Lecture 10

The Grothendieck–Riemann–Roch theorem

In this lecture we will state and prove the Grothendieck–Riemann–Roch theorem. Recall that this theorem involves the γ-filtration on the K-theory of a derived scheme:

Construction 1. Let X be a derived scheme (admitting the resolution property[1]). Recall that $\mathrm{K}_0(\mathrm{X})$ has a λ-ring structure, with augmentation provided by the rank homomorphism $\mathrm{rk} : \mathrm{K}_0(\mathrm{X}) \to \mathrm{H}^0(\mathrm{X}_{\mathrm{Zar}}, \mathbf{Z})$. As we have seen before, one has in this situation a decreasing filtration $\mathrm{Fil}^*_\gamma \mathrm{K}_0(\mathrm{X})$ called the *γ-filtration.* To describe it, set $\gamma^k(x) = \lambda^k(x + k - 1)$ for any $x \in \mathrm{K}_0(\mathrm{X})$ and integer k. Then Fil^1_γ is defined as the kernel of the augmentation, and for $k \geqslant 2$, Fil^k_γ is generated by terms of the form

$$\gamma^{k_0}(x_0) \cdots \gamma^{k_n}(x_n), \qquad (x_i \in \mathrm{Fil}^1_\gamma,\ k_0 + \cdots + k_n \geqslant k).$$

Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion. The GRR theorem concerns more precisely two Gysin homomorphisms

$$i_* : \mathrm{K}_0(\mathrm{Z}) \to \mathrm{K}_0(\mathrm{X})$$
$$i^\gamma_* : \mathrm{Gr}^*_\gamma \mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}} \to \mathrm{Gr}^{*+d}_\gamma \mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}.$$

The first comes from the fact that the functor $i_* : \mathrm{Qcoh}(\mathrm{Z}) \to \mathrm{Qcoh}(\mathrm{X})$ preserves perfect complexes. The second comes from the following theorem, which will be our first goal for this lecture:

Theorem 2. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion of qcqs derived schemes. Suppose that i has virtual codimension d. Then the homomorphism $i_* : \mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}} \to \mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}$ sends $\mathrm{Fil}^k_\gamma \mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}}$ to $\mathrm{Fil}^{k+d}_\gamma \mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}$, for any k.*

The final ingredient in the statement of GRR is a certain map $\mathrm{ch} : \mathrm{K}_0(\mathrm{X}) \to \mathrm{Gr}_\gamma(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}})$. After constructing it, we will proceed to prove:

Theorem 3 (Grothendieck–Riemann–Roch)**.** *Let X be a qcqs derived scheme. Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion of virtual codimension d. Then there is a commutative square*

$$\begin{array}{ccc} \mathrm{K}_0(\mathrm{Z}) & \xrightarrow{\quad i_* \quad} & \mathrm{K}_0(\mathrm{X}) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ \mathrm{Gr}_\gamma(\mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}}) & \xrightarrow{i^\gamma_*(-\cdot \mathrm{Td}(-\mathcal{N}^\vee_{\mathrm{Z}/\mathrm{X}}))} & \mathrm{Gr}_\gamma(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}) \end{array}$$

That is, for any $x \in \mathrm{K}_0(\mathrm{Z})$, we have the identity

$$\mathrm{ch}(i_*(x)) = i^\gamma_*(\mathrm{ch}(x) \cdot \mathrm{Td}(-\mathcal{N}^\vee_{\mathrm{Z}/\mathrm{X}})).$$

Remark 4. Note that $-\mathcal{N}_{\mathrm{Z}/\mathrm{X}}$ is the class in K-theory of the cotangent complex $\mathcal{L}_{\mathrm{Z}/\mathrm{X}} = \mathcal{N}_{\mathrm{Z}/\mathrm{X}}[1]$. One can also prove a GRR theorem for the projection of a projective bundle $\pi : \mathbf{P}_\mathrm{X}(\mathcal{E}) \to \mathrm{X}$. Combining these two variants, one gets a GRR theorem for any quasi-smooth projective morphism of qcqs derived schemes (where $-\mathcal{N}_{\mathrm{Z}/\mathrm{X}}$ is replaced by the relative cotangent complex).

