Lecture 7

Derived blow-ups

In this lecture we will explain how to blow up regular immersions of derived schemes. The interest is that one can extend Thomason's blow-up formula in K-theory to the derived setting, a fact which is an input into the pro-cdh descent theorem of Kerz–Strunk–Tamme.

Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a regular closed immersion of classical schemes. Recall that the blow-up of X in Z is a scheme $\pi : \tilde{\mathrm{X}} \to \mathrm{X}$ which is obtained by "replacing the closed subscheme Z by its projectivized normal bundle". More precisely, π is a proper morphism which is an isomorphism away from Z, and the fibre over Z is $\mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \to \mathrm{Z}$.

Further, the projective bundle $\mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$ sits inside $\tilde{\mathrm{X}}$ as an effective Cartier divisor (i.e. a regular immersion of codimension 1), and we can also think of the blow-up as the universal way to "turn Z into an effective Cartier divisor". More precisely, there is a universal property for the blow-up of the following form: for any morphism $f : \mathrm{S} \to \mathrm{X}$ such that the (classical) base change $f^{-1}(\mathrm{Z}) \hookrightarrow \mathrm{S}$ is an effective Cartier divisor, there is a unique morphism $\mathrm{S} \to \tilde{\mathrm{X}}$ and a morphism of commutative squares

$$\begin{array}{ccc} f^{-1}(\mathrm{Z}) & \hookrightarrow & \mathrm{S} \\ \downarrow & & \downarrow \\ \mathrm{Z} & \xhookrightarrow{i} & \mathrm{X} \end{array} \quad \to \quad \begin{array}{ccc} \mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) & \hookrightarrow & \tilde{\mathrm{X}} \\ \downarrow & & \downarrow \\ \mathrm{Z} & \xhookrightarrow{i} & \mathrm{X} \end{array}$$

We will now extend this construction to the derived setting. In the case of a regular immersion between classical schemes, this will agree with the classical construction. However, even in this case, our construction will actually provide a stronger universal property for the classical blow-up.

1. The construction.

1.1. We begin by defining a derived version of (effective) Cartier divisors.

Definition 1.2. *Let* X *be a derived scheme. A* virtual Cartier divisor *on* X *is the datum of a regular closed immersion* $i : \mathrm{D} \hookrightarrow \mathrm{X}$ *of virtual codimension* 1.

In other words, virtual Cartier divisors are locally of the form

$$\mathrm{Spec}(\mathrm{R}/\!\!/(f)) \hookrightarrow \mathrm{Spec}(\mathrm{R}),$$

where $f \in \mathrm{R}$ is an element and $\mathrm{R}/\!\!/(f)$ is the construction from Lecture 6.

Example 1.3. If R is discrete, recall that $\mathrm{R}/\!\!/(f) \approx \mathrm{R}/(f)$ iff f is regular (i.e. a non-zero-divisor). It follows that for X classical, any classical effective Cartier divisor is a virtual divisor.

Example 1.4. Taking $f = 0$, we get a virtual Cartier divisor $\mathrm{Spec}(\mathrm{R}/\!\!/(0)) \hookrightarrow \mathrm{Spec}(\mathrm{R})$ which is a nil-immersion, i.e. it induces an isomorphism on underlying classical schemes. (It is not an isomorphism: the underlying R-module of $\mathrm{R}/\!\!/(0)$ is given by $\mathrm{R} \oplus \mathrm{R}[1]$.) In particular, it is topologically of codimension 0.

1.5. Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a fixed regular immersion of derived schemes. We define:

Definition 1.6. [1] *Let* S *be a derived scheme and* $f : \mathrm{S} \to \mathrm{X}$ *a morphism. We say that a* virtual Cartier divisor $i_{\mathrm{D}} : \mathrm{D} \hookrightarrow \mathrm{S}$ lying over Z *is the datum of a commutative square*

$$\begin{tikzcd} \mathrm{D} \arrow[r, hook, "i_{\mathrm{D}}"] \arrow[d, "g"] & \mathrm{S} \arrow[d, "f"] \\ \mathrm{Z} \arrow[r, hook, "i"] & \mathrm{X}, \end{tikzcd} \tag{1.1}$$

such that $i_{\mathrm{D}} : \mathrm{D} \hookrightarrow \mathrm{S}$ *is a virtual Cartier divisor, and the induced morphism*

$$g^*(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \to \mathcal{N}_{\mathrm{D}/\mathrm{S}} \tag{1.2}$$

is surjective.

