∞ - Categories = $(\infty, 1)$ - Categories

Principle :

The theory of ∞-categories is a generalisation of category theory and of homotopy theory. It is an <u>extension</u> of category theory: any statement in ordinary category theory should have its counterpart for higher categories (Kan extensions, (co)limits...).

The archetype of an ∞-category which is not a category is the homotopy theory of spaces. Also, some properties only make sense for genuine ∞-categories.

For instance, <u>stable ∞-categories</u>: these are the ∞-categories with finite limits and finite colimits in which a commutative square is cartesian if and only if it is cocartesian (these give rise to all the triangulated categories one finds in nature). But an ordinary category is stable if and only if it is equivalent to $\{0\}$.

∞-groupoids play an essential role here:

$$\{\infty\text{-groupoids}\} \sim \{\text{homotopy types of CW-complexes}\}$$

and : ∞-groupoids are to ∞-categories what sets are to ordinary categories (think of the Yoneda Lemma).

In particular, the free completion of a (small) ∞-category A by colimits is the ∞-category $\underline{\mathrm{Hom}}(A^{op}, S)$ of functors from A^{op} to the ∞-category of ∞-groupoids.

We will first study presentations of cocomplete ∞-categories :

Model categories.

~~A model category structure~~

Consider a category $\mathcal{C}$ endowed with a class of maps W, the elements of which are called <u>weak equivalences</u>. We assume that $\mathcal{C}$ is complete and cocomplete.

This should define an ∞-category $W^{-1}\mathcal{C}$, the universal one obtained from $\mathcal{C}$ by inverting the elements of W. Moreover, we would like the properties (and structures of $\mathcal{C}$) to have their counterpart in $W^{-1}\mathcal{C}$. For instance we would like $W^{-1}C$ to be (co)complete as an ∞-category. Even if this cannot be formulated yet (as we don't know the theory of ∞-categories), this can be formulated in the language of ordinary category theory.

Example: limits.

For a small category I, let $\underline{Hom}(I^{op}, \mathcal{C})$ be the category of functors $I^{op} \to \mathcal{C}$. Let

$$W_I = \{ X \to Y \mid \forall i \quad X_i \to Y_i \in W \}$$

be the class of termwise weak equivalences.

$Ho(\mathcal{C})$ = the universal category obtained from $\mathcal{C}$ by inverting the elements of W.

$$\mathcal{C} \underset{\varprojlim_{I^{op}}}{\overset{c}{\rightleftarrows}} \underline{Hom}(I^{op}, \mathcal{C})$$

One defines $\underset{I^{op}}{\mathrm{holim}}$ as a right adjoint of c:

$$\mathrm{Ho}(\mathcal{C}) \underset{\underset{I^{op}}{\mathrm{holim}}}{\overset{c}{\rightleftarrows}} \mathrm{Ho}(\underline{\mathrm{Hom}}(I^{op}, \mathcal{C}))$$

One defines $\underset{\overleftarrow{I^{op}}}{\mathrm{Rlim}}$ as the right derived functor of $\underset{\overleftarrow{I^{op}}}{\lim}$

$$\begin{tikzcd}
\mathrm{Ho}(\underline{\mathrm{Hom}}(I^{op}, \mathcal{C})) \arrow[r, "\underset{\overleftarrow{I^{op}}}{\mathrm{Rlim}}"] & \mathrm{Ho}(\mathcal{C}) \\
\underline{\mathrm{Hom}}(I^{op}, \mathcal{C}) \arrow[u] \arrow[r, "\underset{\overleftarrow{I^{op}}}{\lim}"'] \arrow[ur, phantom, "\Nearrow\alpha"] & \mathcal{C} \arrow[u]
\end{tikzcd}$$

(*) We would like $\underset{\overleftarrow{I^{op}}}{\mathrm{Rlim}} \simeq \underset{\overleftarrow{I^{op}}}{\mathrm{holim}}$ and that they exist!

Example: $\mathcal{C}$ = complexes in an abelian category $\mathcal{A}$.

W = quasi-isomorphisms

$I = (\bullet \longrightarrow \bullet \longleftarrow \bullet)$

then $\underset{\longleftarrow}{\mathrm{Rlim}} \left(\begin{array}{ccc} & & M' \\ & & \downarrow u \\ M'' & \xrightarrow[v]{} & M \end{array} \right) = \mathrm{Cone}\left(M' \oplus M'' \xrightarrow{(u,v)} M \right)[-1]$

Model structures ensure (*) to hold.

Définition.

Let ~~F be a class~~ $A \xrightarrow{i} B$ and $X \xrightarrow{p} Y$ be two maps.

we write $i \perp p$ (and say that i is left orthogonal to p or p — right ——— i)

if, for any commutative square

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "b"'] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

there exists h such that $hi = a$ and $ph = b$.

If $\mathcal{M}$ is a class of maps

$$\mathcal{M}^{\perp} := \{ p \mid \forall i \in \mathcal{M},\ i \perp p \}$$

$$^{\perp}\mathcal{M} := \{ i \mid \forall p \in \mathcal{M}\ \ i \perp p \}$$

$\mathcal{M}$ is <u>stable by retracts</u> if for any commutative diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "p'"'] \arrow[rr, bend left, "="] & X \arrow[r] \arrow[d, "p"] & X' \arrow[d, "p'"] \\
Y' \arrow[r] \arrow[rr, bend right, "="'] & Y \arrow[r] & Y'
\end{tikzcd}$$

$p \in \mathcal{M} \Rightarrow p' \in \mathcal{M}$.

Definition.

A model category is a quadruple $(\mathcal{C}, W, Cof, Fib)$ where $\mathcal{C}$ is a category and W, Cof, Fib are classes of maps of $\mathcal{C}$ such that:

(~~E~~M1) $\mathcal{C}$ is complete and cocomplete

(~~E~~M2) ~~Dans tout triangle commutatif de~~ In any commutative triangle of $\mathcal{C}$ $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[dr, "h"'] & \cdot \arrow[d, "g"] \\ & \cdot \end{tikzcd}$, ~~si deux parmi~~ f, g, h if two maps out of f, g, h ~~sont dans W~~ are in W, so is the third.

(~~E~~M3) W, Fib, Cof are stable by retracts

(M4) $Cof \cap W \subset {}^{\perp}Fib$ and $Fib \cap W \subset Cof^{\perp}$

(M5) any map f of $\mathcal{C}$ admits factorisations $f = pi = qj$ with $i \in Cof \cap W$, $p \in Fib$, $j \in Cof$, $q \in Fib \cap W$.

Remark: the axioms imply: $\mathrm{Cof} = {}^{\perp}(\mathrm{Fib} \cap W)$ $\quad \mathrm{Cof} \cap W = {}^{\perp}\mathrm{Fib}$

$\mathrm{Fib} = (\mathrm{Cof} \cap W)^{\perp}$ $\quad \mathrm{Fib} \cap W = \mathrm{Cof}^{\perp}$

X fibrant $\quad X \longrightarrow * \in \mathrm{Fib}$

A cofibrant $\quad \emptyset \longrightarrow A \in \mathrm{Cof}$

Small object argument:

An object X of $\mathcal{C}$ is small if $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow \mathrm{Sets}$ commutes with α-filtering colimits for a big enough cardinal α.

α-filtering colimits are colimits indexed by α-filtering categories i.e. small filtering categories I such that for any family of objects $(i_e)_{e \in \Lambda}$ in I with $\mathrm{Card}\,\Lambda \leq \alpha$, there exists an object i and maps $i_e \longrightarrow i$ for each $e \in \Lambda$.

