Exercise sheet 2

1. Prove the Proposition from §1.3 in the lecture: every regular sequence in a ring $A$ is Koszul-regular. (Hint: induction.)

2. Let $k$ be a field, $A = k[x]/\langle x^2 \rangle$. Show that $k$, viewed as an $A$-module, is not perfect.

3. Let $\phi : A \rightarrow B$ be a flat ring homomorphism.

   (i) Show that if a f.g. $A$-module $M$ is of Tor-amplitude $\leq n$, then so is the $B$-module $M \otimes_A B$.

   (ii) Suppose that $\phi$ is faithfully flat, i.e., that a sequence of $A$-modules $M' \rightarrow M \rightarrow M''$ is exact iff $M' \otimes_A B \rightarrow M \otimes_A B \rightarrow M'' \otimes_A B$ is exact. Show that a f.g. $A$-module $M$ is of Tor-amplitude $\leq n$ if and only if $M \otimes_A B$ is of Tor-amplitude $\leq n$.

4. Let $A$ be a noetherian ring and $M$ a finitely generated $A$-module. Show that $M$ is of finite length iff $M_p = 0$ for all non-maximal prime ideals $p$. (Use the Proposition in §1.3 of the lecture.)

   The length of an $A$-module $M$ is the maximal length of a composition series (a filtration where the successive quotients are all simple, i.e., are nonzero and have no non-trivial, non-proper submodules). For example, $A$ has length 1 iff $A$ is a field. For a field, length coincides with dimension of vector spaces.