Lecture 8

$$\mathrm{SCRing} \simeq \mathrm{Fun}_{\pi,\mathrm{fin}}((\mathrm{Poly})^{op}, \mathrm{Spc})$$
$$\parallel \Delta^{op}\mathrm{Set}$$

$R : (\mathrm{Poly})^{op} \to \mathrm{Spc}$ sending finite coproducts in Poly to products of spaces.

Poly = full subcategory of CRing spanned by $\mathbb{Z}[T_1, -, T_n]$, $n \geq 0$.

Prop: $\mathcal{C}$ ∞-Cat admitting sifted colimits.

Then $\mathrm{Fun}_{\mathrm{sift}}(\mathrm{SCRing}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Poly}, \mathcal{C})$.

In other words, SCRing is freely generated under sifted colimits by Poly.

Rmk: Fix $R \in \mathrm{SCRing}$. Then SCRing_R is freely generated by Poly_R under sift. colim.

We can more generally consider $\mathrm{SCRMod} \supset \mathrm{SCRMod}^{cn}$ (connective) where objects of SCRMod are pairs (R, M), $R \in \mathrm{SCRing}$, $M \in \mathrm{Mod}_R$ (resp. $M \in \mathrm{Mod}_R^{cn}$).

Let $\mathcal{C} \subset \mathrm{SCRMod}^{cn}$ be the full subcat spanned by (R, M) such that $R \simeq \mathbb{Z}[T_1, -, T_n]$, $n \geq 0$, and $M \simeq R^{\oplus m}$, $m \geq 0$.

Prop: The inclusion $\mathcal{C} \subset \mathrm{SCRMod}^{cn}$ induces an equivalence
$$\mathrm{Fun}_{\pi,\mathrm{fin}}(\mathcal{C}^{op}, \mathrm{Spc}) \xrightarrow{\sim} \mathrm{SCRMod}^{cn}$$

see SAG 25.2.1.2 / HTT 5.5.8.15

Cor: $\mathcal{E}$ ∞-Cat admitting (small) sifted colimits. Then
$$\mathrm{Fun}_{\mathrm{sift}}(\mathrm{SCRMod}^{cn}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{E}).$$

Informally: to construct a functor $F : \mathrm{SCRMod}^{cn} \to \mathcal{E}$ which commute with sifted colimits, it suffices to specify the value of F on $\mathcal{C}$, i.e. on pairs (R, M) with R polynomial ring, M free module of finite rank. Note that R and M are (by definition) all discrete!

§ Derived symmetric and exterior powers.

We apply the previous construction to produce "derived" versions of Sym^n_R and Λ^n_R

1. Symmetric powers:

$R \in \mathrm{CRing} \hookrightarrow \mathrm{SCRing}$
$M \in \mathrm{Mod}_R$, discrete $\rightsquigarrow$ $\mathrm{Sym}^n_R(M) : \pi_0(M^{\otimes^n_R})/\Sigma_n \in \mathrm{Mod}_R$, discrete.
$n \in \mathbb{Z}_{\geq 0}$

with $\sigma(x_1 \otimes - \otimes x_n) = x_{\sigma(1)} \otimes - \otimes x_{\sigma(n)}$
(coinvariants for the action of Σ_n).

If $M = R^{\oplus m}$, $m \geq 1$, free with basis $\{T_1, -, T_m\}$, then $\mathrm{Sym}^n_R(M)$ is free of rank $\binom{n+m-1}{n}$ with basis the set of monomials $T_1^{d_1} T_2^{d_2} \cdots T_m^{d_m}$ s.that $\sum_{i=1}^m d_i = n$.

This means we have $\mathrm{Sym}^n : \mathcal{C} \to \mathcal{C}$, $(R, M) \mapsto (R, \mathrm{Sym}^n_R(M))$
↑ defined above.

This gives a "unique" extension: $\mathrm{Sym}^n : \mathrm{SCRMod}^{cn} \longrightarrow \mathrm{SCRMod}^{cn}$
$(R,M) \longmapsto (R, \mathrm{Sym}^m_R(M))$
(we use the same notation. In SAG Lurie writes $L\mathrm{Sym}^n_R(M)$ instead). In our terminology, this will be the derived symmetric power of $M \in \mathrm{Mod}_R^{cn}$.

2. Exterior powers.

R $\mathbb{Z}[T_1, \dots, T_k]$ ← or more generally if $R \in \mathrm{CRing}$ discrete.
$M = R^{\oplus m}$.