We let $\mathrm{VCart}_{\mathrm{Z}/\mathrm{X}}(\mathrm{S})$ denote the space of virtual Cartier divisors on S which lie over Z. This construction is functorial in S, and defines a presheaf of spaces

$$(\mathrm{S} \to \mathrm{X}) \mapsto \mathrm{VCart}_{\mathrm{Z}/\mathrm{X}}(\mathrm{S}) \tag{1.3}$$

on the site of derived schemes over X.

Construction 1.7. *Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *be a regular closed immersion of derived schemes. We let* $\pi_{\mathrm{Z}/\mathrm{X}} : \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \to \mathrm{X}$ *denote the derived prestack over* X *given by the presheaf*

$$(\mathrm{DSch}^{\mathrm{aff}}_{/\mathrm{X}})^{\mathrm{op}} \to \mathrm{Spc}$$

which is the restriction of (1.3) to affine derived schemes. We call $\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ *the* derived blow-up *of* $\mathrm{Z} \hookrightarrow \mathrm{X}$, *or simply the derived blow-up of* X *in* Z.

Remark 1.8. In the situation of Definition 1.6, suppose that S is affine. Then the surjectivity condition on

$$g^*(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \to \mathcal{N}_{\mathrm{D}/\mathrm{S}}$$

is equivalent to saying that this morphism exhibits $\mathcal{N}_{\mathrm{D}/\mathrm{S}}$ as a direct summand of $g^*(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$. This follows from the fact that $\mathcal{N}_{\mathrm{D}/\mathrm{S}}$ is projective (so that the map admits a section).

2. First properties.

2.1. We begin with two easy observations:

Proposition 2.2. *The derived prestack* $\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ *is a derived stack, i.e. it satisfies descent.*

This follows from the fact that the notion of virtual Cartier divisor is local, as is the surjectivity condition on the morphism (1.2).

Proposition 2.3. *The construction* $\pi_{\mathrm{Z}/\mathrm{X}} : \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \to \mathrm{X}$ *is stable under base change. That is, for any morphism* $f : \mathrm{X}' \to \mathrm{X}$, *there is a canonical isomorphism*

$$\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \underset{\mathrm{X}}{\times} \mathrm{X}' = \mathrm{Bl}_{\mathrm{Z}'/\mathrm{X}'}\,. \tag{2.1}$$

of derived stacks over X', *where* $i' : \mathrm{Z}' \hookrightarrow \mathrm{X}'$ *denotes the base change of* i *along* f.

We leave the proof as an exercise: the only thing to note is that the derived normal bundle is stable under arbitrary base change.

[1] This definition was suggested to me by David Rydh.

2.4. There is a canonical morphism i_{univ} fitting into a commutative square

$$\begin{tikzcd} \mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \arrow[r, hook, dashed, "i_{\mathrm{univ}}"] \arrow[d] & \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \arrow[d] \\ \mathrm{Z} \arrow[r, hook, "i"] & \mathrm{X}, \end{tikzcd}$$

which we will later identify with the "universal virtual divisor" lying over $\mathrm{Z} \hookrightarrow \mathrm{X}$.

To construct i_{univ}, note that it suffices to define a Z-morphism

$$\mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \to \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \underset{\mathrm{X}}{\times} \mathrm{Z} = \mathrm{Bl}_{\mathrm{Z} \times_{\mathrm{X}} \mathrm{Z}/\mathrm{Z}}\,. \tag{2.2}$$

As we saw last time, for any $z : \mathrm{T} \to \mathrm{Z}$, a T-point of $\mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$ is a pair $(\mathcal{L}, u)$, where $\mathcal{L}$ is a locally free sheaf of rank one on T, and $u : z^*\mathcal{N}_{\mathrm{Z}/\mathrm{X}} \to \mathcal{L}$ is a surjection on π_0 (exhibiting $\mathcal{L}$ as a direct summand). Given such, we can construct a virtual divisor $\mathrm{D}_{\mathcal{L}}$ on T as the derived intersection:

$$\begin{tikzcd} \mathrm{D}_{\mathcal{L}} \arrow[r, hook, "i_{\mathrm{D}}"] \arrow[d, hook, "i_{\mathrm{D}}"] & \mathrm{T} \arrow[d, hook, "s"] \\ \mathrm{T} \arrow[r, hook, "s"] & \mathbf{V}_{\mathrm{T}}(\mathcal{L}), \end{tikzcd}$$

where $\mathbf{V}_{\mathrm{T}}(\mathcal{L})$ is the vector bundle (line bundle) associated to $\mathcal{L}$, and $s : \mathrm{T} \hookrightarrow \mathbf{V}_{\mathrm{T}}(\mathcal{L})$ is the zero section.

