Lecture 6

Proper morphisms in derived algebraic geometry

In the first half of this course, we looked at certain forms of *flat* descent properties satisfied by perfect complexes and algebraic K-theory. In the second half, we will be concerned with *proper* descent formulas. We will begin our study by reviewing in the present lecture the theory of proper morphisms and closed immersions in derived algebraic geometry. Of special interest is the class of derived regular immersions, which is the setting in which one can naturally define derived blow-ups (as we will explain next lecture).

1. Proper morphisms.

1.1. Let $p : \mathrm{Y} \to \mathrm{X}$ be a morphism of derived schemes.

Definition 1.2. *We say that p is* proper *if the following conditions hold:*

(i) *The morphism p is of finite type, i.e.* $p_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ *is of finite type.*

(ii) *The morphism p is separated, i.e.* $p_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ *is separated.*

(iii) *The morphism p satisfies the right lifting property with respect to morphisms of the form* $\mathrm{Spec}(\mathrm{K}) \to \mathrm{Spec}(\mathrm{R})$*, where* $\mathrm{R} \in \mathrm{CRing}$ *is a valuation ring with fraction field* K*. That is, for any solid commutative square*

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{K}) & \longrightarrow & \mathrm{Y} \\ \downarrow & \nearrow & \downarrow f \\ \mathrm{Spec}(\mathrm{R}) & \longrightarrow & \mathrm{X}, \end{array}$$

there exists a dashed arrow making the resulting diagram commute.

Remark 1.3. Separatedness implies that the dotted lift will automatically be unique.

Note that condition (iii) only depends on $p_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$. It follows that:

Proposition 1.4. *A morphism* $p : \mathrm{Y} \to \mathrm{X}$ *is proper iff the morphism* $p_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ *is a proper morphism of classical schemes.*

1.5. From the perspective of quasi-coherent sheaves, the main property of proper morphisms is as follows:

Theorem 1.6 (Grothendieck, Lurie)**.** *Let* $p : \mathrm{Y} \to \mathrm{X}$ *be a proper morphism of derived schemes. Then the direct image functor* $p_* : \mathrm{Qcoh}(\mathrm{Y}) \to \mathrm{Qcoh}(\mathrm{X})$ *preserves almost perfect (resp. coherent) sheaves.*

1.7. The two main sources of examples of proper morphisms are projective bundles $\mathbf{P}_{\mathrm{X}}(\mathcal{E}) \to \mathrm{X}$, and closed immersions $\mathrm{Z} \hookrightarrow \mathrm{X}$. In fact, according to Chow's lemma (of which there is a derived version), these two classes of maps essentially determine all proper morphisms (up to birational morphisms).

2. Vector bundles and projective bundles. Before introducing projective bundles, we take a moment to introduce their affine counterparts (vector bundles).

2.1. Let X be a derived scheme. We can define the ∞-category of *quasi-coherent algebras* over X by the limit

$$\mathrm{QcohAlg}(\mathrm{X}) = \varprojlim_{\mathrm{Spec(R)}\to\mathrm{X}} \mathrm{SCRing}_{\mathrm{R}},$$

indexed over pairs $(\mathrm{R}, \mathrm{Spec(R)} \to \mathrm{X})$ with $\mathrm{R} \in \mathrm{SCRing}$. In particular we have $\mathrm{QcohAlg}(\mathrm{Spec(R)}) = \mathrm{SCRing}_{\mathrm{R}}$ in the affine case.

Construction 2.2. *Let* X *be a derived scheme and* $\mathcal{A} \in \mathrm{QcohAlg}(\mathrm{X})$ *a quasi-coherent* $\mathcal{O}_{\mathrm{X}}$*-algebra. We define a derived scheme* $\mathrm{Spec}_{\mathrm{X}}(\mathcal{A})$ *over* X *as follows. For formal reasons there is an equivalence between (pre)sheaves of spaces on* $\mathrm{DSch}_{/\mathrm{X}}$*, and derived (pre)stacks over* X*. Let* $\mathrm{Spec}_{\mathrm{X}}(\mathcal{A})$ *denote the derived stack corresponding to the sheaf of spaces on* $\mathrm{DSch}_{/\mathrm{X}}$ *given by*

$$(\mathrm{S} \xrightarrow{f} \mathrm{X}) \mapsto \mathrm{Maps}_{\mathrm{QcohAlg(S)}}(f^*\mathcal{A}, \mathcal{O}_{\mathrm{S}}).$$

