8.1. Supports of modules.

**Definition 1.** Let $A$ be a ring and $M \in \text{Mod}^\text{fg}_A$ a f.g. $A$-module. We say that $M$ is **supported at a point** $p \in |\text{Spec}(A)|$ if

$$M \otimes_A \kappa(p) \neq 0.$$ 

We write $\text{Supp}_A(M) \subseteq |\text{Spec}(A)|$ for the subset of points at which $M$ is supported.

**Example 2.** We have $\text{Supp}_A(A) = |\text{Spec}(A)|$.

**Example 3.** Let $I \subseteq A$ be an ideal. We write $V(I) := \text{Supp}_A(A/I)$. This is the set of points $p$ such that $\kappa(p)/I\kappa(p) \neq 0$, i.e., $I \cdot \kappa(p) = 0$. This is equivalent to the condition that $A \to \kappa(p)$ factors through the quotient $A/I$. It follows that there is a canonical bijection $|\text{Spec}(A/I)| \simeq V(I)$.

**Remark 4.** By Nakayama’s lemma, $M \in \text{Mod}^\text{fg}_A$ is supported at a point $p$ iff $M_p \neq 0$, where $p = \text{Ker}(A \to \kappa(p))$. In particular, $\text{Supp}_A(M)$ is empty iff $M$ is zero.

**Proposition 5.**

(i) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of f.g. $A$-modules, then $\text{Supp}_A(M) = \text{Supp}_A(M') \cup \text{Supp}_A(M'')$.

(ii) Let $M$ be a f.g. $A$-module. Suppose $M = \sum_i M_i$ for some family of submodules $(M_i)_i$. Then $\text{Supp}_A(M) = \bigcup_i \text{Supp}_A(M_i)$.

(iii) Let $M$ and $N$ be f.g. $A$-modules. Then $\text{Supp}_A(M \otimes_A N) = \text{Supp}_A(M) \cap \text{Supp}_A(N)$.

**Corollary 6.** Let $A$ be a noetherian ring and $M \in \text{Mod}^\text{fg}_A$. Recall that $M$ admits a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ where the successive quotients $M_i/M_{i-1}$, $1 \leq i \leq n$, are isomorphic to $A/p_i$ where $p_i$ are prime ideals. For any such filtration, we have

$$\text{Supp}_A(M) = \bigcup_i V(p_i).$$

**Proposition 7.**

(i) $V(\langle 0 \rangle) = |\text{Spec}(A)|$.

(ii) For any ideal $I$ we have $V(I) = \emptyset$ iff $I = \langle 1 \rangle$.

(iii) For any two ideals $I$ and $J$ we have $V(I) \cap V(J) = V(I + J)$.

(iv) For any two ideals $I$ and $J$ we have $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.

(v) For any ideal $I$ we have $V(I) = V(\text{rad}(I))$, where $\text{rad}(I) \subseteq A$ denotes the radical of $I$. 

Exercise 8. Let $I$ and $J$ be ideals of $A$. Then $V(I) \subseteq V(J)$ iff $J \subseteq \text{rad}(I)$.

Corollary 9. Let $A$ be a commutative ring. The homomorphism $A \to A_{\text{red}}$ induces a bijection $|\text{Spec}(A)| \simeq |\text{Spec}(A_{\text{red}})|$.

Proof. We have $|\text{Spec}(A)| = V(\langle 0 \rangle) = V(\text{rad}(\langle 0 \rangle)) \simeq |\text{Spec}(A/\text{rad}(\langle 0 \rangle))|$. \hfill \Box

Corollary 10. Let $M$ be a f.g. $A$-module. Then $\text{Supp}_A(M) = V(A/I)$, where $I = \text{Ann}_A(M)$ is the ideal consisting of $a \in A$ such that $ax = 0$ for all $x \in M$.

