1. Let \( A \) be a DVR with uniformizing parameter \( \pi \) (i.e., \( \pi \) is a generator of the maximal ideal).

(a) Show that \( A \) is of dimension 1.
(b) Deduce that \( \dim(A[T]) \geq 2. \)
(c) Show that the ideal of \( A[T] \) generated by the element \( f = \pi T - 1 \in A[T] \) is maximal.
(d) Let \( Y = V(\langle f \rangle) \subset \text{Spec}(A[T]) \). Show that \( \dim(A[T]) \neq \dim(Y) + \text{codim}_{A[T]}(Y) \).
(e) For a general noetherian ring \( A \) and any closed subset \( Y \subseteq \text{Spec}(A) \), show that \( \dim(Y) + \text{codim} A(Y) \leq \dim(A) \).

(a) Note that \( \text{Spec}(A) \) is integral (since \( A \) is an integral domain) and \( V(\langle \pi \rangle) \) is the unique closed point (since \( \langle \pi \rangle \) is the unique maximal ideal). Therefore,
\[
\emptyset \subsetneq V(\langle \pi \rangle) \subseteq V(\langle 0 \rangle) = |\text{Spec}(A)|
\]
is a chain of integral subsets of \( |\text{Spec}(A)| \). It is maximal because if \( V(p) \) is a proper integral subset containing \( V(\langle \pi \rangle) \), then \( p \) is a nonzero prime ideal contained in \( \langle \pi \rangle \). Since \( A \) is a principal ideal domain, \( p = \langle f \rangle \) for some nonzero prime element \( f \). Since PIDs are factorial, \( f \) is irreducible and \( f \in \langle \pi \rangle \) implies \( \langle f \rangle = \langle \pi \rangle \). (To summarize: the maximal ideal is the unique nonzero prime ideal of a DVR.)

(b) We claim that, for any ring \( A \) (possibly even non-noetherian), we have \( \dim(A[T]) \geq \dim(A) + 1 \). Indeed, suppose
\[
\emptyset \subsetneq V(p_0) \subsetneq V(p_1) \subsetneq \cdots \subsetneq V(p_n)
\]
is a maximal chain of integral closed subsets of \( |\text{Spec}(A)| \). Then each extension \( q_i := p_i A[T] \) is a prime ideal of \( A[T] \) (since \( A[T]/p_i A[T] \simeq A/p_i [T] \) is an integral domain). Set \( r := q_0 + \langle T \rangle \). This is also a prime ideal of \( A[T] \) since \( A[T]/r \simeq A/p_0 \), and we have \( q_0 \nsubseteq r \). Thus
\[
\emptyset \subsetneq V(r) \subsetneq V(q_0) \subsetneq V(q_1) \subsetneq \cdots \subsetneq V(q_n)
\]
is a chain of integral subsets of \( |\text{Spec}(A[T])| \).

(c) Note that \( A[\pi^{-1}] \) is the fraction field of \( A \). Consider the unique \( A \)-algebra homomorphism \( \phi : A[T] \to A[\pi^{-1}] \) sending \( T \mapsto 1/\pi \). Then clearly \( \phi \) is surjective, and its kernel is the ideal \( \langle \pi T - 1 \rangle \): for any polynomial \( f \in A[T] \), we may write \( f = g \cdot (\pi T - 1) + r \) where \( g \in A[T] \) and \( r \in A \) (division algorithm), and then \( \phi(f) = f(1/\pi) = r \). It follows that \( \phi \) induces an isomorphism \( A[T]/\langle \pi T - 1 \rangle \simeq A[\pi^{-1}] \), and in particular the ideal \( \langle \pi T - 1 \rangle \) is maximal.

\[1\]In fact, one has \( \dim(A[T]) = \dim(A) + 1 \) for any noetherian ring \( A \), but this is non-trivial; see e.g. [Bourbaki, Comm. alg., §3, no. 4, Cor. 3 to Prop. 7].
(d) By (b) we have dim(A[T]) ≥ 2. By (c), Y is a closed point, so dim(Y) = 0. But codim(Y) = 1 by Krull’s principal ideal theorem (since f is a non-zero-divisor as A[T] is an integral domain).
(e) Easy from the definitions.