A model category is said to be combinatorial if

1) all the objects of $\mathcal{C}$ are small and there exists a small family of objects of $\mathcal{C}$ which generates $\mathcal{C}$ by colimits

2) there exists small sets of maps I, J such that $W \cap \mathrm{Fib} = I^{\perp}$ and $\mathrm{Fib} = J^{\perp}$.

this imply that Cof (resp. $\mathrm{Cof} \cap W$) is the smallest class of maps which contains I (resp. J) and which is closed by retract, pushouts and transfinite composition.

Notation: $\mathrm{sat}(I) = {}^{\perp}(I^{\perp})$

Example:

1) $\mathcal{C} = \mathrm{Top} = \{\text{topological spaces}\}$

$I = \{\, S^{n-1} \hookrightarrow B^n \mid n \geq 0 \,\}$

$J = \{\, I^{n-1} \times \{0\} \longrightarrow I^n \mid n \geq 1 \,\}$ $\qquad I = [0,1]$

$J^{\perp} = \mathrm{Fib} = \{\text{Serre fibrations}\}$

$W = \{\, X \xrightarrow{f} Y \mid \pi_0(X) \xrightarrow{\sim} \pi_0(Y)$ and $\forall x \in X \ \forall n \geq 1 \ \pi_n(X,x) \xrightarrow{\sim} \pi_n(Y,y) \,\}$

2) $\mathcal{C}$ = complexes of abelian groups

W = quasi-isomorphisms

Fib = degree wise surjections

Cof = monomorphisms with cokernel which is degree wise projective

Construction of model categories on presheaf categories (such as simplicial sets).

Let A be a small category. We want to produce combinatorial model category structures on $\mathcal{C} = \widehat{A} = \underline{\mathrm{Hom}}(A^{op}, \mathrm{Sets})$ such that $\mathrm{Cof} = \{\text{monomorphisms}\}$.

Notation: Yoneda as an inclusion: $\mathrm{Hom}_A(-, a) = a$ for $a \in \mathrm{ob}\,A$.

Example to keep in mind

$A = \Delta$.

objects of Δ: non empty finite totally ordered sets

$$\Delta^n = \{0, \ldots, n\} \quad , n \geq 0$$

Maps: order preserving maps.

$\widehat{\Delta} = \mathrm{SSets}$

A <u>cellular model</u> of $\widehat{A}$ is a set $\mathcal{M}$ of monomorphisms of $\widehat{A}$ such that ${}^{\perp}(\mathcal{M}^{\perp}) = \{\text{mono.}\}$.

This always exists: e.g. $\mathcal{M} = \{ U \rightarrowtail Q \mid Q$ quotient of a representable presheaf$\}$

Example:

For $0 \leq i \leq n$ $\delta_i^n : \Delta^{n-1} \longrightarrow \Delta^n$ is the only injective map of Δ such that $i \notin \mathrm{Im}\, \delta_i^n$.

$$\partial\Delta^n = \bigcup_i \mathrm{Im}\,\delta_i^n \rightarrowtail \Delta^n$$

$\mathcal{M} = \{ \partial\Delta^n \rightarrowtail \Delta^n \mid n \geq 0 \}$ is a cellular model of SSets.

Definition. An *interval* of $\hat{A}$ is a presheaf I endowed with two disjoint global sections

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d, "\partial^0"] \\
* \arrow[r, "\partial^1"'] & I
\end{tikzcd}
$$

Notations: $\{e\} = \mathrm{Im}\left(* \xrightarrow{\partial^e} I\right) \subset I$

$$\partial I = \{0\} \cup \{1\} \hookrightarrow I$$

Example: $A = \Delta$, $I = \Delta^1$

$$* = \Delta^0 \underset{\partial^1 = \delta^1_0}{\overset{\partial^0 = \delta^1_1}{\rightrightarrows}} \Delta^1$$

Then $\partial I = \partial\Delta^1$.

Remark: for a pullback square in $\hat{A}$

$$
\begin{tikzcd}
X \arrow[r, hook, "j"] \arrow[d, hook, "i"'] & X' \arrow[d, hook, "i'"] \\
Y \arrow[r, hook, "j'"] & Y'
\end{tikzcd}
$$

with i', j' mono

$Y \coprod_X X' \longrightarrow Y'$ is a monomorphism

and we write $Y \cup X'$ for its image.

For two mono. $X \hookrightarrow Y$ and $K \hookrightarrow L$ we have a pullback square

$$
\begin{tikzcd}
K \times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L \times X \arrow[d] \\
K \times Y \arrow[r] & L \times Y
\end{tikzcd}
$$

and thus an inclusion

$$K \times Y \cup L \times X \hookrightarrow L \times Y.$$

Definition

An <u>homotopy structure</u> on $\hat{A}$ is a couple (I, An) where I is an interval and An is a class of <u>anodyne extensions with respect to I</u>, that is a class of maps of $\hat{A}$ satisfying the following properties:

<u>An0</u> there exists a small set of monic maps Λ in $\hat{A}$ such that $An = Sat(\Lambda)$ (in particular, any element of An is monic).

<u>An1</u> if $K \hookrightarrow L$ is a mono then

$$I \times K \cup \{e\} \times L \hookrightarrow I \times L \in An, \quad e = 0, 1$$

<u>An2</u> if $K \hookrightarrow L \in An$, then

$$I \times K \cup \partial I \times L \hookrightarrow I \times L \in An.$$

Example: Let I be an interval and S a set of monomorphisms in $\hat{A}$. Put Λ for the set of maps of the form

$$I \times K \cup \{e\} \times L \to I \times L \quad \text{for } K \hookrightarrow L \in \mathcal{M}, \ e = 0, 1$$

or of the form

$$I \times K \cup \partial I \times L \to I \times L \quad \text{for } K \hookrightarrow L \in S.$$

Put $An_I(S) = Sat(\Lambda)$ is the class of anodyne extensions with respect to I (and it is in fact the smallest which contains S).

Example: in the previous example, take

$$A = \Delta$$
$$I = \Delta^1$$
$$S = \emptyset$$

this is called the standard homotopy structure on $\hat{\Delta} = \mathrm{SSets}$.

Consider an homotopy structure (I, An) on $\hat{A}$.

Given two maps $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\hat{A}$ define the relation $\sim_I$

$$f \sim_I g$$

if there exists a map $h : I \times X \longrightarrow Y$ such that

$$h|_{\{0\}\times X} = f \quad \text{and} \quad h|_{\{1\}\times X} = g .$$

Define $[X, Y]$ as the quotient of $\mathrm{Hom}_{\hat{A}}(X, Y)$ by the equivalence relation defined by $\sim_I$.

There is unique category $h_I(\hat{A})$ such that

$$\mathrm{ob}\, h_I(\hat{A}) = \mathrm{ob}\, \hat{A}$$

$$\mathrm{Hom}_{h_I(\hat{A})}(X, Y) = [X, Y]$$

and
$$\begin{aligned} \hat{A} &\longrightarrow h_I(\hat{A}) \\ X &\longmapsto X \\ \mathrm{Hom}_{\hat{A}}(X, Y) &\longrightarrow [X, Y] \end{aligned}$$
define a functor.

Definition.

The <u>naive fibrations</u> are the elements of $NFib := An^{\perp} = \Lambda^{\perp}$

A presheaf X is <u>fibrant</u> if $X \longrightarrow *$ is a naive fibration.

A morphism $X \xrightarrow{f} Y$ is a <u>weak equivalence</u> if, for any fibrant object Z, the map

$$[Y, Z] \xrightarrow{f^*} [X, Z]$$

is bijective.

$$\begin{cases} W = \{\text{weak equiv.}\} \\ Cof = \{\text{mono}\} \\ Fib = (Cof \cap W)^{\perp}. \end{cases}$$

Theorem. This defines a (combinatorial) model structure on $\hat{A}$. Moreover, the fibrant objects of this model structure are precisely ~~completely~~ the one defined above, and a map between fibrant objects ~~are~~ is a fibration iff it is a naive fibration.

