
Remark 1. Let $k$ be a field and let $A$ and $B$ be finite type $k$-algebras. If $k$ is perfect, then regularity of $A$ and $B$ implies regularity of $A \otimes_k B$. In general, this is not the case: in fact, $A \otimes_k B$ need not even be reduced. Consider for example the imperfect field $k = F_p(T)$ and the algebras $A = B = k((T^{1/p})) = k[X]/(X^p - T)$.

We introduce a closely related but slightly stronger notion, which will still be closed under tensor products.

Definition 2. Let $k$ be a field and $\overline{k}$ an algebraic closure. We say that a finite type $k$-algebra $A$ is smooth if its extension of scalars $A \otimes_k \overline{k}$ is a regular ring.

Proposition 3. Let $A$ be a $k$-algebra of finite type. If $A$ is smooth, then it is regular. If $k$ is perfect, then the converse also holds.

Definition 4. A homomorphism $\phi : A \to B$ of finite type $k$-algebras is smooth (of relative dimension $d$) if it is flat, and for every closed point $x \in |\text{Spec}(A)|$, $B \otimes_A \kappa(x)$ is a smooth $\kappa(x)$-algebra (of dimension $d$).

Example 5. For every $k$-algebra $A$, the homomorphism $A \to A[T_1, \ldots, T_n]$ is smooth of relative dimension $n$. Indeed $A[T_1, \ldots, T_n] \otimes_A \kappa \simeq \kappa[T_1, \ldots, T_n]$ is of dimension $n$ for every residue field $\kappa$.

Proposition 6. (i) Let $A \to A'$ be a homomorphism of finite type $k$-algebras. Then for every smooth homomorphism $\phi : A \to B$ of relative dimension $d$, the extension of scalars $\phi' : A' \to B \otimes_A A'$ is smooth of relative dimension $d$.

(ii) Let $\phi : A \to B$ and $\psi : B \to C$ be smooth homomorphisms of finite type $k$-algebras, of relative dimensions $d$ and $e$, respectively. Then the composite $\psi \circ \phi : A \to C$ is smooth of relative dimension $d + e$.

Corollary 7. For any two smooth $k$-algebras $A$ and $B$, the tensor product $A \otimes_k B$ is smooth of dimension $\dim(A) + \dim(B)$.

Proof. By the proposition, $k \to A \to A \otimes_k B$ is a smooth homomorphism. \qed

10.2. Chow cohomology.

Definition 8. Let $k$ be a field and $A$ a smooth $k$-algebra of finite type. Assume that $|\text{Spec}(A)|$ is of pure dimension $d$. We define $\text{CH}^k(A) := \text{CH}_{d-k}(A)$ for every $k$, and refer to this as the $k$th Chow cohomology group of $A$.

Our next goal will be to define a product on the graded ring $\text{CH}^*(A) := \bigoplus_k \text{CH}^k(A)$, analogous to cup products in singular cohomology.
10.3. Quasi-smooth homomorphisms.

**Definition 9.** Let $\phi : A \to B$ be a surjective ring homomorphism, with kernel $I$. We say that $\phi$ is quasi-smooth (of relative dimension $-n$) if for every point $x \in \text{Spec}(B)$, the ideal $I_{\phi^{-1}(p(x))} \subseteq A_{\phi^{-1}(p(x))}$ is generated by a Koszul-regular sequence (of length $n$), where $p(x) \subseteq B$ is the prime ideal corresponding to $x$.

**Remark 10.** Since we are only working with noetherian rings, $\phi$ is quasi-smooth iff it is a local complete intersection homomorphism (which is the same definition but where “Koszul-regular” is replaced by “regular”).

**Example 11.** For any ring $A$ and element $a \in A$, the $A$-algebra surjection $A[T] \to A, T \mapsto a$, is quasi-smooth of relative dimension $-1$. Indeed the ideal $\langle T - a \rangle$ is clearly generated by a non-zero-divisor.

