Lecture 3
Algebraic K-theory and G-theory

3.1. Algebraic K-theory
3.2. G-theory
3.1 Algebraic $K$-theory

Construction (Group completion):

$(X, \Theta)$ commutative monoid

\[
X \rightarrow X^{gp} \quad \text{universal monoid homo. with}
\]

\[
X^{gp} \text{ a group (abelian)}
\]

Two constructions of $X^{gp}$:

1) formal differences: \[\{(x, y) \mid x, y \in X^2/\sim\}\]

\[(x, y) \sim (x', y') \iff x \Theta y \Theta z = x \Theta y \Theta z \quad \text{for some } z \in X\]

group operation on $X^{gp}$:

\[(x, y) \Theta (x', y') = (x \Theta x', y \Theta y')\]

2) linear combinations: \[\mathbb{Z}[X]/\sim\]

free abelian group w/ generators $x \in X$ and relations: $[x \Theta y] = [x] + [y]$

Ex: \((\mathbb{Z}, +)^{gp} \cong \mathbb{Z}\)
Exercise: Show 1) and 2) are canonically isomorphic. Show the universal property.

Construction: \( A = \text{comm. ring} \)

\( M(A) = \text{monoid of iso. classes of objects of } \text{Mod}_A \)

operation: direct sum

\( K_0(A) := M(A)^{gp} \)

Example: A local ring, \( (\text{or PID}) \)

\( \Rightarrow \) every \( M \in \text{Mod}_A \) is f.g. free

\( \Rightarrow M(A) \cong (\mathbb{N}, +) \) monoid isomorphism

\( \Rightarrow K_0(A) \cong \mathbb{Z} \)
Lemma: \( A = \text{ring} \)

(i) Every \( x \in \text{Ko}(A) \) can be written as \( \left[ M \right] - \left[ A^0 \right] \), where \( M \in \text{Mod}_A^{\text{proj}} \), \( n \in \text{IN} \)

(ii) \( M, N \in \text{Mod}_A^{\text{proj}} \). Then

\[
[M] = [N] \in \text{Ko}(A)
\]

\( \iff M \oplus A^0 \cong N \oplus A^0 \quad \text{for some } n \geq 0 \)  
(Strictly equivalent)

Proof: Recall every \( M \in \text{Mod}_A^{\text{proj}} \) has

\( M \oplus N' \cong A^0 \) for some \( M \in \text{Mod}_A^{\text{proj}} \), \( n \geq 0 \).

(i) Clear that

\[
x = [M] - [N]
\]

for some \( M, N \in \text{Mod}_A^{\text{proj}} \).

Write \( N \oplus N' \cong A^0 \) \( (N' \in \text{Mod}_A^{\text{proj}}) \).

\[
\Rightarrow [N] = [A^0] - [N']
\]

\[
\Rightarrow x = [M] - [N'] - [A^0] = [M \oplus N'] - [A^0].
\]
(ii) \([M] = [N]\) in \(K_0(A)\)

As formal differences:
\((M, 0) \sim (N, 0)\)

\[\Rightarrow M\oplus 0 \oplus P \cong N\oplus 0 \oplus P\]

for some \(P \in \text{Per}_{\text{per}}(A)\)

\[M\oplus P \cong N\oplus P\]

But \(P \oplus P' \cong A^{\oplus n}\) (\(\exists P', n > 0\))

\[\Rightarrow M\oplus A^{\oplus n} \cong N\oplus A^{\oplus n}\] (Stably equiv.)

\[\boxed{\text{Remark: Ring structure on } K_0(A)}:\]

\([M] \cdot [N] := [M \oplus N]_A\]

unit, add., assoc.,
commutative.
3.2 G-theory

**Definition:** A noetherian ring

\[ G_0(A) = \text{free abelian gp. generated by isomorphism classes of all f.g. modules with relations: for any} \]
\[ 0 \to M' \to M \to M'' \to 0 \text{ short exact,} \]
\[ [M] = [M'] + [M''] \text{ in } G_0(A). \]

**Construction:** \( G_0(A) \) is a module over the comm ring \( K_0(A) \):

\[
K_0(A) \oplus G_0(A) \to G_0(A) \\
[P] \oplus [M] \to [PA \otimes M]
\]

Well-defined: if \( 0 \to M' \to M \to M'' \to 0 \) exact, then \( P \text{ flat}, \) so \( 0 \to PA \otimes M' \to PA \otimes M \to PA \otimes M'' \to 0 \) exact

\[
\Rightarrow [PA \otimes M] = [PA \otimes M'] + [PA \otimes M''].
\]
Remark: $\exists \ K_0(A) \to G_0(A)$ canonical
\[ [M] \mapsto [M] \]

Theorem: If $A$ is regular, then

\[ K_0(A) \to G_0(A) \]

is an isomorphism.

Example: If $A=k$ field, then

\[ \text{Mod}^t_{k^g} = \text{Mod}^t_{k^g \text{ free}} \]

\[ \Rightarrow G_0(k) \cong K_0(k) \cong \mathbb{Z}. \]