~~Remark: if $An = An_I(S)$ for S a set of mono~~
~~This means that, if we write Hom for the internal Hom of $\hat{A}$~~

Example: $A = \Delta$, $I = \Delta^1$, $S = \emptyset$

The standard homotopy structure on SSets defines the standard model structure on SSets whose fibrant objects are Kan complexes and fibrations are the Kan fibrations.

$$|\Delta^n| = \{ (x_0, \dots, x_n) \in [0,1]^{n+1} \mid \sum_i x_i \leq 1 \}$$

$$\mathrm{SSets} \underset{S}{\overset{|-|}{\rightleftarrows}} \mathrm{Top}$$

$$S(X)_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$$

$X \longrightarrow S|X|$ and $|S(Y)| \longrightarrow Y$ are weak equivalences (Milnor) so that the standard model structure on SSets and the model structure on Top define equivalent homotopy theories (in any sense possible...).

Model category of categories.

Cat = category of small categories.

Cat is a model category with:

$W = \{$equivalences of categories$\}$

$Cof = \{$functor $C \to D$ such that $ob C \to ob D$ is injective$\}$

$Fib = \{$isofibrations$\}$

A functor $C \xrightarrow{p} D$ is an <u>iso fibration</u> if for any isomorphism $d_0 \xrightarrow{\delta} d_1$ in D and any object c_0 of C such that $p(c_0) = d_0$, there exists an isomorphism $c_0 \xrightarrow{\gamma} c_1$ in C such that $p(\gamma) = \delta$ (and thus $p(c_1) = d_1$).

The nerve functor

Any poset E is a category with objects the elements of E and

$$\mathrm{Hom}_E(x,y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$

This turns the category of posets into a full subcategory of Cat. In particular, we have a full inclusion

$$\Delta \hookrightarrow Cat$$

which induces an adjunction

$$\hat{\Delta} = SSets \underset{N}{\overset{\tau}{\rightleftarrows}} Cat \quad \text{with } \tau(\Delta^n) = \Delta^n,\ n \geq 0.$$

For a small category C, the nerve of C is $N(C)$ and is described by $N(C)_n = \{c_0 \to c_1 \to \cdots \to c_n\}$.

Define $\Lambda^n_k = \bigcup_{i \neq k} \mathrm{Im}(\Delta^{n-1} \xrightarrow{\delta^n_i} \Delta^n) \hookrightarrow \Delta^n$ for $n \geq 1$ and $0 \leq k \leq n$.

$$I_n = \bigcup_i \mathrm{Im}(\Delta^1 \xrightarrow{u_i} \Delta^n) \hookrightarrow \Delta^n$$

where $u_i : \Delta^1 \to \Delta^n$ $\quad u_i(\varepsilon) = i + \varepsilon$

$0 \mapsto i$

$1 \mapsto i+1$

Theorem (Grothendieck–Segal)

The nerve functor $N : \mathrm{Cat} \to \mathrm{SSets}$ is fully faithful.

For a simplicial set X, the following conditions are equivalent:

(i) X belongs to the essential image of N;

(ii) For any $n \geq 2$, the map

$$X_n = \mathrm{Hom}(\Delta^n, X) \longrightarrow \mathrm{Hom}(I_n, X) = X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is bijective

(iii) For any $n \geq 2$ and $0 < k < n$, the map

$$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$$

is bijective.

Remark: condition (iii) with Λ^n_k allowed with $n \geq 1$ and $0 \leq k \leq n$ characterizes nerves of groupoids.

Definition.

The class of <u>inner anodyne maps</u> is the class

$$\mathrm{InAn} := \mathrm{Sat}(\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2,\ 0 < k < n\})$$

The class of <u>inner fibrations</u> is $\mathrm{InAn}^{\perp} = \{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, 0 < k < n\}^{\perp}$

The class of <u>anodyne maps</u> is

$$An = \mathrm{Sat}\left(\{ \Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 1,\ 0 \leq k \leq n \}\right)$$

The class of <u>Kan fibrations</u> is $An^{\perp}$.

Theorem. $An = An_{\Delta^1}(\emptyset)$ is the smallest class of anodyne maps with respect to the interval Δ^1.
In fact An is precisely the class of trivial cofibrations of the standard model structure.

For inner anodyne maps, that is another story. But we can prove:

Theorem. If $A \longrightarrow B$ is an inner anodyne extension then so is the map $A \times L \cup B \times K \longrightarrow B \times L$ for any monomorphism $K \hookrightarrow L$.

This is a consequence of the fact that:

$$\mathrm{Sat}\left(\{ \partial\Delta^n \times \Delta^2 \cup \Delta^n \times \Lambda^2_1 \longrightarrow \Delta^n \times \Delta^2 \mid n \geq 0 \}\right) = \mathrm{In}\,An.$$

Definition: an <u>∞-category</u> is a simplicial set X such that the map $X \longrightarrow *$ is an inner fibration.

Proposition:
Let X be a simplicial set. The following conditions are equivalent:

(i) X is an ∞-category

(ii) $\underline{Hom}(\Delta^2, X) \to \underline{Hom}(\Lambda^2_1, X)$ is a trivial fibration

(iii) $\underline{Hom}(\Delta^n, X) \to \underline{Hom}(\Lambda^n_k, X)$ is a trivial fibration

for any $n \geq 2$, $0 < k < n$.

(iv) $\underline{Hom}(\Delta^n, X) \to \underline{Hom}(I_n, X) = \underline{Hom}(\Delta^1, X) \times_X \cdots \times_X \underline{Hom}(\Delta^1, X)$

is a trivial fibration for any $n \geq 2$.

(here $\{\text{trivial fibrations}\} = \{\text{monos}\}^{\perp} = \{\partial\Delta^n \to \Delta^n \mid n \geq 0\}^{\perp}$).

<u>Cor</u> For any ∞-category X and any simplicial set A, $\underline{Hom}(A, X)$ is an ∞-category. [Rem: for categories C and D one has $N\,\underline{Hom}(C, D) = \underline{Hom}(NC, ND)$.]

Basic terminology:

Let X be an ∞-category.

An <u>object</u> of X is an element of X_0, or, equivalently a map $\Delta^0 \to X$.

An <u>arrow</u> or a <u>map</u> in X is an element of X_1, or a map $\Delta^1 \xrightarrow{f} X$, which we picture as $a \xrightarrow{f} b$ with $a = (\Delta^0 \xrightarrow{\delta^1_1} \Delta^1 \xrightarrow{f} X)$ and $b = (\Delta^0 \xrightarrow{\delta^1_0} \Delta^1 \to X)$

For a simplicial set A, $Sk^n A = \bigcup_{\substack{\Delta^k \to A \\ k \leq n}} \mathrm{Im}(\Delta^k \to A)$

A triangle in X is a map $\underset{\overset{\shortparallel}{\partial\Delta^2}}{Sk^1 \Delta^2} \xrightarrow{(f,g,h)} X$ i.e. three maps in X

$$(*) \qquad \begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & x_2 \end{tikzcd}$$

We say that the triangle (f, g, h) <u>commutes</u> if

$$\begin{tikzcd} sk^1\Delta^2 = \partial\Delta^2 \arrow[r, "{(f,g,h)}"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed, "\exists h"'] & \end{tikzcd}$$

and we then write $\quad gf \sim h$

For any object $x \in X_0$, the map

$$\Delta^1 \longrightarrow \Delta^0 \xrightarrow{x} X$$

is written 1_x and is called the identity of x.