**Proposition 12.**
(i) Let $A \to A'$ be a flat ring homomorphism. Then for every quasi-smooth surjection $\phi : A \to B$ of relative dimension $-d$, the extension of scalars $\phi' : A' \to B \otimes_A A'$ is a quasi-smooth surjection of relative dimension $-d$.
(ii) Let $\phi : A \to B$ and $\psi : B \to C$ be quasi-smooth surjections of relative dimensions $-d$ and $-e$, respectively. Then the composite $\psi \circ \phi : A \to C$ is quasi-smooth of relative dimension $-(d + e)$.

**Proposition 13.** Let $A \to B$ be a surjective homomorphism of smooth $k$-algebras. Then $A \to B$ is quasi-smooth of relative dimension $-d$, where $d = \dim(A) - \dim(B)$.

**Corollary 14.** Let $A$ be a smooth algebra over a field $k$. Then the canonical ring homomorphism $\delta : A \otimes_k A \to A$ is a quasi-smooth surjection of relative dimension $-d$, where $d = \dim(A)$.

**Proof.** Clearly $\delta$ is surjective (with kernel generated by the elements $a \otimes 1 - 1 \otimes a$, for all $a \in A$). Since we know that $A \otimes_k A$ is a smooth $k$-algebra of dimension $2\dim(A)$, the claim follows from the previous proposition.

10.4. Interlude: more on dimension. We need some more dimension theory. The following statement is a consequence of Krull’s principal ideal theorem.

**Proposition 15.** Let $A$ be a noetherian local ring with maximal ideal $m$. For any element $f \in m$, we have
\[
\dim(A/(f)) \geq \dim(A) - 1
\]
with equality if $f$ is a non-zero-divisor.

The next statement follows from Noether normalization.

**Proposition 16.** Let $k$ be a field and $A$ an integral domain which is of finite type over $k$. Then we have
\[
\dim(A) = \text{trdeg}_k(\text{Frac}(A)),
\]
where trdeg denotes transcendence degree.
Proposition 17. Let $A$ be an integral domain which is of finite type over $k$. Then $\dim(A) = \text{codim}_A(\{x\})$, for any closed point $x \in |\text{Spec}(A)|$.

Proposition 18. Let $A$ and $B$ be finite type $k$-algebras. Then $\dim(A \otimes_k B) = \dim(A) + \dim(B)$.

10.5. Proper intersections.

Lemma 19. Let $k$ be a field and $A$ a finite type $k$-algebra. Let $I$ be an ideal such that $A \to A/I$ is a surjective quasi-smooth homomorphism of relative dimension $-n$. Let $V(p)$ be an irreducible closed subset of dimension $d$. Then each irreducible component of the closed subset $V(I) \cap V(p) = V(I + p)$ is of dimension $\geq d - n$.

Proof. By Exercise 2 on Sheet 9, dimension is “local” in the sense that $\dim(A) = \sup_m \dim(A_m)$ where the supremum is taken over maximal ideals $m$. Therefore, we may as well replace $A$ by $A_m$ to assume that $A$ is a local ring and that the ideal $I$ is generated by a Koszul-regular sequence $(f_1, \ldots, f_n)$ with $f_i \in m$. By the implicit noetherian hypothesis, $(f_1, \ldots, f_n)$ is then a regular sequence.

Let $Y$ be an irreducible component of $V(I + p)$. Let $y \in Y$ be a closed point that is not contained in any other component of $V(I + p)$. Then by §10.4, both $\dim(V(I + p))$ and $\dim(Y)$ are equal to the codimension of $\{y\}$ in $V(I + p)$. The latter is equal to $\dim((A/(I + p))_{p(y)})$ (Sheet 9, Exercise 2), where $p(y)$ is the prime (maximal) ideal corresponding to $y$. Thus we get

$$\dim(Y) = \dim(((A/p)_{p(y)})/(f_1, \ldots, f_n)) = \dim(((A/p)_{p(y)}/(f_1, \ldots, f_n)))$$

The same exercise gives similarly

$$d = \dim(V(p)) = \dim((A/p)_{p(y)}).$$

If $B = (A/p)_{p(y)}$, the claim is now that $\dim(B/(f_1, \ldots, f_n)) \geq \dim(B) - n$. Since the $f_i$ form a regular sequence in $B$, the claim follows by induction from the $n = 1$ case (§10.4).