Define $\Lambda^m_R(M) = (M^{\otimes^n_R})/(\mathrm{sgn}\,\Sigma_n) \in \mathrm{Mod}_R^{cn}$ (could write $\pi_0(M^{\otimes n})/\dots$)
with $\sigma(x_1 \otimes \dots \otimes x_n) = \mathrm{sgn}(\sigma)\, x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$ $\forall \sigma \in \Sigma_n$.

If $\{X_1, \dots, X_m\}$ is a basis of M, then $\Lambda^n_R(M)$ is also free of rk $\binom{m}{m}$.
with basis given by the "ordered tensors".

As above, this defines $\Lambda^n : \mathcal{C} \to \mathcal{C}$, $(R,M) \mapsto (R, \Lambda^n_R(M))$

$\Rightarrow$ get a "unique" extension $\Lambda^n : \mathrm{SCRMod}^{cn} \to \mathrm{SCRMod}^{cn} \rightsquigarrow \Lambda^n_R(M)$ $\forall M$ conn.

We call it the m-th derived exterior power of $M \in \mathrm{Mod}_R^{cn}$.

Properties: (SAG 25.2.3)

a) Base-change: given $A \xrightarrow{\phi} B \in \mathrm{SCRing}$ morphism $\rightsquigarrow \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B$
$M \longmapsto M \otimes_A B$.

Then for every $M \in \mathrm{Mod}_A^{cn}$ we have

i) $B \otimes_A \mathrm{Sym}^n_A(M) \xrightarrow{\sim} \mathrm{Sym}^n_B(B \otimes_A M)$

ii) $B \otimes_A \Lambda^n_A(M) \xrightarrow{\sim} \Lambda^n_B(B \otimes_A M)$.

proof: we always have a comparison map $B \otimes_A \mathrm{Sym}^m_A(M) \xrightarrow{\alpha_M} \mathrm{Sym}^m_B(B \otimes_A M)$, and the construction commutes with sifted colimits. (OK for $B \otimes_A (-)$. For Sym ok by Cor at previous page.) Thus we can assume M is free of finite rank.

$\Rightarrow M = A \otimes_{\mathbb{Z}} M_0$, $M_0 \cong \mathbb{Z}^{\oplus m}$. But then look at

$$\begin{tikzcd}
B \otimes_A A \otimes_{\mathbb{Z}} \mathrm{Sym}^m_{\mathbb{Z}}(M_0) \arrow[rr, "{\alpha''_M \text{ for } B' = B \otimes_A A \text{ and } A' = \mathbb{Z}}"] \arrow[dr, "B \otimes \tilde{\alpha}_M"'] & & \mathrm{Sym}^m_{B \otimes_A A}(B \otimes_A A \otimes_{\mathbb{Z}} M_0) \cong \mathrm{Sym}^m_B(B \otimes_A M) \\
& B \otimes_A \mathrm{Sym}^n_A(A \otimes_{\mathbb{Z}} M_0) \arrow[ur, "\alpha_M"'] &
\end{tikzcd}$$

for $\tilde{\alpha}_M : A \otimes_{\mathbb{Z}} \mathrm{Sym}^m_{\mathbb{Z}}(M_0) \to \mathrm{Sym}^m_A(A \otimes_{\mathbb{Z}} M_0)$

$\Rightarrow$ enough to show the statement when $A = \mathbb{Z}$.

By colimit extension argument, can assume $B = \mathbb{Z}[T_1, \dots, T_k] \Rightarrow$ can use the explicit descript

But then we are saying $B \otimes_{\mathbb{Z}} ((\mathbb{Z}^{\oplus m})^{\otimes^n_{\mathbb{Z}}})/\Sigma_n \xrightarrow{\sim} ((B^{\oplus m})^{\otimes n})/\Sigma_n$ ok.

Same argument works for Λ^n. $\square$

b) $R \in \mathrm{SCRing}$. $M \in \mathrm{Mod}_R^{cn}$ which is locally free of rank $= m$.
($\tilde{M}$ locally free as object of $\mathrm{QCoh}(\mathrm{Spec}(R))$).

$\Rightarrow \mathrm{Sym}^m_R(M)$ and $\Lambda^m_R(M)$ are both locally free of rk $\binom{m+m-1}{m}$, resp. $\binom{m}{m}$

proof: by a) above, we can actually assume that R is $\mathbb{Z}[T_1, T_2, \ldots, T_k]$ and that $M = R \otimes_{\mathbb{Z}} \mathbb{Z}^{\oplus m}$ is free of rank m.
Further using the trick of the previous proof, we can assume $R = \mathbb{Z}$.
But then we have already seen that both $\Lambda^n_R(M)$ and $\mathrm{Sym}^m_R(M)$ are free of the expected rank.