We define 2 ~~equiva~~le relations on maps of X:

$f : a \to b \qquad g : a \to b$

$$f \underset{1}{\sim} g \iff f\,1_a \sim g$$

$$f \underset{2}{\sim} g \iff 1_b\, f \sim g$$

<u>Prop</u>: $\underset{1}{\sim} = \underset{2}{\sim}$ and these are equivalence relation.

Moreover the category τX can be described by

$\mathrm{ob}\,\tau X = \mathrm{ob}\,X = X_0$

$\mathrm{Hom}_{\tau X}(a, b) = \{\text{maps } a \to b \text{ in } X\} / \underset{1}{\sim}$

Composition: $a \xrightarrow{f} b \xrightarrow{g} c$ in X define

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "(f,g)"] \arrow[d, "\text{inner anodyne}"'] & X \arrow[d, "\text{inner fibration}"] \\ \Delta^2 \arrow[r] \arrow[ur, dashed, "B"] & * \end{tikzcd} \qquad \begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "B"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & {} & c \end{tikzcd}$$

one defines $gf := h$ in $\tau(X)$.

Exercise:

For any ∞-category X and any category C, any map $X \to N(C)$ is an inner fibration.

Definition.

For a category C, $k(C)$ is the maximal groupoid contained in C:

$$\mathrm{ob}\, k(C) = \mathrm{ob}\, C$$

$$\mathrm{Hom}_{k(C)}(x,y) = \{\text{isomorphisms } x \xrightarrow{\sim} y \text{ in } C\}.$$

For an ∞-category X we define $k(X)$ by the pullback

$$\begin{tikzcd} k(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Nk\tau(X) \arrow[r, hook] & N\tau(X) \end{tikzcd}$$

$k(X)$ is the maximal ∞-category contained in X such that any map of $k(X)$ is invertible.

An $\underline{\infty\text{-groupoid}}$ is an ∞-category such that $k(X) = X$.

Theorem

A simplicial set X is an <u>∞-groupoid</u> if and only if it is a Kan complex (that is a fibrant object of the standard model structure).

The proof is non trivial though.

Definition.

A map between ∞-categories (= a functor or ∞-functor...)

$$p : X \to Y$$

is an <u>isofibration</u> if it is an inner fibration, and, for any map $y_0 \xrightarrow{v} y_1$ in Y which is invertible (i.e. $v \in k(Y)_1$) and any object x_0 of X such that $p(x_0) = y_0$, there exists an invertible map $x_0 \xrightarrow{u} x_1$ in X such that $p(u) = v$.

Let $J = N\pi_1(\Delta^1)$ where $\pi_1 \Delta^1$ is the groupoid with two objects $0, 1$ equivalent to $*$.

Then J has a natural structure of an interval and $J \to *$ is a trivial fibration (being both a Kan fibration (between fibrant objects) and a weak equivalence of the standard model structure).

Definition: the class of J-<u>anodyne maps</u> is the class

$$\mathrm{An}_J := \mathrm{An}_J\left(\{\Lambda^n_k \to \Delta^n \mid n \geq 2,\ 0 < k < n\}\right)$$

$$= \mathrm{Sat}\left(\{\partial\Delta^n \times J \cup \Delta^n \times \{e\} \hookrightarrow \Delta^n \times J \mid n \geq 0,\ e = 0,1\} \cup \{\partial\Delta^n \times \Delta^2 \cup \Delta^n \times \Lambda^2_1 \to \Delta^n \times \Delta^2,\ n \geq\right.$$

$$J\text{-fibrations} = An_J^{\perp}.$$

X is J-fibrant

$\underset{\text{def.}}{\iff}$ $X \to *$ is a J-fibration iff $ev_0 : \overset{\underline{Hom}(J,X)}{\overset{\shortparallel}{X^J}} \longrightarrow X$ is a trivial fibration

and $\underline{Hom}(\Delta^2, X) \to \underline{Hom}(\Lambda^2_1, X)$ "

(note that J has an automorphism which exchanges its global sections).

More generally a map $X \to Y$ is a J-fibration iff

$$ev_0 : \underline{Hom}(J,X) \longrightarrow \underline{Hom}(J,Y) \times_Y X$$

$$\text{and} \quad \underline{Hom}(\Delta^2, X) \to \underline{Hom}(\Delta^2, Y) \times_{\underline{Hom}(\Lambda^2_1, Y)} \underline{Hom}(\Lambda^2_1, X)$$

are trivial fibrations.

The model category structure associate to the homotopy structure (J, An_J) has J-fibrant objects as fibrant objects and a map between fibrant objects is a fibration iff it is a J-fibration. We call this model structure on SSets the <u>Joyal model category structure</u>..

Theorem (Joyal).
A simplicial set X is J-fibrant iff it is an ∞-category
A morphism of ∞-categories $p: X \to Y$ is a J-fibration iff it is an isofibration.

The proof of Joyal's theorem involves intermediate constructions which are meaningful by themselves.

For an ∞-category X and a simplicial set A write $\mathrm{ob}\,A$ for the set A_0, seen as constant subobject of A, and form the pullback:

$$\begin{tikzcd} k(A,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(A,X) \arrow[d] \\ k\,\underline{\mathrm{Hom}}(\mathrm{ob}A, X) \arrow[r] \arrow[d, equal] & \underline{\mathrm{Hom}}(\mathrm{ob}A, X) \arrow[d, equal] \\ \prod_{A_0} k(X) & \prod_{A_0} X \end{tikzcd}$$

Given an ∞-category Y and a subset $\mathcal{U} \subset Y_0 = \mathrm{ob}\,Y$ write $Y_{\mathcal{U}}$ for the subcomplex defined by the pullbacks

$$\begin{tikzcd} (Y_{\mathcal{U}})_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_n \arrow[d] \\ \mathcal{U}^{n+1} \arrow[r, hook] & Y_0^{n+1} \end{tikzcd}$$

It is easy to see that $Y_{\mathcal{U}}$ is an ∞-category: the full sub-∞-category of Y whose objects are the elements of $\mathcal{U}$.

Define, for a simplicial set B, $h(B,X) = Y_{\mathcal{U}}$ with $Y = \underline{\mathrm{Hom}}(B,X)$ and $\mathcal{U}$ the set of maps $B \to X$ which factor through $k(X)$ (i.e. which send all the maps of B to invertible maps of X).

Then it is easy to see that:

$$(*) \qquad \begin{array}{ccc} \mathrm{Hom}(B, k(A,X)) & \cong & \mathrm{Hom}(A, h(B,X)) \\ \cap & & \cap \\ \mathrm{Hom}(B, \underline{\mathrm{Hom}}(A,X)) & \cong & \mathrm{Hom}(A, \underline{\mathrm{Hom}}(B,X)) \end{array}$$

Let $X \xrightarrow{p} Y$ be an inner fibration between ∞-categories.

We can form the following map, obtained from the inclusion $\{0\} \hookrightarrow \Delta^1$:

$$ev_0 : h(\Delta^1, X) \longrightarrow h(\Delta^1, Y) \times_Y X$$

(identifying X with $\underline{\mathrm{Hom}}(\Delta^0, X) = h(\Delta^0, X)$).

Remark: p is an iso fibration $\iff$

$$\begin{tikzcd} \emptyset = \partial\Delta^0 \arrow[d] \arrow[r, phantom, "\perp"] & ev_0 \\ \Delta^0 & \end{tikzcd}$$

the main step in The proof of Joyal's theorem is:

Theorem:
$$\begin{tikzcd} \partial\Delta^n \arrow[d] \arrow[r, phantom, "\perp"] & ev_0 \\ \Delta^n & \end{tikzcd} \quad \text{for any } n > 0.$$

Corollary: p is an iso fibration $\iff$ ev_0 is a trivial fibration

$\iff h(B,X) \to h(A,X) \times_{h(A,Y)} h(B,Y)$ is a trivial fibration for any anodyne map $A \hookrightarrow B$.