Proposition 20. Assume the field $k$ is algebraically closed. Let $A$ be a smooth $k$-algebra of dimension $d$. Let $V(p)$ and $V(q)$ be integral closed subsets of $|\text{Spec}(A)|$ of dimensions $m$ and $n$, respectively. Then every irreducible component of $V(p) \cap V(q) = V(p + q)$ is of dimension $\geq m + n - d$.

Proof. Then the canonical map $\delta : A \otimes_k A \to A$ is a quasi-smooth surjection of relative dimension $-d$; let $K$ be its kernel. The ideal $r = p \otimes_k A + A \otimes_k q \subset A \otimes_k A$ corresponds to the subset $V(r) \subset |\text{Spec}(A \otimes_k A)|$, which is integral of dimension $m + n$ since

$$(A \otimes_k A)/(p \otimes_k A + A \otimes_k q) \simeq (A/p) \otimes_k (A/q)$$

is an integral domain of dimension $m + n$ (this requires that $k$ is algebraically closed). Moreover, we have

$$(A \otimes_k A)/(K + r) \simeq A/(p + q).$$
Thus, the claim follows from the previous statement applied to the finite type $k$-algebra $A \otimes_k A$, with quasi-smooth surjection $\delta : A \otimes_k A \to A$ and integral subset $V(\mathfrak{r})$. □

**Remark 21.** The argument used in the proof is called “reduction to the diagonal” (Serre). Geometrically, $\delta : A \otimes_k A \to A$ corresponds to the diagonal morphism $X \to X \times_k X$ of the $k$-scheme $X = \text{Spec}(A)$. The isomorphism $(A \otimes_k A)/(K + \mathfrak{r}) \simeq A/(p + q)$ can be written as

$$(A/p \otimes_k A/q) \otimes_{A \otimes_k A} A \simeq A/(p + q).$$

This corresponds to the formula

$$(Y \times_k Z) \cap \Delta = Y \cap Z,$$

where $Y = V(p)$ and $Z = V(q)$ and $\Delta = V(K) \subset X \times_k X$ is the image of the diagonal immersion.

**Definition 22.** Let $A$ be a finite type $k$-algebra of dimension $d$. Let $Y = V(p)$ and $Z = V(q)$ be integral closed subsets of $|\text{Spec}(A)|$ of dimensions $m$ and $n$, respectively. We say that $Y$ and $Z$ intersect properly, or without excess, if every irreducible component of $Y \cap Z$ is of dimension $\leq m + n - d$. (An excess component is an irreducible component of dimension $> m + n - d$.)

**Remark 23.** If $A$ is smooth and $k$ is algebraically closed, then by the proposition, $Y$ and $Z$ intersect properly iff every irreducible component of $Y \cap Z$ is of dimension exactly $m + n - d$, i.e., iff $Y \cap Z$ is of pure dimension $m + n - d$.

### 10.6. Intersection products.

**Construction 24.** Let $A$ be a smooth $k$-algebra of dimension $d$. Let $Y = V(p)$ and $Z = V(q)$ be integral subsets of $|\text{Spec}(A)|$ of codimensions $p$ and $q$, respectively. Assume that $Y$ and $Z$ intersect properly, so that $Y \cap Z$ is of pure dimension $(d - p) + (d - q) - d = d - (p + q)$. We define the intersection product of $[Y]$ and $[Z]$ as the cycle

$$[V(p)] \cup [V(q)] = \sum_i (-1)^i [\text{Tor}_i^A(A/p, A/q)]_{d-p-q}$$

in $\text{CH}_{d-p-q}(A) \simeq \text{CH}^{p+q}(A)$.

We will not prove the following (difficult) theorem.

**Theorem 25.** Let $A$ be a smooth $k$-algebra. There exists an intersection product

$$\cup : \text{CH}^p(A) \otimes \text{CH}^q(A) \to \text{CH}^{p+q}(A)$$

that agrees with the above construction in case of proper intersections, and turns $\text{CH}^*(A)$ into a graded ring (associative, commutative, unital).