Claim 2.5. *We have* $\mathrm{D}_{\mathcal{L}} \in \mathrm{VCart}_{\mathrm{Z} \times_{\mathrm{X}} \mathrm{Z}/\mathrm{Z}}(\mathrm{T})$.

The assignment $(\mathcal{L}, u) \mapsto \mathrm{D}_{\mathcal{L}}$ is clearly functorial and gives a map (2.2).

3. Comparison with classical blow-ups.

3.1. Our goal for the remainder of this lecture is to prove the following:

Theorem 3.2. *Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *be a regular closed immersion of derived schemes.*

(i) *The derived stack* $\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ *is schematic.*

(ii) *If i is a regular immersion between* classical *schemes, then the derived scheme* $\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ *is classical, and coincides with the classical blow-up* $\mathrm{Bl}^{\mathrm{cl}}_{\mathrm{Z}/\mathrm{X}}$ *as constructed in EGA II.*

In the last lecture we showed that Zariski-locally on X, the regular immersion $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ can be written as a derived base change of the zero section in n-dimensional affine space (where n is the virtual codimension in the given neighbourhood). Therefore, in order to prove the first claim, it follows from Proposition 2.3 that we may assume that i is the zero section $i : \{0\} \hookrightarrow \mathbf{A}^n$ for some $n \geqslant 0$. In fact, we will show:

Claim 3.3 (Main Claim)**.** *Let* $i : \{0\} \hookrightarrow \mathbf{A}^n$ *and consider the derived blow-up* $\mathcal{B} := \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$. *Then* $\mathcal{B}$ *is schematic, classical, and moreover isomorphic to the classical blow-up of* i.

This will in fact also show the second assertion of Theorem 3.2. Indeed there is a canonical morphism

$$\varepsilon : \mathrm{Bl}^{\mathrm{cl}}_{\mathrm{Z}/\mathrm{X}} \to \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}, \tag{3.1}$$

classifying the Cartier divisor $\mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \hookrightarrow \mathrm{Bl}^{\mathrm{cl}}_{\mathrm{Z}/\mathrm{X}}$. In order to show that ε is invertible, we can reduce to the case of the origin in affine space again, since a regular immersion between classical schemes can be written locally as a flat base change of $\{0\} \hookrightarrow \mathbf{A}^n$ (and the classical blow-up is stable under flat base change).

3.4. Claim 3.3 will also show the following properties of the derived blow-up:

Corollary 3.5. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a regular immersion of derived schemes.*

(i) *The structural morphism*

$$\pi_{\mathrm{Z}/\mathrm{X}} : \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \to \mathrm{X}$$

is proper.

(ii) *The morphism*

$$i_{\mathrm{univ}} : \mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \hookrightarrow \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \tag{3.2}$$

exhibits $\mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}})$ as the universal virtual Cartier divisor lying over $\mathrm{Z} \hookrightarrow \mathrm{X}$. That is, given a morphism $\mathrm{S} \to \mathrm{X}$, any virtual divisor $\mathrm{D} \in \mathrm{VCart}_{\mathrm{Z}/\mathrm{X}}(\mathrm{S})$ can be written as the base change of i_{univ} along the morphism $\mathrm{S} \to \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ classifying D.

(iii) *The morphism $\pi_{\mathrm{Z}/\mathrm{X}}$ induces an isomorphism*

$$\pi_{\mathrm{Z}/\mathrm{X}} : \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} - \mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \xrightarrow{\sim} \mathrm{X} - \mathrm{Z}. \tag{3.3}$$

Proof. For (i), note that the property of being proper is local on the target and stable under base change. Thus the assertion follows from the analogous statement for the classical blow-up (of the affine space in the origin).

For (ii), the fact that $\mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \hookrightarrow \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ is a virtual Cartier divisor can also be checked locally, so it follows similarly from the fact that $\mathbf{P}^{n-1} \hookrightarrow \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$ is a Cartier divisor (in the classical sense, hence also in the virtual sense). The existence of the *universal* virtual divisor $\mathrm{D}_{\mathrm{univ}} \hookrightarrow \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ lying over $\mathrm{Z} \hookrightarrow \mathrm{X}$ is formal: it comes from the identity of $\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}$ under the tautological identification

$$\mathrm{VCart}_{\mathrm{Z}/\mathrm{X}}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}) = \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}} \xrightarrow{\pi_{\mathrm{Z}/\mathrm{X}}} \mathrm{X}) = \mathrm{Maps}_{\mathrm{X}}(\mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}, \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}}).$$

By universality there is a canonical morphism $\mathbf{P}_{\mathrm{Z}}(\mathcal{N}_{\mathrm{Z}/\mathrm{X}}) \to \mathrm{D}_{\mathrm{univ}}$, which is an isomorphism locally, by universality of the Cartier divisor $\mathbf{P}^{n-1} \hookrightarrow \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$.