Our proof of GRR will use the following derived version of the excess intersection formula (which we will not have time to prove today):

[1] There exists a *sheaf of spectra* K such that $\mathrm{K}_0(\mathrm{X}) \simeq \pi_0\Gamma(\mathrm{X}, \mathrm{K})$. If one admits the existence of this K, then the λ-ring structure can be defined without the extra hypothesis on X. In this lecture, we will either assume that this has been done, or that X admits the resolution property.

Theorem 5 (Excess intersection formula)**.** *Suppose given an* excessive square *of derived schemes, i.e. a commutative square*

$$\begin{array}{ccc} \mathrm{Z}' & \xrightarrow{i'} & \mathrm{X}' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ \mathrm{Z} & \xrightarrow{i} & \mathrm{X} \end{array}$$

satisfying the following conditions:

(a) *The morphisms i and i' are quasi-smooth closed immersions, of virtual codimensions d and d', respectively.*

(b) *The square is cartesian on underlying classical schemes. That is, the morphism $\mathrm{Z}'_{\mathrm{cl}} \to (\mathrm{Z} \times_{\mathrm{X}} \mathrm{X}')_{\mathrm{cl}}$ is invertible.*

(c) *The map $g^*\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \to \mathcal{N}_{\mathrm{Z}'/\mathrm{X}'}$ is surjective on π_0.*

Let $\mathcal{E}$ denote the excess sheaf, *i.e., the fibre of the map $g^*\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \to \mathcal{N}_{\mathrm{Z}'/\mathrm{X}'}$. By the assumptions, $\mathcal{E}$ is locally free of rank $d - d' \geqslant 0$. Then we have*

$$f^* i_*(x) = i'_*(g^*(x) \cdot \lambda_{-1}(\mathcal{E}))$$
$$f^* i^\gamma_*(x) = (i')^\gamma_*(g^*(x) \cdot c^{d-d'}(\mathcal{E})),$$

for all $x \in \mathrm{K}_0(\mathrm{Z})$ and for all $x \in \mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}}$, respectively. Here $\lambda_{-1}(\mathcal{E}) = \sum_i (-1)^i [\bigwedge^i(\mathcal{E})]$ and $c^{d-d'}(\mathcal{E})$ is the top Chern class (to be defined).

Example 6. Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion. Then we have an excessive square

$$\begin{array}{ccc} \mathrm{Z} & = & \mathrm{Z} \\ \| & & \big\downarrow{\scriptstyle i} \\ \mathrm{Z} & \xrightarrow{i} & \mathrm{X}. \end{array}$$

In this case there is "maximal excess", i.e., $\mathcal{E} = \mathcal{N}_{\mathrm{Z}/\mathrm{X}}$. Thus Theorem 5 gives the formulas

$$i^* i_*(x) = x \cdot \lambda_{-1}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$$
$$i^* i^\gamma_*(x) = x \cdot c^d(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}).$$

Example 7. Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion of virtual codimension d. Let $\tilde{\mathrm{X}} \to \mathrm{X}$ be the blow-up and let $i_{\mathrm{E}} : \mathrm{E} \hookrightarrow \tilde{\mathrm{X}}$ be the virtual exceptional divisor. Then the blow-up square

$$\begin{array}{ccc} \mathrm{E} & \xrightarrow{i_{\mathrm{E}}} & \tilde{\mathrm{X}} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ \mathrm{Z} & \xrightarrow{i} & \mathrm{X} \end{array}$$

is excessive. Thus Theorem 5 gives the formulas

$$f^* i_*(x) = (i_{\mathrm{E}})_*(g^*(x) \cdot \lambda_{-1}(\mathcal{E}))$$
$$f^* i^\gamma_*(x) = (i_{\mathrm{E}})^\gamma_*(g^*(x) \cdot c^{d-1}(\mathcal{E})).$$

