Lecture 8

K-theory with supports and intersection numbers

8.1. Supports of modules.

Definition 1. Let A be a ring and $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$ a f.g. A-module. We say that M is *supported at* a point $p \in \left|\mathrm{Spec}(\mathrm{A})\right|$ if

$$\mathrm{M} \otimes_{\mathrm{A}} \kappa(p) \neq 0.$$

We write $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}) \subseteq \left|\mathrm{Spec}(\mathrm{A})\right|$ for the subset of points at which M is supported.

Example 2. We have $\mathrm{Supp}_{\mathrm{A}}(\mathrm{A}) = \left|\mathrm{Spec}(\mathrm{A})\right|$.

Example 3. Let $\mathrm{I} \subseteq \mathrm{A}$ be an ideal. We write $\mathrm{V}(\mathrm{I}) := \mathrm{Supp}_{\mathrm{A}}(\mathrm{A}/\mathrm{I})$. This is the set of points p such that $\kappa(p)/\mathrm{I}\kappa(p) \neq 0$, i.e., $\mathrm{I} \cdot \kappa(p) = 0$. This is equivalent to the condition that $\mathrm{A} \to \kappa(p)$ factors through the quotient A/I. It follows that there is a canonical bijection

$$\left|\mathrm{Spec}(\mathrm{A}/\mathrm{I})\right| \simeq \mathrm{V}(\mathrm{I}).$$

Remark 4. By Nakayama's lemma, $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$ is supported at a point p iff $\mathrm{M}_{\mathfrak{p}} \neq 0$, where $\mathfrak{p} = \mathrm{Ker}(\mathrm{A} \to \kappa(p))$. In particular, $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M})$ is empty iff M is zero.

Proposition 5.

(i) *If* $0 \to \mathrm{M}' \to \mathrm{M} \to \mathrm{M}'' \to 0$ *is a short exact sequence of f.g.* A*-modules, then* $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}) = \mathrm{Supp}_{\mathrm{A}}(\mathrm{M}') \cup \mathrm{Supp}_{\mathrm{A}}(\mathrm{M}'')$.

(ii) *Let* M *be a f.g.* A*-module. Suppose* $\mathrm{M} = \sum_i \mathrm{M}_i$ *for some family of submodules* $(\mathrm{M}_i)_i$. *Then* $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}) = \bigcup_i \mathrm{Supp}_{\mathrm{A}}(\mathrm{M}_i)$.

(iii) *Let* M *and* N *be f.g.* A*-modules. Then* $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{N}) = \mathrm{Supp}_{\mathrm{A}}(\mathrm{M}) \cap \mathrm{Supp}_{\mathrm{A}}(\mathrm{N})$.

Corollary 6. *Let* A *be a noetherian ring and* $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$. *Recall that* M *admits a filtration* $0 = \mathrm{M}_0 \subset \mathrm{M}_1 \subset \cdots \subset \mathrm{M}_n = \mathrm{M}$ *where the successive quotients* $\mathrm{M}_i/\mathrm{M}_{i-1}$, $1 \leqslant i \leqslant n$, *are isomorphic to* $\mathrm{A}/\mathfrak{p}_i$ *where* $\mathfrak{p}_i$ *are prime ideals. For any such filtration, we have*

$$\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}) = \bigcup_i \mathrm{V}(\mathfrak{p}_i).$$

Proposition 7.

(i) $\mathrm{V}(\langle 0 \rangle) = \left|\mathrm{Spec}(\mathrm{A})\right|$.

(ii) *For any ideal* I *we have* $\mathrm{V}(\mathrm{I}) = \varnothing$ *iff* $\mathrm{I} = \langle 1 \rangle$.

(iii) *For any two ideals* I *and* J *we have* $\mathrm{V}(\mathrm{I}) \cap \mathrm{V}(\mathrm{J}) = \mathrm{V}(\mathrm{I} + \mathrm{J})$.

(iv) *For any two ideals* I *and* J *we have* $\mathrm{V}(\mathrm{I}) \cup \mathrm{V}(\mathrm{J}) = \mathrm{V}(\mathrm{I} \cap \mathrm{J}) = \mathrm{V}(\mathrm{IJ})$.

