1. (a) Show that every commutative ring $A$ is a filtered colimit (inductive limit) of its finitely generated subrings (i.e., subrings that are finitely generated as $\mathbb{Z}$-algebras).

(b) Suppose given a diagram of commutative rings $(A_\alpha)_{\alpha \in \Lambda}$ indexed by a poset (partially ordered set) $\Lambda$. Let $A$ denote the filtered colimit $\varinjlim_{\alpha \in \Lambda} A_\alpha$ (in the category of commutative rings). Show that there is a canonical isomorphism

$$K_0(A) \simeq \lim_{\alpha \in \Lambda} K_0(A_\alpha).$$

(c) Deduce from (a) and (b) that, for every commutative ring $A$, there is a canonical isomorphism $K_0(A) \simeq \lim_{\alpha \in \Lambda} K_0(A_\alpha)$ where $\Lambda$ is a poset and $A_\alpha$ are noetherian.

2. (a) Let $A$ be a commutative ring and $\phi : M_* \to N_*$ a quasi-isomorphism of chain complexes over $A$. Let $P_*$ be a chain complex of projective $A$-modules. Suppose that $M_*$, $N_*$ and $P_*$ are all bounded below. Show that any morphism $\beta : P_* \to N_*$ lifts to a morphism $\alpha : P_* \to M_*$, such that the diagram

$$\begin{array}{ccc}
M_* & \xrightarrow{\alpha} & M_* \\
\downarrow{\phi} & & \downarrow{\phi} \\
N_* & \xrightarrow{\beta} & N_*
\end{array}$$

commutes up to homotopy (i.e., the morphisms $\beta$ and $\phi \circ \alpha$ are homotopic).

(b) Let $M_*$ and $N_*$ be bounded below complexes over $A$. If $N_*$ is projective, show that any quasi-isomorphism $\phi : M_* \to N_*$ admits a section up to homotopy, which is also a quasi-isomorphism.

(c) Let $M_*$ and $N_*$ be bounded below complexes over $A$. Suppose they are quasi-isomorphic (in the sense that there exists a zig-zag of quasi-isomorphisms between them). Show that if $M_*$ is projective, then there exists a quasi-isomorphism $M_* \to N_*$. Give an example to show that this is not true if $M_*$ is not projective.

3. Let $A$ be a commutative ring and let $M_* \to N_* \to K_*$ be an exact triangle of chain complexes of $A$-modules. Show that if any two of the terms are perfect, so is the third.

4. Let $A$ be a commutative ring. A chain complex $M_*$ is called **connective** if it is 0-connective, i.e., $H_i(M_*) = 0$ for $i < 0$. Imitate the construction of $K_0(\text{Perf}_A)$ to define a variant $K_0(\text{Perf}^\text{cn}_A)$ using (quasi-isomorphism classes of) connective perfect complexes. Show that there is a canonical isomorphism

$$K_0(\text{Perf}^\text{cn}_A) \xrightarrow{\sim} K_0(\text{Perf}_A).$$