The proof of (iii) follows similarly from the analogous property of the classical blow-up of $\{0\} \hookrightarrow \mathbf{A}^n$. □

3.6. The next remark explains the relationship between the derived blow-ups described here and the construction of [2].

Remark 3.7. Let $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ be an affine classical scheme and let $f_1, \ldots, f_n$ be a sequence of elements. These define a derived regular subscheme $\mathrm{Z} = \mathrm{Spec}(\mathrm{R} /\!\!/ (f_i)_i)$, which is the base change of $\{0\} \hookrightarrow \mathbf{A}^n$ along the morphism $f : \mathrm{X} \to \mathbf{A}^n$ determined by the f_i's.

In this setting, Kerz–Strunk–Tamme [2] defined the derived blow-up of X in Z as

$$\mathrm{Bl}^{\mathrm{KST}}_{\mathrm{Z}/\mathrm{X}} := \mathrm{Bl}^{\mathrm{cl}}_{\{0\}/\mathbf{A}^n} \underset{\mathbf{A}^n}{\times} \mathrm{X},$$

where $\mathrm{Bl}^{\mathrm{cl}}_{\{0\}/\mathbf{A}^n}$ denotes the classical blow-up. It will thus follow from Claim 3.3 that there are isomorphisms

$$\mathrm{Bl}^{\mathrm{KST}}_{\mathrm{Z}/\mathrm{X}} \approx \mathrm{Bl}_{\{0\}/\mathbf{A}^n} \underset{\mathbf{A}^n}{\times} \mathrm{X} \approx \mathrm{Bl}_{\mathrm{Z}/\mathrm{X}},$$

where the second isomorphism is the stability under base change (Proposition 2.3).

3.8. The rest of the lecture will be dedicated to the proof of Claim 3.3. Let $\mathcal{B} := \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$ and $\mathrm{B} = \mathrm{Bl}^{\mathrm{cl}}_{\{0\}/\mathbf{A}^n}$ be the derived and classical blow-ups, respectively. In order to show that the canonical morphism (3.1)

$$\varepsilon : \mathrm{B} \to \mathcal{B}$$

is invertible, we will use the following general fact: suppose we have a morphism $f : \mathcal{Y} \to \mathcal{X}$ of derived stacks that admits a relative cotangent complex; then f is an isomorphism iff $\mathcal{L}_{\mathcal{Y}/\mathcal{X}} = 0$ and f induces an isomorphism of underlying classical stacks $\mathcal{Y}_{\mathrm{cl}} \to \mathcal{X}_{\mathrm{cl}}$.

4. Deformation theory of virtual divisors.

4.1. We begin by showing that the morphism ε has relative cotangent complex equal to zero. The main point is to compute the relative cotangent complex $\mathcal{L}_{\mathcal{B}/\mathbf{A}^n}$ (and then to observe that it is canonically isomorphic, through the morphism ε, to the cotangent complex $\mathcal{L}_{\mathrm{B}/\mathbf{A}^n}$).

Proposition 4.2. *The cotangent complex $\mathcal{L}_{\mathcal{B}/\mathbf{A}^n}$ is canonically isomorphic to*

$$\mathcal{L}_{\mathcal{B}/\mathbf{A}^n} = (i_{\mathrm{univ}})_*(\mathcal{L}_{\mathbf{P}^{n-1}}). \tag{4.1}$$

In particular, for any $(\mathrm{S} \xrightarrow{f} \mathbf{A}^n) \in \mathrm{DSch}^{\mathrm{aff}}_{/\mathbf{A}^n}$, and any point $\eta \in \mathcal{B}(\mathrm{S} \xrightarrow{f} \mathbf{A}^n)$ classifying a virtual divisor $i_{\mathrm{D}} : \mathrm{D} \hookrightarrow \mathrm{S}$, we have

$$\eta^*\mathcal{L}_{\mathcal{B}/\mathbf{A}^n} = (i_{\mathrm{D}})_*(\mathcal{F} \otimes \mathcal{N}^{\otimes -1}_{\mathrm{D}/\mathrm{S}}),$$

where $\mathcal{F}$ denotes the locally free sheaf $\mathrm{Fib}(\delta^*\mathcal{N}_{\mathrm{W}/\mathrm{S}} \to \mathcal{N}_{\mathrm{D}/\mathrm{S}}) = \mathcal{L}_{\mathrm{D}/\mathrm{W}}[-2]$*, where $\delta : \mathrm{D} \hookrightarrow \mathrm{W} = \mathrm{S} \times_{\mathbf{A}^n} \{0\}$.*

4.3. To prove Proposition 4.2 we will begin by realizing $\mathcal{B} := \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$ as an open substack of a bigger derived stack.