(v) *For any ideal* I *we have* $\mathrm{V}(\mathrm{I}) = \mathrm{V}(\mathrm{rad}(\mathrm{I}))$, *where* $\mathrm{rad}(\mathrm{I}) \subseteq \mathrm{A}$ *denotes the radical of* I.

Exercise 8. Let I and J be ideals of A. Then $\mathrm{V(I)} \subseteq \mathrm{V(J)}$ iff $\mathrm{J} \subseteq \mathrm{rad(I)}$.

Corollary 9. *Let* A *be a commutative ring. The homomorphism* $\mathrm{A} \twoheadrightarrow \mathrm{A_{red}}$ *induces a bijection* $|\mathrm{Spec(A)}| \simeq |\mathrm{Spec(A_{red})}|$.

Proof. We have $|\mathrm{Spec(A)}| = \mathrm{V}(\langle 0\rangle) = \mathrm{V}(\mathrm{rad}(\langle 0\rangle)) \simeq |\mathrm{Spec}(\mathrm{A}/\mathrm{rad}(\langle 0\rangle))|$. □

Corollary 10. *Let* M *be a f.g.* A*-module. Then* $\mathrm{Supp_A(M)} = \mathrm{V(A/I)}$*, where* $\mathrm{I} = \mathrm{Ann_A(M)}$ *is the ideal consisting of* $a \in \mathrm{A}$ *such that* $ax = 0$ *for all* $x \in \mathrm{M}$.

Proof. Choose an A-linear surjection $\mathrm{A}^{\oplus n} \twoheadrightarrow \mathrm{M}$, corresponding to elements $x_i \in \mathrm{M}$. Then we have $\mathrm{M} = \sum_i \mathrm{A}x_i$ so by the Proposition,

$$\mathrm{Supp_A(M)} = \bigcup_i \mathrm{Supp_A}(\mathrm{A}x_i).$$

Note that $\mathrm{A}x_i \simeq \mathrm{A/I}_i$ where

$$\mathrm{I}_i = \mathrm{Ann}(x_i) = \mathrm{Ker}(\mathrm{A} \xrightarrow{x_i} \mathrm{M}) \subseteq \mathrm{A}.$$

Thus we have

$$\mathrm{Supp_A(M)} = \bigcup_i \mathrm{V}(\mathrm{I}_i) = \mathrm{V(I)}$$

by the Example above and the fact that $\mathrm{I} = \bigcap_i \mathrm{I}_i$. □

Corollary 11. *Let* A *be a noetherian ring and* $\mathrm{M} \in \mathrm{Mod_A^{fg}}$*. Then for any ideal* $\mathrm{I} \subseteq \mathrm{A}$*, we have* $\mathrm{Supp_A(M)} \subseteq \mathrm{V(I)}$ *iff* M *is* I^∞*-torsion.*

Proof. Let $\mathrm{J} = \mathrm{Ann_A(M)}$. Then $\mathrm{Supp_A(M)} = \mathrm{V(J)}$, so the condition is equivalent to $\mathrm{V(J)} \subseteq \mathrm{V(I)}$. This is equivalent to $\mathrm{I} \subseteq \mathrm{rad(J)}$, and since A is noetherian, to $\mathrm{I}^k \subseteq \mathrm{J}$ for some $k \geqslant 0$. But this is the same as $\mathrm{I}^k\mathrm{M} = 0$. □

8.2. G-theory with supports. We consider a variant of G-theory where the modules have prescribed support.

Construction 12. Let A be a noetherian ring and $\mathrm{Y} \subseteq |\mathrm{Spec(A)}|$ a subset. We denote by $\mathrm{G_0^Y(A)}$ the free abelian group on isomorphism classes of f.g. A-modules M which are supported on Y, i.e., for which $\mathrm{Supp_A(M)} \subseteq \mathrm{Y}$, modulo relations given by short exact sequences.

Since M is supported on V(I) iff it is I^∞-torsion, $\mathrm{G_0^{V(I)}(A)}$ is just another notation for $\mathrm{K_0(Mod_A^{fg}(I^\infty))}$. In particular, the dévissage isomorphism can be re-interpreted as the assertion that G-theory does not see the difference between the category of A/I-modules and that of A-modules supported on $\mathrm{V(I)} \simeq |\mathrm{Spec(A/I)}|$.