We now proceed towards the proof of Theorem 2. We begin with the following observation:

Lemma 8. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion of qcqs derived schemes. Then the homomorphism $i_* : \mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}} \to \mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}$ has image contained in the subgroup $\mathrm{Fil}^1_\gamma \mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}$.*

Proof. Given $\mathcal{F} \in \mathrm{Perf}(\mathrm{Z})$, the claim is that the virtual rank of $i_*(\mathcal{F})$ is zero (as a locally constant function on $\mathrm{X}_{\mathrm{Zar}}$). The claim being local on X, we may assume that X is affine, say

$\mathrm{X} = \mathrm{Spec}(\mathrm{R})$, and that Z is the derived zero-locus of functions $f_1, \ldots, f_n \in \pi_0(\mathrm{R})$. Let M denote the $\mathrm{R}/\!\!/(f_1, \ldots, f_n)$-module $\Gamma(\mathrm{Z}, \mathcal{F})$. It will suffice to show that the R-module

$$\Gamma(\mathrm{X}, i_*(\mathcal{F})) \simeq \bigotimes_{i=1}^{n} \mathrm{Cofib}(\mathrm{M} \xrightarrow{f_i} \mathrm{M})$$

has virtual rank 0, which is clear. □

We will need the following construction from the theory of λ-rings (see [1, Exp. V, 5.3]):

Construction 9. Let A be a λ-ring. Suppose that $\mathrm{N} \in \mathrm{A}$ is an element such that $\lambda^k(\mathrm{N}) = 0$ for all $k > d$ (for some d). Then there exist unique elements $\lambda^p(\mathrm{N}, x) \in \mathrm{A}$ for all $x \in \mathrm{A}$, $p \geqslant 1$, satisfying

$$\lambda^p(\mathrm{N}, x) \cdot \lambda_{-1}(\mathrm{N}) = \lambda^p(x \cdot \lambda_{-1}(\mathrm{N})).$$

Similarly we have

$$\gamma^p(\mathrm{N}, x) \cdot \lambda_{-1}(\mathrm{N}) = \gamma^p(x \cdot \lambda_{-1}(\mathrm{N})).$$

Lemma 10. *For any $x \in \mathrm{A}$ and any $p \geqslant 1$, we have*

$$\gamma^p(\mathrm{N}, x) \in \mathrm{Fil}^{p-d}(\mathrm{A}).$$

For any $x \in \mathrm{Fil}^k_\gamma(\mathrm{A})$, we have

$$\gamma^{k+d}(\mathrm{N}, x) - (-1)^{k+d-1}(k+d-1)! \cdot x \in \mathrm{Fil}^{k+1}_\gamma(\mathrm{A}).$$

The key ingredient in the proof of Theorem 2 is the following:

Proposition 11. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion of qcqs derived schemes. Then for any $x \in \mathrm{K}_0(\mathrm{Z})$ and any $p \geqslant 1$, we have an equality*

$$i_*(\gamma^p(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}, x)) = \gamma^p(i_*(x))$$

in $\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}$.

To prove it we will need the following lemma.

Lemma 12. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a quasi-smooth closed immersion of virtual codimension d. Consider the blow-up square*

$$\begin{array}{ccc} \mathrm{E} & \xrightarrow{i_{\mathrm{E}}} & \tilde{\mathrm{X}} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \mathrm{Z} & \xrightarrow{i} & \mathrm{X} \end{array}$$

Denote by $\mathcal{L}$ the conormal sheaf of the immersion i_{E}, and by $\mathcal{E}$ the excess sheaf. Suppose there exists a locally free $\mathcal{O}_{\mathrm{Z}}$-module $\mathcal{N}'$ such that $[\mathcal{N}_{\mathrm{Z}/\mathrm{X}}] = [\mathcal{N}'] + 2$ in $\mathrm{K}_0(\mathrm{Z})$. Then one has the identity

$$\lambda_{-1}(\mathcal{E}) \equiv 0 \pmod{1 - \mathcal{L}}$$

in $\mathrm{K}_0(\mathrm{Z})$.