Proof. Choose an $A$-linear surjection $A^{\oplus n} \twoheadrightarrow M$, corresponding to elements $x_i \in M$. Then we have $M = \sum_i Ax_i$ so by the Proposition,
$$\text{Supp}_A(M) = \bigcup_i \text{Supp}_A(Ax_i).$$
Note that $Ax_i \simeq A/I_i$ where
$$I_i = \text{Ann}(x_i) = \text{Ker}(A \xrightarrow{z_i} M) \subseteq A.$$ Thus we have
$$\text{Supp}_A(M) = \bigcup_i V(I_i) = V(I)$$ by the Example above and the fact that $I = \bigcap_i I_i$. \hfill \Box

Corollary 11. Let $A$ be a noetherian ring and $M \in \text{Mod}_{fg}^A$. Then for any ideal $I \subseteq A$, we have $\text{Supp}_A(M) \subseteq V(I)$ iff $M$ is $I^\infty$-torsion.

Proof. Let $J = \text{Ann}_A(M)$. Then $\text{Supp}_A(M) = V(J)$, so the condition is equivalent to $V(J) \subseteq V(I)$. This is equivalent to $I \subseteq \text{rad}(J)$, and since $A$ is noetherian, to $I^k \subseteq J$ for some $k \geq 0$. But this is the same as $I^k M = 0$. \hfill \Box

8.2. G-theory with supports. We consider a variant of G-theory where the modules have prescribed support.

Construction 12. Let $A$ be a noetherian ring and $Y \subseteq |\text{Spec}(A)|$ a subset. We denote by $G_0^Y(A)$ the free abelian group on isomorphism classes of f.g. $A$-modules $M$ which are supported on $Y$, i.e., for which $\text{Supp}_A(M) \subseteq Y$, modulo relations given by short exact sequences.

Since $M$ is supported on $V(I)$ iff it is $I^\infty$-torsion, $G_0^{V(I)}(A)$ is just another notation for $K_0(\text{Mod}_{fg}^A(I^\infty))$. In particular, the d\'evissage isomorphism can be re-interpreted as the assertion that G-theory does not see the difference between the category of $A/I$-modules and that of $A$-modules supported on $V(I) \simeq |\text{Spec}(A/I)|$.

Corollary 13. Let $A$ be a noetherian ring and $I$ an ideal. Then we have a canonical isomorphism of abelian groups
$$G_0(A/I) \simeq G_0^{V(I)}(A).$$
8.3. K-theory with supports.

**Definition 14.** Let $A$ be a ring. A perfect complex $M_\bullet$ is **supported at a point** $p \in \text{Spec}(A)$ if at least one homology group $H_i(M_\bullet)$ is supported at $p$. We let $\text{Supp}_A(M_\bullet) \subseteq \text{Spec}(A)$ denote the subset of points where $M_\bullet$ is supported. By definition,

$$\text{Supp}_A(M_\bullet) = \bigcup_{i \in \mathbb{Z}} \text{Supp}_A(H_i(M_\bullet)).$$

This is a finite union since $M_\bullet$ is perfect.

**Remark 15.** The same definition also makes sense more generally for coherent complexes.

**Remark 16.** The support of $M_\bullet$ only depends on its quasi-isomorphism class. In particular, $M_\bullet$ has empty support iff it is acyclic.

**Construction 17.** Let $A$ be a ring and $Y \subseteq \text{Spec}(A)$ a subset. Denote by $\text{Perf}^Y_A$ the category of perfect complexes $M_\bullet$ whose support $\text{Supp}_A(M_\bullet)$ is contained in $Y$. Denote by $K_0(\text{Perf}^Y_A)$, or simply $K^Y_0(A)$, the free abelian group on quasi-isomorphism classes of perfect complexes $M_\bullet \in \text{Perf}^Y_A$, modulo relations given by exact triangles.

**Proposition 18.** Let $A$ be a noetherian ring and $I \subseteq A$ an ideal. There is a canonical homomorphism

$$K^V_0(A) \to G^V_0(A) \simeq G_0(A/I)$$

which is an isomorphism if $A$ is regular.