Using $(*)$, we also get:

Corollary.

For any ∞-category X and any simplicial set A we have

$$k(A, X) = k\left(\underline{\mathrm{Hom}}(A, X)\right)$$

(in other words: a natural transformation is invertible iff it is termwise invertible)

For any isofibration between ∞-categories $p : X \to Y$ and any monomorphism $A \xrightarrow{u} B$ in SSets, the map

$$k(B, X) \longrightarrow k(A, X) \times_{k(A,Y)} k(B, Y)$$

is a Kan fibration, and it is a trivial fibration whenever u is a trivial cofibration of the Joyal model structure.

<u>Cor.</u> Let X be an ∞-category and $x, y \in \mathrm{ob}\, X = X_0$.

$$\begin{tikzcd}
X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & k(\Delta^1, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d] \\
\Delta^0 = * \arrow[r, "{(x,y)}"'] & k(X) \times k(X) \arrow[r] & X \times X = \underline{\mathrm{Hom}}(\partial\Delta^1, X)
\end{tikzcd}$$

Then $X(x,y)$ is a Kan complex.

~~Rectification~~

Definition.

Let $u: X \to Y$ be a functor between ∞-categories.

u is <u>fully faithful</u> if, for any $a, b \in \mathrm{Ob}X$, the induced map

$$u: X(a,b) \to Y(u(a), u(b))$$

is an equivalence of ∞-groupoids
($\Leftrightarrow$ w.e. of the standard model structure
$\Leftrightarrow$ w.e of the Joyal model structure
$\Leftrightarrow$ Δ^1-homotopy equivalence
$\Leftrightarrow$ J-homotopy equivalence ...)

u is <u>essentially surjective</u> if, for any $y \in \mathrm{Ob}Y = Y_0$ there exists $x \in \mathrm{Ob}X = X_0$ as well as an invertible map

$u(x) \to y$ in Y ($\Leftrightarrow$ $\tau X \to \tau Y$ is essentially surjective)

u is an <u>equivalence of ∞-categories</u> if it is fully faithful and ess. surj.

Theorem.

A functor between ∞-categories is a weak equivalence of the Joyal model structure iff it is an equivalence of ∞-categories

($\Leftrightarrow$ a J-homotopy equivalence)

Rectification of ∞-categories.

Simplicial categories.

$$\mathrm{Cat}_\Delta = \text{category of} \begin{cases} \text{categories enriched in simplicial sets} \\ \text{simplicial categories} \end{cases}$$

A simplicial category C is thus given by a set of objects $\mathrm{Ob}\,C$, and for each $x, y \in \mathrm{Ob}\,C$, a simplicial set $\mathrm{Hom}_C(x,y)$, together with morphisms of simplicial sets

$$\mathrm{Hom}_C(x,y) \times \mathrm{Hom}_C(y,z) \longrightarrow \mathrm{Hom}_C(x,z)$$

and identities $1_x \in \mathrm{Hom}_C(x,x)_0$ which turn C into a category...

A simplicial functor $F: C \longrightarrow D$ is a map $\mathrm{Ob}\,C \longrightarrow \mathrm{Ob}\,D$, $x \longmapsto F(x)$

together with morphisms of simplicial sets

$$(*) \qquad \mathrm{Hom}_C(x,y) \longrightarrow \mathrm{Hom}_D(F(x), F(y))$$

which are compatible with composition laws and identities.

Such a functor is a <u>weak equivalence</u> (or <u>Dwyer-Kan equivalence</u>) if it is fully faithful (in the sense that $(*)$ is a weak equivalence of the standard model structure on sSets) and essentially surjective in the following sense.

A map $x \to y$ in C is invertible if, for any z,

$$\mathrm{Hom}_C(y,z) \longrightarrow \mathrm{Hom}_C(y,x) \text{ is a weak equivalence.}$$

F is essentially surjective if, for any $y \in \mathrm{Ob}\,D$, there exists $x \in \mathrm{Ob}\,C$ and an invertible map $F(x) \longrightarrow y$.

A simplicial functor $F: \mathcal{C} \to \mathcal{D}$ is an iso fibration if:

a) For any $x, y \in \text{Ob}\,\mathcal{C}$ the map

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$

is a Kan fibration.

b) For any invertible map $y_0 \xrightarrow{g} y_1$ in $\mathcal{D}$, if there is an object $x_0 \in \text{Ob}\,\mathcal{C}$ such that $F(x_0) = y_0$, then there is an invertible map $x_0 \xrightarrow{f} x_1$ in $\mathcal{C}$ such that $F(f) = g$.

$$\mathrm{Cof} = \left(\{\text{isofibrations}\} \cap \{\text{D.-K. equiv.}\}\right)^{\perp}$$

Theorem (Bergner) This defines a model category structure on Cat_{Δ}.

Let P be a partially ordered set.

Define $\mathbb{C}[P]$ as the category below:

$$\text{ob}\,\mathbb{C}[P] = P$$

For $x, y \in P$

$$\mathrm{Hom}_{\mathbb{C}[P]}(x,y) = \{\text{totally ordered subsets } S \subset P \text{ with } \min S = x \text{ and } \max S = y\}$$

Composition

$$\mathrm{Hom}_{\mathbb{C}[P]}(x,y) \times \mathrm{Hom}_{\mathbb{C}[P]}(y,z) \to \mathrm{Hom}_{\mathbb{C}[P]}(x,z)$$

$$(S \quad , \quad T) \longmapsto S \cup T$$

$$1_x = \{x\} \in \mathrm{Hom}_{\mathbb{C}[P]}(x,x).$$

If $P \xrightarrow{u} Q$ is an order preserving map, we have a functor

$$\mathbb{C}[P] \longrightarrow \mathbb{C}[Q]$$

induced by $\mathrm{Hom}_{\mathbb{C}[P]}(x,y) \longrightarrow \mathrm{Hom}_{\mathbb{C}[Q]}(u(x),u(y))$

$$S \longmapsto u(S)$$

As the $\mathrm{Hom}_{\mathbb{C}[P]}(x,y)$'s are posets, it makes sense to take their nerves and we define the simplicial category $\mathbb{C}[NP]$ ~~by~~ $\mathrm{Ob}\,\mathbb{C}[NP] = P$ and

$$\mathrm{Hom}_{\mathbb{C}[NP]}(x,y) = N\,\mathrm{Hom}_{\mathbb{C}[P]}(x,y)$$

This defines a functor

$$\Delta \longrightarrow \mathrm{Cat}_{\Delta}$$
$$\Delta^n \longmapsto \mathbb{C}[\Delta^n]$$

and thus an adjunction

$$\mathrm{SSets} \underset{N}{\overset{\mathbb{C}[-]}{\rightleftarrows}} \mathrm{Cat}_{\Delta}$$

where $\mathbb{C}[X] = \varinjlim_{\Delta^n \to X} \mathbb{C}[\Delta^n]$ and, for a simplicial category C,

$$N(C)_n = \mathrm{Hom}_{\mathrm{Cat}_{\Delta}}(\mathbb{C}[\Delta^n], C).$$

For C an ordinary category seen as a simplicial category, $N(C)$ is the usual nerve (because $\mathrm{Hom}_{\mathbb{C}[\Delta^n]}(x,y) \to \mathrm{Hom}_{\Delta^n}(x,y)$ is an homotopy equivalence for $0 \le x, y \le n$).

Quillen adjunctions.

Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

An adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ is said to be a <u>Quillen adjunction</u> if it satisfies one of the equivalent conditions below.

