Exercise sheet 5

1. (a) Show that every commutative ring A is a filtered colimit (inductive limit) of its finitely generated subrings (i.e., subrings that are finitely generated as **Z**-algebras).

(b) Suppose given a diagram of commutative rings $(\mathrm{A}_\alpha)_{\alpha\in\Lambda}$ indexed by a poset (partially ordered set) Λ. Let A denote the filtered colimit $\varinjlim_\alpha \mathrm{A}_\alpha$ (in the category of commutative rings). Show that there is a canonical isomorphism

$$\mathrm{K}_0(\mathrm{A}) \simeq \varinjlim_{\alpha\in\Lambda} \mathrm{K}_0(\mathrm{A}_\alpha).$$

(c) Deduce from (a) and (b) that, for every commutative ring A, there is a canonical isomorphism $\mathrm{K}_0(\mathrm{A}) \simeq \varinjlim_{\alpha\in\Lambda} \mathrm{K}_0(\mathrm{A}_\alpha)$ where Λ is a poset and A_α are noetherian.

2. (a) Let A be a commutative ring and $\phi : \mathrm{M}_\bullet \to \mathrm{N}_\bullet$ a quasi-isomorphism of chain complexes over A. Let $\mathrm{P}_\bullet$ be a chain complex of projective A-modules. Suppose that $\mathrm{M}_\bullet$, $\mathrm{N}_\bullet$ and $\mathrm{P}_\bullet$ are all bounded below. Show that any morphism $\beta : \mathrm{P}_\bullet \to \mathrm{N}_\bullet$ lifts to a morphism $\alpha : \mathrm{P}_\bullet \to \mathrm{M}_\bullet$, such that the diagram

$$\begin{array}{ccc} & & \mathrm{M}_\bullet \\ & \overset{\alpha}{\nearrow} & \downarrow \phi \\ \mathrm{P}_\bullet & \xrightarrow{\beta} & \mathrm{N}_\bullet \end{array}$$

commutes up to homotopy (i.e., the morphisms β and $\phi \circ \alpha$ are homotopic).

(b) Let $\mathrm{M}_\bullet$ and $\mathrm{N}_\bullet$ be bounded below complexes over A. If $\mathrm{N}_\bullet$ is projective, show that any quasi-isomorphism $\phi : \mathrm{M}_\bullet \to \mathrm{N}_\bullet$ admits a section up to homotopy, which is also a quasi-isomorphism.

(c) Let $\mathrm{M}_\bullet$ and $\mathrm{N}_\bullet$ be bounded below complexes over A. Suppose they are quasi-isomorphic (in the sense that there exists a zig-zag of quasi-isomorphisms between them). Show that if $\mathrm{M}_\bullet$ is projective, then there exists a quasi-isomorphism $\mathrm{M}_\bullet \to \mathrm{N}_\bullet$. Give an example to show that this is not true if $\mathrm{M}_\bullet$ is not projective.

3. Let A be a commutative ring and let $\mathrm{M}_\bullet \to \mathrm{N}_\bullet \to \mathrm{K}_\bullet$ be an exact triangle of chain complexes of A-modules. Show that if any two of the terms are perfect, so is the third.

4. Let A be a commutative ring. A chain complex $\mathrm{M}_\bullet$ is called *connective* if it is 0-connective, i.e., $\mathrm{H}_i(\mathrm{M}_\bullet) = 0$ for $i < 0$. Imitate the construction of $\mathrm{K}_0(\mathrm{Perf}_\mathrm{A})$ to define a variant $\mathrm{K}_0(\mathrm{Perf}^{\mathrm{cn}}_\mathrm{A})$ using (quasi-isomorphism classes of) connective perfect complexes. Show that there is a canonical isomorphism

$$\mathrm{K}_0(\mathrm{Perf}^{\mathrm{cn}}_\mathrm{A}) \xrightarrow{\sim} \mathrm{K}_0(\mathrm{Perf}_\mathrm{A}).$$