Proof. In $\mathrm{K}_0(\mathrm{Z})$ we have the identities:

$$\begin{aligned} \lambda_{-1}(\mathcal{E}) &= \sum_{k \geqslant 0} (-1)^k \lambda^k(\mathcal{E}) \\ &= (-1)^{d-1}\lambda^{d-1}(\mathcal{E} - 1) \\ &= (-1)^{d-1}\lambda^{d-1}(\mathcal{E} + \mathcal{L} - 1 - \mathcal{L}) \\ &= (-1)^{d-1}\lambda^{d-1}(g^*\mathcal{N}_{\mathrm{Z}/\mathrm{X}} - 2 + 1 - \mathcal{L}). \end{aligned}$$

We claim that $\lambda^k(1-\mathcal{L})$ is divisible by $1-\mathcal{L}$ for all $k \geqslant 1$. Indeed, it is the coefficient of t^k in the power series $\lambda_t(1-\mathcal{L}) = \lambda_t(1)/\lambda_t(\mathcal{L}) = (1+t)/(1+\mathcal{L}t)$. Therefore, reducing modulo $(1-\mathcal{L})$, we get:

$$\begin{aligned}\lambda_{-1}(\mathcal{E}) &\equiv (-1)^{d-1}\lambda^{d-1}(g^*\mathcal{N}_{\mathrm{Z/X}} - 2)\\ &\equiv (-1)^{d-1}\lambda^{d-1}(g^*\mathcal{N}')\\ &\equiv 0\end{aligned}$$

since $\mathcal{N}'$ is of rank $d-2$. □

Proof of Proposition 11. The statement will follow from the analogous formula for the λ^p:

$$i_*(\lambda^p(\mathcal{N}_{\mathrm{Z/X}}, x)) = \lambda^p(i_*(x)).$$

We can guarantee that the assumption of Lemma 12 holds by replacing i with the composite $i' : \mathrm{Z} \hookrightarrow \mathrm{X} \hookrightarrow \mathbf{P}^1_{\mathrm{X}} \hookrightarrow \mathbf{P}^1_{\mathbf{P}^1_{\mathrm{X}}}$ (note that the statement for i' will imply it for i). Recall that $f^* : \mathrm{Qcoh}(\mathrm{X}) \to \mathrm{Qcoh}(\tilde{\mathrm{X}})$ is fully faithful, i.e. $f_*f^* \simeq \mathrm{id}$, and both functors f^* and f_* preserve perfect complexes. In particular $f^* : \mathrm{K}_0(\mathrm{X}) \to \mathrm{K}_0(\tilde{\mathrm{X}})$ admits a retraction, so it will suffice to show

$$f^*i_*(\lambda^p(\mathcal{N}_{\mathrm{Z/X}}, x)) = f^*\lambda^p(i_*(x)).$$

Using the excess intersection formula (Example 7), one reduces to showing the identity

$$(i_{\mathrm{E}})_*(\lambda^p(\mathcal{L}, x)) = \lambda^p((i_{\mathrm{E}})_*(x)).$$