Proposition 2.3. *The derived stack* $\mathrm{Spec}_{\mathrm{X}}(\mathcal{A})$ *is schematic. Further, the morphism of derived schemes* $\mathrm{Spec}_{\mathrm{X}}(\mathcal{A}) \to \mathrm{X}$ *is* affine *(which means that* $\mathrm{Spec}_{\mathrm{X}}(\mathcal{A}) \times_{\mathrm{X}} \mathrm{S}$ *is affine for any* $\mathrm{S} \in \mathrm{DSch}^{\mathrm{aff}}$*).*

Proof. Since the construction is stable under base change, we can choose an affine Zariski cover of X and thereby reduce to the affine case. If $\mathrm{X} = \mathrm{Spec(R)}$ and $\mathrm{A} = \Gamma(\mathrm{X}, \mathcal{A})$, then we have $\mathrm{Spec}_{\mathrm{X}}(\mathcal{A}) = \mathrm{Spec(A)}$. Note that this also shows that $\mathrm{Spec}_{\mathrm{X}}(\mathcal{A}) \to \mathrm{X}$ is affine. □

Construction 2.4. *Let* $\mathcal{E}$ *be a locally free sheaf of finite rank n, i.e. a quasi-coherent sheaf such that there exists a Zariski cover* $\mathrm{X} = \bigcup_i \mathrm{U}_i$ *with* $\mathcal{E}|_{\mathrm{U}_i}$ *free of rank n for each i (in the sense that* $\mathcal{E}|_{\mathrm{U}_i} \approx \mathcal{O}_{\mathrm{U}_i}^{\oplus n}$*). The* vector bundle *associated to* $\mathcal{E}$ *is the derived scheme over* X

$$\mathbf{V}_{\mathrm{X}}(\mathcal{E}) := \mathrm{Spec}_{\mathrm{X}}(\mathrm{Sym}_{\mathcal{O}_{\mathrm{X}}}(\mathcal{E})).$$

It is easy to compute the cotangent complex:

Exercise 2.5. *At any point* $x : \mathrm{Spec(R)} \to \mathbf{V}_{\mathrm{X}}(\mathcal{E})$*, we have*

$$x^*\mathcal{L}_{\mathbf{V}_{\mathrm{X}}(\mathcal{E})/\mathrm{X}} = x^*\mathcal{E}.$$

In particular, it follows that $\mathbf{V}_{\mathrm{X}}(\mathcal{E}) \to \mathrm{X}$ *is smooth.*

Example 2.6. Taking $\mathcal{E} = \mathcal{O}_{\mathrm{X}}^{\oplus n}$ to be *free* gives us n-dimensional affine space $\mathbf{A}^n_{\mathrm{X}} := \mathbf{V}_{\mathrm{X}}(\mathcal{O}_{\mathrm{X}}^{\oplus n})$.

2.7. We now proceed to the projective version.

Construction 2.8. *Let* X *be a derived scheme and* $\mathcal{E} \in \mathrm{Qcoh}(\mathrm{X})$ *a locally free sheaf of finite rank n. We define a derived stack* $\mathbf{P}_{\mathrm{X}}(\mathcal{E})$ *over* X *corresponding to the following presheaf on* $\mathrm{DSch}_{/\mathrm{X}}$*. We send any* $\mathrm{S} \xrightarrow{f} \mathrm{X}$ *to the space of pairs* $(\mathcal{L}, u)$*, where* $\mathcal{L}$ *is a locally free sheaf of rank 1 on* S*, and* $u : f^*(\mathcal{E}) \to \mathcal{L}$ *is a morphism in* $\mathrm{Qcoh(S)}$ *which is surjective on* π_0*.*

Proposition 2.9. *The derived stack* $\mathbf{P}_{\mathrm{X}}(\mathcal{E})$ *is schematic, and the morphism of derived schemes* $\mathbf{P}_{\mathrm{X}}(\mathcal{E}) \to \mathrm{X}$ *is smooth and proper. Further, at any point* $x : \mathrm{S} = \mathrm{Spec(R)} \to \mathbf{P}_{\mathrm{X}}(\mathcal{E})$*, classifying a pair* $(\mathcal{L}, u)$*, the relative cotangent complex is given by*

$$x^*\mathcal{L}_{\mathbf{P}_{\mathrm{X}}(\mathcal{E})/\mathrm{X}} = \mathcal{M} \otimes \mathcal{L}^{\otimes -1},$$

where $\mathcal{M}$ *denotes the fibre of u.*

Proof. One can imitate the usual proof of representability [1, Thm. 9.7.4] to show that $\mathbf{P}_{\mathrm{X}}(\mathcal{E})$ is a derived scheme. Smoothness follows from the computation of the cotangent complex, which we omit (see [2, § 19.2.6]).