(i) F preserves cofibrations and trivial cofibrations

(ii) G —— fibrations —— fibrations

(iii) F preserves cofibrations and weak equivalences between cofibrant objects

(iv) G preserves fibrations and weak equivalences between fibrant objects

(v) F preserves cofibrations and G preserves fibrations between fibrant objects

In this situation, let

$\mathcal{C}_c$ = cofibrant objects of $\mathcal{C}$

$\mathcal{D}_f$ = fibrant objects of $\mathcal{D}$.

$$
\begin{tikzcd}
\mathcal{C}_c \arrow[r, hook] \arrow[drr] & \mathcal{C} \arrow[d] \arrow[drr] & & \\
& \mathcal{D} \arrow[drr] & Ho(\mathcal{C}_c) \arrow[r, "\sim"] \arrow[d, dotted, "\bar{F}"'] & Ho\,\mathcal{C} \arrow[dl, dashed, "LF"] \\
& & & Ho(\mathcal{D})
\end{tikzcd}
$$

A quasi-inverse

$$Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{C}_c)$$
$$X \longmapsto X_{cof}$$

where $\emptyset \rightarrowtail X_{cof} \xrightarrow{\sim} X$.

$LF : Ho(\mathcal{C}) \to Ho(\mathcal{D})$ is the total left derived functor of F.

Similarly, one has the total right derived functor

$$RG: Ho(\mathcal{D}) \to Ho(\mathcal{C})$$

Theorem (Quillen)

One has an adjunction

$$Ho(\mathcal{C}) \underset{RG}{\overset{LF}{\rightleftarrows}} Ho(\mathcal{D}).$$

(F, G) is called a <u>Quillen equivalence</u> if LF (or equivalently RG) is an equivalence of categories.

Theorem (Joyal-Lurie)

$$sSets \underset{N}{\overset{\mathfrak{C}[-]}{\rightleftarrows}} Cat_{\Delta}$$

is a Quillen equivalence.

Idea of proof.

First, one checks that it is a Quillen adjunction.

$$\mathfrak{C}[\Delta^0] = \Delta^0 = *$$

$$\mathfrak{C}[\Delta^1] = \Delta^1 = (0 \to 1)$$

For $n \geq 2$ and x, y in Δ^n

$$\partial(\Delta^1)^{n-1} = \mathrm{Hom}_{\mathfrak{C}[\partial\Delta^n]}(x,y) \rightarrowtail \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(x,y) = (\Delta^1)^{n-1}$$

On the other hand, trivial fibrations of Cat_{Δ} are maps $F: \mathcal{C} \to \mathcal{D}$ which are surjective on objects and such that $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ is a trivial fibration

$\Rightarrow \mathfrak{C}[-]$ preserves cofibrations.

For $n \geq 2$ and $0 < k < n$, the map

$$\mathrm{Hom}_{\mathbb{C}[\Lambda^n_k]}(x,y) \longrightarrow \mathrm{Hom}_{\mathbb{C}[\Delta^n]}(x,y)$$

$$\begin{array}{ccc} \cong & & \cong \end{array}$$

$$(\Delta^1)^{n-2} \times \{\varepsilon\} \cup \partial(\Delta^1)^{n-2} \times \Delta^1 \hookrightarrow (\Delta^1)^{n-1}$$

is anodyne. One deduces easily that N preserves isofibrations.

In particular, for any fibrant simplicial category C (i.e such that $\mathrm{Hom}_C(x,y)$ is a Kan complex for any x,y) $N(C)$ is an ∞-category.

One deduces that N preserves <u>and detects</u> weak equivalences between fibrant objects. The theorem thus follows from

Note that $\mathrm{ob}\, C = \mathrm{ob}\, N(C)$ for any simplicial category C.

Theorem:

For any (fibrant) ~~\infty-cat~~ simplicial category C and any objects x, y in C, the map

$$\mathrm{Hom}_{\mathbb{C}[N(C)]}(x,y) \longrightarrow \mathrm{Hom}_C(x,y)$$

is a weak equivalence.

Example.

Kan is the simplicial category whose objects are Kan complexes and $\mathrm{Hom}_{Kan}(X,Y) = \underline{\mathrm{Hom}}(X,Y)$ (with $\underline{\mathrm{Hom}}$ the internal Hom of sSets).

Define $S = N(Kan)$.

We also define $S_\bullet = N(Kan_\bullet)$ where $Kan_\bullet$ is the simplicial category of pointed Kan complexes.

We thus have a functor $\begin{array}{c} S_{\bullet} \\ \downarrow \\ S \end{array}$

Remark, for any small category C we have a fully faithful functor which preserves both limits and colimits

$$\hat{C} \longrightarrow Cat/_C$$

$$F \longmapsto C/_F$$

where $C/_F$ is the category whose objects are pairs (x,s) with $x \in Ob\, C$ and $s \in F(x) \Leftrightarrow x \xrightarrow{s} F$.

maps $(x,s) \to (y,t)$ are given by
$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "s"'] & \circlearrowleft & y \arrow[dl, "t"] \\ & C & \end{tikzcd}$$

In other words, one can identify presheaves over C with Grothendieck fibrations $X \to C$ whose fibres are sets (i.e. small discrete categories). We have a pull back

$$\begin{tikzcd} C/_F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (Sets_{\bullet})^{op} \arrow[d] \\ C \arrow[r, "F"] & Sets^{op} \end{tikzcd} \quad \text{i.e.} \quad \begin{array}{c} Sets_{\bullet}^{op} \\ \downarrow \\ Sets \end{array}$$ is the universal Grothendieck fibration whose fibers are small sets.

Definition

The class of left (right) anodyne maps is

$$An_{left} = Sat\left(\{ \Lambda^n_k \hookrightarrow \Delta^n \mid 0 \leq k < n,\ n \geq 1 \}\right)$$

$$\left(An_{right} = Sat\left(\{ \text{———} \mid 0 < k \leq n,\ \text{———} \}\right)\right)$$

The class of left fibrations (right fibrations) are

$$Fib_{left} = An_{left}^{\perp} \quad \left(Fib_{right} = An_{right}^{\perp} \right).$$

Let C be a simplicial set.

$$\mathrm{sSets}/C \cong \widehat{\Delta/C}$$

$$J_C := \begin{array}{c} J\times C \\ \downarrow \\ C \end{array}$$ is an interval of $\widehat{\Delta/C}$.

An_{J_C} = the smallest class of anodyne extensions with respect to J_C which contains maps of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[rr] \arrow[dr] & & \Delta^n \arrow[dl] \\ & C & \end{tikzcd}$$ with $n \geq 1$ and $0 < k \leq n$.

(J_C, An_{J_C}) defines a model category structure on sSets/C which we call the <u>contravariant model structure over C</u>.

Theorem:
The fibrant objects of the contravariant model structure over C are precisely the right fibrations $F \to C$.
Moreover, for any right fibration $F \to C$ and any object $x \in ob C$ the fiber F_x at x is a Kan complex, and a map between right fibrations

$$\begin{tikzcd} F \arrow[rr] \arrow[dr, "f"'] & & G \arrow[dl, "g"] \\ & C & \end{tikzcd}$$ is a weak equivalence

iff $\forall x \in ob C$ $F_x \to G_x$ is an equivalence of ∞-groupoids

iff $F \to G$ is a weak equivalence in the Joyal model structure.

$$
\begin{tikzcd}
\mathrm{Map}_C(F,G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(F,G) \arrow[d, "g_*"] \\
\Delta^0 \arrow[r, "f"'] & \underline{\mathrm{Hom}}(F,C)
\end{tikzcd}
$$

g_* is a right fibration

Let $\mathcal{P}(C)$ be the (fibrant) simplicial category whose objects are the right fibrations with

$$\mathrm{Hom}_{\mathcal{P}(C)}(F,G) = \mathrm{Map}_C(F,G).$$

Define $\hat{C} = N(\mathcal{P}(C))$.