By Lemma 12 the element $\lambda_{-1}(\mathcal{E}) \in \mathrm{K}_0(\mathrm{Z})$ is divisible by $1-[\mathcal{L}]$, so there exists $x' \in \mathrm{K}_0(\mathrm{Z})$ such that $x = x' \cdot (1-[\mathcal{L}])$. Then by the self-intersection formula (Example 6), we have $x = (i_{\mathrm{E}})^*(i_{\mathrm{E}})_*(x')$. In other words, the relation in question can be rewritten as

$$(i_{\mathrm{E}})_*(i_{\mathrm{E}})^*(\lambda^p(\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E}), y')) = \lambda^p((i_{\mathrm{E}})_*(i_{\mathrm{E}})^*(y')),$$

where $y' = (i_{\mathrm{E}})_*(x')$ (since $(i_{\mathrm{E}})^*(\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E})) \simeq \mathcal{L}$). The exact triangle

$$\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E}) \to \mathcal{O}_{\tilde{\mathrm{X}}} \to (i_{\mathrm{E}})_*\mathcal{O}_{\mathrm{E}},$$

gives the equality $(i_{\mathrm{E}})_*(1) = 1 - [\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E})]$ and hence

$$(i_{\mathrm{E}})^*(i_{\mathrm{E}})_*(1) = 1 - [\mathcal{L}].$$

Using the projection formula we reduce to showing the relation

$$\lambda^p([\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E})], y') = \lambda^p(y' \cdot (1 - [\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E})]))$$

which holds by construction of the left-hand side, since $\lambda_{-1}[\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E})] = 1 - [\mathcal{O}_{\tilde{\mathrm{X}}}(-\mathrm{E})]$. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $x \in \mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}}$ and suppose that $x \in \mathrm{Fil}^k_\gamma \mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}}$ for some k, so that x is a sum of elements of the form

$$a \cdot \gamma^{i_1}(x_1) \cdots \gamma^{i_n}(x_n)$$

with $a \in \mathbf{Q}$, $i_1 + \cdots + i_n \geqslant k$, and such that each $x_j \in \mathrm{Fil}^1_\gamma \mathrm{K}_0(\mathrm{Z})$ for each j. Let R denote the sub-$\mathbf{Q}$-λ-algebra of $\mathrm{K}_0(\mathrm{Z})_{\mathbf{Q}}$ generated by the class $[\mathcal{N}_{\mathrm{Z/X}}]$ and the classes x_j. Then $x \in \mathrm{Fil}^k_\gamma(\mathrm{R})$ and it will suffice to show that $i_*(\mathrm{Fil}^k_\gamma(\mathrm{R})) \subset \mathrm{Fil}^{k+d}_\gamma \mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}$. Since $\mathrm{K}_0(-)$ commutes with finite direct sums, we may replace X by a connected component to assume that x_j are represented by perfect complexes of constant virtual rank r_j. Choosing m such that $\mathrm{Fil}^m(\mathrm{R}) = 0$, we now argue by induction on k (the case $k = m$ being trivial). Let $b_s = (-1)^{s-1}(s-1)!$ for each s. By Lemma 10 we have

$$\gamma^{k+d}(\mathcal{N}_{\mathrm{Z/X}}, x) - b_{k+d} \cdot x \in \mathrm{Fil}^{k+1}(\mathrm{R}).$$

Therefore, by the induction hypothesis we have

$$i_*(\gamma^{k+d}(\mathcal{N}_{\mathrm{Z/X}}, x) - b_{k+d} \cdot x) \in \mathrm{Fil}^{k+d+1}(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}).$$

From Proposition 11 we deduce that

$$\gamma^{k+d}(i_*(x)) - b_{k+d} \cdot i_*(x) \in \mathrm{Fil}^{k+d+1}(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}).$$

By Lemma 8, $i_*(x) \in \mathrm{Fil}^1(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}})$, so $\gamma^{k+d}(i_*(x)) \in \mathrm{Fil}^{k+d}(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}})$. It follows that $i_*(x) \in \mathrm{Fil}^{k+d}(\mathrm{K}_0(\mathrm{X})_{\mathbf{Q}})$, as claimed. □

Our next goal is to define the Chern character map $\mathrm{K}_0(\mathrm{X}) \to \mathrm{Gr}_\gamma \mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}$. This is a construction that makes sense for rather general λ-rings.

