Lecture 10

Smooth algebras and their Chow cohomology

10.1. Smooth algebras.

Remark 1. Let k be a field and let A and B be finite type k-algebras. If k is perfect, then regularity of A and B implies regularity of $\mathrm{A} \otimes_k \mathrm{B}$. In general, this is not the case: in fact, $\mathrm{A} \otimes_k \mathrm{B}$ need not even be reduced. Consider for example the imperfect field $k = \mathbf{F}_p(\mathrm{T})$ and the algebras $\mathrm{A} = \mathrm{B} = k(\mathrm{T}^{1/p}) = k[\mathrm{X}]/(\mathrm{X}^p - \mathrm{T})$.

We introduce a closely related but slightly stronger notion, which will still be closed under tensor products.

Definition 2. Let k be a field and $\overline{k}$ an algebraic closure. We say that a finite type k-algebra A is *smooth* if its extension of scalars $\mathrm{A} \otimes_k \overline{k}$ is a regular ring.

Proposition 3. *Let* A *be a k-algebra of finite type. If* A *is smooth, then it is regular. If k is perfect, then the converse also holds.*

Definition 4. A homomorphism $\phi : \mathrm{A} \to \mathrm{B}$ of finite type k-algebras is *smooth* (of relative dimension d) if it is flat, and for every closed point $x \in \left|\mathrm{Spec}(\mathrm{A})\right|$, $\mathrm{B} \otimes_{\mathrm{A}} \kappa(x)$ is a smooth $\kappa(x)$-algebra (of dimension d).

Example 5. For every k-algebra A, the homomorphism $\mathrm{A} \to \mathrm{A}[\mathrm{T}_1, \dots, \mathrm{T}_n]$ is smooth of relative dimension n. Indeed $\mathrm{A}[\mathrm{T}_1, \dots, \mathrm{T}_n] \otimes_{\mathrm{A}} \kappa \simeq \kappa[\mathrm{T}_1, \dots, \mathrm{T}_n]$ is of dimension n for every residue field κ.

Proposition 6.

(i) *Let* $\mathrm{A} \to \mathrm{A}'$ *be a homomorphism of finite type k-algebras. Then for every smooth homomorphism* $\phi : \mathrm{A} \to \mathrm{B}$ *of relative dimension d, the extension of scalars* $\phi' : \mathrm{A}' \to \mathrm{B} \otimes_{\mathrm{A}} \mathrm{A}'$ *is smooth of relative dimension d.*

(ii) *Let* $\phi : \mathrm{A} \to \mathrm{B}$ *and* $\psi : \mathrm{B} \to \mathrm{C}$ *be smooth homomorphisms of finite type k-algebras, of relative dimensions d and e, respectively. Then the composite* $\psi \circ \phi : \mathrm{A} \to \mathrm{C}$ *is smooth of relative dimension* $d + e$*.*

Corollary 7. *For any two smooth k-algebras* A *and* B*, the tensor product* $\mathrm{A} \otimes_k \mathrm{B}$ *is smooth of dimension* $\dim(\mathrm{A}) + \dim(\mathrm{B})$.

Proof. By the proposition, $k \to \mathrm{A} \to \mathrm{A} \otimes_k \mathrm{B}$ is a smooth homomorphism. □

10.2. Chow cohomology.

Definition 8. Let k be a field and A a smooth k-algebra of finite type. Assume that $\left|\mathrm{Spec}(\mathrm{A})\right|$ is of pure dimension d. We define $\mathrm{CH}^k(\mathrm{A}) := \mathrm{CH}_{d-k}(\mathrm{A})$ for every k, and refer to this as the kth *Chow cohomology* group of A.

Our next goal will be to define a product on the graded ring $\mathrm{CH}^*(\mathrm{A}) := \bigoplus_k \mathrm{CH}^k(\mathrm{A})$, analoguous to cup products in singular cohomology.

10.3. Quasi-smooth homomorphisms.

Definition 9. Let $\phi : \mathrm{A} \to \mathrm{B}$ be a surjective ring homomorphism, with kernel I. We say that ϕ is *quasi-smooth* (of relative dimension $-n$) if for every point $x \in \left|\mathrm{Spec(B)}\right|$, the ideal $\mathrm{I}_{\phi^{-1}(\mathfrak{p}(x))} \subseteq \mathrm{A}_{\phi^{-1}(\mathfrak{p}(x))}$ is generated by a Koszul-regular sequence (of length n), where $\mathfrak{p}(x) \subseteq \mathrm{B}$ is the prime ideal corresponding to x.

