Lecture 12
More on divisors

12.1. Regular local rings of dimension 1. Recall that DVR's are regular (§2.3) and of dimension 1 (Sheet 9, Exercise 1). The converse is also true: any 1-dimensional regular local ring is a DVR.

Let's review how this works. Let A be a regular local ring of dimension 1.

Claim 1. *The maximal ideal* $\mathfrak{m}$ *is principal.*

Proof. Since A is local, $V(\mathfrak{m}) \subseteq |\mathrm{Spec}(A)|$ is the unique closed point. It is of codimension 1 by Exercise 2 on Sheet 9. Since regular rings are factorial, it follows that $\mathfrak{m}$ is principal (by the Lemma used in the solution of Sheet 8, Exercise 3). □

We fix a uniformizer, i.e., a generator $\pi \in \mathfrak{m}$.

Claim 2. *For every nonzero element* $a \in A$*, there is a unique* $n \geqslant 0$ *such that* $a = u\pi^n$ *for some unit* $u \in A^\times$*.*

Proof. By Krull's intersection theorem, the intersection $\bigcap_{i=1}^{\infty} \mathfrak{m}^i$ is zero. This means that there is a maximal $n \geqslant 0$ such that $a \in \mathfrak{m}^n$ and $a \notin \mathfrak{m}^{n+1}$. Then we have $a = u\pi^n$ for some $u \notin \mathfrak{m}$, which is necessarily a unit. □

Construction 3. We define a (discrete) valuation $v : K^\times \to \mathbf{Z}$, where $K = \mathrm{Frac}(A)$. Take $x \in K^\times$ and choose a fraction f/g representing it. Then by above we may write $f = u\pi^m$, $g = v\pi^n$ with $u, v \in A^\times$ and $m, n \geqslant 0$. Then $f/g = (u/v)\pi^{m-n}$ and we define $v(f/g) = m - n$. One checks that this is well-defined (doesn't depend on the fraction representing x) and indeed defines a valuation.

Claim 4. *Let* A *be a regular local ring of dimension* 1*, with maximal ideal* $\mathfrak{m}$*. Then every nonzero ideal* $I \subseteq A$ *is of the form* $\langle \pi^n \rangle$*, where* $n = \min_{a \in I} v(a)$*.*

Proof. Choose a nonzero element $a \in I$ of minimal valuation n. Then we may write $a = u\pi^n$ with $u \in A^\times$ and $n \geqslant 0$, and every other element of I is divisible by a (since it is either 0 or $v\pi^m$ for $m \geqslant n$), and hence by π^n. Thus $I \subseteq \langle \pi^n \rangle$. Conversely, $\pi^n = u^{-1}a \in \langle a \rangle \subseteq I$. □

12.2. Effective vs. non-effective Cartier divisors. By multiplying with an appropriate element $a \in A$, we can turn any Cartier divisor M into an effective one aM:

Lemma 5. *Let* A *be a ring and* $(M, M \hookrightarrow \mathrm{Frac}(A))$ *a Cartier divisor on* A*. Then there exists a nonzero element* $a \in A$ *such that the submodule* $aM \subset \mathrm{Frac}(A)$ *is contained in the subring* $A \subset \mathrm{Frac}(A)$*.*

Proof. Since M is invertible, it is finitely generated. Identifying M with a submodule of Frac(A), we can choose a finite set of generators and view them as elements $f_1/g_1, \dots, f_n/g_n \in \mathrm{Frac}(A)$. Then $a = g_1 \cdots g_n$ does the job. □

Every Cartier divisor is a difference of two *effective* Cartier divisors:

Lemma 6. *Let* $(M, M \hookrightarrow \mathrm{Frac}(A))$ *be a Cartier divisor on* A. *Let* $a \in A$ *be nonzero such that* $aM \subset A$. *Then we have*

$$(M, M \hookrightarrow \mathrm{Frac}(A)) = (aM, aM \hookrightarrow A \hookrightarrow \mathrm{Frac}(A)) - \mathrm{div}_A(a)$$

in the group Cart(A).

Exercise 7. Let $\mathrm{Cart}^+(A)$ denote the set of effective Cartier divisors. This admits a canonical monoid structure and there is a canonical injective homomorphism

$$\mathrm{Cart}^+(A) \to \mathrm{Cart}(A)$$

which exhibits Cart(A) as the group completion of $\mathrm{Cart}^+(A)$.

