Exercise sheet 9

1. Let A be a DVR with uniformizing parameter π (i.e., π is a generator of the maximal ideal).

(a) Show that A is of dimension 1.

(b) Deduce that $\dim(A[T]) \geqslant 2$.[1]

(c) Show that the ideal of A[T] generated by the element $f = \pi T - 1 \in A[T]$ is maximal.

(d) Let $Y = V(\langle f \rangle) \subset |\mathrm{Spec}(A[T])|$. Show that $\dim(A[T]) \neq \dim(Y) + \mathrm{codim}_{A[T]}(Y)$.

(e) For a general noetherian ring A and any closed subset $Y \subseteq |\mathrm{Spec}(A)|$, show that $\dim(Y) + \mathrm{codim}_A(Y) \leqslant \dim(A)$.

(a) Note that $|\mathrm{Spec}(A)|$ is integral (since A is an integral domain) and $V(\langle \pi \rangle)$ is the unique closed point (since $\langle \pi \rangle$ is the unique maximal ideal). Therefore,

$$\varnothing \subsetneq V(\langle \pi \rangle) \subsetneq V(\langle 0 \rangle) = |\mathrm{Spec}(A)|$$

is a chain of integral subsets of $|\mathrm{Spec}(A)|$. It is maximal because if $V(\mathfrak{p})$ is a proper integral subset containing $V(\langle \pi \rangle)$, then $\mathfrak{p}$ is a nonzero prime ideal contained in $\langle \pi \rangle$. Since A is a principal ideal domain, $\mathfrak{p} = \langle f \rangle$ for some nonzero prime element f. Since PIDs are factorial, f is irreducible and $f \in \langle \pi \rangle$ implies $\langle f \rangle = \langle \pi \rangle$. (To summarize: the maximal ideal is the unique nonzero prime ideal of a DVR.)

(b) We claim that, for any ring A (possibly even non-noetherian), we have $\dim(A[T]) \geqslant \dim(A) + 1$. Indeed, suppose

$$\varnothing \subsetneq V(\mathfrak{p}_0) \subsetneq V(\mathfrak{p}_1) \subsetneq \cdots \subsetneq V(\mathfrak{p}_n)$$

is a maximal chain of integral closed subsets of $|\mathrm{Spec}(A)|$. Then each extension $\mathfrak{q}_i := \mathfrak{p}_i A[T]$ is a prime ideal of A[T] (since $A[T]/\mathfrak{p}_i A[T] \simeq A/\mathfrak{p}_i[T]$ is an integral domain). Set $\mathfrak{r} := \mathfrak{q}_0 + \langle T \rangle$. This is also a prime ideal of A[T] since $A[T]/\mathfrak{r} \simeq A/\mathfrak{p}_0$, and we have $\mathfrak{q}_0 \subsetneq \mathfrak{r}$. Thus

$$\varnothing \subsetneq V(\mathfrak{r}) \subsetneq V(\mathfrak{q}_0) \subsetneq V(\mathfrak{q}_1) \subsetneq \cdots \subsetneq V(\mathfrak{q}_n)$$

is a chain of integral subsets of $|\mathrm{Spec}(A[T])|$.

(c) Note that $A[\pi^{-1}]$ is the fraction field of A. Consider the unique A-algebra homomorphism $\phi : A[T] \to A[\pi^{-1}]$ sending $T \mapsto 1/\pi$. Then clearly ϕ is surjective, and its kernel is the ideal $\langle \pi T - 1 \rangle$: for any polynomial $f \in A[T]$, we may write $f = g \cdot (\pi T - 1) + r$ where $g \in A[T]$ and $r \in A$ (division algorithm), and then $\phi(f) = f(1/\pi) = r$. It follows that ϕ induces an isomorphism $A[T]/\langle \pi T - 1 \rangle \simeq A[\pi^{-1}]$, and in particular the ideal $\langle \pi T - 1 \rangle$ is maximal.

[1] In fact, one has $\dim(A[T]) = \dim(A) + 1$ for any noetherian ring A, but this is non-trivial; see e.g. [Bourbaki, Comm. alg., §3, no. 4, Cor. 3 to Prop. 7].

(d) By (b) we have $\dim(A[T]) \geqslant 2$. By (c), Y is a closed point, so $\dim(Y) = 0$. But $\operatorname{codim}(Y) = 1$ by Krull's principal ideal theorem (since f is a non-zero-divisor as A[T] is an integral domain).

(e) Easy from the definitions.

2. Let A be a noetherian ring.

(a) The codimension of any integral closed subset $V(\mathfrak{p}) \subset |\operatorname{Spec}(A)|$ is given by

$$\operatorname{codim}_A(V(\mathfrak{p})) = \dim(A_{\mathfrak{p}}).$$

(b) Show that the dimension of A is given by the formula

$$\dim(A) = \sup_x \operatorname{codim}_A(\{x\}),$$

where the supremum is taken over all closed points x of $|\operatorname{Spec}(A)|$.

3. Let A be a noetherian ring. Define a homomorphism

$$\gamma_A : Z_*(A) \to G_0(A)$$

by sending the class of an integral subset $V(\mathfrak{p})$ to the class $[A/\mathfrak{p}]$.

(a) Let k be an algebraically closed field and $A = k[T, U]$. Show that γ_A descends to a homomorphism

$$\gamma_A : CH_*(A) \to G_0(A)$$

which is invertible.

