Exercise sheet 11

1. Let A be a ring.

(a) Let M and N be invertible A-modules. Show that $[\mathrm{M}] = [\mathrm{N}]$ in $\mathrm{K}_0(\mathrm{A})$ iff $\mathrm{M} \simeq \mathrm{N}$ as A-modules.

(b) Let M be a f.g. projective A-module such that $[\mathrm{M}] \in \mathrm{K}_0(\mathrm{A})$ is a unit. Show that M is invertible as an A-module.

(c) Show that the group homomorphism $\mathrm{Pic}(\mathrm{A}) \to \mathrm{K}_0(\mathrm{A})^\times$ is injective but not always bijective.

(a) The condition is obviously sufficient. Necessity follows from the fact that the composite

$$\mathrm{Pic}(\mathrm{A}) \to \mathrm{K}_0(\mathrm{A}) \xrightarrow{\det_{\mathrm{A}}} \mathrm{Pic}(\mathrm{A})$$

is the identity.

(b) Let $x \in \mathrm{K}_0(\mathrm{A})$ be an element such that $[\mathrm{M}] \cdot x = 1$ in $\mathrm{K}_0(\mathrm{A})$. We may write $x = [\mathrm{N}] - [\mathrm{A}^{\oplus n}]$ for some $\mathrm{N} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fgproj}}$ and $n \geqslant 0$ (Lecture 3), and $[\mathrm{M}] \cdot x = 1$ is equivalent to

$$[\mathrm{M} \otimes_{\mathrm{A}} \mathrm{N}] = [\mathrm{A} \oplus \mathrm{M}^{\oplus n}]$$

in $\mathrm{K}_0(\mathrm{A})$. It follows that $\mathrm{M} \otimes_{\mathrm{A}} \mathrm{N}$ and $\mathrm{A} \oplus \mathrm{M}^{\oplus n}$ are stably equivalent (Lecture 3), i.e.,

$$(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{N}) \oplus \mathrm{A}^{\oplus k} \simeq (\mathrm{A} \oplus \mathrm{M}^{\oplus n}) \oplus \mathrm{A}^{\oplus k}$$

for some $k \geqslant 0$. Localizing at any prime ideal $\mathfrak{p}$ and using Nakayama, we find that

$$\mathrm{rk}_{\mathrm{A}_\mathfrak{p}}(\mathrm{M}_\mathfrak{p}) \cdot \mathrm{rk}_{\mathrm{A}_\mathfrak{p}}(\mathrm{N}_\mathfrak{p}) + k = 1 + n \cdot \mathrm{rk}_{\mathrm{A}_\mathfrak{p}}(\mathrm{M}_\mathfrak{p}) + k,$$

hence $\mathrm{rk}_{\mathrm{A}_\mathfrak{p}}(\mathrm{M}_\mathfrak{p}) \cdot (\mathrm{rk}_{\mathrm{A}_\mathfrak{p}}(\mathrm{N}_\mathfrak{p}) - n) = 1$ and in particular $\mathrm{M}_\mathfrak{p}$ is of rank one for every $\mathfrak{p}$. Thus M is invertible (Lecture §11.1).

(c) If $[\mathrm{M}] \in \mathrm{Pic}(\mathrm{A})$ is in the kernel, then $[\mathrm{M}] = [\mathrm{A}]$ in $\mathrm{K}_0(\mathrm{A})$ and hence $[\mathrm{M}] = [\mathrm{A}]$ in $\mathrm{Pic}(\mathrm{A})$ by (a). So the map is injective.

Let A be a PID, so that every f.g. projective A-module is free. Then $\mathrm{K}_0(\mathrm{A}) \simeq \mathbf{Z}$. Similarly $\mathrm{Pic}(\mathrm{A})$ is the trivial group, since every invertible A-module is free of rank one. Thus the map is identified in this case with $1 \hookrightarrow \{\pm 1\}$.

2. Let A be a regular ring. Show that the homomorphism $\mathrm{Pic}(\mathrm{A}) \to \mathrm{Pic}(\mathrm{A}[\mathrm{T}_1, \ldots, \mathrm{T}_n])$, induced by extension of scalars, is bijective for all $n > 0$.

The map is a retraction of the map $\mathrm{K}_0(\mathrm{A}) \to \mathrm{K}_0(\mathrm{A}[\mathrm{T}_1, \ldots, \mathrm{T}_n])$ (in the category of sets). The latter is an isomorphism since A is regular (Lecture §5.3), hence so is the former.

In other words, we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Pic}(\mathrm{A}) & \longrightarrow & \mathrm{K}_0(\mathrm{A}) & \xrightarrow{\det} & \mathrm{Pic}(\mathrm{A}) \\
\downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} \\
\mathrm{Pic}(\mathrm{A}[\mathrm{T}_1,\dots,\mathrm{T}_n]) & \longrightarrow & \mathrm{K}_0(\mathrm{A}[\mathrm{T}_1,\dots,\mathrm{T}_n]) & \xrightarrow{\det} & \mathrm{Pic}(\mathrm{A}[\mathrm{T}_1,\dots,\mathrm{T}_n])
\end{array}$$

where the vertical arrows are inverse image along $\phi : \mathrm{A} \to \mathrm{A}[\mathrm{T}_1, \dots, \mathrm{T}_n]$. Since the middle vertical arrow is bijective, it follows that the left-hand vertical arrow is injective and the right-hand vertical arrow is surjective.