12.3. Multiplicities of Cartier divisors. We want to define a map from Cartier divisors to Weil divisors. This amounts to assigning *multiplicities* to a Cartier divisors for every integral subset $V(\mathfrak{p}) \subset \left|\mathrm{Spec}(A)\right|$ of codimension 1.

Remark 8. Let A be a regular ring. For every integral subset $V(\mathfrak{p}) \subset \left|\mathrm{Spec}(A)\right|$ of codimension 1, the localization $A_\mathfrak{p}$ is a 1-dimensional regular local ring (since $\dim(A_\mathfrak{p}) = \mathrm{codim}(V(\mathfrak{p}))$ by Sheet 9, Exercise 2). By §11.6, it is a DVR; in particular, every nonzero ideal of $A_\mathfrak{p}$ is of the form $\langle \pi^n \rangle$ where π is a uniformizer and $n \geqslant 0$.

Claim 9. *Let* A *and* $\mathfrak{p}$ *as above. Every Cartier divisor on* $A_\mathfrak{p}$ *is of the form* $(\pi^n A_\mathfrak{p}, \pi^n A_\mathfrak{p} \hookrightarrow \mathrm{Frac}(A_\mathfrak{p}))$, *where* $n \in \mathbf{Z}$ *(possibly negative).*

Proof. Let $(M, M \hookrightarrow \mathrm{Frac}(A_\mathfrak{p}))$ be a Cartier divisor. Consider the ideal of $A_\mathfrak{p}$ generated by elements $a \in A_\mathfrak{p}$ such that the submodule $aM \subset \mathrm{Frac}(A_\mathfrak{p})$ is contained in $A_\mathfrak{p}$. By §12.2 it is nonzero, and as such it must be of the form $\pi^r A_\mathfrak{p}$, $r \geqslant 0$ (previous remark). Then by construction, the submodule $\pi^r M \subset \mathrm{Frac}(A_\mathfrak{p})$ is contained in $A_\mathfrak{p}$, hence is also of the form $\pi^s A_\mathfrak{p}$, $s \geqslant 0$. We deduce that M is the submodule of $\mathrm{Frac}(A_\mathfrak{p})$ given by $\pi^{s-r} A_\mathfrak{p}$. □

Definition 10. Let A be a regular ring and $V(\mathfrak{p}) \subset \left|\mathrm{Spec}(A)\right|$ an integral subset of codimension 1. For a Cartier divisor $(M, M \hookrightarrow \mathrm{Frac}(A))$ on A, its *multiplicity* along $\mathfrak{p}$ is the integer $n_\mathfrak{p}$ such that $M_\mathfrak{p}$ is of the form $\pi^{n_\mathfrak{p}} A_\mathfrak{p}$. When there is possible ambiguity, we may write $n_\mathfrak{p}(M)$.

Lemma 11. *Let* $(I, I \hookrightarrow A)$ *be an effective Cartier divisor on* A. *Let* $V(\mathfrak{p}) \subset \left|\mathrm{Spec}(A)\right|$ *be an integral subset of codimension* 1.

(a) *The multiplicity at* $\mathfrak{p}$ *is zero unless* $V(\mathfrak{p}) \subseteq V(I)$.

(b) *We have* $V(\mathfrak{p}) \subseteq V(I)$ *iff* $V(\mathfrak{p})$ *is an irreducible component of* $V(I)$.

Proof.

(a) By definition, $I_{\mathfrak{p}} = \langle \pi^{n_{\mathfrak{p}}} \rangle$ for every $\mathfrak{p}$. If $V(\mathfrak{p}) \not\subseteq V(I)$, i.e., $I \not\subseteq \operatorname{rad}(\mathfrak{p}) = \mathfrak{p}$, take some $x \in I \setminus \mathfrak{p}$. Then $1 = x/x \in I_{\mathfrak{p}}$, so $n_{\mathfrak{p}} = 0$.

(b) Suppose $V(\mathfrak{p}) \subseteq V(I)$. Since I is invertible, we have $V(I) \subsetneq V(0)$. If there was an integral subset $V(\mathfrak{q})$ with $V(\mathfrak{p}) \subsetneq V(\mathfrak{q}) \subseteq V(I)$, then

$$V(\mathfrak{p}) \subsetneq V(\mathfrak{q}) \subsetneq V(0) = \left|\operatorname{Spec}(A)\right|$$

would be a chain of integral subsets of length 2, in contradiction with $\operatorname{codim}(V(\mathfrak{p})) = 1$. ☐