Properness can be deduced as follows. It suffices to show that $\mathbf{P}_{\mathrm{X}}(\mathcal{E})_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ is proper. Smoothness implies flatness so we have $\mathbf{P}_{\mathrm{X}}(\mathcal{E})_{\mathrm{cl}} = \mathbf{P}_{\mathrm{X}}(\mathcal{E}) \times_{\mathrm{X}} \mathrm{X}_{\mathrm{cl}} = \mathbf{P}_{\mathrm{X}_{\mathrm{cl}}}(i^*\mathcal{E})$ where $i : \mathrm{X}_{\mathrm{cl}} \hookrightarrow \mathrm{X}$. Then the claim follows from the fact that the classical projective bundle $\mathbf{P}_{\mathrm{X}_{\mathrm{cl}}}(i^*\mathcal{E}) \to \mathrm{X}_{\mathrm{cl}}$ is proper. □

3. Closed immersions. We now discuss the other main class of examples of proper morphisms.

3.1. Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a morphism of derived schemes. We define:

Definition 3.2.

(i) *If* X *and* Z *are affine, we say that* i *is a* closed immersion *if it corresponds to a homomorphism of simplicial commutative rings* $\mathrm{A} \to \mathrm{B}$ *which induces a* surjection *on* π_0.

(ii) *In general, we say that* i *is a* closed immersion *if for any affine derived scheme* $\mathrm{S} = \mathrm{Spec(R)}$ *and any morphism* $\mathrm{S} \to \mathrm{X}$*, the derived scheme* $\mathrm{Y} \times_{\mathrm{X}} \mathrm{S}$ *is affine, and the morphism of affine derived schemes* $\mathrm{Y} \times_{\mathrm{X}} \mathrm{S} \to \mathrm{S}$ *is a closed immersion.*

3.3. We have:

Proposition 3.4. *Let* $i : \mathrm{Z} \to \mathrm{X}$ *be a morphism of derived schemes. Then the following are equivalent:*

(i) *The morphism* i *is a closed immersion.*

(ii) *The morphism* i_{cl} *is a closed immersion of classical schemes.*

Proof. If X is affine, then this is true by definition. Recall that (ii) is equivalent to the condition that for any classical affine scheme S and any morphism $\mathrm{S} \to \mathrm{X}$, the morphism $(\mathrm{Z} \times_{\mathrm{X}} \mathrm{S})_{\mathrm{cl}} \to \mathrm{S}_{\mathrm{cl}}$ is a closed immersion of classical affine schemes. The claim follows therefore from the affine case, and the following observation. □

Proposition 3.5. *Let* X *be a derived scheme. Then* X *is affine iff* X_{cl} *is affine.*

In particular, Proposition 3.4 has the obvious corollary:

Corollary 3.6. *Any closed immersion of derived schemes is proper.*

Example 3.7. A special example of a closed immersion is the canonical morphism

$$\mathrm{X}_{\mathrm{cl}} \hookrightarrow \mathrm{X}.$$

More generally, we refer to any closed immersion $\mathrm{Z} \hookrightarrow \mathrm{X}$ which induces an isomorphism $\mathrm{Z}_{\mathrm{cl}} \approx \mathrm{X}_{\mathrm{cl}}$ as a *nil-immersion.*

4. Regular closed immersions. In classical algebraic geometry, we can view closed subschemes as being "cut out" by functions (or by the ideal they generate). In derived geometry, not all closed immersions are of this form (Example 3.7). In this section we look at a special class of closed immersions which are "defined by equations".

4.1. We begin with an important construction in derived commutative algebra:

Construction 4.2. *Let* R *be a simplicial commutative ring. Let* $f_1, \ldots, f_n$ *be a sequence of points of the underlying space* $\mathrm{R}_{\mathrm{Spc}}$. *We define a new simplicial commutative ring* $\mathrm{R}/\!\!/(f_i)_i$ *by the cocartesian square in* SCRing*:*

$$\begin{array}{ccc} \mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n] & \longrightarrow & \mathbf{Z}[\mathrm{T}_1, \ldots, \mathrm{T}_n]/(\mathrm{T}_1, \ldots, \mathrm{T}_n) \\ \big\downarrow {\scriptstyle \mathrm{T}_i \mapsto f_i} & & \big\downarrow \\ \mathrm{R} & \longrightarrow & \mathrm{R}/\!\!/(f_i)_i. \end{array}$$

