1. Show that for any noetherian ring $A$ and every $n \geq 0$, inverse image along $\phi : A \to A[T_1, \ldots, T_n]$ induces an isomorphism

$$\phi^* : G_0(A) \to G_0(A[T_1, \ldots, T_n]).$$

(We showed this when $A$ is a field and $n = 1$. For the case of an integral domain, use a noetherian induction argument and the localization sequence to reduce to the fraction field.)

The homomorphism $\phi^*$ exists since $\phi$ is flat. By induction we may assume $n = 1$. Let $\rho : A \to A_{\text{red}}$ be the reduction homomorphism. Since $\phi : A \to A[T]$ is flat, the base change formula ($\S 6.3$) implies that the square

$$\begin{array}{ccc}
G_0(A_{\text{red}}) & \xrightarrow{\phi^*} & G_0(A_{\text{red}}[T]) \\
\downarrow{\rho^*} & & \downarrow{\rho^*} \\
G_0(A) & \xrightarrow{\phi^*} & G_0(A[T])
\end{array}$$

commutes. We have abused notation by writing $\phi$ also for $A_{\text{red}} \to A_{\text{red}}[T]$, and $\rho$ also for $A[T] \to (A[T])_{\text{red}} \simeq A_{\text{red}}[T]$. By nil-invariance ($\S 6.4$), the vertical arrows are invertible. Therefore, we may replace $A$ by $A_{\text{red}}$ (which is still noetherian) and thereby assume that $A$ is reduced.

Since $A$ is noetherian, it has finitely many minimal primes $p_1, \ldots, p_n$. Let $B$ denote the total ring of fractions of $A$, i.e., the localization at the set of zero divisors. Recall:

**Lemma 1.** Let $A$ be a reduced ring with finitely many minimal prime ideals $p_1, \ldots, p_n$. If $B$ denotes the total ring of fractions of $A$, then we have a canonical isomorphism

$$B \simeq \prod_i B/p_iB.$$

Moreover, each $B/p_iB$ is the field of fractions of the integral domain $A/p_i$.

Therefore, we have

$$G_0(B) \simeq \bigoplus_i G_0(K_i)$$

where $K_i = \text{Frac}(A/p_i)$, since it is easy to check that $G_0$ sends finite products of rings to direct sums.

We now want to apply the localization sequence ($\S 7.3$), except that we are not in the case of a localization at a single element. Nevertheless, it’s easy to check that our proof generalizes to show:
Theorem 2. Let $A$ be a ring and $S$ a multiplicative subset. Then there is a canonical exact sequence

$$\lim_{s \in S} G_0(A/\langle s \rangle) \to G_0(A) \to G_0(A[S^{-1}]) \to 0$$

(Implicit here is a filtered colimit argument,

$$\text{K}_0(\text{Mod}^{fg}_A(S)) \simeq \lim_{s \in S} G_0(A/\langle s \rangle),$$

where $\text{Mod}^{fg}_A(S)$ denotes the category of f.g. $A$-modules that are $S$-torsion, i.e., $s$-torsion for some element $s \in S$.)

In our case, we get the localization sequence

$$\lim_{s \in S} G_0(A/\langle s \rangle) \to G_0(A) \to \bigoplus_i G_0(K_i) \to 0,$$

where the colimit is taken over zero-divisors $s$. We also have

$$G_0(B[T]) \simeq \bigoplus_i G_0(K_i[T])$$

as $B[T] \simeq \prod_i K_i[T]$, so this fits into the commutative diagram with exact rows:

$$\begin{array}{ccc}
\lim_{s \in S} G_0(A/\langle s \rangle) & \to & G_0(A) \\
\downarrow \phi^* & & \downarrow \phi^* \\
\lim_{s \in S} G_0(A/\langle s \rangle)[T] & \to & G_0(A[T])
\end{array}$$

\hskip 1in

$$\bigoplus_i G_0(K_i) \to 0$$

where we again use $\phi$ to denote each of the ring homomorphisms $? \to ?[T]$.

Since the $K_i$ are fields, we already know that the right-hand vertical arrow is invertible (Sheet 7, Exercise 3). We also know that our middle vertical arrow is injective, by the existence of a morphism $\psi : A[T] \to A$, which is of Tor-amplitude $\leq 1$ (use the Koszul complex $\text{Kosz}_A[T](T)$ to resolve $A$ as an $A$-module), and is a retraction of $\phi$ (so that $\psi^* \phi^* = \text{id}$). Thus it remains to show surjectivity. For that we could apply the five lemma if we only knew that the left-hand vertical arrow was invertible.

We can finish the proof by a noetherian induction argument. Call an ideal $I \subseteq A$ good if the inverse image map $G_0(A/I) \to G_0(A/I[T])$ is invertible. The conclusion of the discussion above can be summarized as: if every nonzero ideal of $A$ is good, then the zero ideal is also good. This holds for any reduced noetherian ring $A$.