Notation 13. Let A be an **N**-graded commutative ring. Assume $\mathrm{A}^0 = \mathbf{Z}$ or more generally that $\mathrm{A}^0 = \mathrm{K}$ is a *binomial ring* (which essentially means that $\lambda^n(x) = \binom{x}{n}$ defines a λ-structure on K). Denote by $\hat{\mathrm{A}}$ the product $\prod_{i\geqslant 0} \mathrm{A}^i$, viewed as a unital commutative ring. There is a canonical augmentation homomorphism $\hat{\mathrm{A}} \to \mathrm{A}^0 = \mathrm{K}$, whose kernel we denote by $\hat{\mathrm{A}}^+$. We denote by $1 + \hat{\mathrm{A}}^+$ the subgroup of the multiplicative group of units in $\hat{\mathrm{A}}$, consisting of elements of augmentation 1.

Construction 14. Let A be an **N**-graded commutative ring as in Notation 13. The *Chern ring* $\mathrm{Chern}_\mathrm{K}(\mathrm{A})$ associated to A has underlying abelian group $\mathrm{K} \times (1 + \hat{\mathrm{A}}^+)$. Its elements will be denoted by $[n, x]$ with $n \in \mathrm{K}$ and $x = 1 + \sum_{i\geqslant 1} x^i \in 1 + \hat{\mathrm{A}}^+$, with $x^i \in \mathrm{A}^i$. The addition is defined by

$$[n, x] + [n', x'] = [n + n', xx'].$$

We refer to [1, Exp. 0, Appendix, § 3] for a description of the multiplicative structure. Briefly speaking, $\mathrm{Chern}_\mathrm{K}(\mathrm{A})$ can be viewed as the result of adjoining a unit to the nonunital commutative ring $1 + \hat{\mathrm{A}}^+$. Moreover, the λ-structure on K induces a λ-structure on $\mathrm{Chern}_\mathrm{K}(\mathrm{A})$ (see *loc. cit.*). Note that there is an augmentation $\mathrm{Chern}_\mathrm{K}(\mathrm{A}) \to \mathrm{K}$ given by $[n, x] \mapsto n$.

Construction 15. Let K be a binomial ring, and Λ an augmented K-λ-algebra. Let $\mathrm{Gr}_\gamma \Lambda$ be the associated graded K-algebra. For each $x \in \Lambda$ and $i > 0$, the *ith Chern class* $c^i(x) \in \mathrm{Gr}^i_\gamma \Lambda$ is the class of the element $\gamma^i(x - \varepsilon(x)) \in \mathrm{Fil}^i_\gamma(\Lambda)$. We set $\tilde{c}(x) = [\varepsilon(x), 1 + \sum_{i>0} c^i(x)]$ for each x. This defines a homomorphism of K-λ-algebras

$$\tilde{c} : \Lambda \to \mathrm{K} \times (1 + \widehat{\mathrm{Gr}_\gamma(\Lambda)})^+ = \mathrm{Chern}_\mathrm{K}(\mathrm{Gr}_\gamma \Lambda)$$

called the *completed Chern character*.

Construction 16. Let A an **N**-graded commutative ring as in Notation 13. Write $\mathrm{A}_{\mathbf{Q}} := \mathrm{A} \otimes \mathbf{Q}$. The *Chern homomorphism* is a morphism of augmented K-algebras

$$\mathrm{ch} : \mathrm{Chern}_\mathrm{K}(\mathrm{A}) \to \widehat{\mathrm{A}_{\mathbf{Q}}}$$

which is determined by the following properties: it is additive, sends $1 \mapsto 1$, the positive-degree components of $\mathrm{ch}(x)$ are given by homogeneous universal polynomials in the components of x, and finally

$$\mathrm{ch}[1, 1 + x^1] = \exp(x^1) = \sum_{n\geqslant 0} (x^1)^n/n!.$$

Construction 17. For any formal power series $f \in \mathbf{Q}[\![t]\!]$, there is an associated additive homomorphism

$$\mathcal{T}_f : 1 + \hat{\mathrm{A}}^+ \to 1 + \widehat{\mathrm{A}_{\mathbf{Q}}}^+$$

defined using Hirzebruch polynomials. For example, for $f(t) = t/(1 - \exp(-t))$, the construction $\mathcal{T}_f$ is called the *Todd operator* and denoted Td.

