Exercise sheet 9

1. Let A be a DVR with uniformizing parameter π (i.e., π is a generator of the maximal ideal).

(a) Show that A is of dimension 1.

(b) Deduce that $\dim(A[T]) \geqslant 2$.[1]

(c) Show that the ideal of A[T] generated by the element $f = \pi T - 1 \in A[T]$ is maximal.

(d) Let $Y = V(\langle f\rangle) \subset |\mathrm{Spec}(A[T])|$. Show that $\dim(A[T]) \neq \dim(Y) + \mathrm{codim}_{A[T]}(Y)$.

(e) For a general noetherian ring A and any closed subset $Y \subseteq |\mathrm{Spec}(A)|$, show that $\dim(Y) + \mathrm{codim}_A(Y) \leqslant \dim(A)$.

2. Let A be a noetherian ring.

(a) The codimension of any integral closed subset $V(\mathfrak{p}) \subset |\mathrm{Spec}(A)|$ is given by

$$\mathrm{codim}_A(V(\mathfrak{p})) = \dim(A_{\mathfrak{p}}).$$

(b) Show that the dimension of A is given by the formula

$$\dim(A) = \sup_x \mathrm{codim}_A(\{x\}),$$

where the supremum is taken over all closed points x of $|\mathrm{Spec}(A)|$.

3. Let A be a noetherian ring. Define a homomorphism

$$\gamma_A : Z_*(A) \to G_0(A)$$

by sending the class of an integral subset $V(\mathfrak{p})$ to the class $[A/\mathfrak{p}]$.

(a) Let k be an algebraically closed field and $A = k[T, U]$. Show that γ_A descends to a homomorphism

$$\gamma_A : CH_*(A) \to G_0(A)$$

which is invertible.

(b) Let A be any noetherian ring and $\phi : A \twoheadrightarrow A/I$ a surjective ring homomorphism. Show that the square

$$\begin{array}{ccc} Z_*(A/I) & \xrightarrow{\phi_*} & Z_*(A) \\ \downarrow{\scriptstyle \gamma_{A/I}} & & \downarrow{\scriptstyle \gamma_A} \\ G_0(A/I) & \xrightarrow{\phi_*} & G_0(A) \end{array}$$

commutes.

[1] In fact, one has $\dim(A[T]) = \dim(A) + 1$ for any noetherian ring A, but this is non-trivial; see e.g. [Bourbaki, Comm. alg., §3, no. 4, Cor. 3 to Prop. 7].

4. Let A be a noetherian ring and let $\mathrm{V}(\mathfrak{p})$ and $\mathrm{V}(\mathfrak{q})$ be distinct integral closed subsets of $\left|\mathrm{Spec}(\mathrm{A})\right|$, both of dimension d. Prove the formula

$$[\mathrm{A}/(\mathfrak{p} \cap \mathfrak{q})]_d = [\mathrm{V}(\mathfrak{p})] + [\mathrm{V}(\mathfrak{q})]$$

in $\mathrm{CH}_d(\mathrm{A})$.