2. Let A be a noetherian ring.
(a) The codimension of any integral closed subset V(p) ⊂ Spec(A) is given by codim_A(V(p)) = dim(A_p).
(b) Show that the dimension of A is given by the formula
\[ \dim(A) = \sup_x \text{codim}_A(\{x\}), \]
where the supremum is taken over all closed points x of |Spec(A)|.

3. Let A be a noetherian ring. Define a homomorphism
\[ \gamma_A : \mathbb{Z}^*(A) \to G_0(A) \]
by sending the class of an integral subset V(p) to the class [A/p].
(a) Let k be an algebraically closed field and A = k[T, U]. Show that γ_A descends to a homomorphism
\[ \gamma_A : \text{CH}_*(A) \to G_0(A) \]
which is invertible.
(b) Let A be any noetherian ring and φ : A ↠ A/I a surjective ring homomorphism. Show that the square
\[ \begin{array}{ccc}
\mathbb{Z}^*(A/I) & \xrightarrow{\phi_*} & \mathbb{Z}^*(A) \\
\downarrow{\gamma_{A/I}} & & \downarrow{\gamma_A} \\
G_0(A/I) & \xrightarrow{\phi_*} & G_0(A)
\end{array} \]
commutes.
(a) Let’s first describe Z_k(A) for all k. Since A is of dimension 2, Z_k(A) = 0 for k ≥ 3. Since A is an integral domain and hence irreducible, Z_2(A) is free abelian on the generator V((0)) = |Spec(A)|. By definition Z_1(A) is free abelian on the generators V(p), integral closed subsets of dimension 1, or equivalently\(^2\) of codimension 1. Finally Z_0(A) is free abelian on integral closed subsets of dimension 0, i.e., closed points of |Spec(A)|, which are in bijection with pairs (x_1, x_2) ∈ k^2 since k is algebraically closed (Sheet 8, Exercise 1).
Note that R_2(A) = 0 since there are no 3-dimensional closed subsets of |Spec(A)|, so CH_2(A) ≃ Z.

\(^2\)Though the equality codim(V(p)) + dim(V(p)) = dim(A) does not hold in general (see Sheet 8, Exercise 1), it does hold when A is an integral domain of finite type over a field k.
Since $A$ is factorial, given $[V(p)] \in Z_1(A)$, we may write $p = \langle f \rangle$, for some (nonzero) element $f \in A$, by the Lemma in the proof of Sheet 8, Exercise 3. Then we have $\text{div}_{V(0)}(f) = [A/(\langle f \rangle)]_1 = [V(\langle f \rangle)] = [V(p)]$ in $Z_1(A)$. Thus every $[V(p)] \in Z_1(A)$ is rationally equivalent to zero, and $\text{CH}_1(A) \simeq 0$.

Take an element $[V(m)] \in Z_0(A)$, corresponding to a pair $(x_1, x_2) \in k^2$ (so that $m = \langle T - x_1, U - x_2 \rangle$). Then we have $\text{div}_{V(\langle T - x_1 \rangle)}(U - x_2) = [(A/(T - x_1))/(U - x_2)]_0 = [A/m]_0 = [V(m)]$.

Thus $\text{CH}_0(A) \simeq 0$.

Using these descriptions of the subgroups $R_k(A)$, it is easy to check that $\gamma_A : Z_k(A) \to G_0(A)$ sends $R_k(A)$ to zero. To show that the induced map $\text{CH}_k(A) \to G_0(A)$ is bijective, we can use homotopy invariance of $G$-theory to observe that $G_0(A) \simeq G_0(k) \simeq \mathbb{Z}$ is free abelian on the single generator $[A]$ (which is the image of $[V(0)] \in \text{CH}_2(A)$).

4. Let $A$ be a noetherian ring and let $V(p)$ and $V(q)$ be distinct integral closed subsets of $|\text{Spec}(A)|$, both of dimension $d$. Prove the formula

$$[A/(p \cap q)]_d = [V(p)] + [V(q)]$$

in $\text{CH}_d(A)$.

The construction $[-]_d$ is additive in short exact sequences. Apply this to the short exact sequence

$$0 \to A/(p \cap q) \to A/p \oplus A/q \to A/(p + q) \to 0.$$ 

Observe that $[A/(p + q)]_d = 0$ since $\text{Supp}_A(A/(p + q)) = V(p + q) = V(p) \cap V(q)$ is of dimension strictly less than $d$ (since $V(p)$ and $V(q)$ are distinct).