Corollary 12. *Let* $(I, I \hookrightarrow A)$ *be an effective Cartier divisor on* A. *As* $V(\mathfrak{p}) \subset \left|\operatorname{Spec}(A)\right|$ *ranges over integral subsets of codimension* 1*, only finitely many of the multiplicities* $n_{\mathfrak{p}}$ *are nonzero.*

Proof. If $n_{\mathfrak{p}}$ is nonzero, then $V(\mathfrak{p})$ is an irreducible component of $V(I)$. But since A/I is noetherian, there are only finitely many such. ☐

Corollary 13. *Let* $(M, M \hookrightarrow \operatorname{Frac}(A))$ *be a Cartier divisor on* A. *As* $V(\mathfrak{p}) \subset \left|\operatorname{Spec}(A)\right|$ *ranges over integral subsets of codimension* 1*, only finitely many of the multiplicities* $n_{\mathfrak{p}}$ *are nonzero.*

Proof. Write M as a difference of two effective Cartier divisors. It follows from the definitions that multiplicities are additive, so we reduce to the effective case. ☐

12.4. Examples of multiplicities.

Example 14. Let $A = k[T]$ for a field k and $f = T - 1 \in k[T]$. Consider the principal Cartier divisor $\operatorname{div}_{k[T]}(f) \in \operatorname{Cart}(k[T])$. This is an effective Cartier divisor corresponding to the ideal $\mathfrak{p} = \langle T - 1 \rangle \subset k[T]$. Let's compute its multiplicities. Given an integral subset $V(\mathfrak{q})$ of codimension 1, we know that the multiplicity $n_{\mathfrak{q}}$ is zero unless $V(\mathfrak{q})$ is an irreducible component of $V(\mathfrak{p})$. But the latter is integral since $\mathfrak{p}$ is prime. Thus the only nonzero multiplicity is at the prime $\mathfrak{p}$. Clearly the ideal $\mathfrak{p}_{\mathfrak{p}} \subset A_{\mathfrak{p}} = k[T]_{\langle T-1 \rangle}$ is generated by π where $\pi = T - 1$ is the uniformizer, so $n_{\mathfrak{p}} = 1$.

Example 15. Let $A = k[T]$ and $f = T^2 \in k[T]$. Then $\operatorname{div}_{k[T]}(f) \in \operatorname{Cart}(k[T])$ is the effective Cartier divisor corresponding to the ideal $I = \langle T^2 \rangle \subset k[T]$. We have $V(I) = V(T)$ since $\operatorname{rad}(I) = \langle T \rangle$, so $\mathfrak{p} = \langle T \rangle$ is the only prime with nonzero multiplicity. At $\mathfrak{p}$, the DVR $A_{\mathfrak{p}} = k[T]_{\langle T \rangle}$ has uniformizer $\pi = T$ and $I_{\mathfrak{p}} = \langle T^2 \rangle = \langle \pi^2 \rangle$, so the multiplicity is $n_{\mathfrak{p}} = 2$.

Example 16. Let $A = k[X, Y]$ and $f = XY \in k[X, Y]$. Then $\operatorname{div}(f)$ is the effective Cartier divisor corresponding to the ideal $I = \langle XY \rangle \subset k[X, Y]$. In this case V(I) is not integral but rather has two irreducible components: $V(I) = V(X) \cup V(Y)$, since $\langle XY \rangle = \langle X \rangle \cap \langle Y \rangle$. Let's compute the multiplicity at $\mathfrak{p} = \langle X \rangle$. The uniformizer of $A_{\mathfrak{p}}$ is X, since its maximal ideal is $\mathfrak{p}_{\mathfrak{p}} = (X)A_{\mathfrak{p}}$. We also have $I_{\mathfrak{p}} = (XY)A_{\mathfrak{p}} = (X)A_{\mathfrak{p}}$

(since Y is a unit in $A_{\mathfrak{p}}$), so we find that the multiplicity $n_{\mathfrak{p}}$ is 1. By symmetry, the multiplicity at $\mathfrak{q} = \langle Y \rangle$ is also 1.

Example 17. Let $A = k[T]$ and $f = T^2/(T-1) \in \mathrm{Frac}(k[T])$. Then $\mathrm{div}(f)$ is the (non-effective!) Cartier divisor given by the A-submodule I of $\mathrm{Frac}(k[T])$ generated by f. We can write it as the difference of two effective Cartier divisors:

$$\mathrm{div}(f) = \mathrm{div}(T^2) - \mathrm{div}(T-1).$$

So it has multiplicity 2 at $\langle T \rangle$, -1 at $\langle T-1 \rangle$, and zero everywhere else.

