## Exercise sheet 4

1. Let A be a commutative ring and M<sub>•</sub> a chain complex of A-modules. Show that if M<sub>•</sub> is acyclic, then it is perfect.

Note that the acyclicity of  $M_{\bullet}$  means that the unique morphism  $0 \to M_{\bullet}$  is a quasi-isomorphism (where 0 is the zero complex). Since 0 is a bounded complex of f.g. projectives, this means that  $M_{\bullet}$  is perfect.

**2.** Let A be a commutative ring and  $M_{\bullet}$  a chain complex of A-modules. Suppose that  $M_{\bullet}$  is *n*-connective for some integer *n*, i.e.,  $H_i(M_{\bullet}) = 0$  for i < n. Then there is a diagram of chain complexes

$$\mathcal{M}_{\bullet} \xleftarrow{\operatorname{qus}} \tau_{\geq n}(\mathcal{M}_{\bullet}) \to \mathcal{H}_n(\mathcal{M}_{\bullet})[n].$$

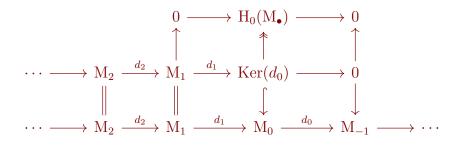
Here  $\tau_{\geq n}(\mathbf{M}_{\bullet})$  denotes the truncated complex

$$\cdots \to \mathcal{M}_{n+2} \xrightarrow{d_{n+2}} \mathcal{M}_{n+1} \to \operatorname{Ker}(d_n) \to 0,$$

where  $\operatorname{Ker}(d_n)$  is in degree n (and the differential  $\operatorname{M}_{n+1} \to \operatorname{Ker}(d_n)$  factors through  $\operatorname{Im}(d_{n+1}) \subseteq \operatorname{Ker}(d_n)$ ).

Note that replacing  $M_{\bullet}$  by  $M_{\bullet}[n]$  has the effect of replacing  $H_n(M_{\bullet})[n]$  by  $H_n(M_{\bullet}[n])[n] = H_0(M_{\bullet})[n]$ , and  $\tau_{\leq n}(M_{\bullet})$  by  $\tau_{\leq n}(M_{\bullet}[n]) = \tau_{\leq 0}(M_{\bullet})$ . Therefore, we may as well assume that  $M_{\bullet}$  is 0-connective. (This simplifies nothing except the notation.)

The morphisms  $M_{\bullet} \leftarrow \tau_{\geq 0}(M_{\bullet}) \rightarrow H_0(M_{\bullet})[0]$  are defined as



It is clear that  $M_{\bullet} \leftarrow \tau_{\geq 0}(M_{\bullet})$  is a quasi-isomorphism.

- **3.** Let A be a commutative ring and M<sub>•</sub> a chain complex of A-modules. Show that the following conditions are equivalent:
  - (a)  $H_i(M_{\bullet}) \neq 0$  for exactly one  $i \in \mathbb{Z}$ .

(b)  $M_{\bullet}$  is quasi-isomorphic to  $H_i(M_{\bullet})[i]$ , via a zig-zag  $M_{\bullet} \leftarrow ? \rightarrow H_i(M_{\bullet})[i]$ , where both arrows are quasi-isomorphisms.

Since the complex  $H_i(M_{\bullet})[i]$  is concentrated in degree *i*, it is clear that it has exactly one non-vanishing homology group. Since the condition in (a) is preserved by quasi-isomorphisms, it follows that (b) implies (a).

Suppose (a), and let  $H_i(M_{\bullet})$  be the only non-vanishing homology group. Then  $M_{\bullet}$  is in particular *i*-connective, so by Exercise 1 there exists a zig-zag

$$\mathcal{M}_{\bullet} \xleftarrow{q_{\mathrm{ls}}} \tau_{\geq i}(\mathcal{M}_{\bullet}) \to \mathcal{H}_{i}(\mathcal{M}_{\bullet})[i].$$

By definition,  $\tau_{\geq i}(\mathbf{M}_{\bullet})$  and  $\mathbf{H}_{i}(\mathbf{M}_{\bullet})[i]$  are both bounded on the right by *i* (below *i*, all their terms vanish). At *i*, the map clearly induces an isomorphism on  $\mathbf{H}_{i}$ . To the left (above *i*),  $\tau_{\geq i}(\mathbf{M}_{\bullet})$  has the same homology groups as  $\mathbf{M}_{\bullet}$  and is therefore acyclic. Thus we see that  $\tau_{\geq i}(\mathbf{M}_{\bullet}) \to \mathbf{H}_{i}(\mathbf{M}_{\bullet})[i]$  is also a quasi-isomorphism.

4. (i) Give an example of a perfect complex  $P_{\bullet}$  over some ring A which is unbounded  $(P_i \neq 0 \text{ for infinitely many } i \in \mathbf{Z}).$ 

(ii) Give an example of a perfect complex  $Q_{\bullet}$  over some ring A which has  $H_i(Q_{\bullet}) \neq 0$  for at least two  $i \in \mathbb{Z}$ , and which is not a bounded complex of f.g. projective modules.

(i) For example,

$$\mathbf{P}_{\bullet} = \left( \cdots \xrightarrow{\mathbf{0}} \mathbf{A} \xrightarrow{\mathrm{id}} \mathbf{A} \xrightarrow{\mathbf{0}} \mathbf{A} \xrightarrow{\mathrm{id}} \mathbf{A} \xrightarrow{\mathbf{0}} \cdots \right)$$

for any commutative ring A. Since  $H_i(P_{\bullet}) = 0$  for all *i*, this complex is acyclic. By exercise 1, it is perfect.

(ii) For example, the complex of **Z**-modules

$$\mathbf{Q}_{\bullet} = \left( 0 \to \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \to 0 \right)$$

where the map is multiplication by 2. Certainly  $\mathbf{Z}/4\mathbf{Z}$  is not a projective  $\mathbf{Z}$ -module, and the complex has non-vanishing H<sub>0</sub> and H<sub>1</sub> (both isomorphic to  $\mathbf{Z}/2\mathbf{Z}$ ). But it is indeed perfect:

(0)

This diagram depicts a quasi-isomorphism between the upper row, a finite complex of f.g. free **Z**-modules, and the lower row,  $Q_{\bullet}$ .