Now let X be a derived scheme and consider the λ-ring $\mathrm{K}_0(\mathrm{X})$ (augmented over the binomial ring $\mathrm{H}^0(\mathrm{X}_{\mathrm{Zar}}, \mathbf{Z})$). We simplify the notation by writing

$$\mathrm{ch} : \mathrm{K}_0(\mathrm{X}) \xrightarrow{\tilde{c}} \mathrm{Chern}(\mathrm{Gr}_\gamma \mathrm{K}_0(\mathrm{X})) \xrightarrow{\mathrm{ch}} \mathrm{Gr}_\gamma \mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}.$$

We can now make sense of the statement of GRR (Theorem 3).

The following is essentially formal:

Lemma 18. *Let* X *be a qcqs derived scheme. If* $\mathcal{F}$ *is a locally free sheaf of rank* n *on* X*, then we have*

$$\mathrm{ch}(\lambda_{-1}[\mathcal{F}]) = c^n(\mathcal{F}^\vee)\,\mathrm{Td}(-\mathcal{F}^\vee)$$

in $\mathrm{Gr}^*_\gamma \mathrm{K}_0(\mathrm{X})_{\mathbf{Q}}$*, where* $\mathrm{Td}(-\mathcal{F}^\vee) = \mathrm{Td}(\mathcal{F}^\vee)^{-1}$.

Exercise 19. Let $i_1 : \mathrm{Z} \hookrightarrow \mathrm{Y}$ and $i_2 : \mathrm{Y} \hookrightarrow \mathrm{X}$ be quasi-smooth closed immersions of quasi-compact derived schemes, of virtual codimensions d_1 and d_2, respectively. Suppose that Theorem 3 holds for i_1 with respect to an element $x \in \mathrm{K}_0(\mathrm{Z})$, and for i_2 with respect to the element $(i_1)_*(x)$. Then it holds for $i_2 \circ i_1$ with respect to the element x.

Proof of Theorem 3. Consider the composite $i' : \mathrm{X} \hookrightarrow \mathbf{P}^1_{\mathrm{X}} \hookrightarrow \mathbf{P}^1_{\mathbf{P}^1_{\mathrm{X}}}$. Using Exercise 19, we may replace i by i' and assume that the condition of Lemma 12 is satisfied. Consider the blow-up square:

$$\begin{array}{ccc} \mathrm{E} & \xrightarrow{i_{\mathrm{E}}} & \tilde{\mathrm{X}} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \mathrm{Z} & \xrightarrow{i} & \mathrm{X} \end{array}$$

and adopt the notation of Lemma 12. As in the proof of Proposition 11, it will suffice to apply f^* and demonstrate the relation

$$f^*\,\mathrm{ch}(i_*(x)) = f^* i^\gamma_*(\mathrm{ch}(x)\cdot \mathrm{Td}(-\mathcal{N}_{\mathrm{Z}/\mathrm{X}}))$$

for any $x \in \mathrm{K}_0(\mathrm{Z})$. Using the excess intersection formula (Theorem 5) and the fact that f^* commutes with ch, this is equivalent to the relation

$$\mathrm{ch}((i_{\mathrm{E}})_*(g^*(x)\cdot\lambda_{-1}(\mathcal{E}))) = (i_{\mathrm{E}})^\gamma_*(\mathrm{ch}(g^*x)\cdot \mathrm{Td}(-g^*\mathcal{N}_{\mathrm{Z}/\mathrm{X}})\cdot c^{d-1}(\hat{\mathcal{E}})).$$