Remark 10. Since we are only working with noetherian rings, ϕ is quasi-smooth iff it is a *local complete intersection homomorphism* (which is the same definition but where "Koszul-regular" is replaced by "regular").

Example 11. For any ring A and element $a \in \mathrm{A}$, the A-algebra surjection $\mathrm{A[T]} \to \mathrm{A}$, $\mathrm{T} \mapsto a$, is quasi-smooth of relative dimension -1. Indeed the ideal $\langle \mathrm{T} - a \rangle$ is clearly generated by a non-zero-divisor.

Proposition 12.

(i) *Let* $\mathrm{A} \to \mathrm{A}'$ *be a* flat *ring homomorphism. Then for every quasi-smooth surjection* $\phi : \mathrm{A} \to \mathrm{B}$ *of relative dimension* $-d$*, the extension of scalars* $\phi' : \mathrm{A}' \to \mathrm{B} \otimes_{\mathrm{A}} \mathrm{A}'$ *is a quasi-smooth surjection of relative dimension* $-d$*.*

(ii) *Let* $\phi : \mathrm{A} \to \mathrm{B}$ *and* $\psi : \mathrm{B} \to \mathrm{C}$ *be quasi-smooth surjections of relative dimensions* $-d$ *and* $-e$*, respectively. Then the composite* $\psi \circ \phi : \mathrm{A} \to \mathrm{C}$ *is quasi-smooth of relative dimension* $-(d + e)$*.*

Proposition 13. *Let* $\mathrm{A} \to \mathrm{B}$ *be a surjective homomorphism of smooth* k*-algebras. Then* $\mathrm{A} \to \mathrm{B}$ *is quasi-smooth of relative dimension* $-d$*, where* $d = \dim(\mathrm{A}) - \dim(\mathrm{B})$*.*

Corollary 14. *Let* A *be a smooth algebra over a field* k*. Then the canonical ring homomorphism* $\delta : \mathrm{A} \otimes_k \mathrm{A} \to \mathrm{A}$ *is a quasi-smooth surjection of relative dimension* $-d$*, where* $d = \dim(\mathrm{A})$*.*

Proof. Clearly δ is surjective (with kernel generated by the elements $a \otimes 1 - 1 \otimes a$, for all $a \in \mathrm{A}$). Since we know that $\mathrm{A} \otimes_k \mathrm{A}$ is a smooth k-algebra of dimension $2\dim(\mathrm{A})$, the claim follows from the previous proposition. □

10.4. Interlude: more on dimension. We need some more dimension theory. The following statement is a consequence of Krull's principal ideal theorem.

Proposition 15. *Let* A *be a noetherian local ring with maximal ideal* $\mathfrak{m}$*. For any element* $f \in \mathfrak{m}$*, we have*

$$\dim(\mathrm{A}/\langle f \rangle) \geqslant \dim(\mathrm{A}) - 1$$

with equality if f *is a non-zero-divisor.*

The next statement follows from Noether normalization.

Proposition 16. *Let* k *be a field and* A *an integral domain which is of finite type over* k*. Then we have*

$$\dim(\mathrm{A}) = \mathrm{trdeg}_k(\mathrm{Frac(A)}),$$

where trdeg *denotes transcendence degree.*

Proposition 17. *Let* A *be an integral domain which is of finite type over* k*. Then* $\dim(\mathrm{A}) = \mathrm{codim}_{\mathrm{A}}(\{x\})$*, for any closed point* $x \in |\mathrm{Spec}(\mathrm{A})|$*.*

Proposition 18. *Let* A *and* B *be finite type* k*-algebras. Then* $\dim(\mathrm{A} \otimes_k \mathrm{B}) = \dim(\mathrm{A}) + \dim(\mathrm{B})$*.*

10.5. Proper intersections.

Lemma 19. *Let* k *be a field and* A *a finite type* k*-algebra. Let* I *be an ideal such that* $\mathrm{A} \to \mathrm{A}/\mathrm{I}$ *is a surjective quasi-smooth homomorphism of relative dimension* $-n$*. Let* $\mathrm{V}(\mathfrak{p})$ *be an integral closed subset of dimension* d*. Then each irreducible component of the closed subset* $\mathrm{V}(\mathrm{I}) \cap \mathrm{V}(\mathfrak{p}) = \mathrm{V}(\mathrm{I} + \mathfrak{p})$ *is of dimension* $\geqslant d - n$*.*