12.5. From Cartier divisors to Weil divisors.

Construction 18. Let A be a regular ring. Define a homomorphism

$$\mathrm{Cart}(A) \to Z^1(A)$$

by sending $(M, M \hookrightarrow \mathrm{Frac}(A))$ to

$$\sum_{V(\mathfrak{p})} n_{\mathfrak{p}}[V(\mathfrak{p})],$$

where $V(\mathfrak{p}) \subset |\mathrm{Spec}(A)|$ ranges over integral subsets of codimension 1, and $n_{\mathfrak{p}}$ is the multiplicity of M at $\mathfrak{p}$.

Example 19. The Weil divisor associated to the Cartier divisor $\mathrm{div}(T^2/(T-1))$ on $k[T]$ is

$$2[V(T)] - [V(T-1)] \in Z^1(k[T]) = Z_0(k[T]).$$

Thus, the Weil divisor encodes the fact that the rational function $T^2/(T-1)$ has a zero of order 2 at $T = 0$ and a pole of order 1 at $T = 1$.

Theorem 20. *Let* A *be a regular ring. The canonical homomorphisms*

$$Z^1(A) \to \mathrm{Cart}(A), \qquad \mathrm{Cart}(A) \to Z^1(A)$$

are inverse to each other.

Proof. Let $V(\mathfrak{p}) \subset |\mathrm{Spec}(A)|$ be an integral subset of codimension 1. The left-hand map sends $[V(\mathfrak{p})]$ to the effective Cartier divisor $(\mathfrak{p}, \mathfrak{p} \hookrightarrow \mathrm{Frac}(A))$. Clearly, this has multiplicity 1 at $\mathfrak{p}$ and zero multiplicity elsewhere, so the right-hand map sends it back to $[V(\mathfrak{p})]$. This shows that one composite is the identity.

It will suffice to show that the right-hand map is injective. Let $(M, M \hookrightarrow \mathrm{Frac}(A))$ be a Cartier divisor and suppose that it goes to zero, i.e., that all the multiplicities $n_{\mathfrak{p}}$ are zero. Identify M with a sub-A-module of Frac(A). We'll show that $M = A$ (which is the zero Cartier divisor), or equivalently $M_{\mathfrak{m}} = A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$.

Since M is invertible, $M_{\mathfrak{m}} \subset \mathrm{Frac}(A)_{\mathfrak{m}} = \mathrm{Frac}(A)$ is a principal ideal. If $x = f/g$ is a generator, then by choosing prime factorizations of f and g we may write $x = up_1^{e_1} \cdots p_n^{e_n}$ for some unit u, primes p_i, and integers $e_n \in \mathbf{Z}$. Then M has multiplicity e_i at $\langle p_i \rangle$. But since all the multiplicities are zero by assumption, we deduce that x is a unit, hence $M_{\mathfrak{m}} = A_{\mathfrak{m}}$. ☐

12.6. The Cartier divisor class group.

Definition 21. Two Cartier divisors on A are *linearly equivalent* if their difference is a principal Cartier divisor. We let CartCl(A) denote the quotient of Cart(A) by the subgroup of principal Cartier divisors. This is called the *Cartier divisor class group.*

Proposition 22. *Let* A *be an integral domain. Then the forgetful map* $\mathrm{Cart}(A) \to \mathrm{Pic}(A)$ *induces a canonical isomorphism*

$$\mathrm{CartCl}(A) \simeq \mathrm{Pic}(A).$$

Proof. Follows from the exact sequence (Lecture §11.4)

$$\mathrm{Frac}(A)^{\times} \xrightarrow{\mathrm{div}_A} \mathrm{Cart}(A) \twoheadrightarrow \mathrm{Pic}(A) \to 0$$

as the image of div_A is precisely the subgroup of principal Cartier divisors. □

Exercise 23. The isomorphism $\mathrm{Cart}(A) \to Z^1(A) \simeq Z_{d-1}(A)$, where $d = \dim(A)$, sends the subgroup of principal Cartier divisors to the subgroup $R_{d-1}(A)$ of principal Weil divisors.

Corollary 24. *Let* A *be a regular ring. Then there are canonical isomorphisms*

$$\mathrm{Pic}(A) \simeq \mathrm{CartCl}(A) \simeq \mathrm{CH}^1(A).$$

We have finally achieved our goal of giving an explicit description of the group $\mathrm{CH}^1(A) \simeq \mathrm{CH}_{d-1}(A)$.