Remark: $S_\bullet \to S$ is a left fibration so that $S_\bullet^{op} \to S^{op}$ is a right fibration.

Theorem: there is a canonical weak equivalence of the Joyal model structure

$$(*) \qquad \hat{C} \xrightarrow{\sim} \underline{\mathrm{Hom}}(C^{op}, S)$$

In other words, for any map $C^{op} \xrightarrow{F} S$, we can form the pullback

$$
\begin{tikzcd}
C/F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S_\bullet^{op} \arrow[d] \\
C \arrow[r, "F"'] & S^{op}
\end{tikzcd}
$$

so that $C/F \to C$ is a right fibration.

Any object of $\hat{C}$ is obtained in this way up to a unique invertible map

("unique" means that the space of choices is contractible).

Remark: to define (*) one restricts to the case where $C = \Delta^n$

We have pull back functors induced by $\Delta^n \longrightarrow C$

$$\hat{C} \longrightarrow \hat{\Delta^n}$$

and
$$\Delta^n \times P(\Delta^n) \xrightarrow{ev} Kan$$
$$\Big(i\, ,\ \begin{array}{c} F \\ \downarrow \\ \Delta^n \end{array}\Big) \longmapsto Map_{\Delta^n}(\Delta^n/i\, , F)$$

as well as
$$\begin{tikzcd}
\mathbb{C}[\Delta^n \times \hat{\Delta^n}] \arrow[r] \arrow[dddd] & \mathbb{C}[\Delta^n] \times \mathbb{C}[\hat{\Delta^n}] \arrow[d] \\
 & \mathbb{C}[\Delta^n] \times P(\Delta^n) \arrow[d] \\
 & \Delta^n \times P(\Delta^n) \arrow[d, "ev"] \\
 & Kan \\
S &
\end{tikzcd}$$

thus giving $\hat{C} \longrightarrow \underline{Hom}(\Delta^{n\,op}, S)$

if $\hat{C}'$ denotes $N\, P'(C)$ with $P'(C) =$ objects of $P(C)$ + choice of pull backs along maps $\Delta^n \to C$

we have

$$\hat{C} \xleftarrow{\sim} \hat{C}' \longrightarrow \operatorname*{holim}_{\Delta^n \to C} \underline{Hom}(\Delta^{n\,op}, S) \xleftarrow{\sim} \underline{Hom}(C^{op}, S)$$

from which we can get $\hat{C} \longrightarrow \underline{Hom}(C^{op}, S)$.

Limits and colimits.

Let C be an ∞-category.

A right fibration $\begin{matrix} F \\ \downarrow \\ C \end{matrix}$ is <u>representable</u> ~~represents an object~~

there exists an commutative diagram

$$\begin{tikzcd} & F \arrow[d] \\ \Delta^0 \arrow[ur, "u"] \arrow[r, "x"'] & C \end{tikzcd}$$

with u a contravariant weak equivalence.

An object x of C is called <u>terminal</u> if the map

$$\Delta^0 \xrightarrow{x} C$$

is a contravariant weak equivalence

We define $t(C)$ as the full ∞-subcategory of C whose objects are terminal.

Theorem. $t(C)$ is either empty or a contractible ∞-groupoid.

~~In other words, for any~~

This comes from the fact that, if x is terminal then, for any object y of C, the ∞-groupoid

$$\mathrm{Hom}_C(y, x) \quad \text{is contractible}$$

i.e $\mathrm{Hom}_C(y, x) \to \Delta^0$ is a trivial fibration.

For an object x of C we write
$$\begin{tikzcd} \Delta^0 \arrow[d] \arrow[dd, bend right, "x"'] \\ C/x \arrow[d] \\ C \end{tikzcd}$$
for any choice of a factorisation of $\Delta^0 \xrightarrow{x} C$ by C/x a contravariant weak equivalence followed by a right fibration.

Let I be a $\left[\begin{array}{l}\text{simplicial set} \\ \text{small}\end{array}\right.$ ∞-category and $F: I \to C$ a functor. We consider the pullback square

$$\begin{tikzcd} \{\text{Cones of } F\} := C/_F \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(I,C)/_F \arrow[d] \\ C \arrow[r] & \underline{\mathrm{Hom}}(I,C) \end{tikzcd}$$

where $C \longrightarrow \underline{\mathrm{Hom}}(I,C)$ is induced by the projection $C \times I \xrightarrow{pr_1} C$ (i.e. is the "constant" functor).

We say that F admits a limit in C if $C/_F$ is representable, i.e. it has a terminal object. Such an object is called a terminal cone of F and its image in C is written $\varprojlim_I F = \varprojlim_{i \in I} F_i = \varprojlim F$.

Definition.
An ∞-category has limits of type I if any functor $F: I \to C$ has a limit in C.
An ∞-category has colimits of type I if C^{op} has limits of type I^{op}.

An ∞-category has <u>finite limits</u> if it has limits of type I ~~Δ~~ for any <u>finite</u> simplicial set (= simplicial sets of finite presentation).

An ∞-category has <u>finite colimits</u> if ~~it has~~ C^{op} has finite limits.

Warning: if I is a finite category, the nerve $N(I)$ is not necessarily a finite simplicial set.

Example: ~~if I is the category~~ finite groups G (such as $\mathbb{Z}/2\mathbb{Z}$), seen as finite categories with one object do not have finite nerves (otherwise their classifying space BG would have the homotopy types of finite CW-complexes and thus G ~~$\Rightarrow$~~ $\bigoplus_{n\in\mathbb{Z}} H^n(G,\mathbb{Z})$ would be an abelian group of finite type ...)

Theorem

An ∞-category has finite limits if and only if it has finite products (i.e. limits of type I for I any finite constant simplicial set) as well as fiber products (i.e limits of type $I = \left(\begin{array}{cc} \cdot & \longrightarrow \cdot \\ \downarrow & \\ \cdot & \end{array} \right)$).

An ∞-category has arbitrary small limits iff it has finite limits as well as small products.

Example: S has small limits an colimits.

If C has limits of type I then so does $\underline{\mathrm{Hom}}(A,C)$ for any A.

and we have all the usual formulas...

Let C be a small ∞-category and D an ∞-category which is cocomplete (i.e. which has small colimits).

$\underline{\mathrm{Hom}}_!(\hat{C}, D)$ is the full ∞-subcategory of $\underline{\mathrm{Hom}}(\hat{C}, D)$ which consists of colimit preserving functors $\hat{C} \to D$.

Theorem: the Yoneda embedding $C \xrightarrow{h} \hat{C}$ is fully faithful and $\underline{\mathrm{Hom}}_!(\hat{C}, D) \xrightarrow{h^*} \underline{\mathrm{Hom}}(C, D)$ is an equivalence of ∞-categories.

~~$\underline{\mathrm{Hom}}_!(\hat{C}, D) \xrightarrow{h^*} \underline{\mathrm{Hom}}_!(\hat{C}, D)$~~

$\square = \Delta^1 \times \Delta^1$.

$$\ulcorner = \begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (0,1) \\ (1,0) & \end{tikzcd} \qquad \lrcorner = \begin{tikzcd} & (0,1) \arrow[d] \\ (1,0) \arrow[r] & (1,1) \end{tikzcd}$$

$$\ulcorner \xrightarrow{i} \square \qquad \lrcorner \xhookrightarrow{j} \square$$

A commutative square $\square \xrightarrow{F} \mathcal{C}$ is <u>cartesian</u> :)

$F_{0,0} \to \varprojlim j^* F$ is invertible.

$$\varprojlim \begin{pmatrix} F_{0,0} = F_{0,0} \\ \| \quad\quad \| \\ F_{0,0} = F_{0,0} \end{pmatrix}$$

We have $\square^{op} \simeq \square$. A commutative square $\square \xrightarrow{F} \mathcal{C}$ is cocartesian if $\square^{op} \simeq \square \xrightarrow{F^{op}} \mathcal{C}^{op}$ is cartesian.

bicartesian = cartesian and cocartesian.