Construction 4.4 (Moduli of closed subschemes)**.** *Let $\mathcal{M}$ denote the derived stack*

$$\mathcal{M} : \mathrm{S} \mapsto (\mathrm{DSch}^{\mathrm{closed,lfp,lfta}}_{/\mathrm{S}})^{\approx},$$

which sends $\mathrm{S} \in \mathrm{DSch}^{\mathrm{aff}}$ to the space of closed derived subschemes $\mathrm{Z} \hookrightarrow \mathrm{S}$ which are locally of finite presentation (= lfp) and locally of finite tor-amplitude (= lfta).

Construction 4.5 (Weil restriction)**.** *Let $\mathcal{Y}$ be a derived stack over X and let $f : \mathrm{X} \to \mathrm{X}'$ be a morphism. The* Weil restriction *of $\mathcal{Y}$ along f is the prestack $\mathrm{Res}_{\mathrm{X}/\mathrm{X}'}(\mathcal{Y})$ over X' defined by the assignment*

$$\mathrm{Res}_{\mathrm{X}/\mathrm{X}'}(\mathcal{Y}) : (\mathrm{S}' \mapsto \mathrm{X}') \mapsto \mathcal{Y}(\mathrm{S}' \underset{\mathrm{X}'}{\times} \mathrm{X}).$$

We can view $\mathcal{M}$ as a derived stack over the terminal scheme $\mathrm{Spec}(\mathbf{Z}) \approx \{0\}$; we let

$$\mathcal{R} := \mathrm{Res}_{\{0\}/\mathbf{A}^n}(\mathcal{M})$$

denote the Weil restriction along $\{0\} \hookrightarrow \mathbf{A}^n$. This is defined by the assignment

$$(\mathrm{S} \xrightarrow{f} \mathbf{A}^n) \mapsto (\mathrm{DSch}^{\mathrm{closed,lfp,lfta}}_{/\mathrm{S} \times_{\mathbf{A}^n} \{0\}})^{\approx}.$$

By definition, we have an inclusion

$$\mathcal{B} \hookrightarrow \mathcal{R} \tag{4.2}$$

of derived stacks over $\mathbf{A}^n$. This amounts to viewing a virtual divisor $\mathrm{D} \in \mathrm{VCart}_{\{0\}/\mathbf{A}^n}(\mathrm{S})$ as a morphism $\delta : \mathrm{D} \to \mathrm{W} := \mathrm{S} \times_{\mathbf{A}^n} \{0\}$, which is in particular a closed immersion that is lfp and lfta.

Claim 4.6. *The morphism (4.2) is an open immersion.*

Proof. Let η be a point of $\mathcal{R}$ classifying a closed immersion $\delta : \mathrm{D} \hookrightarrow \mathrm{W} = \mathrm{S} \times_{\mathbf{A}^n} \{0\}$ and let $i_{\mathrm{D}} : \mathrm{D} \hookrightarrow \mathrm{W} \to \mathrm{S}$. The cotangent complex $\mathcal{L}_{\mathrm{D/S}}$ is automatically perfect (since i_{D} is lfp) and 0-connected (since it is a closed immersion). The inclusion $\mathcal{B} \hookrightarrow \mathcal{R}$ is defined by imposing the further conditions:

(i) The sheaf $\mathcal{L}_{\mathrm{D/S}}[-1]$ is locally free and of rank 1 (so that $i_{\mathrm{D}} : \mathrm{D} \hookrightarrow \mathrm{S}$ is a virtual divisor).

(ii) The morphism $\delta^*\mathcal{N}_{\mathrm{W/S}} \to \mathcal{N}_{\mathrm{D/S}}$ is surjective, or equivalently its cofibre $\mathcal{N}_{\mathrm{D/W}}$ is 0-connected.

In the presence of (i), condition (ii) is also equivalent to saying that the fibre of that map, $\mathcal{L}_{\mathrm{D/W}}[-2]$, is locally free (of rank $n-1$). Thus both conditions are "open", see [3, Prop. 2.9.3.2, Lem. 2.9.3.3]. □

4.7. It follows that there is a canonical isomorphism $\mathcal{L}_{\mathcal{B}/\mathbf{A}^n} = \mathcal{L}_{\mathcal{R}/\mathbf{A}^n}|_{\mathcal{B}}$. In order to compute the cotangent complex of $\mathcal{R}$, we can compute the cotangent complex $\mathcal{L}_{\mathcal{M}}$ and then apply the general deformation theory of Weil restrictions.