We conclude by quoting the following

Prop (SAG 25.2.3.4). Let R be a discrete SCRing, and let $M \in \mathrm{Mod}_R$ be a flat R-module. Then:

1) $\mathrm{Sym}^m_R(M) \cong \mathrm{Sym}'^n_R(M)$, where $\mathrm{Sym}'^n_R(M) := \pi_0(M^{\otimes^n_R}) / \Sigma_n$ (non derived version)
2) $\Lambda^n_R(M) \cong \Lambda'^n_R(M)$, where $(\Lambda')^n_R(M) = \pi_0(M^{\otimes^n_R}) / \mathrm{sgn}\, \Sigma_n$ (non derived version)

In particular, $\mathrm{Sym}^m_R(M)$ and $\Lambda^n_R(M)$ are discrete.

Warning: if $M \in \mathrm{Mod}_R$ discrete, with R discrete, $\mathrm{Sym}^m_R(M)$ and $\Lambda^n_R(M)$ are not discrete, in general, unless M is flat. In particular, they don't agree with the "classical constructions". However, the classical construction agrees with $\pi_0(\mathrm{Sym}^m_R(M))$ and $\pi_0(\Lambda^n_R(M))$ resp.

From $\mathrm{Sym}^m_R(-)$ to $\Lambda^n_R(-)$: the 2 functors are related as follows:

Prop (DAG 3.1.1). Let $R \in \mathrm{SCRing}$, $M \in \mathrm{Mod}_R^{cn}$. The functor $M \mapsto (\mathrm{Sym}^m_R(M[1]))[-m]$ agrees with $\Lambda^n_R(M)$, up to equivalence.

proof: One has to show that the functor $T^n_R : M \mapsto \mathrm{Sym}^m_R(M[1])[-n]$ agrees with the "non abelian left derived functor" (in Lurie's sense) of Λ^n_R.
Thus we have to show $\Lambda^n_R(M) \cong T^n_R(M)$ if $R \in \mathrm{CRing}$ is discrete and $M = R \otimes_{\mathbb{Z}} \mathbb{Z}^m$ is free. In fact we can use the previous prop (a) to reduce to the case $R = \mathbb{Z}$.
The case $m \leq 1$ is clear, so assume $m \geq 2$. Note that if M and N are free, we have:

$$\mathrm{Sym}^m_R((M \oplus N)[1]) \cong \bigoplus_{i+j=n} \mathrm{Sym}^i_R(M[1]) \otimes_R \mathrm{Sym}^j_R(N[1]) \quad (**)$$

see below

and so we can assume $M \cong R$ (i.e. R is free of rank $= 1$).
In this case $\Lambda^m_R(M) = 0$, So it's enough to show $\mathrm{Sym}^m_R(M[1])[-n] = 0$.

Claim: $R \oplus R[1] \cong \bigoplus_{n \geq 0} \mathrm{Sym}^m_R(R[1])$. In particular, $\mathrm{Sym}^m_R(R[1]) = 0$ $m \geq 2$.

(Note that Sym^n_R can be computed degreewise on cofibrant connective R-modules for R discrete). See SAG 25.2.4.2. □

Another useful ~~[illegible]~~ property:

Prop: $R \in \mathrm{SCRing}$, $M \in \mathrm{Mod}_R^{cn, perf}$. Then $\mathrm{Sym}^m_R(M)$ and $\Lambda^n_R(M)$ are perfect.

Note: $\mathrm{Mod}_R^{proj} \subset \mathrm{Mod}_R^{cn, perf} \subset \mathrm{Mod}_R^{perf}$, and Sym and Λ preserve projectivity.

We wish now to use the operation $\mathrm{Mod}_R^{proj} \xrightarrow[\Lambda^n_R(-)]{} \mathrm{Mod}_R^{proj}$

To define a λ-ring structure on

$\uparrow$ proj. R-modules $\iff$ locally free of finite rank

$$K_0^{\oplus}(\mathrm{Mod}_R^{proj}) \cong K_0(R), \quad R \in \mathrm{SCRing}$$

$\uparrow$ Lecture 5

Recall before the following classical definition.

<u>Def</u>: A commutative ring K is called a pre-λ-ring (SGA 6 terminology) if we are given a family of operations $\lambda^k : K \to K$, $k \geq 0$ such that $\forall\, x, y \in K$, $\lambda^k(x+y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y)$ $(*)$

$$\lambda^0(x) = 1, \quad \lambda^1(x) = x.$$

We therefore have to define $\lambda^k : K_0(R) \to K_0(R)$, $R \in \mathrm{SCRing}$.