**Proof.** Let $M_\bullet$ be a perfect complex supported on $V(I)$. Then $H_i(M_\bullet)$ is supported on $V(I)$ for all $i$. Thus the homomorphism

$$K^V_0(A) \to K_0(A) \to G_0(A)$$

sending $[M_\bullet] \mapsto \sum_i (-1)^i [H_i(M_\bullet)]$, factors through $G^V_0(A)$ and induces a homomorphism

$$K^V_0(A) \to G^V_0(A) \simeq G_0(A/I)$$

via the dévissage isomorphism ($\S 8.4$).

If $A$ is regular, then for every $M \in \text{Mod}^f_{A/I}$, $M|_A \in \text{Mod}^f_A$ is perfect. Thus there is a map

$$G_0(A/I) \to K_0(\text{Perf}_A) \simeq K_0(A).$$

By methods that we are by now familiar with, one checks that this is well-defined and is inverse to the map in question. \qed
8.4. Cup products in G-theory.

Remark 19. Let A be a regular ring. Via the canonical “Poincaré duality” isomorphism

\[ K_0(A) \cong G_0(A), \]

the abelian group \( G_0(A) \) inherits a product. To describe it explicitly, recall that there is an isomorphism

\[ G_0(A) \cong K_0(\text{Perf}_A) \]

given by \([M] \mapsto [M[0]]\) (see the proof of the Theorem in §5.3). Its inverse is given by \([M \cdot] \mapsto \sum_{i \geq 0} (-1)^i [H_i(M \otimes A N)]\). The product on the ring \( K_0(\text{Perf}_A) \) (see §6.1) is computed by the derived tensor product. Thus for \( M, N \in \text{Mod}^g \), the product \([M] \cup [N] \in G_0(A)\) is computed by the formula

\[ [M] \cup [N] = \sum_{i \geq 0} (-1)^i [H_i(M \otimes A N)] = \sum_{i \geq 0} (-1)^i [\text{Tor}^A_i(M, N)]. \]

In particular, when the higher Tors vanish, we have:

Proposition 20. Let A be a regular ring. Let I and J be ideals such that the square

\[
\begin{array}{ccc}
A & \rightarrow & A/I \\
\downarrow & & \downarrow \\
A/J & \rightarrow & A/I \otimes_A A/J \cong A/(I + J)
\end{array}
\]

is Tor-independent (equivalently, \( \text{Tor}^A_i(A/I, A/J) = 0 \) for all \( i > 0 \)). Then we have

\[ [A/I] \cup [A/J] = [A/(I + J)] \]

in \( G_0(A) \).

Example 21. The Tor-independence condition holds when I is generated by a Koszul-regular sequence \((f_1, \ldots, f_m)\) and J is generated by a Koszul-regular sequence \((g_1, \ldots, g_n)\) such that \((f_1, \ldots, f_m, g_1, \ldots, g_n)\) is a Koszul-regular sequence. Indeed in that case we have quasi-isomorphism

\[ A/I \otimes_A^L A/J \cong \text{Kosz}_A(f_1)_i \otimes_A^L \text{Kosz}_A(g_1)_j \]

\[ \cong \text{Kosz}_A(f_1, \ldots, f_m, g_1, \ldots, g_n) \]

\[ \cong A/(I + J). \]

The first quasi-isomorphism holds because of the Koszul-regularity of \((f_i)_i\) and \((g_j)_j\). The second is clear from the definition of the Koszul complex. The third holds because of the Koszul-regularity of \((f_1, \ldots, f_m, g_1, \ldots, g_n)\).

8.5. Cup products in K-theory with supports.

Lemma 22. Let A be a noetherian ring and \( M_1, M_2, M_3 \) an exact triangle of coherent complexes. If any two of these three complexes is supported on a subset \( Y \subseteq |\text{Spec}(A)| \), then so is the third.
Proof. By rotating the triangle, we can assume without loss of generality that \(M'\) and \(M''\) are the complexes supported on \(Y\) (note that shifting a complex has no effect on its support).