The homomorphism $\mathrm{R} \to \mathrm{R}/\!\!/(f_i)_i$ *induces on* π_0 *the surjection*

$$\pi_0(\mathrm{R}) \to \pi_0(\mathrm{R})/(f_i)_i,$$

the usual quotient by the ideal generated by the f_i*'s.*

Exercise 4.3. *The underlying* R*-module of* $\mathrm{R}/\!\!/(f_i)_i$ *can be computed as the "Koszul complex"*

$$\bigotimes_i \mathrm{R} \xrightarrow{f_i} \mathrm{R}.$$

Example 4.4. Suppose that R is discrete and that $(f_i)_i$ form a *regular sequence*: f_1 is a non-zero-divisor, and each f_i is a non-zero-divisor in the ring $\mathrm{R}/(f_1, \ldots, f_{i-1})$ for $i > 1$. In this case, the Koszul complex defines a free resolution of $\mathrm{R}/(f_i)_i$, and in particular we find that $\mathrm{R}/\!\!/(f_i)_i$ is discrete (and in fact $\mathrm{R}/\!\!/(f_i)_i \approx \mathrm{R}/(f_i)_i$).

4.5. We now define:

Definition 4.6. *Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *be a closed immersion of derived schemes. The closed immersion* i *is* regular *(or a* local complete intersection*) if the shifted cotangent complex* $\mathcal{L}^*_{\mathrm{Z}/\mathrm{X}}[-1]$ *is a locally free* $\mathcal{O}_{\mathrm{Z}}$*-module of finite rank.*

Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a regular closed immersion. We define the *conormal sheaf* $\mathcal{N}^*_{\mathrm{Z}/\mathrm{X}}$ as the shifted cotangent complex

$$\mathcal{N}^*_{\mathrm{Z}/\mathrm{X}} := \mathcal{L}^*_{\mathrm{Z}/\mathrm{X}}[-1],$$

which is by assumption a locally free $\mathcal{O}_{\mathrm{Z}}$-module of finite rank. The associated vector bundle, which we denote $\mathbf{N}^*_{\mathrm{Z}/\mathrm{X}}$, is called the *conormal bundle* of i.

Definition 4.7. *The* virtual codimension *of* i*, defined Zariski-locally on* Z*, is the rank of the locally free* $\mathcal{O}_{\mathrm{Z}}$*-module* $\mathcal{N}^*_{\mathrm{Z}/\mathrm{X}}$*.*

Remark 4.8. If X and Z are *smooth* over some base S, then any closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ is regular. This follows from the exact triangle

$$i^*\mathcal{L}^*_{\mathrm{X}/\mathrm{S}} \to \mathcal{L}^*_{\mathrm{Z}/\mathrm{S}} \to \mathcal{L}^*_{\mathrm{Z}/\mathrm{X}}.$$

4.9. We have the following geometric characterization of regular immersions.

Proposition 4.10. *Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *a closed immersion of derived schemes. Then* i *is regular if and only if, Zariski-locally on* X*, there exists a morphism* $f : \mathrm{X} \to \mathbf{A}^n$ *and a cartesian square*

$$\begin{array}{ccc} \mathrm{Z} & \hookrightarrow & \mathrm{X} \\ \downarrow & & \downarrow f \\ \{0\} & \hookrightarrow & \mathbf{A}^n, \end{array}$$

where $\{0\} \hookrightarrow \mathbf{A}^n$ *denotes the inclusion of the origin into* n*-dimensional affine space over* $\mathrm{Spec}(\mathbf{Z})$*.*

In other words, the statement is that regular immersions are locally defined by Construction 4.2.

Remark 4.11. Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a closed immersion between classical schemes. If it is regular in the usual sense (i.e. locally cut out by a regular sequence), then we have $\mathcal{L}_{\mathrm{Z}/\mathrm{X}} = \mathcal{I}/\mathcal{I}^2[1]$, where $\mathcal{I} \subset \mathcal{O}_{\mathrm{X}}$ is the ideal of definition, with $\mathcal{I}/\mathcal{I}^2$ locally free; in particular it follows that i is regular the sense of Definition 4.6. One can also show the converse direction: the question being Zariski-local, one can reduce to the case where X is the spectrum of a (discrete) local ring R and $\mathrm{Z} = \mathrm{Spec}(\mathrm{R}/\!\!/(f_i)_i)$, in which case the discreteness of the $\mathrm{R}/\!\!/(f_i)_i$ (or its underlying R-module, the Koszul complex), implies that the sequence $(f_i)_i$ is regular.

Further, it is easy to prove a classical variant of Proposition 4.10 where the morphism f is required to be flat.