The first part follows more or less directly from Lurie's version of Kodaira–Spencer theory, i.e. the moduli of derived Deligne–Mumford stacks; see [3, § 19.4.3]. We get:

Proposition 4.8. *The cotangent complex $\mathcal{L}_{\mathcal{M}}$ exists and is perfect. Let $\mathrm{S} \in \mathrm{DSch}^{\mathrm{aff}}$ and $\xi : \mathrm{S} \to \mathcal{M}$ be a morphism classifying an lfp closed immersion $i : \mathrm{Z} \hookrightarrow \mathrm{S}$. Then there is a canonical isomorphism*

$$\xi^*\mathcal{L}_{\mathcal{M}} = i_{\mathrm{L}!}\mathcal{L}_{\mathrm{Z/S}}[-1],$$

where $i_{\mathrm{L}!} := i_!(- \otimes \omega_{\mathrm{Z/S}})$, $\omega_{\mathrm{Z/S}}$ being the relative dualizing complex.

4.9. Applying the deformation theory of Weil restrictions (see [3, Prop. 19.1.4.3]), we deduce:

Corollary 4.10. *The cotangent complex $\mathcal{L}_{\mathcal{R}/\mathbf{A}^n}$ exists and is perfect. Moreover, for any $(\mathrm{S} \xrightarrow{f} \mathbf{A}^n) \in \mathrm{DSch}^{\mathrm{aff}}_{/\mathbf{A}^n}$, and any point $\eta \in \mathcal{R}(\mathrm{S} \xrightarrow{f} \mathbf{A}^n)$ classifying a closed immersion $\delta : \mathrm{Z} \hookrightarrow \mathrm{W} := \mathrm{S} \times_{\mathbf{A}^n} \{0\}$, there is a canonical isomorphism*

$$\eta^*\mathcal{L}_{\mathcal{R}/\mathbf{A}^n} = i_*(\mathcal{F} \otimes \mathcal{N}_{\mathrm{Z/S}}^{\otimes -1}),$$

where $i : \mathrm{Z} \hookrightarrow \mathrm{W} \hookrightarrow \mathrm{S}$ and $\mathcal{F}$ denotes the locally free sheaf $\mathrm{Fib}(\mathcal{O}_{\mathrm{Z}}^n \to \mathcal{N}_{\mathrm{Z/S}}) = \mathcal{L}_{\mathrm{Z/W}}[-2]$.

5. Chart-by-chart comparison.

5.1. Define the affine schemes

$$\mathrm{U}_i = \mathrm{Spec}(\mathbf{Z}[x_1/x_i, \ldots, x_n/x_i, x_i]),$$
$$\mathrm{E}_i = \mathrm{Spec}(\mathbf{Z}[x_1/x_i, \ldots, x_n/x_i])$$

for each $1 \leqslant i \leqslant n$. The schemes U_i provide an affine Zariski cover for the classical blow-up B, and the E_i provide a cover for the exceptional divisor $\mathbf{P}^{n-1}$ (see e.g. [1, Example IV-17]). Our goal next is to show that they also provide a affine Zariski cover for the derived blow-up.

The evident closed immersions $\mathrm{E}_i \hookrightarrow \mathrm{U}_i$ define Cartier divisors lying over $\{0\} \hookrightarrow \mathbf{A}^n$, and thus provide canonical morphisms

$$\mathrm{U}_i \to \mathcal{B} \tag{5.1}$$

for each i. Let $\mathcal{B}_i$ denote the image of the map (5.1), fitting in a factorization

$$\mathrm{U}_i \twoheadrightarrow \mathcal{B}_i \hookrightarrow \mathcal{B}, \tag{5.2}$$

where the first arrow is an effective epimorphism of Zariski sheaves and the second is a monomorphism. In other words, the derived stack $\mathcal{B}_i$ can be described as follows: for any

$\mathrm{S} = \mathrm{Spec}(\mathrm{R}) \in \mathrm{DSch}^{\mathrm{aff}}$ and any morphism $f : \mathrm{S} \to \mathbf{A}^n$, the S-points of $\mathcal{B}_i$ are the virtual Cartier divisors lying over $\{0\} \hookrightarrow \mathbf{A}^n$ of the form

$$\begin{tikzcd} \mathrm{Z} \arrow[r, hook] \arrow[d] & \mathrm{S} \arrow[d] \\ \mathrm{E}_i \arrow[r, hook] \arrow[d] & \mathrm{U}_i \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbf{A}^n \end{tikzcd} \tag{5.3}$$

where the upper square is cartesian. In particular, if f corresponds to points $f_1, \ldots, f_n \in \mathrm{R}_{\mathrm{Spc}}$, then we have an isomorphism $\mathrm{Z} = \mathrm{Spec}(\mathrm{R}/\!\!/f_i)$.

