Exercise sheet 1

1. Let A be a ring. Suppose that every A-module is flat. Show that every ideal I is idempotent, i.e., $I^2 = I$. Deduce that every prime ideal of A is maximal.

Let $I \subseteq A$ be an ideal. Tensoring the exact sequence $0 \to I \hookrightarrow A \to A/I \to 0$ with the flat module A/I gives the exact sequence

$$0 \to I/I^2 \to A/I \to A/I \otimes_A A/I \to 0.$$

Since the second map is invertible, $I/I^2 = 0$.

Let $\mathfrak{p}$ be a prime ideal. We want to show that the integral domain $A/\mathfrak{p}$ is a field. Let $a \in A$ be a nonzero element. By above, $\langle a^2 \rangle = \langle a \rangle$, so there exists an element $b \in A$ with $a^2 b = a$. Supposing for a moment that A is an integral domain, we would have $ab = 1$, hence A would be a field. We would be done if we could apply this argument to the integral domain $A/\mathfrak{p}$ instead of A.

Thus let A be a ring such that every A-module is flat, and let's show that A/I has the same property for every ideal $I \subseteq A$. Let N be an A/I-module. Then $N \simeq N_{[A]} \otimes_A A/I$. But $N_{[A]}$ is a flat A-module by assumption, so N is also flat.

2. Let A be a ring and M a finitely generated A-module. Show that the following two conditions are equivalent:

(i) M is finitely presented and for every prime ideal $\mathfrak{p}$, the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is free.

(ii) There exists a set of elements $\{f_i\}_i \subseteq A$ generating the unit ideal, i.e., $\langle f_i \rangle_i = A$, such that $M[f_i^{-1}]$ is a free $A[f_i^{-1}]$-module for each i.

If A is noetherian, then we can drop the "finitely presented" hypothesis in (i).

Hint: you can use (or prove) the following fact: If M is an A-module, then it is finitely generated iff there are elements $\{f_i\}_i \subseteq A$ generating the unit ideal such that $M[f_i^{-1}]$ are all finitely generated. More to the point, you can use this to deduce the analogue for finite presentation.

Assume (i). Let $\{\mathfrak{m}_i\}_i$ be the set of maximal ideals. Since each $M_{\mathfrak{m}_i}$ is a finite free $A_\mathfrak{m}$-module, it follows that there exists $f_i \notin \mathfrak{m}_i$ such that $M[f_i^{-1}] = M \otimes_A A[f_i^{-1}]$ is a finite free $A[f_i^{-1}]$-module. Then $\langle f_i \rangle_i$ is not contained in any $\mathfrak{m}_i$ so it is the unit ideal.

Assume (ii). Since M is f.g. there is a surjection $A^{\oplus n} \to M$. Let K be its kernel. Since $A[f_i^{-1}]$ are flat A-modules, we have exact sequences $0 \to K \otimes_A A[f_i^{-1}] \to A[f_i^{-1}] \to M \otimes_A A[f_i^{-1}] \to 0$. In particular, $K \otimes A[f_i^{-1}]$ is f.g. for each i. This implies that K is f.g. (needs justification). Thus we get a surjection $A^{\oplus m} \to K$ and a finite presentation $A^{\oplus m} \to A^{\oplus n} \to M \to 0$ of M.

Consider a prime ideal $\mathfrak{p}$. It does not contain all the f_i's, so there exists at least one $f \notin \mathfrak{p}$ such that $M[f^{-1}]$ is free over $A[f^{-1}]$. Then $M_\mathfrak{p} \simeq M[f^{-1}] \otimes_{A[f^{-1}]} A_\mathfrak{p}$ is also free.

If A noetherian, then f.g. = f.p.

3. Let A be a ring and M a finitely presented A-module. Show that M is flat iff it is locally free, i.e., satisfies the conditions of Exercise 2.

More generally, if M *is only assumed finitely generated, then one can show:* M *is flat iff* $M_{\mathfrak{p}}$ *is free for all prime ideals* $\mathfrak{p}$*. See [Matsumura, CRT, Thm. 7.10].*

Since flatness can be checked on the localizations $M_{\mathfrak{p}}$, it is clear that the condition is sufficient. Suppose M is flat. Then $M_{\mathfrak{p}}$ is also flat, so we may as well assume A is local with maximal ideal $\mathfrak{m}$ and residue field $\kappa = A/\mathfrak{m}$. Since M f.p., $M \otimes_A \kappa = M/\mathfrak{m}M$ is a finite-dimensional κ-vector space. Choosing a basis and lifting it to M gives rise to an A-linear map $A^{\oplus n} \to M$, which is surjective by Nakayama. Consider the exact sequence

$$0 \to K \to A^{\oplus n} \to M \to 0$$

and the induced long-exact Tor sequence

$$\mathrm{Tor}^1_A(M, \kappa) \to K \otimes_A \kappa \to \kappa^{\oplus n} \to M \otimes_A \kappa \to 0.$$

Since M is flat the Tor vanishes and we conclude that $K \otimes_A \kappa = K/\mathfrak{m}K = 0$. Since K is finitely generated, Nakayama now implies $K = 0$. So $M \simeq A^{\oplus n}$ is free.

4. Let A be a ring and M a finitely generated A-module. Show that $d(\kappa) = \dim_\kappa(M \otimes_A \kappa)$ is finite for all $A \to \kappa$ with κ a field. If A is an integral domain, show that $d(\kappa) \geqslant d(K)$ where $A \to K$ is the field of fractions.

If M is f.g. then so is every extension of scalars.

Assume A is an integral domain. Since κ is local, the ring homomorphism $\phi : A \to \kappa$ factors through a local homomorphism $A_{\mathfrak{p}} \to \kappa$, where $\mathfrak{p} = \mathrm{Ker}(\phi)$. Since it kills the maximal ideal, it factors further through the residue field $\kappa(\mathfrak{p})$. We have $d(\kappa) = d(\kappa(\mathfrak{p}))$.

Consider the f.g. $A_{\mathfrak{p}}$-module $M \otimes_A A_{\mathfrak{p}}$. Choose a surjection $A_{\mathfrak{p}}^{\oplus m} \to M_{\mathfrak{p}}$ with m minimal; by Nakayama, $m = d(\kappa(\mathfrak{p}))$. Extending scalars along $A_{\mathfrak{p}} \to A_{\langle 0 \rangle} = K$, we get a surjection $K^{\oplus m} \to M \otimes_A K$. The target is a K-vector space of dimension $d(K)$, so the claim follows.