4.12. For the proof of Proposition 4.10, we will need the following very useful lemma about the cotangent complex:

Lemma 4.13. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of affine derived schemes. Let $\mathcal{K}$ be the cofibre of the morphism $f^\sharp : \mathcal{O}_\mathrm{X} \to f_*(\mathcal{O}_\mathrm{Y})$ in the category of quasi-coherent $\mathcal{O}_\mathrm{X}$-modules. Then there is a canonical morphism of quasi-coherent $\mathcal{O}_\mathrm{Y}$-modules*

$$\varphi : f^*(\mathcal{K}) \to \mathcal{L}^*_{\mathrm{Y}/\mathrm{X}}.$$

Further, if $f^\sharp$ is n-connected for some $n \geqslant 2$, then φ is $(n+2)$-connected.

Recall that a morphism of spectra is *n-connected* if its cofibre is n-connected, or equivalently if its fibre is $(n-1)$-connected.

We will also need the following fact, which can also be proved using Lemma 4.13:

Proposition 4.14. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of derived schemes. Then f is an isomorphism iff the following hold:*

(i) *The morphism $f_{\mathrm{cl}} : \mathrm{Y}_{\mathrm{cl}} \to \mathrm{X}_{\mathrm{cl}}$ is an isomorphism.*

(ii) *The morphism f is formally étale, i.e. $\mathcal{L}_{\mathrm{Y}/\mathrm{X}} = 0$.*

Proof of Proposition 4.10. We can assume that X and Z are affine. Also, note that it suffices to show the claim Zariski-locally on X (since it is obviously true on the complement $\mathrm{X} - \mathrm{Z}$).

Since the morphism $i^\sharp : \mathcal{O}_\mathrm{X} \to i_*(\mathcal{O}_\mathrm{Z})$ is 0-connected (as i is a closed immersion), we obtain by Lemma 4.13 a canonical 1-connected morphism

$$\varphi : i^*(\mathcal{I}) \to \mathcal{N}^*_{\mathrm{Z}/\mathrm{X}}$$

where $\mathcal{I}$ is the fibre of $i^\sharp$ (which is (-1)-connected, i.e. connective). In particular it induces isomorphisms $\pi_0(i^*\mathcal{I}) \simeq \pi_0(\mathcal{N}^*_{\mathrm{Z}/\mathrm{X}})$. Locally on Z we can assume $\mathcal{N}_{\mathrm{Z}/\mathrm{X}}$ is free and choose a basis $f_1, \ldots, f_n$; these give rise to global sections of $i_*i^*(\mathcal{I})$. Since the map $\mathcal{I} \to i_*i^*(\mathcal{I})$ is surjective (follows from the fact that $i^\sharp$ is), we can lift these to sections of $\mathcal{I}$, or equivalent to sections of $\mathcal{O}_\mathrm{X}$ which vanish on Z. These determine a morphism $f : \mathrm{X} \to \mathbf{A}^n$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Z} & \hookrightarrow & \mathrm{X} \\ \downarrow & & \downarrow f \\ \{0\} & \overset{z}{\hookrightarrow} & \mathbf{A}^n. \end{array}$$

It remains to show that the morphism

$$\delta : \mathrm{Z} \to \mathrm{X} \underset{\mathbf{A}^n_\mathrm{S}}{\times} \mathrm{S}$$

is invertible. By Proposition 4.14 it suffices to show that it induces an isomorphism of underlying classical schemes, and is formally étale. The first assertion follows from the fact that the ideal defining the classical closed immersion $\mathrm{Z}_{\mathrm{cl}} \hookrightarrow \mathrm{X}_{\mathrm{cl}}$ is the image of $\pi_0(\mathcal{I}) \to \pi_0(\mathcal{O}_\mathrm{X})$, and the elements f_i generate $\pi_0(\mathcal{I})$ (by inspection of their construction). For the second assertion, one uses the exact triangle

$$\delta^*\mathcal{L}^*_{\mathrm{X} \times_{\mathbf{A}^n_\mathrm{S}} \mathrm{S}/\mathrm{X}} \to \mathcal{L}^*_{\mathrm{Z}/\mathrm{X}} \to \mathcal{L}^*_{\mathrm{Z}/\mathrm{X} \times_{\mathbf{A}^n_\mathrm{S}} \mathrm{S}}$$

which is identified with

$$\mathcal{O}_\mathrm{Z}^{\oplus n}[1] \to \mathcal{N}^*_{\mathrm{Z}/\mathrm{X}}[1] \to \mathcal{L}^*_{\mathrm{Z}/\mathrm{X} \times_{\mathbf{A}^n_\mathrm{S}} \mathrm{S}},$$

and one shows that the first arrow is an isomorphism. □

References.

[1] *EGA I*, Springer edition.
[2] Jacob Lurie, *Spectral algebraic geometry*, version of 2017-10-13, available at http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf.