Exercise sheet 7

1. (i) Let \( F : \mathcal{A} \to \mathcal{A}' \) be an exact functor between abelian categories. Show that \( \operatorname{Ker}(F) \subseteq \mathcal{A} \), the full subcategory spanned by objects \( A \) such that \( F(A) \cong 0 \), is a Serre subcategory.

(ii) Let \( F : \mathcal{A} \to \mathcal{A}' \) be an exact functor between abelian categories. Suppose that \( F \) admits a right adjoint \( G \) such that the co-unit transformation \( FG \to \text{id} \) is invertible (equivalently, \( G \) is fully faithful). Show that there is a canonical equivalence

\[ \mathcal{A}/\operatorname{Ker}(F) \to \mathcal{A}'. \]

(iii) Let \( \mathcal{A} \) be an abelian category and \( \mathcal{B} \subseteq \mathcal{A} \) a Serre subcategory. Let \( \mathcal{A}_0 \subseteq \mathcal{A} \) be a full subabelian subcategory such that if \( A \in \mathcal{A}_0 \) and \( B \in \mathcal{B} \) is a subobject or quotient of \( A \) then also \( B \in \mathcal{A}_0 \). Show that the canonical functor

\[ \mathcal{A}_0/(\mathcal{B} \cap \mathcal{A}_0) \to \mathcal{A}/\mathcal{B} \]

is fully faithful.

(i) Let \( 0 \to X' \to X \to X'' \to 0 \) be a short exact sequence in \( \mathcal{A} \). Since \( F \) is exact, \( 0 \to F(X') \to F(X) \to F(X'') \to 0 \) is still exact. Thus \( F(X) \cong 0 \) iff \( F(X') \cong 0 \) and \( F(X'') \cong 0 \).

(ii) We show that the functor \( F : \mathcal{A} \to \mathcal{A}' \) satisfies the universal property of the quotient \( \gamma : \mathcal{A} \to \mathcal{A}/\operatorname{Ker}(F) \).

\[ \xymatrix{ \mathcal{A} \ar[r]^-\alpha \ar[d]^-F & \mathcal{B} \\
\mathcal{A}' \ar@{-->}[ru]^-\beta } \]

Thus, let \( \alpha : \mathcal{A} \to \mathcal{B} \) be a functor such that \( \alpha(\operatorname{Ker}(F)) = 0 \), i.e., \( \operatorname{Ker}(F) \subseteq \operatorname{Ker}(\alpha) \). We first note that if \( \beta \) exists (making the diagram commute), then we have a canonical isomorphism

\[ \beta \simeq \beta F \simeq \alpha G, \]

where the first isomorphism is induced by the co-unit \( \cong \) \( FG \). Thus \( \beta \) is unique up to isomorphism if it exists. For existence, it will suffice to show that the only possible candidate \( \beta := \alpha G \) does satisfy \( \beta F \simeq \alpha \). For an object \( X \in \mathcal{A} \), consider the unit morphism \( \eta_X : X \to GF(X) \) and let \( K \) be its kernel. The triangle identities for the adjunction \( (F, G) \) imply that the composite

\[ F(X) \xrightarrow{F(\eta_X)} FGF(X) \xrightarrow{\varepsilon_{F(X)}} F(X) \]

is the identity, where \( \varepsilon \) is the co-unit. Since the latter is invertible by assumption, so is \( F(\eta_X) : F(X) \to FGF(X) \). In particular \( F(K) \cong 0 \), hence also \( \alpha(K) \cong 0 \) by the assumption on \( \alpha \). The same argument applies to the cokernel so we find that \( \alpha(\eta_X) \) is an isomorphism \( \beta F(X) = \alpha GF(X) \cong \alpha(X) \). The argument is natural in \( X \) so we get an isomorphism of functors \( \beta F \simeq \alpha \) as desired.
(iii) [Thanks to V. Sosnilo for this argument.] The existence of the functor comes from the universal property: the inclusion functor \( A_0 \to A \) clearly sends \( B \cap A_0 \) to \( B \). For objects \( X \) and \( Y \) of \( A_0 \), we need to show that the map

\[
\text{Hom}_{A_0/(B \cap A_0)}(X, Y) \to \text{Hom}_{A/B}(X, Y)
\]

is bijective. An element of the target can be represented by a zig-zag in \( A \):

\[
\begin{array}{ccc}
& X & \\
Z & f & g \\
& Y & \\
\end{array}
\]

where \( f \) is a \( B \)-isomorphism, i.e., \( \ker(f) \) and \( \coker(f) \) are contained in \( B \). This represents the morphism \( g \circ f^{-1} : X \to Y \) in \( A/B \).

Consider the short exact sequence

\[
0 \to \text{Im}(f) \to X \to \coker(f) \to 0.
\]

Since \( X \in A_0 \) and \( \coker(f) \in B \), the assumption implies \( \coker(f) \in A_0 \) and hence also \( Z/\ker(f) = \text{Im}(f) \in A_0 \). Consider the commutative diagram of short exact sequences

\[
\begin{array}{cccc}
0 & \to & \ker(f) & \to Z & \to Z/\ker(f) & \to 0 \\
0 & \downarrow & \downarrow g & \downarrow \overline{g} & & \\
0 & \to & g(\ker(f)) & \to Y & \to Y/g(\ker(f)) & \to 0
\end{array}
\]

Since \( \ker(f) \in B \), also \( g(\ker(f)) \in B \) since \( B \) is a Serre subcategory. Since \( Y \in A_0 \) it also follows by the assumption that \( g(\ker(f)) \in A_0 \). Since \( A_0 \) is abelian, then \( Y' := Y/g(\ker(f)) \) is also in \( A_0 \). In particular, \( q \) is a \( (B \cap A_0) \)-isomorphism between objects of \( A_0 \). The commutative diagram

\[
\begin{array}{ccc}
& Z & \\
X & f & \overline{g} \\
& Z/\ker(f) & \\
\end{array}
\]

exhibits an equivalence between the two zig-zags \( X \leftarrow Z \to Y' \) and \( X \leftarrow Z/\ker(f) \to Y' \). In particular, \( qg f^{-1} \) and \( \overline{g} f^{-1} \) represent the same morphism \( X \to Y' \) in \( A/B \). It follows that \( q^{-1} qg f^{-1} = g f^{-1} \) and \( q^{-1} \overline{g} f^{-1} \) represent the same morphism \( X \to Y \) in \( A/B \). In other words, the zig-zags \( X \leftarrow Z/\ker(f) \to Y' \) and \( Y' \leftarrow Y \to Y \) both represent morphisms in \( A_0/(A_0 \cap B) \) which compose to a morphism whose image in \( A/B \) is equivalent to our original morphism \( g \circ f^{-1} \).

2. Let \( A \) be a ring and \( f \in A \) an element.
(i) Let $\text{Mod}_A(f^\infty) \subseteq \text{Mod}_A$ denote the full subcategory of $A$-modules $M$ that are $f^\infty$-torsion (i.e., for every $x \in M$, $f^kx = 0$ for $k \gg 0$). Show that this is a Serre subcategory and that the canonical functor

$$\text{Mod}_A/\text{Mod}_A(f^\infty) \to \text{Mod}_A[f^{-1}]$$

is an equivalence.

(ii) Assume that $A$ is noetherian. Show that the canonical functor

$$\text{Mod}_{fg,A}^{fg}/\text{Mod}_{fg,A}^{fg}(f^\infty) \to \text{Mod}_{fg,A}^{fg}[f^{-1}]$$

is fully faithful.

(iii) Let $B = A[f^{-1}]$. Show that every f.g. $B$-module $N$ lifts to a f.g. $A$-module $M$ such that $M \otimes_A B \simeq N$. Deduce that the canonical functor

$$\text{Mod}_{fg,A}^{fg}/\text{Mod}_{fg,A}^{fg}(f^\infty) \to \text{Mod}_{fg,A}^{fg}[f^{-1}]$$

is an equivalence. (Hint: consider $N_{[A]} \in \text{Mod}_A$, which may not be f.g. However you can find a surjection $A^{\oplus(I)} \to N_{[A]}$ from a free $A$-module indexed on a (possibly infinite) set $I$...)

(i) Consider the exact functor

$$(-) \otimes_A A[f^{-1}] : \text{Mod}_A \to \text{Mod}_A[f^{-1}].$$

Its kernel consists of $A$-modules $M$ such that $M[f^{-1}] = 0$, or equivalently, $M$ is $f^\infty$-torsion. In other words, this is the full subcategory $\text{Mod}_A(f^\infty)$. Thus by Exercise 1(i), the latter is a Serre subcategory. Recall that $(-) \otimes_A A[f^{-1}]$ is left adjoint to the restriction of scalars functor $(-)|_{[A]}$. The latter is fully faithful (note that $A[f^{-1}] \otimes_A A[f^{-1}] \simeq A[f^{-1}]$ and then argue as in the proof that restriction of scalars along $A \to A/I$ is fully faithful, §1.2). Now the claim follows from Exercise 1(ii).