(b) Let A be any noetherian ring and $\phi : A \twoheadrightarrow A/I$ a surjective ring homomorphism. Show that the square

$$\begin{array}{ccc} Z_*(A/I) & \xrightarrow{\phi_*} & Z_*(A) \\ \downarrow{\scriptstyle \gamma_{A/I}} & & \downarrow{\scriptstyle \gamma_A} \\ G_0(A/I) & \xrightarrow{\phi_*} & G_0(A) \end{array}$$

commutes.

(a) Let's first describe $Z_k(A)$ for all k. Since A is of dimension 2, $Z_k(A) = 0$ for $k \geqslant 3$. Since A is an integral domain and hence irreducible, $Z_2(A)$ is free abelian on the generator $V(\langle 0 \rangle) = |\operatorname{Spec}(A)|$. By definition $Z_1(A)$ is free abelian on the generators $V(\mathfrak{p})$, integral closed subsets of dimension 1, or equivalently[2] of codimension 1. Finally $Z_0(A)$ is free abelian on integral closed subsets of dimension 0, i.e., closed points of $|\operatorname{Spec}(A)|$, which are in bijection with pairs $(x_1, x_2) \in k^2$ since k is algebraically closed (Sheet 8, Exercise 1).

Note that $R_2(A) = 0$ since there are no 3-dimensional closed subsets of $|\operatorname{Spec}(A)|$, so $CH_2(A) \simeq \mathbf{Z}$.

[2]Though the equality $\operatorname{codim}(V(\mathfrak{p})) + \dim(V(\mathfrak{p})) = \dim(A)$ does not hold in general (see Sheet 8, Exercise 1), it does hold when A is an integral domain of finite type over a field k.

Since A is factorial, given $[V(\mathfrak{p})] \in Z_1(A)$, we may write $\mathfrak{p} = \langle f \rangle$, for some (nonzero) element $f \in A$, by the Lemma in the proof of Sheet 8, Exercise 3. Then we have $\mathrm{div}_{V(0)}(f) = [A/\langle f \rangle]_1 = [V(\langle f \rangle)] = [V(\mathfrak{p})]$ in $Z_1(A)$. Thus every $[V(\mathfrak{p})] \in Z_1(A)$ is rationally equivalent to zero, and $CH_1(A) \simeq 0$.

Take an element $[V(\mathfrak{m})] \in Z_0(A)$, corresponding to a pair $(x_1, x_2) \in k^2$ (so that $\mathfrak{m} = \langle T - x_1, U - x_2 \rangle$). Then we have

$$\begin{aligned}\mathrm{div}_{V(\langle T - x_1 \rangle)}(U - x_2) &= [(A/(T - x_1))/\langle U - x_2 \rangle]_0 \\ &= [A/\mathfrak{m}]_0 = [V(\mathfrak{m})].\end{aligned}$$

Thus $CH_0(A) \simeq 0$.

Using these descriptions of the subgroups $R_k(A)$, it is easy to check that $\gamma_A : Z_k(A) \to G_0(A)$ sends $R_k(A)$ to zero. To show that the induced map $CH_k(A) \to G_0(A)$ is bijective, we can use homotopy invariance of G-theory to observe that $G_0(A) \simeq G_0(k) \simeq \mathbf{Z}$ is free abelian on the single generator $[A]$ (which is the image of $[V(0)] \in CH_2(A)$.)

(b) Let $V_{A/I}(\mathfrak{p})$ be an integral closed subset of $|\mathrm{Spec}(A/I)|$ (where $\mathfrak{p}$ is a prime ideal of A/I). Under the bijection $|\mathrm{Spec}(A/I)| \simeq V_A(I)$, it corresponds to the integral closed subset $V_A(\mathfrak{q})$, where $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$ is the contraction of $\mathfrak{p}$. Therefore the clockwise composite sends $[V(\mathfrak{p})]$ to

$$\gamma_A \phi_* [V_{A/I}(\mathfrak{p})] = \gamma_A [V_A(\mathfrak{q})] = [A/\mathfrak{q}].$$

Since $(A/I)/\mathfrak{p} \simeq A/\mathfrak{q}$, the counter-clockwise composite is given by

$$\phi_* \gamma_{A/I} [V_{A/I}(\mathfrak{p})] = \phi_* [(A/I)/\mathfrak{p}] = [A/\mathfrak{q}].$$

4. Let A be a noetherian ring and let $V(\mathfrak{p})$ and $V(\mathfrak{q})$ be distinct integral closed subsets of $|\mathrm{Spec}(A)|$, both of dimension d. Prove the formula

$$[A/(\mathfrak{p} \cap \mathfrak{q})]_d = [V(\mathfrak{p})] + [V(\mathfrak{q})]$$

in $CH_d(A)$.

The construction $[-]_d$ is additive in short exact sequences. Apply this to the short exact sequence

$$0 \to A/(\mathfrak{p} \cap \mathfrak{q}) \to A/\mathfrak{p} \oplus A/\mathfrak{q} \to A/(\mathfrak{p} + \mathfrak{q}) \to 0.$$

Observe that $[A/(\mathfrak{p} + \mathfrak{q})]_d = 0$ since $\mathrm{Supp}_A(A/(\mathfrak{p} + \mathfrak{q})) = V(\mathfrak{p} + \mathfrak{q}) = V(\mathfrak{p}) \cap V(\mathfrak{q})$ is of dimension strictly less than d (since $V(\mathfrak{p})$ and $V(\mathfrak{q})$ are distinct).