Corollary 13. *Let* A *be a noetherian ring and* I *an ideal. Then we have a canonical isomorphism of abelian groups*

$$\mathrm{G_0(A/I)} \simeq \mathrm{G_0^{V(I)}(A)}.$$

8.3. K-theory with supports.

Definition 14. Let A be a ring. A perfect complex $\mathrm{M}_\bullet$ is *supported at* a point $p \in \left|\mathrm{Spec}(\mathrm{A})\right|$ if at least one homology group $\mathrm{H}_i(\mathrm{M}_\bullet)$ is supported at p. We let $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}_\bullet) \subseteq \left|\mathrm{Spec}(\mathrm{A})\right|$ denote the subset of points where $\mathrm{M}_\bullet$ is supported. By definition,

$$\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}_\bullet) = \bigcup_{i \in \mathbf{Z}} \mathrm{Supp}_{\mathrm{A}}(\mathrm{H}_i(\mathrm{M}_\bullet)).$$

This is a finite union since $\mathrm{M}_\bullet$ is perfect.

Remark 15. The same definition also makes sense more generally for coherent complexes.

Remark 16. The support of $\mathrm{M}_\bullet$ only depends on its quasi-isomorphism class. In particular, $\mathrm{M}_\bullet$ has empty support iff it is acyclic.

Construction 17. Let A be a ring and $\mathrm{Y} \subseteq \left|\mathrm{Spec}(\mathrm{A})\right|$ a subset. Denote by $\mathrm{Perf}_{\mathrm{A}}^{\mathrm{Y}}$ the category of perfect complexes $\mathrm{M}_\bullet$ whose support $\mathrm{Supp}_{\mathrm{A}}(\mathrm{M}_\bullet)$ is contained in Y. Denote by $\mathrm{K}_0(\mathrm{Perf}_{\mathrm{A}}^{\mathrm{Y}})$, or simply $\mathrm{K}_0^{\mathrm{Y}}(\mathrm{A})$, the free abelian group on quasi-isomorphism classes of perfect complexes $\mathrm{M}_\bullet \in \mathrm{Perf}_{\mathrm{A}}^{\mathrm{Y}}$, modulo relations given by exact triangles.

Proposition 18. *Let* A *be a noetherian ring and* $\mathrm{I} \subseteq \mathrm{A}$ *an ideal. There is a canonical homomorphism*

$$\mathrm{K}_0^{\mathrm{V(I)}}(\mathrm{A}) \to \mathrm{G}_0^{\mathrm{V(I)}}(\mathrm{A}) \simeq \mathrm{G}_0(\mathrm{A}/\mathrm{I})$$

which is an isomorphism if A *is regular.*

Proof. Let $\mathrm{M}_\bullet$ be a perfect complex supported on V(I). Then $\mathrm{H}_i(\mathrm{M}_\bullet)$ is supported on V(I) for all i. Thus the homomorphism

$$\mathrm{K}_0^{\mathrm{V(I)}}(\mathrm{A}) \to \mathrm{K}_0(\mathrm{A}) \to \mathrm{G}_0(\mathrm{A})$$

sending $[\mathrm{M}_\bullet] \mapsto \sum_i (-1)^i [\mathrm{H}_i(\mathrm{M}_\bullet)]$, factors through $\mathrm{G}_0^{\mathrm{V(I)}}(\mathrm{A})$ and induces a homomorphism

$$\mathrm{K}_0^{\mathrm{V(I)}}(\mathrm{A}) \to \mathrm{G}_0^{\mathrm{V(I)}}(\mathrm{A}) \simeq \mathrm{G}_0(\mathrm{A}/\mathrm{I})$$

via the dévissage isomorphism (§8.4).

If A is regular, then for every $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}/\mathrm{I}}^{\mathrm{fg}}$, $\mathrm{M}_{[\mathrm{A}]} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$ is perfect. Thus there is a map

$$\mathrm{G}_0(\mathrm{A}/\mathrm{I}) \to \mathrm{K}_0(\mathrm{Perf}_{\mathrm{A}}) \simeq \mathrm{K}_0(\mathrm{A}).$$

By methods that we are by now familiar with, one checks that this is well-defined and is inverse to the map in question. □

8.4. Cup products in G-theory.

Remark 19. Let A be a *regular* ring. Via the canonical "Poincaré duality" isomorphism

$$\mathrm{K}_0(\mathrm{A}) \simeq \mathrm{G}_0(\mathrm{A}),$$

the abelian group $\mathrm{G}_0(\mathrm{A})$ inherits a product. To describe it explicitly, recall that there is an isomorphism

$$\mathrm{G}_0(\mathrm{A}) \xrightarrow{\sim} \mathrm{K}_0(\mathrm{Perf}_\mathrm{A})$$

given by $[\mathrm{M}] \mapsto [\mathrm{M}[0]]$ (see the proof of the Theorem in §5.3). Its inverse is given by $[\mathrm{M}_\bullet] \mapsto \sum_i (-1)^i [\mathrm{H}_i(\mathrm{M}_\bullet)]$. The product on the ring $\mathrm{K}_0(\mathrm{Perf}_\mathrm{A})$ (see §6.1) is computed by the derived tensor product. Thus for $\mathrm{M}, \mathrm{N} \in \mathrm{Mod}_\mathrm{A}^{\mathrm{fg}}$, the product $[\mathrm{M}] \cup [\mathrm{N}] \in \mathrm{G}_0(\mathrm{A})$ is computed by the formula

$$[\mathrm{M}] \cup [\mathrm{N}] = \sum_{i \geqslant 0} (-1)^i [\mathrm{H}_i(\mathrm{M} \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N})] = \sum_{i \geqslant 0} (-1)^i [\mathrm{Tor}_i^\mathrm{A}(\mathrm{M}, \mathrm{N})].$$

In particular, when the higher Tors vanish, we have:

Proposition 20. *Let* A *be a regular ring. Let* I *and* J *be ideals such that the square*

$$\begin{array}{ccc} \mathrm{A} & \longrightarrow & \mathrm{A}/\mathrm{I} \\ \downarrow & & \downarrow \\ \mathrm{A}/\mathrm{J} & \longrightarrow & \mathrm{A}/\mathrm{I} \otimes_\mathrm{A} \mathrm{A}/\mathrm{J} \simeq \mathrm{A}/(\mathrm{I}+\mathrm{J}) \end{array}$$

is Tor-independent (equivalently, $\mathrm{Tor}_i^\mathrm{A}(\mathrm{A}/\mathrm{I}, \mathrm{A}/\mathrm{J}) = 0$ *for all* $i > 0$*). Then we have*

$$[\mathrm{A}/\mathrm{I}] \cup [\mathrm{A}/\mathrm{J}] = [\mathrm{A}/(\mathrm{I}+\mathrm{J})]$$

in $\mathrm{G}_0(\mathrm{A})$*.*

Example 21. The Tor-independence condition holds when I is generated by a Koszul-regular sequence $(f_1, \ldots, f_m)$ and J is generated by a Koszul-regular sequence $(g_1, \ldots, g_n)$ such that $(f_1, \ldots, f_m, g_1, \ldots, g_n)$ is a Koszul-regular sequence. Indeed in that case we have quasi-isomorphism

$$\begin{aligned} \mathrm{A}/\mathrm{I} \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{A}/\mathrm{J} &\simeq \mathrm{Kosz}_\mathrm{A}(f_i)_i \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{Kosz}_\mathrm{A}(g_j)_j \\ &\simeq \mathrm{Kosz}_\mathrm{A}(f_1, \ldots, f_m, g_1, \ldots, g_n) \\ &\simeq \mathrm{A}/(\mathrm{I}+\mathrm{J}). \end{aligned}$$

The first quasi-isomorphism holds because of the Koszul-regularity of $(f_i)_i$ and $(g_j)_j$. The second is clear from the definition of the Koszul complex. The third holds because of the Koszul-regularity of $(f_1, \ldots, f_m, g_1, \ldots, g_n)$.

8.5. Cup products in K-theory with supports.

Lemma 22. *Let* A *be a noetherian ring and* $\mathrm{M}'_\bullet \to \mathrm{M}_\bullet \to \mathrm{M}''_\bullet$ *an exact triangle of coherent complexes. If any two of these three complexes is supported on a subset* $\mathrm{Y} \subseteq \left|\mathrm{Spec}(\mathrm{A})\right|$*, then so is the third.*

Proof. By rotating the triangle, we can assume without loss of generality that $\mathrm{M}'_\bullet$ and $\mathrm{M}''_\bullet$ are the complexes supported on Y (note that shifting a complex has no effect on its support).

Consider the long exact sequence in homology.

$$\cdots \xrightarrow{\partial} \mathrm{H}_i(\mathrm{M}'_\bullet) \xrightarrow{\phi} \mathrm{H}_i(\mathrm{M}_\bullet) \xrightarrow{\psi} \mathrm{H}_i(\mathrm{M}''_\bullet) \xrightarrow{\partial} \cdots$$

From the short exact sequence

$$0 \to \mathrm{Im}(\phi) \hookrightarrow \mathrm{H}_i(\mathrm{M}_\bullet) \twoheadrightarrow \mathrm{Im}(\psi) \to 0$$

we see that $\mathrm{Supp}_\mathrm{A}(\mathrm{H}_i(\mathrm{M}_\bullet)) = \mathrm{Supp}_\mathrm{A}(\mathrm{Im}(\phi)) \cup \mathrm{Supp}_\mathrm{A}(\mathrm{Im}(\psi))$. Since $\mathrm{Im}(\psi) \subseteq \mathrm{H}_i(\mathrm{M}''_\bullet)$ we have

$$\mathrm{Supp}_\mathrm{A}(\mathrm{Im}(\psi)) \subseteq \mathrm{Supp}_\mathrm{A}(\mathrm{H}_i(\mathrm{M}''_\bullet)) \subseteq \mathrm{Y}.$$

Similarly since $\mathrm{Im}(\phi)$ is a quotient of $\mathrm{H}_i(\mathrm{M}'_\bullet)$, we have

$$\mathrm{Supp}_\mathrm{A}(\mathrm{Im}(\phi)) \subseteq \mathrm{Supp}_\mathrm{A}(\mathrm{H}_i(\mathrm{M}'_\bullet)) \subseteq \mathrm{Y}.$$

The claim follows. □

Lemma 23. *Let* A *be a noetherian ring. If* $\mathrm{M}_\bullet$ *is a coherent complex supported on a subset* $\mathrm{Y} \subseteq \left|\mathrm{Spec}(\mathrm{A})\right|$ *and* $\mathrm{N}_\bullet$ *is a coherent complex supported on* $\mathrm{Z} \subseteq \left|\mathrm{Spec}(\mathrm{A})\right|$, *then the derived tensor product*

$$\mathrm{M}_\bullet \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N}_\bullet$$

is supported on $\mathrm{Y} \cap \mathrm{Z}$.

Proof. First suppose that $\mathrm{M}_\bullet$ and $\mathrm{N}_\bullet$ both have homology concentrated in degree zero. Then replacing $\mathrm{M}_\bullet$ by the quasi-isomorphic complex $\mathrm{H}_0(\mathrm{M}_\bullet)[0]$, and similarly for $\mathrm{N}_\bullet$, we reduce to the analogous question for finitely generated modules instead of coherent complexes. Thus let M and N be f.g. A-modules. For a prime ideal $\mathfrak{p} \subseteq \mathrm{A}$, we have

$$\mathrm{H}_i(\mathrm{M} \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N})_\mathfrak{p} \simeq \mathrm{H}_i(\mathrm{M}_\mathfrak{p} \otimes^{\mathbf{L}}_{\mathrm{A}_\mathfrak{p}} \mathrm{N}_\mathfrak{p})$$

by exactness of localization. Thus we see that $\mathrm{H}_i(\mathrm{M} \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N})$ has support contained inside that of M and N, whence the claim.

