1. Let $A$ be a commutative ring.

(a) Show that $A$ is a filtered colimit (inductive limit) of its finitely generated subrings (i.e., subrings that are finitely generated as $\mathbb{Z}$-algebras).

(b) Suppose given a diagram of commutative rings $(A_\lambda)_{\lambda \in \Lambda}$ indexed by a poset (partially ordered set) $\Lambda$. Let $A$ denote the filtered colimit $\text{lim}_{\lambda \rightarrow A} A_\lambda$ (in the category of commutative rings). Show that there is a canonical isomorphism

$$K_0(A) \simeq \lim_{\lambda \in \Lambda} K_0(A_\lambda).$$

(c) Deduce from (a) and (b) that, for every commutative ring $A$, there is a canonical isomorphism $K_0(A) \simeq \lim_{\lambda \in \Lambda} K_0(A_\lambda)$ where $\Lambda$ is a poset and $A_\lambda$ are noetherian.

(a) The set $\Lambda$ of finitely generated subrings of $A$ becomes a poset with the inclusion order. Any poset can be regarded as a filtered category whose objects are the elements of the poset, and there is at most one morphism between any two objects. We then have a canonical functor $\Lambda \rightarrow \text{CRing}$, sending a f.g. subring $A_\lambda$ to itself. Since $A$ is the union of these subrings, it is easy to see that $A$ is the colimit of this diagram.

(b) We first show that the canonical homomorphism of monoids

$$\lim_{\lambda} M(A_\lambda) \rightarrow M(A),$$

induced by the extension of scalars maps $M(A_\lambda) \rightarrow M(A)$, is invertible. For surjectivity, we need to show that every f.g. projective $A$-module $M$ descends to a f.g. projective $A_\alpha$-module $M_\alpha$, for some sufficiently large $\alpha \in \Lambda$. Note that $M$ is a direct summand of some $A^{\oplus n}$, so it is the image of some projector (idempotent endomorphism) $\phi : A^{\oplus n} \rightarrow A^{\oplus n}$. This corresponds to a matrix $(\phi_{i,j})_{i,j}$ with entries $\phi_{i,j} \in A$. Since $A$ is the colimit of $(A_\lambda)_\lambda$, there exists some $\alpha \in \Lambda$ and a matrix $(\psi_{i,j})_{i,j}$ with entries in $A_\alpha$, such that $\psi_{i,j} \rightarrow \phi_{i,j}$ for all $i,j$. We let $M_\alpha$ be the image of the corresponding projector $A^{\oplus n}_\alpha \rightarrow A^{\oplus n}_\alpha$. Then we have $M_\alpha \otimes_{A_\alpha} A \simeq M$ by construction.

For injectivity, we need to show that if $M$ and $N$ are $A_\lambda$ modules, then any $A$-module isomorphism $M \otimes_{A_\lambda} A \simeq N \otimes_{A_\lambda} A$ descends to an $A_\alpha$-module isomorphism $M \otimes_{A_\lambda} A_\alpha \simeq N \otimes_{A_\lambda} A_\alpha$ for some $\alpha \geq \lambda$. The argument is similar to above, noting that such an $A$-module isomorphism is determined by (finitely many) elements of $A$, which must come from some $A_\alpha$.

Now, apply group completion $(-)^{gp}$ to the above monoid isomorphism. Since $(-)^{gp}$ is left adjoint to the functor from abelian groups to commutative monoids, it commutes with colimits. Thus the claim follows.

(c) We only need to show that finitely generated $\mathbb{Z}$-algebras are noetherian. This follows from the Hilbert basis theorem.
2. (a) Let $A$ be a commutative ring and $\phi : M_* \rightarrow N_*$ a quasi-isomorphism of chain complexes over $A$. Let $P_*$ be a chain complex of projective $A$-modules. Suppose that $M_*$, $N_*$ and $P_*$ are all bounded below. Show that any morphism $\beta : P_* \rightarrow N_*$ lifts to a morphism $\alpha : P_* \rightarrow M_*$, such that the diagram

\[
\begin{array}{ccc}
M_* & \xrightarrow{\alpha} & N_* \\
\downarrow & & \downarrow \\
P_* & \xrightarrow{\beta} & N_*
\end{array}
\]

commutes up to homotopy (i.e., the morphisms $\beta$ and $\phi \circ \alpha$ are homotopic).