5.2. We first observe that U_i do indeed provide a Zariski atlas for $\mathcal{B}$.

Claim 5.3. *The induced morphism*

$$\coprod_i \mathrm{U}_i \to \mathcal{B} \tag{5.4}$$

is an effective epimorphism of Zariski sheaves.

Proof. Let $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ be an affine derived scheme, $f : \mathrm{S} \to \mathbf{A}^n$ a morphism, and $\eta \in \mathcal{B}(\mathrm{S} \to \mathbf{A}^n)$ a point. It suffices to show that, Zariski-locally on S, η lifts to a point $\tilde{\eta} \in \mathrm{U}_i(\mathrm{S} \to \mathbf{A}^n)$ for some i. The point η classifies a virtual Cartier divisor $i_{\mathrm{D}} : \mathrm{D} \hookrightarrow \mathrm{S}$ lying over $\{0\} \hookrightarrow \mathbf{A}^n$. By the geometric characterization of derived regular immersions we proved last lecture (Prop. 4.10), D is locally of the form $\mathrm{Spec}(\mathrm{R}/\!\!/f)$ for some point $f \in \mathrm{R}_{\mathrm{Spc}}$. Moreover, the proof shows that f can be taken to be any point such that $df \in \mathcal{N}_{\mathrm{D}/\mathrm{S}}$ is a generator. By assumption, the canonical morphism

$$\langle df_1, \ldots, df_n \rangle = \mathcal{N}_{\mathrm{W}/\mathrm{S}} = g^*\mathcal{N}_{\{0\}/\mathbf{A}^n} \to \mathcal{N}_{\mathrm{D}/\mathrm{S}},$$

is a projection onto a direct summand, where $g : \mathrm{D} \to \{0\}$ and $\mathrm{W} := \mathrm{S} \times_{\mathbf{A}^n} \{0\} = \mathrm{Spec}(\mathrm{R}/\!\!/(f_i)_i)$. In particular, we can take $f = f_i$ for some i, so that η lifts to $\mathrm{U}_i(\mathrm{S} \to \mathbf{A}^n)$ as desired. □

5.4. Next we would like to show that $\mathrm{U}_i \to \mathcal{B}_{\mathrm{cl}}$ is a monomorphism for each i. It will be useful to make the following preliminary observation:

Claim 5.5. *Let* $(\mathrm{S} \xrightarrow{f} \mathbf{A}^n) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathbf{A}^n}$ *be a* classical *affine scheme over* $\mathbf{A}^n$*. Then the space* $\mathcal{B}_i(\mathrm{S})$ *is discrete for each* i*.*

Proof. Let $\eta \in \mathcal{B}_i(\mathrm{S})$ be a point corresponding to a virtual Cartier divisor $i_{\mathrm{D}} : \mathrm{D} \hookrightarrow \mathrm{S}$, and let us show that its connected component is contractible, i.e. that the loop space $\Omega(\mathcal{B}(\mathrm{S}), \eta) = \mathrm{Aut}_{\mathcal{B}(\mathrm{S})}(\eta)$ is discrete.

Let $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$, $\mathrm{D} = \mathrm{Spec}(\mathrm{R}/\!\!/f_i)$; we can think of the datum of the virtual divisor D as a morphism $\delta : \mathrm{D} \to \mathrm{W} = \mathrm{S} \times_{\mathbf{A}^n} \{0\} = \mathrm{Spec}(\mathrm{R}/\!\!/(f_j)_j)$, or equivalently a homomorphism $\delta : \mathrm{R}/\!\!/(f_j)_j \to \mathrm{R}/\!\!/f_i$ fitting into a commutative diagram

$$\begin{tikzcd} \mathbf{Z}[x_1, \ldots, x_n] \arrow[r] \arrow[d, "(f_j)_j"] & \mathbf{Z}[x_1, \ldots, x_n]/(x_1, \ldots, x_n) \arrow[d] \arrow[ddr, bend left] & \\ \mathrm{R} \arrow[r] \arrow[drr, bend right] & \mathrm{R}/\!\!/(f_j)_j \arrow[dr, dashed, "\delta"] & \\ & & \mathrm{R}/\!\!/f_i \end{tikzcd}$$

where the square is cocartesian. The commutativity of the "outer square" amounts to the data of paths $\alpha_j : f_j \approx 0$ in $\mathrm{R}/\!\!/f_i$ for each j, which thus determine δ uniquely.