Next suppose that $\mathrm{M}_\bullet$ is an arbitrary coherent complex but $\mathrm{N}_\bullet$ still has homology concentrated in degree zero. Again we reduce to considering $\mathrm{M}_\bullet \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N}$ for $\mathrm{M}_\bullet \in \mathrm{Coh}_\mathrm{A}$ and $\mathrm{N} \in \mathrm{Mod}^{\mathrm{fg}}_\mathrm{A}$. We want to show that this is "good", where good means it has support contained in the support of $\mathrm{M}_\bullet$ and the support of N. Let $[a, b]$ be the range where the homology of $\mathrm{M}_\bullet$ is concentrated. Recall the exact triangles

$$\mathrm{H}_i(\mathrm{M}_\bullet)[i] \to \tau_{\leqslant i}(\mathrm{M}_\bullet) \to \tau_{\leqslant i-1}(\mathrm{M}_\bullet)$$

for each i. These remain exact after applying $(-) \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N}$:

$$\mathrm{H}_i(\mathrm{M}_\bullet)[i] \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N} \to \tau_{\leqslant i}(\mathrm{M}_\bullet) \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N} \to \tau_{\leqslant i-1}(\mathrm{M}_\bullet) \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N}.$$

For $i = a$, the right-most term is acyclic and $\mathrm{H}_a(\mathrm{M}_\bullet)[a] \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N} \to \tau_{\leqslant a}(\mathrm{M}_\bullet) \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N}$ is a quasi-isomorphism. By the first case above, $\mathrm{H}_a(\mathrm{M}_\bullet)[a] \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N}$ is good. Hence so is $\tau_{\leqslant a}(\mathrm{M}_\bullet) \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N}$. For any i, if $\tau_{\leqslant i-1}(\mathrm{M}_\bullet) \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N}$ is good, then by the previous lemma it follows that $\tau_{\leqslant i}(\mathrm{M}_\bullet) \otimes^{\mathbf{L}}_\mathrm{A} \mathrm{N}$ is also good. By induction we conclude that $\tau_{\leqslant b}(\mathrm{M}_\bullet) \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{N}$ is good. This is quasi-isomorphic to $\mathrm{M}_\bullet \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{N}$ since M is b-coconnective. Thus $\mathrm{M}_\bullet \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{N}$ is good as desired.

Finally, one extends to any coherent complex $\mathrm{N}_\bullet$ by a symmetric argument. □

Construction 24 (Cup product with supports)**.** It follows from the lemma that there is a canonical product

$$\cup : \mathrm{K}_0^{\mathrm{Y}}(\mathrm{A}) \otimes \mathrm{K}_0^{\mathrm{Z}}(\mathrm{A}) \to \mathrm{K}_0^{\mathrm{Y}\cap\mathrm{Z}}(\mathrm{A})$$

defined by $[\mathrm{M}_\bullet] \otimes [\mathrm{N}_\bullet] \mapsto [\mathrm{M}_\bullet \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{N}_\bullet]$.

8.6. Intersection numbers.

Remark 25. Let A be a regular ring. Then via the isomorphisms $\mathrm{K}_0^{\mathrm{V(I)}}(\mathrm{A}) \simeq \mathrm{G}_0(\mathrm{A}/\mathrm{I})$, for any ideal I, the cup product with supports induces a product of the form

$$\mathrm{G}_0(\mathrm{A}/\mathrm{I}) \otimes \mathrm{G}_0(\mathrm{A}/\mathrm{J}) \to \mathrm{G}_0(\mathrm{A}/(\mathrm{I}+\mathrm{J})).$$

Remark 26. Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Then there is a unique closed point $x \in |\mathrm{Spec}(\mathrm{A})|$ (with residue field $\kappa(x) = \mathrm{A}/\mathfrak{m}$). Then $\mathrm{V}(\mathfrak{m}) = \{x\}$, so dévissage yields the isomorphism

$$\mathrm{G}_0^{\{x\}}(\mathrm{A}) \simeq \mathrm{G}_0(\kappa(x)) \simeq \mathbf{Z}.$$

Definition 27. Let A be a regular local ring. Let M and N be f.g. A-modules with supports V(I) and V(J), respectively, such that $\mathrm{V}(\mathrm{I}+\mathrm{J}) = \{x\}$ (where x is the closed point). Consider the pairing

$$\chi_{\mathrm{A}} : \mathrm{G}_0(\mathrm{A}/\mathrm{I}) \otimes \mathrm{G}_0(\mathrm{A}/\mathrm{J}) \to \mathrm{G}_0(\mathrm{A}/(\mathrm{I}+\mathrm{J})) \simeq \mathrm{G}_0(\kappa(x)) \simeq \mathbf{Z}.$$