Consider the long exact sequence in homology.
\[
\cdots \to \partial H_i(M') \xrightarrow{\varphi} H_i(M) \xrightarrow{\psi} H_i(M'') \xrightarrow{\partial} \cdots
\]
From the short exact sequence
\[
0 \to \text{Im}(\varphi) \to H_i(M) \xrightarrow{\psi} \text{Im}(\psi) \to 0
\]
we see that \(\text{Supp}_A(H_i(M)) = \text{Supp}_A(\text{Im}(\varphi)) \cup \text{Supp}_A(\text{Im}(\psi))\). Since \(\text{Im}(\psi) \subseteq H_i(M'')\) we have
\[
\text{Supp}_A(\text{Im}(\psi)) \subseteq \text{Supp}_A(H_i(M'')) \subseteq Y.
\]
Similarly since \(\text{Im}(\varphi)\) is a quotient of \(H_i(M')\), we have
\[
\text{Supp}_A(\text{Im}(\varphi)) \subseteq \text{Supp}_A(H_i(M')) \subseteq Y.
\]
The claim follows. \(\square\)

Lemma 23. Let \(A\) be a noetherian ring. If \(M_*\) is a coherent complex supported on a subset \(Y \subseteq |\text{Spec}(A)|\) and \(N_*\) is a coherent complex supported on \(Z \subseteq |\text{Spec}(A)|\), then the derived tensor product
\[
M_* \otimes^L_A N_*
\]
is supported on \(Y \cap Z\).

Proof. First suppose that \(M_*\) and \(N_*\) both have homology concentrated in degree zero. Then replacing \(M_*\) by the quasi-isomorphic complex \(H_0(M_*)[0]\), and similarly for \(N_*\), we reduce to considering \(M_* \otimes^L_A N\) for \(M_* \in \text{Coh}_A\) and \(N \in \text{Mod}^f_A\). We want to show that this is “good”, where good means it has support contained in the support of \(M_*\) and the support of \(N\). Let \([a, b]\) be the range where the homology of \(M_*\) is concentrated. Recall the exact triangles
\[
H_i(M_*[i]) \to \tau_{<i}(M_*) \to \tau_{\leq i-1}(M_*)
\]
for each \(i\). These remain exact after applying \((-) \otimes^L_A N\):
\[
H_i(M_*[i]) \otimes^L_A N \to \tau_{<i}(M_*) \otimes^L_A N \to \tau_{\leq i-1}(M_*) \otimes^L_A N.
\]
For \(i = a\), the right-most term is acyclic and \(H_a(M_*)[a] \otimes^L_A N \to \tau_{<a}(M_*) \otimes^L_A N\) is a quasi-isomorphism. By the first case above, \(H_a(M_*)[a] \otimes^L_A N\) is good. Hence so is \(\tau_{<a}(M_*) \otimes^L_A N\). For any \(i\), if \(\tau_{\leq i-1}(M_*) \otimes^L_A N\) is good, then by the previous lemma it follows that \(\tau_{<i}(M_*) \otimes^L_A N\) is also good. By induction we conclude
that \( \tau_{<b}(M_\bullet) \otimes^L_A N \) is good. This is quasi-isomorphic to \( M_\bullet \otimes^L_A N \) since \( M \) is \( b \)-coconnective. Thus \( M_\bullet \otimes^L_A N \) is good as desired.

Finally, one extends to any coherent complex \( N_\bullet \) by a symmetric argument. \( \square \)

**Construction 24** (Cup product with supports). It follows from the lemma that there is a canonical product

\[
\bigcup : K^Y_0(A) \otimes K^Z_0(A) \to K^{Y \cap Z}_0(A)
\]
defined by \([M_\bullet] \otimes [N_\bullet] \mapsto [M_\bullet \otimes^L_A N_\bullet] \).

