$ is a weak Kan complex X.

Objects of $\mathbf{C}$ are 0-simplices of X. Morphisms of $\mathbf{C}$ are 1-simplices. A homotopy between two morphisms f and g is a 2-simplex $\Delta^2 \to \mathrm{X}$ of the form

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & & \overset{\mathrm{id}}{\searrow} & \\ x & & \xrightarrow{\quad g \quad} & & y \end{array}$$

We say that f and g are homotopic if there exists a homotopy as above; this defines an equivalence relation on the set of morphisms $x \to y$.

Given two morphisms $f : x \to y$ and $g : y \to z$ in $\mathbf{C}$, it follows from the definition of weak Kan complex that there exists a 2-simplex of X

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ x & & \xrightarrow{\quad h \quad} & & z \end{array}$$

We refer to h as the composite of f and g, denoted $g \circ f$. While h is not unique, one can derive from the definition the fact that it is unique *up to homotopy*, or more precisely, up to a contractible space of choices. Similarly, one can see that composition is associative up to homotopy.

Given an ∞-category $\mathbf{C}$, we can define an ordinary category $\mathrm{Ho}(\mathbf{C})$, called the *homotopy category* of $\mathbf{C}$, with the same objects, and Hom-sets $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(x, y)$ given by sets of homotopy classes of morphisms $x \to y$ in $\mathbf{C}$.

We will say that a morphism $f : x \to y$ in $\mathbf{C}$ is an *isomorphism* if it induces an isomorphism in the homotopy category $\mathrm{Ho}(\mathbf{C})$. Equivalently, f is *invertible*; that is, there exists a morphism $g : y \to x$ and a homotopy $f \circ g \Rightarrow \mathrm{id}_y$ and $g \circ f \Rightarrow \mathrm{id}_x$.

The theory of ∞-categories was developed by André Joyal and Jacob Lurie; we refer the reader to [1] for details.

Remark 26. A *Kan complex* is a simplicial set X satisfying a lifting condition for all horns:

($*$) Any map $\Lambda^n_i \to \mathrm{X}$, with $n \geqslant 0$ and $0 \leqslant i \leqslant n$, lifts to a map $\Delta^n \to \mathrm{X}$.

The nerve of any groupoid is then a Kan complex, and we take Kan complexes as a definition of the term "∞-groupoid". It is possible to show that an ∞-groupoid is the same thing as an ∞-category where all morphisms are invertible.

Example 27. Let X be a nice topological space (say, a CW-complex). Then one can define a simplicial set $\mathrm{Sing}(\mathrm{X})_\bullet$, whose n-simplices are continuous maps $\Delta^n_\top \to \mathrm{X}$ (where $\Delta^n_\top$ is the topological version of the standard n-simplex). This is a Kan complex, and moreover the assignment $\mathrm{X} \mapsto \mathrm{Sing}(\mathrm{X})_\bullet$ gives rise to an equivalence between the homotopy theories of nice spaces and Kan complexes. For this reason, we will use the terms "space" and "∞-groupoid" interchangeably.

The ∞-category of spaces plays the role of the category of sets; for example, for any two objects x and y of an ∞-category $\mathbf{C}$, there is a space $\mathrm{Maps}_{\mathbf{C}}(x, y)$, called the *mapping space.* In order to define the ∞-category of spaces, it will be useful to have a non-linear version of the notion of dg-category. Namely, define a *simplicial category* $\mathbf{C}$ (or *simplicially enriched category*) to be a category enriched in the category of simplicial sets (i.e., there is a simplicial Hom-set $\mathrm{Hom}_{\mathbf{C}}(x, y)$ of maps between any two objects).

Example 28. The category of simplicial sets is simplicially enriched. For any two simplicial sets X and Y, there is a simplicial set $\underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{Y})$ with n-simplices

$$\underline{\mathrm{Hom}}(\mathrm{X}, \mathrm{Y})_n = \mathrm{Hom}(\Delta^n \times \mathrm{X}, \mathrm{Y}).$$

There is an analogue of Construction 16, called the *simplicial nerve* $\mathrm{N}_{\boldsymbol{\Delta}}(\mathbf{C})$ (where we use simplicial homotopies instead of chain homotopies). One can show that this is a weak Kan complex as long as the simplicial Hom-sets are Kan complexes.