If $M \in \mathrm{Mod}_R^{proj}$, we can set $\lambda^k(M) = [\Lambda^k_R(M)]$, which make sense, since $\Lambda^k_R(M) \in \mathrm{Mod}_R^{proj}$ $\forall\, k \geq 0$, thanks to property b) discussed above.

We need then to prove the relation $(*)$: to do so, we essentially reduce to the $(**)$ decomposition above, that we need to explain more generally for any $M, N \in \mathrm{Mod}_R^{proj}$.

<u>Construction</u>: $R \in \mathrm{CRing}$ discrete; consider a sequence

SGA 6 V.2.2.1

$0 \to M' \xrightarrow[\rho]{} M \to M'' \to 0$ of finitely generated and free R-modules.

$(M' \xrightarrow[\rho]{} M \to \mathrm{Cof}(\rho) = M'' \to +)$

We define a filtration on $\mathrm{Sym}^m_R(M)$ ($= \mathrm{Sym}'^m_R(M)$, with the notation introduced above)

$$0 = F^{-1,m}(\rho) \subseteq F^{0,n}(\rho) \subseteq F^{1,n}(\rho) \subseteq \dots \subseteq F^{n,n}(\rho) = \mathrm{Sym}^m_R(M)$$

$(**)$

where

$F^{d,m}(\rho) =$ submodule generated by symmetric powers of M' of homogeneous degree $= d$.

$=$ Image of the canonical morphism:

$$\mathrm{Sym}^{m-d}_R(M') \otimes_R \mathrm{Sym}^d_R(M) \longrightarrow \mathrm{Sym}^m_R(M).$$

<u>Claim</u>: $F^{d,m}(\rho) / F^{d-1,m}(\rho) \cong \mathrm{Sym}^d_R(M'') \otimes_R \mathrm{Sym}^{n-d}_R(M')$. (Both terms discrete)

This is completely classical.

We can extend this construction: let $\mathcal{E} = \mathrm{Fun}(\Delta^1, \mathrm{SCRMod}^{cn}) \times_{\mathrm{Fun}(\Delta^1, \mathrm{SCRing})} \mathrm{SCRing}$

objects: $(R, \rho : M' \to M)$, $R \in \mathrm{SCRing}$, $M, M' \in \mathrm{Mod}_R^{cn}$

$\mathcal{E}_0 \subseteq \mathcal{E}$ full subcategory spanned by $(\mathbb{Z}[T_1, \dots, T_r], \rho: M' \to M)$ with $M', M^{\#}$ finitely generated and free ($\to$ discrete).

Then
$$\mathrm{Fun}_{\mathrm{sift}}(\mathcal{E}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{E}_0, \mathcal{C}) \quad \forall\ \infty\text{-Cat } \mathcal{C} \text{ admitting sifted colimits.}$$
$(\mathcal{E} \xrightarrow{\sim} \mathrm{Fun}_{\Pi, \mathrm{fin}}(\mathcal{E}_0, \mathrm{Spc}))$.

This gives an extension of $F^{d,n}(R, \rho)$, any $R \in \mathrm{SCRing}$

By construction, we have $\mathrm{Cofib}(F^{d-1,n}(\rho) \to F^{d,n}(\rho)) \cong \mathrm{Sym}^d_R(\mathrm{Cof}\,\rho) \otimes_R \mathrm{Sym}^{n-d}_R(M')$.

Write $\mathrm{gr}^d(\mathrm{Sym}^n_R(M))$ for this cofiber.

<u>Consequence</u>: Suppose $M' \xrightarrow{\rho} M \to M''$ cofiber sequence, with $M', M, M'' \in \mathrm{Mod}_R^{\mathrm{proj}}$.
$$\Rightarrow \bigoplus_{d \geq 0} \mathrm{gr}^d(\mathrm{Sym}^n_R(M)) \cong \bigoplus_{d \geq 0} \mathrm{Sym}^d_R(M'') \otimes \mathrm{Sym}^{n-d}_R(M')$$
and
$$[\bigoplus_{d \geq 0} \mathrm{gr}^d(\mathrm{Sym}^n_R(M))] = \sum_{d \geq 0} [\mathrm{gr}^d(\mathrm{Sym}^n_R(M))] = [\mathrm{Sym}^n_R(M)]$$
in $K_0(R)$.

by the property of the filtration (***)

Using now the equivalence $\Lambda^n_R(M) = \mathrm{Sym}^n_R(M[1])[-n]$, we get the required formula $(*)$, since the product structure on $K_0(R)$ is induced exactly by $\otimes_R$.