(ii) We want to apply Exercise 1(iii) to the Serre subcategory $\text{Mod}_A(f^\infty) \subseteq \text{Mod}_A$ and the subcategory $\text{Mod}_{fg,A}^{fg} \subseteq \text{Mod}_A$. The condition is that if $M \in \text{Mod}_{fg,A}^{fg}$ and $N \in \text{Mod}_A(f^\infty)$ is a subobject or quotient of $M$, then $N$ is also f.g. This is clear since $A$ is noetherian. Thus Exercise 1(iii) yields that

$$\text{Mod}_{fg,A}^{fg}/\text{Mod}_{fg,A}^{fg}(f^\infty) \to \text{Mod}_A/\text{Mod}_A(f^\infty)$$

is fully faithful. By (i) the target is equivalent to $\text{Mod}_A[f^{-1}]$, so we have shown that

$$\text{Mod}_{fg,A}^{fg}/\text{Mod}_{fg,A}^{fg}(f^\infty) \to \text{Mod}_A[f^{-1}]$$

is fully faithful. But this functor lands in the full subcategory $\text{Mod}_{fg,A}^{fg}[f^{-1}]$ and the induced functor

$$\text{Mod}_{fg,A}^{fg}/\text{Mod}_{fg,A}^{fg}(f^\infty) \to \text{Mod}_{fg,A}^{fg}[f^{-1}]$$

must then also be fully faithful.

(iii) Consider the $A$-module $N_{[A]}$. We can find a surjection $\phi : A^{\oplus(I)} \to N_{[A]}$ from a free $A$-module indexed on a (possibly infinite) set $I$ (for example, take $I$ to be the set of elements of $N$). These correspond to elements $\phi_i \in N$ for $i \in I$. Since
N_{[A]} \otimes_A B \simeq N \text{ is f.g., we know that there exists a finite subset } J \subset I \text{ such that the induced map } B^{\oplus(J)} \to N \text{ is surjective. Let } M \subset N_{[A]} \text{ be the image of the map } A^{\oplus(J)} \to N_{[A]}. \text{ It is then f.g. and satisfies } M \otimes_A B \simeq N \text{ by construction. This shows that the functor in question is essentially surjective, and it was already shown to be fully faithful in part (ii).}

3. Let A be a noetherian ring.

(i) Show that \( \phi : A \to A[T] \) induces an injective homomorphism

\[ \phi^* : G_0(A) \to G_0(A[T]). \]

(Hint: Note that \( \phi \) admits a retraction in the category of commutative rings...)

(ii) If A is a field \( k \), show that \( \phi^* : G_0(k) \to G_0(k[T]) \) is an isomorphism.

(i) Note that \( \phi : A \to A[T] \) is flat and in particular of finite Tor-amplitude. Therefore there is a well-defined homomorphism \( \phi^* : G_0(A) \to G_0(A[T]) \) (see §6.3). Let \( \sigma : A[T] \to A \) be the ring homomorphism \( T \mapsto 0 \). Since \( \sigma \circ \phi = \text{id} \), we have (see §6.3)

\[ \sigma^* \phi^* = \text{id} : G_0(A) \to G_0(A). \]

In particular, \( \phi^* \) is injective.

(ii) It remains to show that \( \phi^* \) is surjective. Note that we have a commutative square

\[
\begin{array}{ccc}
K_0(k) & \xrightarrow{\phi^*} & K_0(k[T]) \\
\downarrow & & \downarrow \\
G_0(k) & \xrightarrow{\phi^*} & G_0(k[T]).
\end{array}
\]

Since \( k \) and \( k[T] \) are regular rings (see §2.3 in the lecture), the vertical arrows are invertible. The upper horizontal arrow is also invertible: for both \( k \) and \( k[T] \), every f.g. projective module is free, so the map is identified with the identity \( \text{id} : \mathbb{Z} \to \mathbb{Z} \). It follows that the lower horizontal arrow is also invertible.

4. Let A be an integral domain. Given an element \( f \in A \) and a point \( p \in |\text{Spec}(A)| \), the value of \( f \) at \( p \), denoted \( f(p) \), is the image of \( f \) by the homomorphism \( \phi : A \to \kappa(p) \) (Elements of A are thought of as “algebraic functions” on Spec(A).)

(i) Show that if an element \( f \) vanishes at the generic point \( \eta \) then \( f = 0 \).

(ii) Give an example to show that if A is not an integral domain, then an element \( f \in A \) can vanish at every point without being zero.

(Use the definition of \( |\text{Spec}(A)| \) given in the lecture, not the one using prime ideals.)

(i) Since \( A \hookrightarrow \kappa(\eta) = \text{Frac}(A) \) is injective, we have \( f(\eta) = 0 \) iff \( f = 0 \).
(ii) Consider the ring of dual numbers $A = k[\varepsilon]/\langle \varepsilon^2 \rangle$ over a field $k$. Recall that $A$ has a single point $p = [A \rightarrow k]$. The element $\varepsilon \in A$ has value $\varepsilon(p) = 0$ at this point, but $\varepsilon \neq 0$. 