Proof. By Exercise 2 on Sheet 9, dimension is "local" in the sense that $\dim(\mathrm{A}) = \sup_{\mathfrak{m}} \dim(\mathrm{A}_{\mathfrak{m}})$ where the supremum is taken over maximal ideals $\mathfrak{m}$. Therefore, we may as well replace A by $\mathrm{A}_{\mathfrak{m}}$ to assume that A is a local ring and that the ideal I is generated by a Koszul-regular sequence $(f_1, \ldots, f_n)$ with $f_i \in \mathfrak{m}$. By the implicit noetherian hypothesis, $(f_1, \ldots, f_n)$ is then a regular sequence.

Let Y be an irreducible component of $\mathrm{V}(\mathrm{I} + \mathfrak{p})$. Let $y \in \mathrm{Y}$ be a closed point that is not contained in any other component of $\mathrm{V}(\mathrm{I} + \mathfrak{p})$. Then by §10.4, both $\dim(\mathrm{V}(\mathrm{I} + \mathfrak{p}))$ and $\dim(\mathrm{Y})$ are equal to the codimension of $\{y\}$ in $\mathrm{V}(\mathrm{I} + \mathfrak{p})$. The latter is equal to $\dim((\mathrm{A}/(\mathrm{I} + \mathfrak{p}))_{\mathfrak{p}(y)})$ (Sheet 9, Exercise 2), where $\mathfrak{p}(y)$ is the prime (maximal) ideal corresponding to y. Thus we get

$$\dim(\mathrm{Y}) = \dim(((\mathrm{A}/\mathfrak{p})/\langle f_1, \ldots, f_n \rangle)_{\mathfrak{p}(y)}) = \dim(((\mathrm{A}/\mathfrak{p})_{\mathfrak{p}(y)}/\langle f_1, \ldots, f_n \rangle))$$

The same exercise gives similarly

$$d = \dim(\mathrm{V}(\mathfrak{p})) = \dim((\mathrm{A}/\mathfrak{p})_{\mathfrak{p}(y)}).$$

If $\mathrm{B} = (\mathrm{A}/\mathfrak{p})_{\mathfrak{p}(y)}$, the claim is now that $\dim(\mathrm{B}/\langle f_1, \ldots, f_n \rangle) \geqslant \dim(\mathrm{B}) - n$. Since the f_i form a regular sequence in B, the claim follows by induction from the $n = 1$ case (§10.4). □

Proposition 20. *Assume the field* k *is algebraically closed. Let* A *be a smooth* k*-algebra of dimension* d*. Let* $\mathrm{V}(\mathfrak{p})$ *and* $\mathrm{V}(\mathfrak{q})$ *be integral closed subsets of* $|\mathrm{Spec}(\mathrm{A})|$ *of dimensions* m *and* n*, respectively. Then every irreducible component of* $\mathrm{V}(\mathfrak{p}) \cap \mathrm{V}(\mathfrak{q}) = \mathrm{V}(\mathfrak{p} + \mathfrak{q})$ *is of dimension* $\geqslant m + n - d$*.*

Proof. Then the canonical map $\delta : \mathrm{A} \otimes_k \mathrm{A} \to \mathrm{A}$ is a quasi-smooth surjection of relative dimension $-d$; let K be its kernel. The ideal $\mathfrak{r} = \mathfrak{p} \otimes_k \mathrm{A} + \mathrm{A} \otimes_k \mathfrak{q} \subset \mathrm{A} \otimes_k \mathrm{A}$ corresponds to the subset $\mathrm{V}(\mathfrak{r}) \subset |\mathrm{Spec}(\mathrm{A} \otimes_k \mathrm{A})|$, which is integral of dimension $m + n$ since

$$(\mathrm{A} \otimes_k \mathrm{A})/(\mathfrak{p} \otimes_k \mathrm{A} + \mathrm{A} \otimes_k \mathfrak{q}) \simeq (\mathrm{A}/\mathfrak{p}) \otimes_k (\mathrm{A}/\mathfrak{q})$$

is an integral domain of dimension $m + n$ (this requires that k is algebraically closed). Moreover, we have

$$(\mathrm{A} \otimes_k \mathrm{A})/(\mathrm{K} + \mathfrak{r}) \simeq \mathrm{A}/(\mathfrak{p} + \mathfrak{q}).$$