Now let’s show that every nonzero ideal of $A$ is good. Suppose that isn’t the case. Then since $A$ is noetherian, we can choose a maximal one out of the bad ideals, say $I$, and then every nonzero ideal of $A/I$ is good. Then the above argument yields that the zero ideal of $A/I$ is good, i.e., $G_0(A/I) \to G_0(A/I[T])$ is invertible. But that contradicts the assumption that $I$ was bad. Thus, every nonzero ideal of $A$ is good, so by above, the zero ideal is good, as desired.
2. Let \( k \) be an algebraically closed field and \( A = k[X, Y, Z]/\langle XZ, Z(Z^2 - Y^3) \rangle \). Show that \( |\text{Spec}(A)| \) has two irreducible components, \( Y_1 = V(\langle Z \rangle) \) and \( Y_2 = V((X, Z^2 - Y^3)) \), and is not of pure dimension.

Let \( I_1 = \langle Z \rangle \) and \( I_2 = \langle X, Z^2 - Y^3 \rangle \) as ideals of \( B = k[X, Y, Z] \). Then \( I_1I_2 = \langle ZX, Z(Z^2 - Y^3) \rangle \), so \( V_B(I_1I_2) = V_B(I_1I_2) \) as subsets of \( |\text{Spec}(B)| \). Under the identification \( V_B(I_1I_2) \cong |\text{Spec}(A)| \), we get \( Y_1 \cup Y_2 = V_A(I_1A) \cup V_A(I_2A) = |\text{Spec}(A)| \). Moreover, \( Y_1 \) and \( Y_2 \) are irreducible components of \( |\text{Spec}(A)| \). Moreover, the computations above show that

\[
A/I_1A = k[X, Y, Z]/\langle Z, ZX, Z(Z^2 - Y^3) \rangle \cong k[X, Y] \\
A/I_2A = k[X, Y, Z]/\langle X, Z^2 - Y^3, XZ, Z(Z^2 - Y^3) \rangle \cong k[Y, Z]/\langle Z^2 - Y^3 \rangle
\]

are integral domains. (For the second, it suffices to show that the polynomial \( Z^2 - Y^3 \) is irreducible, since \( k[Y, Z] \) is factorial. Regarding it as a polynomial in \( k[Z][Y] \), it suffices by Gauss’s lemma to show that it is irreducible in \( k(Z)[Y] \). But it is a degree 3 polynomial with no root in \( k(Z) \), so it is irreducible.) In particular, \( Y_1 \) and \( Y_2 \) are the irreducible components of \( |\text{Spec}(A)| \).

Moreover, the computations above show that

\[
\dim(Y_1) = \dim(A/I_1A) = \dim(k[X, Y]) = 2, \\
\dim(Y_2) = \dim(A/I_2A) = \dim(k[Y, Z]/\langle Z^2 - Y^3 \rangle) = 1
\]

where the last equality on the second line follows from Krull’s principal ideal theorem and the fact that \( Z^2 - Y^3 \) is a zero divisor in the integral domain \( k[Y, Z] \).

(Since \( \dim(A) = \sup_m \dim(A_m) \) by Sheet 9, Exercise 3, we can localize at a maximal ideal and then apply Krull’s theorem.)

3. Let \( k \) be an algebraically closed field and \( A = k[X, Y] \). Let \( f, g \in A \) be polynomials.

In each of the following examples, determine whether \( V(f) \) and \( V(g) \) intersect properly, and compute the cycle \( [A/\langle f, g \rangle]_d \in \mathbb{Z}_d(A) \), where \( d = \dim(V(f)) + \dim(V(g)) - \dim(A) \).

(a) \( f = X, \ g = Y \)
(b) \( f = X, \ g = X \)
(c) \( f = Y - X^2, \ g = Y \)
(d) \( f = XY, \ g = Y^2 \)

(Note: we only defined properness of intersection between irreducible subsets. However the same definition makes sense as long as both subsets are of pure dimension.)

(a) We have \( V(X) \cap V(Y) = V(\langle X, Y \rangle) \) which is a closed point \( x \), since \( \langle X, Y \rangle \) is maximal. In particular it is of dimension 0 which is the same as \( d = 1 + 1 - 2 = 0 \). So the intersection is proper.