### 8.6. Intersection numbers.

**Remark 25.** Let \( A \) be a regular ring. Then via the isomorphisms \( K^V_0(A) \simeq G_0(A/I) \), for any ideal \( I \), the cup product with supports induces a product of the form

\[
G_0(A/I) \otimes G_0(A/J) \to G_0(A/(I+J)).
\]

**Remark 26.** Let \( A \) be a noetherian local ring with maximal ideal \( m \). Then there is a unique closed point \( x \in |\text{Spec}(A)| \) (with residue field \( \kappa(x) = A/m \)). Then \( V(m) = \{x\} \), so dévissage yields the isomorphism

\[
G^*_0(A) \simeq G_0(\kappa(x)) \simeq \mathbb{Z}.
\]

**Definition 27.** Let \( A \) be a regular local ring. Let \( M \) and \( N \) be f.g. \( A \)-modules with supports \( V(I) \) and \( V(J) \), respectively, such that \( V(I + J) = \{x\} \) (where \( x \) is the closed point). Consider the pairing

\[
\chi_A : G_0(A/I) \otimes G_0(A/J) \to G_0(A/(I+J)) \simeq G_0(\kappa(x)) \simeq \mathbb{Z}.
\]

Consider the classes \([M] \in G^V_0(A) \simeq G_0(A/I)\), \([N] \in G^V_0(A) \simeq G_0(A/J)\). These give rise via \( \chi_A \) to an integer \( \chi_A(M, N) \in \mathbb{Z} \) called the *intersection multiplicity* of \( M \) and \( N \).

**Exercise 28.** Let \( A \) be a noetherian local ring and \( M \in \text{Mod}^A_{f.g.} \).

(i) Show that \( M \) is supported on \( V(m) \simeq \{x\} \) iff it is of finite length.

(ii) Show that the isomorphism \( G^*_0(A) \simeq \mathbb{Z} \) above sends \([M] \mapsto \ell_A(M)\), where \( \ell_A(M) \) denotes the length of \( M \).

(iii) If \( A \) is regular, show that the intersection multiplicity is computed by the formula

\[
\chi_A(M, N) = \sum_i (-1)^i \ell_A(\text{Tor}^A_i(M, N)).
\]

(This is Serre’s intersection number.)
8.7. Irreducible subsets of the Zariski spectrum.

**Definition 29.** Let $A$ be a commutative ring. A *closed subset* of $|\text{Spec}(A)|$ is a subset of the form $V(I)$, where $I$ is an ideal. An *irreducible closed subset* is a subset of the form $V(I)$, where $\text{rad}(I)$ is a prime ideal. An *integral closed subset* is a subset of the form $V(p)$, where $p$ is a prime ideal.

**Example 30.** Note that $|\text{Spec}(A)|$ is irreducible (as a subset of itself) iff the nilradical is a prime ideal, i.e., iff $A_{\text{red}}$ is an integral domain. Equivalently, $A$ contains exactly one minimal prime ideal. In this case we also say that $A$ is irreducible.

**Definition 31.** Note that every integral closed subset $V(p)$ is contained in an integral closed subset $V(q)$ where $q$ is a *minimal* prime ideal. Subsets of the latter form are called *irreducible components* of $|\text{Spec}(A)|$.

**Remark 32.** Let $I \subseteq A$ be an ideal and consider the subset $V(I) \subseteq |\text{Spec}(A)|$. Via the canonical bijection $V(I) \simeq |\text{Spec}(A/I)|$, we can regard any subset $Y \subseteq V(I)$ as a subset of $|\text{Spec}(A/I)|$. We say $Y$ is closed/integral/irreducible in $V(I)$, or an irreducible component of $V(I)$, if it is such as a subset of $|\text{Spec}(A/I)|$. Similarly we say a point $\eta \in V(I)$ is a generic point of $V(I)$ if it is a generic point of $|\text{Spec}(A/I)|$.

**Definition 33.** The *codimension* of an irreducible closed subset $Y \subseteq |\text{Spec}(A)|$ is the maximal length $n$ of a chain

$$Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$$

of irreducible closed subsets of $|\text{Spec}(A)|$. More generally, if $Y \subseteq |\text{Spec}(A)|$ is a closed subset, we say that its codimension is the infimum of the codimensions of all irreducible closed subsets contained in $Y$. We denote this natural number by $\text{codim}(Y)$, or $\text{codim}_A(Y)$ when there is potential ambiguity. We say $Y$ is of *pure codimension* $n$ if all its irreducible components are of codimension $n$.

**Example 34.** The irreducible subsets of codimension 0 are the irreducible components.