Construction 29. Consider the simplicial category of Kan complexes (with the simplicial enrichment as above). Its simplicial Hom-sets are Kan complexes, and its simplicial nerve is called the *∞-category of spaces*, denoted Spc.

The notion of *simplicial commutative ring*, which will form the basic building blocks of derived schemes, can be thought of as a space (in the above sense), equipped with operations of addition and multiplication (that are commutative and associative in a much stricter sense than the notion of $\mathcal{E}_\infty$-space).

Definition 30. A *simplicial commutative ring* A is a simplicial object in the category of commutative rings. In other words, it is a simplicial set such that each term A_n is equipped with a structure of (unital associative) commutative ring, and all maps $\alpha^* : \mathrm{A}_n \to \mathrm{A}_m$ (for any $\alpha : [m] \to [n]$) are ring homomorphisms.

Any ordinary commutative ring A can be viewed as a constant simplicial commutative ring $c(\mathrm{A})$; we will usually omit c from the notation.

Definition 31. A *trivial Kan fibration* of simplicial sets is a map $p : \mathrm{X} \to \mathrm{Y}$ that satisfies the following lifting property. For any diagram of solid arrows

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathrm{X} \\ \big\downarrow & \overset{h}{\nearrow} & \big\downarrow p \\ \Delta^n & \longrightarrow & \mathrm{Y}, \end{array}$$

there exists a dashed arrow h making the diagram commute.

Any trivial Kan fibration is a weak homotopy equivalence and is degreewise surjective. The following notion can therefore be viewed as a simplicial version of "K-projective":

Definition 32. A simplicial commutative ring A is *cofibrant* if for any map of simplicial commutative rings $\mathrm{R} \to \mathrm{R}'$ which is a trivial Kan fibration, and any morphism of simplicial commutative rings $\mathrm{A} \to \mathrm{R}'$, there exists a unique lift $\mathrm{A} \to \mathrm{R}$.

The most important example of cofibrant simplicial commutative rings is as follows:

Example 33. The polynomial algebras $\mathbf{Z}[\mathrm{T}_0, \ldots, \mathrm{T}_n]$ are cofibrant (when viewed as constant simplicial commutative rings).

Example 34. Any commutative ring A admits a standard cofibrant resolution $\tilde{\mathrm{A}}$. The 0th term $\tilde{\mathrm{A}}_0$ is the polynomial $\mathbf{Z}$-algebra generated by the elements of A. The 1st term $\tilde{\mathrm{A}}_1$ is the polynomial $\mathbf{Z}$-algebra generated by the elements of $\tilde{\mathrm{A}}_0$, and so on.

Construction 35. The category of simplicial commutative rings admits a canonical simplicial enrichment, since for any simplicial commutative ring B, the simplicial Hom-sets $\underline{\mathrm{Hom}}(\mathrm{A}, \mathrm{B})$ inherit a structure of simplicial commutative ring from B. The ∞-category SCRing is the simplicial nerve of the simplicially enriched category of *cofibrant* simplicial commutative rings.

From this point on, we will use the term "simplicial commutative ring" as a synonym for "object of the ∞-category SCRing"; that is, we will implicitly replace any simplicial commutative ring in sight by a cofibrant resolution.

There is a *derived tensor product* on the ∞-category SCRing.

Example 36. Given two commutative rings A and B, the derived tensor product $\mathrm{A} \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathrm{B}$ can be computed by taking a cofibrant resolution $\tilde{\mathrm{A}}$ and forming the levelwise tensor product $\tilde{\mathrm{A}} \otimes_{\mathbf{Z}} \mathrm{B}$.

References

[1] J. Lurie, *Higher topos theory.*
[2] J. Lurie, *Higher algebra.*