(b) Let $M_*$ and $N_*$ be bounded below complexes over $A$. If $N_*$ is projective, show that any quasi-isomorphism $\phi : M_* \rightarrow N_*$ admits a section up to homotopy, which is also a quasi-isomorphism.

(c) Let $M_*$ and $N_*$ be bounded below complexes over $A$. Suppose they are quasi-isomorphic (in the sense that there exists a zig-zag of quasi-isomorphisms between them). Show that if $M_*$ is projective, then there exists a quasi-isomorphism $M_* \rightarrow N_*$. Give an example to show that this is not true if $M_*$ is not projective.

(a) We use the following observation:

**Lemma 1.** Let $P_*$ be a complex of projectives and $Q_*$ an acyclic complex. Let $\alpha$ and $\beta$ be morphisms $P_* \rightarrow Q_*$. Suppose given, for some integer $n$, a collection of morphisms $h_i : P_i \rightarrow Q_{i+1}$ for $i \leq n$, satisfying

$$\alpha_i - \beta_i = h_{i-1}d_i + d_{i+1}h_i$$

for each $i \leq n - 1$. Then the $h_i$ extend to a homotopy $\alpha \simeq \beta$.

**Proof.** Let $\phi_n : P_n \rightarrow Q_n$ be the morphism $(\alpha_n - \beta_n) - h_{n-1}d_n$. Note that $\phi_n$ lands in $\text{Ker}(d_n)$. Indeed from the relation

$$\alpha_{n-1} - \beta_{n-1} = d_nh_{n-1} + h_{n-2}d_{n-1}$$

we derive, by composing with $d_n$ on the right (and using the fact that $\alpha$ and $\beta$ commute with $d$ and that $d^2 = 0$),

$$d_n(\alpha_n - \beta_n) = d_nh_{n-1}d_n.$$ 

In other words, $d_n\phi_n = 0$. Now since $Q_*$ is acyclic and in particular $H_n(Q_*) = 0$, it follows that $\phi_n \in \text{Im}(d_{n+1})$. Since $P_*$, we can find a lift in the diagram

\[
\begin{array}{ccc}
Q_{n+1} & \xrightarrow{h_n} & Q_n \\
\downarrow & & \downarrow \\
P_n & \xrightarrow{\phi_n} & \text{Im}(d_{n+1})
\end{array}
\]

The relation

$$d_{n+1}h_n = \phi_n = (\alpha_n - \beta_n) - h_{n-1}d_n$$

translates to

$$\alpha_n - \beta_n = h_{n-1}d_n + d_{n+1}h_n.$$ 

The claim follows by induction. \qed
Let $K_\bullet = \text{Cone}(\phi)_\bullet$. Since $\phi$ is a quasi-isomorphism, $K_\bullet$ is acyclic. Applying the Lemma (we can take $h_i = 0$ for $i \ll 0$ since the complexes are bounded below), we deduce that the morphism

$$P_\bullet \xrightarrow{\beta} N_\bullet \rightarrow K_\bullet$$

is homotopic to zero through a homotopy $(h_i)_i$. The morphisms $h_i : P_i \rightarrow K_{i+1} = N_{i+1} \oplus M_i$ give rise in particular to morphisms $\alpha_i : P_i \rightarrow M_i$. From the homotopy relation satisfied by $h_i$, one reads off that $\alpha$ defines a morphism of chain complexes and that $\beta$ is homotopic to $\phi \circ \alpha$.