Using the equality $[g^*\mathcal{N}_{\mathrm{Z}/\mathrm{X}}] = -([\mathcal{E}]+[\mathcal{L}])$ we get $\mathrm{Td}(g^*\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) = \mathrm{Td}(-\mathcal{N}_{\mathrm{E}/\tilde{\mathrm{X}}})\cdot\mathrm{Td}(-\mathcal{E})$. By Lemma 18 we reduce to showing

$$\mathrm{ch}((i_{\mathrm{E}})_*(g^*(x)\cdot\lambda_{-1}(\mathcal{E}))) = (i_{\mathrm{E}})^\gamma_*(\mathrm{ch}(g^*x\cdot\lambda_{-1}(\mathcal{E}))\cdot \mathrm{Td}(-\mathcal{N}^\vee_{\mathrm{E}/\tilde{\mathrm{X}}}).$$

Now replacing x with $g^*(x)\cdot\lambda_{-1}(\mathcal{E})$, and i with i_{E}, we may reduce to the case where i is of virtual codimension 1. Moreover, since $\lambda_{-1}(\mathcal{E})$ is divisible by $1-[\mathcal{L}]$ (Lemma 12), we may reduce to the case where $x = (i_{\mathrm{E}})^*(y)$ for some $y \in \mathrm{K}_0(\mathrm{X})$. Thus, we need to show

$$\mathrm{ch}(i_* i^*(y)) = i^\gamma_*(\mathrm{ch}(i^*(y))\cdot \mathrm{Td}(\mathcal{N}^\vee_{\mathrm{Z}/\mathrm{X}})^{-1}).$$

Using the projection formula on both sides, we reduce to showing

$$\mathrm{ch}(i_*(1)) = i^\gamma_*(\mathrm{Td}(\mathcal{N}^\vee_{\mathrm{Z}/\mathrm{X}})^{-1}). \tag{0.1}$$

The exact triangle $\mathcal{O}_{\mathrm{X}}(-\mathrm{Z}) \to \mathcal{O}_{\mathrm{X}} \to i_*\mathcal{O}_{\mathrm{Z}}$ gives $i_*(1) = 1-[\mathcal{L}]$, where $\mathcal{L} := \mathcal{O}_{\mathrm{X}}(-\mathrm{Z})$. Since $\mathcal{L}$ is of rank 1, we have $\tilde{c}(\mathcal{L}) = [1, 1+c^1(\mathcal{L})]$, hence $\mathrm{ch}(\mathcal{L}) = \exp(c^1(\mathcal{L}))$. Thus the left-hand side of (0.1) is given by

$$\mathrm{ch}(i_*(1)) = 1-\exp(c^1(\mathcal{L})).$$

For the right-hand side, note that since $-[\mathcal{N}_{\mathrm{Z}/\mathrm{X}}] = -i^*[\mathcal{L}]$, we have

$$i^\gamma_*(\mathrm{Td}(\mathcal{N}^\vee_{\mathrm{Z}/\mathrm{X}})^{-1}) = i^\gamma_* i^*(\mathrm{Td}(-\mathcal{L}^\vee)) = i^\gamma_*(1)\cdot\mathrm{Td}(-\mathcal{L}^\vee)$$

by the projection formula. We have $i^\gamma_*(1) = -c^1(\mathcal{L})$, by definition of $c^1(\mathcal{L})$, since $i^\gamma_*(1)$ is the image of $i_*(1) = 1-[\mathcal{L}]$ in $\mathrm{Gr}^1_\gamma \mathrm{K}_0(\mathrm{X})$. Thus we have

$$i^\gamma_*(\mathrm{Td}(\mathcal{N}^\vee_{\mathrm{Z}/\mathrm{X}})^{-1}) = -c^1(\mathcal{L})\cdot\frac{1-\exp(c^1(\mathcal{L}))}{-c^1(\mathcal{L})} = 1-\exp(c^1(\mathcal{L})),$$

as desired. ∎

References

[1] SGA 6.