In summary:

<u>Prop</u>: $K_0(R)$ is a pre-λ-ring.

We have then $\lambda^k: K_0(R) \to K_0(R)$.

Suppose $R \xrightarrow{f} R'$ is a morphism in SCRing. The compatibility of Λ^n_R with base change, discussed above, shows that λ^k is compatible with
$$K_0(R) \xrightarrow{f^*} K_0(R'), \quad M \mapsto M \otimes_R R'.$$

<u>Rmk</u>: we don't discuss, for now, the proof of the fact that $K_0(R)$ has in fact the structure of λ-ring (aka special λ-ring). See SGA 6, VI for classical proof.

§ Globalization:

$X \in \mathrm{DSch}$. We would like to extend $\lambda^K: K_0(R) \to K_0(R)$ to $K_0(X)$.

<u>Def</u>: Let $\mathrm{Perf}(X) \subset \mathrm{QCoh}(X)$ be the subcategory spanned by perfect complexes. We have $\mathrm{Vect}(X) \subset \mathrm{Perf}(X)$, where $\mathrm{Vect}(X)$ is the subcat. generated by locally free sheaves. We have
$$K_0^{\mathrm{naive}}(X) = K_0(\mathrm{Vect}(X)) \xrightarrow{i} K_0(\mathrm{Perf}(X)) = K_0(X) \quad \text{canonical map.}$$
Recall that if $X = \mathrm{Spec}(R)$, $R \in \mathrm{SCRing}$, then i is an isomorphism.

We say that X satisfy the global resolution property if i is an equivalence

<u>Recall</u>: if X is a classical scheme, this property is satisfied if X is quasi compact and quasi separated, admitting an ample family of line bundles (SGA 6 & Thomason). See SGA 6, II. 2.2.9.

Assume X has global resolution. Then we could replace $\mathrm{Perf}(X)$ with $\mathrm{Vect}(X)$

Then:

$$\mathrm{Vect}(X) \simeq \lim_{\substack{\longleftarrow \\ \mathrm{Spec}(A) \hookrightarrow X}} \mathrm{Mod}_A^{proj}$$

$$\Rightarrow \quad \begin{array}{ccc} \lambda^K : \mathrm{Vect}(X) & \longrightarrow & \mathrm{Vect}(X) \\ \wr\wr & & \wr\wr \\ \varprojlim \mathrm{Mod}_A^{proj} & \longrightarrow & \varprojlim \mathrm{Mod}_A^{proj} \end{array}$$

$\Rightarrow$ get the map $\lambda^K : K_0(\mathrm{Vect}(X)) \longrightarrow K_0(\mathrm{Vect}(X))$, $\forall K \geq 0$

The identity making $K_0(X)$ into a pre-λ-ring can be checked locally. Hence they are automatically satisfied, thanks to the discussion above.

§ γ-operations.

This can be done axiomatically on any pre λ-ring K.

$\gamma^K : K \to K$. Take $x \in K$.

Write $\lambda_s(x) = \sum_{i \geq 0} \lambda^i(x) s^i \in K[\![s]\!]$.

Change variables: $s = \frac{t}{1-t}$. Then we can rewrite $\lambda_s(x)$ as $\gamma_t(x) = \sum_{K \geq 0} \gamma^K(x) t^K$

$\Rightarrow$ $\gamma^K(x) =$ coefficient of t^K.

<u>Ex</u>: $\gamma^K(x) = \lambda^K(x + K - 1)$.

Properties: $\gamma_t(x+y) = \gamma_t(x)\,\gamma_t(y)$

$$\Rightarrow \gamma^0(x) = 1, \quad \gamma^1(x) = x, \quad \gamma^K(x+y) = \sum \gamma^i(x)\,\gamma^{K-i}(y).$$

Thus, the γ-operations satisfy the axioms of a λ-ring structure on K.

§ Adams operations.

Assume that K is augmented $\varepsilon : K \to H^0$, $H^0 \subset K$ binomial ring.

Then we can define: $\quad (x \in H^0 \Rightarrow \binom{x}{K} \in H^0, \forall K \geq 1)$

$\psi^0(x) = \varepsilon(x)$

$\psi^1(x) = x$

$\psi^2(x) = x^2 - 2\lambda^2(x)$

$$\psi^K(x) = \lambda^1(x)\psi^{K-1}(x) - \lambda^2(x)\psi^{K-2}(x) + \cdots + (-1)^{K-1} K \cdot \lambda^K(x).$$

The operations are defined to satisfy $\psi^j \psi^K = \psi^{jK}$ $\quad j, K \geq 0$ if K satisfy the "splitting principle".