We compute \([A/\langle f, g \rangle]_0 = [k[X, Y]/\langle X, Y \rangle]_0 = [k]_0 \). The support of \( k \) is \( V((X, Y)) = \{ x \} \) which has a single irreducible component corresponding to the generic
point \(x\). The multiplicity at \(x\) is 1 because \(k\) is a simple \(A_{p(x)}\) module. So 
\([A/\langle f, g \rangle]_0 = [V((X, Y))].\)
(b) We have \(V(X) \cap V(X) = V(X)\) which is of dimension 1, while \(d = 0\). So the intersection is not proper.
We have \([A/\langle f, g \rangle]_0 = [k[X, Y]/\langle X \rangle]_0 = 0\) because \(V(X)\) has no irreducible component of dimension 0.
(c) We have \(V(Y - X^2) \cap V(Y) = V((Y, Y - X^2)) = V((Y, X^2)).\) Since the radical of \(\langle X^2, Y \rangle\) is \(\langle X, Y \rangle\), this is again the closed point \(\{x\}\) of (a). The intersection is proper as \(d = 1 + 1 - 2\) again.
We have \([A/\langle f, g \rangle]_0 = [k[X, Y]/\langle X^2, Y \rangle]_0 = [k[X]/\langle X^2 \rangle]_0.\) Since \(V_{k[X]}(X^2) = V_{k[X]}(X)\) is a closed point, it has a single irreducible component of dimension 0. The multiplicity at that point is given by the length of \(k[X]/\langle X^2 \rangle\) as a module over \(k[X]/\langle X \rangle\), which is 2. So we get \([A/\langle f, g \rangle]_0 = 2 \cdot [V((X, Y))].\) Note that this is double the cycle in (a), which reflects that the parabola \(Y = X^2\) intersects the line \(Y = 0\) in a double point at \((0, 0)\).
(d) We have \(V(XY) \cap V(Y^2) = V(XY) \cap V(Y) = V(Y)\) which is of dimension 1 (note \(V(Y) \subset V(XY))\). But \(d = 1 + 1 - 2\) (note \(V(XY) = V(X) \cup V(Y)\) has two irreducible components both of dimension 1). So the intersection is not proper.
We have \([A/\langle f, g \rangle]_0 = [k[X, Y]/\langle XY, Y^2 \rangle]_0 = 0\) because \(V(Y)\) has no irreducible component of dimension 0.

4. Let \(A\) be a noetherian ring and \(f \in A\) an element. Let \(\phi : A \rightarrow A[f^{-1}]\) and \(\psi : A \rightarrow A/\langle f \rangle\). Show that there is an exact sequence
\[
\text{CH}_n(A/\langle f \rangle) \xrightarrow{\psi_*} \text{CH}_n(A) \xrightarrow{\rho^*} \text{CH}_n(A[f^{-1}]) \rightarrow 0
\]
for every \(n\).
We have \(\phi^* \psi_* = 0\) since for any prime ideal \(p\) of \(A/\langle f \rangle\),
\[
\phi^* \psi_* [V_{A/\langle f \rangle}(p)] = \phi^* [V_A(q)] = [A[f^{-1}]/qA[f^{-1}]]_n = 0
\]
where \(q = \psi^{-1}(p)\) (since \(f \in q\)).
For surjectivity of \(\phi^*,\) let \(V(p)\) be an integral subset of \(|\text{Spec}(A[f^{-1}])|\) of dimension \(n\). The contraction of the prime ideal \(p\) is a prime ideal \(q = \phi^{-1}(p) \subset A\) not containing \(f,\) and whose extension \(qA[f^{-1}]\) recovers \(p\). Thus \(\phi^* [V(q)] = [A[f^{-1}]/qA[f^{-1}]]_n = [A[f^{-1}]/p]_n = [V(p)]\).
Finally let \(\alpha\) be an \(n\)-cycle on \(A\) such that \(\phi^*(\alpha) \in Z_n(A[f^{-1}])\) is rationally equivalent to zero. We need to show that there exists a cycle \(\hat{\alpha} \in Z_n(A/\langle f \rangle)\) such that \(\psi_*(\hat{\alpha}) - \alpha\) is rationally equivalent to zero. Write \(\alpha\) as a linear combination
\[
\alpha = \sum_i n_i [V(p_i)]
\]
where the $p_i$ are prime ideals of $A$. By assumption, we have
\[ \sum_i n_i[V(p_i)] = \sum_j \text{div}_{V(q_j)}(g_j) \]
in $Z_n(A[f^{-1}])$, where $V(q_j)$ are integral subsets of $|\text{Spec}(A[f^{-1}])|$ of dimension $n + 1$ and $g_j$ are elements of $A[f^{-1}]$ with $g_j \not\in q_j$. Each $q_j$ is an extension of a prime of $A$ (which we denote by $q_j$ again). We can also assume that $g_j$ come from $A$ by clearing denominators (multiply by a large enough power of $f$). Then the difference
\[ \beta = \sum_i n_i[V(p_i)] - \sum_j \text{div}_{V(q_j)}(g_j) \]
may be viewed as an element of $Z_n(A)$ which goes to zero in $Z_k(A[f^{-1}])$. Since the latter is a free abelian group, this means that $\beta$ can have nonzero multiplicity at an integral subset $V(p) \subseteq |\text{Spec}(A)|$ only if $\phi^*[V(p)] = 0 \in Z_n(A[f^{-1}])$, hence $pA[f^{-1}] = A[f^{-1}]$, hence $f \in p$. In particular, $\beta$ lifts to an element $\tilde{\alpha} \in Z_n(A/\langle f \rangle)$ such that $\psi_*(\tilde{\alpha}) = \beta$. The claim follows.