Consider the classes $[\mathrm{M}] \in \mathrm{G}_0^{\mathrm{V(I)}}(\mathrm{A}) \simeq \mathrm{G}_0(\mathrm{A}/\mathrm{I})$, $[\mathrm{N}] \in \mathrm{G}_0^{\mathrm{V(J)}}(\mathrm{A}) \simeq \mathrm{G}_0(\mathrm{A}/\mathrm{J})$. These give rise via χ_{A} to an integer $\chi_{\mathrm{A}}(\mathrm{M},\mathrm{N}) \in \mathbf{Z}$ called the *intersection multiplicity* of M and N.

Exercise 28. Let A be a noetherian local ring and $\mathrm{M} \in \mathrm{Mod}_{\mathrm{A}}^{\mathrm{fg}}$.

(i) Show that M is supported on $\mathrm{V}(\mathfrak{m}) \simeq \{x\}$ iff it is of finite length.

(ii) Show that the isomorphism $\mathrm{G}_0^{\{x\}}(\mathrm{A}) \simeq \mathbf{Z}$ above sends $[\mathrm{M}] \mapsto \ell_{\mathrm{A}}(\mathrm{M})$, where $\ell_{\mathrm{A}}(\mathrm{M})$ denotes the length of M.

(iii) If A is regular, show that the intersection multiplicity is computed by the formula

$$\chi_{\mathrm{A}}(\mathrm{M},\mathrm{N}) = \sum_i (-1)^i \, \ell_{\mathrm{A}}(\mathrm{Tor}_i^{\mathrm{A}}(\mathrm{M},\mathrm{N})).$$

(This is Serre's intersection number.)

8.7. Irreducible subsets of the Zariski spectrum.

Definition 29. Let A be a commutative ring. A *closed subset* of $|\mathrm{Spec}(A)|$ is a subset of the form $V(I)$, where I is an ideal. An *irreducible closed subset* is a subset of the form $V(I)$, where $\mathrm{rad}(I)$ is a prime ideal. An *integral closed subset* is a subset of the form $V(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal.

Example 30. Note that $|\mathrm{Spec}(A)|$ is irreducible (as a subset of itself) iff the nilradical is a prime ideal, i.e., iff A_{red} is an integral domain. Equivalently, A contains exactly one minimal prime ideal. In this case we also say that A is irreducible.

Definition 31. Note that every integral closed subset $V(\mathfrak{p})$ is contained in an integral closed subset $V(\mathfrak{q})$ where $\mathfrak{q}$ is a *minimal* prime ideal. Subsets of the latter form are called *irreducible components* of $|\mathrm{Spec}(A)|$.

Remark 32. Let $I \subseteq A$ be an ideal and consider the subset $V(I) \subseteq |\mathrm{Spec}(A)|$. Via the canonical bijection $V(I) \simeq |\mathrm{Spec}(A/I)|$, we can regard any subset $Y \subseteq V(I)$ as a subset of $|\mathrm{Spec}(A/I)|$. We say Y is closed/integral/irreducible in $V(I)$, or an irreducible component of $V(I)$, if it is such as a subset of $|\mathrm{Spec}(A/I)|$. Similarly we say a point $\eta \in V(I)$ is a generic point of $V(I)$ if it is a generic point of $|\mathrm{Spec}(A/I)|$.

Definition 33. The *codimension* of an irreducible closed subset $Y \subseteq |\mathrm{Spec}(A)|$ is the maximal length n of a chain

$$Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$$

of irreducible closed subsets of $|\mathrm{Spec}(A)|$. More generally, if $Y \subseteq |\mathrm{Spec}(A)|$ is a closed subset, we say that its codimension is the infimum of the codimensions of all irreducible closed subsets contained in Y. We denote this natural number by $\mathrm{codim}(Y)$, or $\mathrm{codim}_A(Y)$ when there is potential ambiguity. We say Y is of *pure codimension n* if all its irreducible components are of codimension n.

Example 34. The irreducible subsets of codimension 0 are the irreducible components.