3. Let A be an integral domain. Recall the rank homomorphism $\mathrm{rk} : \mathrm{G}_0(\mathrm{A}) \to \mathbf{Z}$ (Sheet 3, Exercise 4), which we regard as a homomorphism $\mathrm{rk} : \mathrm{K}_0(\mathrm{A}) \to \mathbf{Z}$ by restricting along the canonical homomorphism $\mathrm{K}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$. Let $x \in \mathrm{K}_0(\mathrm{A})$ be a class of positive rank ($\mathrm{rk}(x) > 0$). Show that

$$n \cdot x = [\mathrm{M}]$$

for some $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fgproj}}$ and integer $n \geqslant 0$.

Hint: reduce to the case where A is of finite dimension d, and use a theorem of Serre which states that any projective A-module is the direct sum of a free module and a projective module of rank $\leqslant d$ (see [Serre, *Modules projectifs et espaces fibrés à fibre vectorielle*]).

Using (Sheet 5, Exercise 1), we can reduce to the case where A is a finite type $\mathbf{Z}$-algebra, and in particular of some finite dimension d. Note that it is harmless to replace x by $m \cdot x$ for an integer $m > 0$. By performing such a replacement if necessary, we can assume that $\mathrm{rk}(n \cdot x) \geqslant d$. Write $n \cdot x = [\mathrm{N}] - [\mathrm{A}^{\oplus k}]$ for some $\mathrm{N} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fgproj}}$ and $k \geqslant 0$ (Lecture 3), i.e.,[1]

$$n \cdot x + k = [\mathrm{N}].$$

Now N has rank equal to $\mathrm{rk}(n \cdot x + k) \geqslant d + k$. By Serre's theorem, it is the direct sum of a free module of rank $\geqslant k$ and a projective module. In particular, we may write $\mathrm{N} \simeq \mathrm{A}^{\oplus k} \oplus \mathrm{M}$ where M is projective. Then $n \cdot x + k = [\mathrm{N}] = k + [\mathrm{M}]$, so $n \cdot x = [\mathrm{M}]$ as claimed.

4. Let k be an algebraically closed field and $\mathrm{A} = k[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2 - \mathrm{Y}^3 \rangle$ (an integral domain of dimension 1). Let $f \in \mathrm{Frac}(\mathrm{A})^\times$ denote the rational function $(\mathrm{X} - \mathrm{Y})/\mathrm{Y}$. For the closed point $x_0 = \mathrm{V}(\langle \mathrm{X}, \mathrm{Y} \rangle)$ in $|\mathrm{Spec}(\mathrm{A})|$, show that $f_{\mathfrak{p}(x_0)} \in \mathrm{Frac}(\mathrm{A}_{\mathfrak{p}(x_0)})$ is not contained in the subring $\mathrm{A}_{\mathfrak{p}(x_0)}$. For every other closed point $x \neq x_0$, show that $f_{\mathfrak{p}(x)}$ is even contained in the subgroup of units $\mathrm{A}_{\mathfrak{p}(x)}^\times$. Deduce that the principal Cartier divisor $\mathrm{div}_{\mathrm{A}}(f) \in \mathrm{Cart}(\mathrm{A})$ is nonzero.

We saw that A is an integral domain of dimension 1 in the proof of Exercise 2 on Sheet 10.

[1] Recall 1 denotes the unit $[\mathrm{A}] \in \mathrm{K}_0(\mathrm{A})$, so $k = 1 + \cdots + 1 = [\mathrm{A}] + \cdots + [\mathrm{A}] = [\mathrm{A}^{\oplus k}]$.

Let $\mathfrak{m}$ be a maximal ideal of A. Since k is algebraically closed, this corresponds by Exercise 1 on Sheet 8 to a maximal ideal $\langle \mathrm{X} - a, \mathrm{Y} - b \rangle \subset k[\mathrm{X}, \mathrm{Y}]$ (where $a, b \in k$) which contains $\langle \mathrm{X}^2 - \mathrm{Y}^3 \rangle$. The latter condition means $a^2 = b^3$.

Note that the only way $f = (\mathrm{X} - \mathrm{Y})/\mathrm{Y} \in \mathrm{Frac}(\mathrm{A})$ could belong to $\mathrm{A}_{\mathfrak{m}} \subset \mathrm{Frac}(\mathrm{A}_{\mathfrak{m}}) = \mathrm{Frac}(\mathrm{A})$ is if Y becomes a unit in $\mathrm{A}_{\mathfrak{m}}$, i.e., if $\mathrm{Y} \notin \mathfrak{m} = \langle \mathrm{X} - a, \mathrm{Y} - b \rangle$. This is equivalent to $b \neq 0$, or, since $a^2 = b^3 = 0$, to $(a, b) \neq (0, 0)$. Moreover, if $(a, b) \neq (0, 0)$ then $f \in \mathrm{A}_{\langle \mathrm{X} - a, \mathrm{Y} - b \rangle}$ is even a unit because $\mathrm{X} - \mathrm{Y} \notin \mathfrak{m}$ also.