Furthermore, since $\eta \in \mathcal{B}_i(\mathrm{S} \to \mathbf{A}^n)$, the outer square factors as

$$\begin{tikzcd}
\mathbf{Z}[x_1, \ldots, x_n] \arrow[r] \arrow[d] & \mathbf{Z}[x_1, \ldots, x_n]/(x_1, \ldots, x_n) \arrow[d] \\
\mathbf{Z}[x_1/x_i, \ldots, x_n/x_i, x_i] \arrow[r] \arrow[d] & \mathbf{Z}[x_1/x_i, \ldots, x_n/x_i] \arrow[d] \\
\mathrm{R} \arrow[r] & \mathrm{R}/\!\!/f_i
\end{tikzcd}$$

This implies that the path $\alpha_i : f_i \approx 0$ is isomorphic to the "canonical path" $f_i \approx 0$. Since R is discrete, $\mathrm{R}/\!\!/f_i$ is 1-truncated, and therefore the space of paths in $\mathrm{R}/\!\!/f_i$ is discrete (i.e. any isomorphism between paths is the identity).

Now, an automorphism of η is the same thing as an automorphism of D over W, or equivalently an $\mathrm{R}/\!\!/(f_j)_j$-algebra automorphism of $\mathrm{R}/\!\!/f_i$. The latter is the same thing as an automorphism φ of R-algebras equipped with isomorphisms of paths $\varphi(\alpha_j) \approx \alpha_j$ for each j; since the space of paths is discrete, these are properties rather than additional structure on φ. Therefore it suffices to show that φ is trivial as an automorphism of R-algebras. An R-algebra automorphism of $\mathrm{R}/\!\!/f_i$ is uniquely determined by a path $f_i \approx 0$ in $\mathrm{R}/\!\!/f_i$ (which is the image by the automorphism of the canonical path $f_i \approx 0$). Since the automorphism φ preserves the path α_i, which is the canonical path $f_i \approx 0$, this determines it uniquely as the identity automorphism. $\square$

Claim 5.6. *The canonical morphism* $\mathrm{U}_i \to (\mathcal{B}_i)_{\mathrm{cl}}$ *is a monomorphism.*

Proof. The claim is that for every *classical* affine scheme $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ and every morphism $f : \mathrm{S} \to \mathbf{A}^n$, the map

$$\mathrm{U}_i(\mathrm{S} \to \mathbf{A}^n) \to \mathcal{B}_i(\mathrm{S} \to \mathbf{A}^n)$$

is a monomorphism of spaces. Since both spaces are in fact discrete by Claim 5.5, this is just a map of sets and we will show that it is injective.

A point $\eta \in \mathcal{B}_i(\mathrm{S})$ is determined by a set of paths $(\alpha_j : f_j \approx 0)_j$ in $\mathrm{R}/\!\!/f_i$. Each of these paths amount to the datum of an element $a_j \in \mathrm{R}$ with $a_j f_i = f_j$. A point $\tilde{\eta} \in \mathrm{U}_i(\mathrm{S})$ corresponds to a set of elements $\tilde{a}_j \in \mathrm{R}$ with $\tilde{a}_j f_i = f_j$, for each $j \neq i$. Now if $\tilde{\eta}$ lifts η, then we clearly have $\tilde{a}_j = a_j$ for each $j \neq i$, so $\tilde{\eta}$ is unique. $\square$

5.7. Putting everything together, we get:

Claim 5.8. *The canonical morphism* $\mathrm{U}_i \to \mathcal{B}_i$ *is an isomorphism for each* i.

Proof. Each $\mathrm{U}_i \to \mathcal{B}_i$ is formally étale, as the restriction of the formally étale morphism $\varepsilon : \mathrm{B} \to \mathcal{B}$. Therefore it suffices to show that $\mathrm{U}_i \to (\mathcal{B}_i)_{\mathrm{cl}}$ is invertible. We showed that it is an effective epimorphism and a monomorphism, so the claim follows. $\square$

5.9. Finally we can deduce our main claim, that $\mathrm{B} \to \mathcal{B}$ is invertible (Claim 3.3). Indeed, we have exhibited an affine Zariski cover for $\mathcal{B}$ which is canonically isomorphic to the "standard" cover for the classical blow-up B.

References.

[1] Eisenbud, Harris, *The geometry of schemes.*
[2] M. Kerz, F. Strunk, G. Tamme *Algebraic K-theory and descent for blow-ups.*
[3] Jacob Lurie, *Spectral algebraic geometry*, version of 2017-10-13, available at `http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf`.