Thus, the claim follows from the previous statement applied to the finite type k-algebra $\mathrm{A} \otimes_k \mathrm{A}$, with quasi-smooth surjection $\delta : \mathrm{A} \otimes_k \mathrm{A} \to \mathrm{A}$ and integral subset $\mathrm{V}(\mathfrak{r})$. □

Remark 21. The argument used in the proof is called "reduction to the diagonal" (Serre). Geometrically, $\delta : \mathrm{A} \otimes_k \mathrm{A} \to \mathrm{A}$ corresponds to the diagonal morphism $\mathrm{X} \to \mathrm{X} \times_k \mathrm{X}$ of the k-scheme $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$. The isomorphism $(\mathrm{A} \otimes_k \mathrm{A})/(\mathrm{K} + \mathfrak{r}) \simeq \mathrm{A}/(\mathfrak{p} + \mathfrak{q})$ can be written as

$$(\mathrm{A}/\mathfrak{p} \otimes_k \mathrm{A}/\mathfrak{q}) \otimes_{\mathrm{A} \otimes_k \mathrm{A}} \mathrm{A} \simeq \mathrm{A}/(\mathfrak{p} + \mathfrak{q}).$$

This corresponds to the formula

$$(\mathrm{Y} \times_k \mathrm{Z}) \cap \Delta = \mathrm{Y} \cap \mathrm{Z},$$

where $\mathrm{Y} = \mathrm{V}(\mathfrak{p})$ and $\mathrm{Z} = \mathrm{V}(\mathfrak{q})$ and $\Delta = \mathrm{V}(\mathrm{K}) \subset \mathrm{X} \times_k \mathrm{X}$ is the image of the diagonal immersion.

Definition 22. Let A be a finite type k-algebra of dimension d. Let $\mathrm{Y} = \mathrm{V}(\mathfrak{p})$ and $\mathrm{Z} = \mathrm{V}(\mathfrak{q})$ be integral closed subsets of $|\mathrm{Spec}(\mathrm{A})|$ of dimensions m and n, respectively. We say that Y and Z *intersect properly*, or *without excess*, if every irreducible component of $\mathrm{Y} \cap \mathrm{Z}$ is of dimension $\leqslant m + n - d$. (An *excess component* is an irreducible component of dimension $> m + n - d$.)

Remark 23. If A is smooth and k is algebraically closed, then by the proposition, Y and Z intersect properly iff every irreducible component of $\mathrm{Y} \cap \mathrm{Z}$ is of dimension exactly $m + n - d$, i.e., iff $\mathrm{Y} \cap \mathrm{Z}$ is of pure dimension $m + n - d$.

10.6. Intersection products.

Construction 24. Let A be a smooth k-algebra of dimension d. Let $\mathrm{Y} = \mathrm{V}(\mathfrak{p})$ and $\mathrm{Z} = \mathrm{V}(\mathfrak{q})$ be integral subsets of $|\mathrm{Spec}(\mathrm{A})|$ of codimensions p and q, respectively. Assume that Y and Z intersect properly, so that $\mathrm{Y} \cap \mathrm{Z}$ is of pure dimension $(d - p) + (d - q) - d = d - (p + q)$. We define the intersection product of [Y] and [Z] as the cycle

$$[\mathrm{V}(\mathfrak{p})] \cup [\mathrm{V}(\mathfrak{q})] = \sum_i (-1)^i [\mathrm{Tor}_i^{\mathrm{A}}(\mathrm{A}/\mathfrak{p}, \mathrm{A}/\mathfrak{q})]_{d-p-q}$$

in $\mathrm{CH}_{d-p-q}(\mathrm{A}) \simeq \mathrm{CH}^{p+q}(\mathrm{A})$.

We will not prove the following (difficult) theorem.

Theorem 25. *Let* A *be a smooth k-algebra. There exists an intersection product*

$$\cup : \mathrm{CH}^p(\mathrm{A}) \otimes \mathrm{CH}^q(\mathrm{A}) \to \mathrm{CH}^{p+q}(\mathrm{A})$$

that agrees with the above construction in case of proper intersections, and turns $\mathrm{CH}^*(\mathrm{A})$ *into a graded ring (associative, commutative, unital).*