(b) Apply (a) to the diagram

(c) If $M_\bullet$ and $N_\bullet$ are quasi-isomorphic, there exists a zig-zag

$$(0.1) \quad M_\bullet = L_0^0 \leftarrow L_1^1 \rightarrow L_2^2 \leftarrow \cdots \rightarrow L_n^n = N_\bullet$$

where all arrows are quasi-isomorphisms. Note that the $L_i^i$ are not necessarily bounded below (and not necessarily projective). However, since $M_\bullet$ is bounded below, it is $n$-connective for some $n$, hence so are all $L_i^i$. For each diagram

$$L_{i-1}^i \leftarrow L_i^i \rightarrow L_{i+1}^i,$$

we can expand this to a diagram

$$\xymatrix{ & L_{i-1}^i \ar[rr] & & L_i^i \ar[rr] & & L_{i+1}^i \ar[ll] \ar[ll]_{\tau \geq n}(L_i^i) }$$

where all arrows are still quasi-isomorphisms. For every odd $i \geq 1$, replace $L_i^i$ in (0.1) by $\tau_{\geq n}(L_i^i) \rightarrow L_{i+1}^i$. For every odd $i \geq 1$, we can also find a quasi-isomorphism from a bounded below complex $P_i^i$ of projectives, and replace every $L_i^i$ by $P_i^i$. Applying (b), we can now reverse the wrong-way arrows in the zigzag (0.1).

3. Let $A$ be a commutative ring and let $M_\bullet \rightarrow N_\bullet \rightarrow K_\bullet$ be an exact triangle of chain complexes of $A$-modules. Show that if any two of the terms is perfect, so is the third.

Given the technology we have, the proof is much easier in the case where $A$ is noetherian, which we assume. From the long exact sequence in homology we see that if any two of the terms is coherent, then so is the third. Thus it will suffice to show that if any two of the terms is of finite Tor-amplitude, then so is the third. This follows by inspecting the long exact sequence associated to the exact triangle $M_\bullet \otimes_A^L E \rightarrow N_\bullet \otimes_A^L E \rightarrow K_\bullet \otimes_A^L E$, for every $A$-module $E$. 

Here is another argument which works if we assume that the complexes are bounded below. By rotating the triangle, we may assume that $M\cdot$ and $N\cdot$ are perfect (since shifting has no effect on perfectness). We may assume that $K\cdot$ is the cone of $\phi : M\cdot \to N\cdot$. Let $P\cdot \to M\cdot$ and $Q\cdot \to N\cdot$ be quasi-isomorphisms with $P\cdot, Q\cdot \in \text{Proj}_A^b$. Then by Exercise 2 (a), we can find a morphism $\psi$ making the left-hand square below commute up to homotopy:

$$
\begin{array}{ccc}
P\cdot & \xrightarrow{\psi} & Q\cdot \\
\downarrow & & \downarrow \\
M\cdot & \xrightarrow{\phi} & N\cdot \\
\end{array}
$$

Then there is an induced quasi-isomorphism between the cones. Since $P_i$ and $Q_i$ are all projectives, we also have $\text{Cone}(\psi) \in \text{Proj}_A^b$. Thus $K\cdot = \text{Cone}(\phi)\cdot$ is perfect.

4. Let $A$ be a commutative ring. A chain complex $M\cdot$ is called connective if it is 0-connective, i.e., $H_i(M\cdot) = 0$ for $i < 0$. Imitate the construction of $K_0(\text{Perf}_A)$ to define a variant $K_0(\text{Perf}_{\text{cn}}^A)$ using (quasi-isomorphism classes of) connective perfect complexes. Show that there is a canonical isomorphism $K_0(\text{Perf}_{\text{cn}}^A) \cong K_0(\text{Perf}_A)$.

Note that there is a commutative diagram

$$
\begin{array}{ccc}
K_0(A) & \xrightarrow{\sim} & K_0(\text{Perf}_A) \\
& (\dagger) & \\
& K_0(\text{Perf}_{\text{cn}}^A) \\
\end{array}
$$

where the upper arrow is known to be an isomorphism from Lecture 4. This already implies that the arrow in question is surjective. There are various ways to proceed. One is to show that the map $(\dagger)$ is surjective, repeating the proof for the upper arrow. Another is to write down a retraction $K_0(\text{Perf}_A) \to K_0(\text{Perf}_{\text{cn}}^A)$. Given a perfect complex $M\cdot \in \text{Perf}_A$, send $[M\cdot] \to [M\cdot[k]]$ where $k$ is an arbitrary even shift large enough so that $M\cdot[k]$ is connective.