Definition. An ∞-category $\mathcal{C}$ is <u>stable</u> if it has a zero object (i.e. an object which is both terminal and initial), it has finite limits and colimits and a commutative square of $\mathcal{C}$ is cartesian iff it is cocartesian.

For $x \in ob\,\mathcal{C}$ $\quad \Sigma x = \operatorname{colim}\begin{pmatrix} x \to 0 \\ \downarrow \\ 0 \end{pmatrix}$ and $\Omega x = \lim \begin{pmatrix} 0 \\ \downarrow \\ 0 \to x \end{pmatrix}$

then $\mathrm{Hom}_{\mathcal{C}}(\Sigma x, y) \simeq \mathrm{Hom}_{\mathcal{C}}(x, \Omega y)$ and $\Sigma\Omega x \simeq x$

as well as $x \simeq \Omega\Sigma x$.

For a map $x \xrightarrow{f} y$ in $\mathcal{C}$, define $\mathrm{Cone}(f)$ by

$$\mathrm{Cone}(f) = \mathrm{colim}\left(\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ 0 & \end{tikzcd}\right).$$

$$\mathrm{Cone}(y \to \mathrm{Cone}(f)) \simeq \Sigma x$$

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{Cone}(f) \arrow[r] & \Sigma x \end{tikzcd}$$

In general:
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$
(1) cocartesian

(1)∘(2) cocartesian $\Leftrightarrow$ (2) cocartesian.

$ho(\mathcal{C}) := \tau(\mathcal{C})$.

$\Sigma : ho(\mathcal{C}) \to ho(\mathcal{C})$ is an equivalence.

$ho(\mathcal{C})$ has finite sums and is additive.

distinguished triangles: those isomorphic to $x \xrightarrow{f} y \to \mathrm{Cone}(f) \to \Sigma x$

Theorem: $ho(\mathcal{C})$ is triangulated.

Theorem: Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories.
Then ~~u preserves~~ finite the following conditions are equivalent:

(i) u preserves finite limits
(ii) ———————— colimits
(iii) $ho(u) : ho(\mathcal{C}) \to h(\mathcal{D})$ is triangulated (i.e. $\Sigma u(x) \simeq u(\Sigma x)$ and u preserves distinguished triangles).
~~We then~~

If $ho(u): ho(C) \to ho(D)$ is triangulated, then

$u: C \to D$ is an equivalence of ∞-categories iff $ho(u)$ is an equivalence of (triangulated categories).

Theorem. Let $\mathcal{C}$ be a stable ∞-category.

Consider, for each finite poset E, an arbitrary triangulated structure on $ho(\underline{Hom}(NE, \mathcal{C}))$, such that, for each map of posets $E \xrightarrow{u} F$, the functor

$$ho\,\underline{Hom}(NF, \mathcal{C}) \to ho\,\underline{Hom}(NE, \mathcal{C})$$

is triangulated.

Then the triangulated structure on $ho\,\mathcal{C} = ho\,\underline{Hom}(\Delta^0, \mathcal{C})$ must be the ~~can~~ canonical one.

Definition. A stable ∞-category is $\mathbb{Q}$-linear if $ho\,\mathcal{C}$ is a $\mathbb{Q}$-linear additive category.

Dold-Kan correspondence (simplicial homology):

Ab = abelian groups.

$Ab^{\Delta^{op}} = \{\text{functors } \Delta^{op} \to Ab\} = \{\text{abelian groups objects in } SSets\}$

$Ab^{\Delta^{op}} \longrightarrow Comp(Ab) = C(Ab)$

$M \longmapsto (M^n, d^n)$ with $M^n = M_{-n}$ (and $M_{-k} = 0$ for $k > 0$)

$d: M^n \to M^{n+1}$ $\quad d = \sum_i (-1)^i d^i$

with d^i induced by $\delta_i^n: \Delta^{n-1} \to \Delta^n$.

$$\mathrm{SSets} \longrightarrow Ab^{\Delta^{op}} \longrightarrow C_-(Ab) \underset{\text{inclusion}}{\overset{\mathbb{Q}\otimes(-)}{\rightleftarrows}} C(\mathbb{Q}\text{-Vect})$$

$$X \longmapsto \mathbb{Z}(X)$$

and $\mathbb{Z}(X)\otimes\mathbb{Z}(Y) \xrightarrow{q\text{-is}} \mathbb{Z}(X\times Y)$ (natural in X, Y).

induces

$$\mathrm{Cat}_\Delta \underset{\mathcal{N}}{\overset{\mathbb{Q}[-]}{\rightleftarrows}} \mathrm{dgCat}_{\mathbb{Q}}$$

$\mathbb{Q}[C]$ object: those of C

$$\mathrm{Hom}_{\mathbb{Q}[C]}(x,y)^n = \mathbb{Q}[\mathrm{Hom}_C(x,y)_{-n}]$$

Theorem. For any pre-triangulated dg category C $N\mathcal{N}(C)$ is a stable ∞-category.

Furthermore $h_0(N\mathcal{N}(C)) \simeq H^0(C)$

In $dgcat_{\mathbb{Q}}$: Model structure for which

W.e.: maps $C \to D$ such that $H^0\mathrm{pre\,tr}(C) \to H^0\mathrm{pre\text{-}tr}(D)$ is an equiv. of categories.

$\mathrm{Stab}_{\mathbb{Q}}$: objects: stable $\mathbb{Q}$-linear ∞-categories

maps ~~from~~ stable functors

(= limits preserving functors

Thm: $\mathrm{Ho}(\mathrm{Stab}_{\mathbb{Q}}) \simeq \mathrm{Ho}(\text{pre-triangulated } \mathbb{Q}\text{-linear dg categories})$

Note: $\mathrm{Ho}(\mathrm{Stab}_{\mathbb{Q}})$ concrete: $\mathrm{Hom}(C,D) = \pi_0\, k\,\underline{\mathrm{Hom}}_{st}(C,D)$

where $\underline{\mathrm{Hom}}_{st}(C,D)$ = full ∞-subcat. of $\underline{\mathrm{Hom}}(C,D)$ spanned by finite limit preserving functors).

$\underline{\mathrm{Hom}}_{st}(C,D)$ is an internal Hom in Stab $\Rightarrow$ such a thing exists in dg cat (cf. Toën's construction).

Symmetric monoidal ∞-categories:

$\Gamma = \{\text{pointed finite sets}\}^{op}$

$\Delta \xrightarrow{i} \Gamma \iff \Delta^{op} \xrightarrow{S^1} \{\text{pointed finite sets}\}$

$\Delta^n \mapsto n_+ \qquad S^1 = \Delta^1/\partial\Delta^1 = \Delta^1/_{0=1}$

$\Gamma \xrightarrow{\underline{C}} \infty\text{-Cat}$

s.t. $\underline{C}(0_+) \sim *$ $\quad C := \underline{C}(1_+)$

$\underline{C}_n = \underline{C}(n_+) \longrightarrow \underbrace{\underline{C}_1 \times \cdots \times \underline{C}_1}_{n+1 \text{ times}}$ equivalence of ∞-categories.

$\underline{C}$ is a symmetric monoidal structure on C.

$* \simeq \underline{C}(0_+) \longrightarrow \underline{C}(n_+)$ provides the unit $1 \in ob\, C$. $\quad \Delta^1 \xrightarrow{\delta_1^2} \Delta^2$ induces

$\Rightarrow ho(C)$ is symmetric monoidal category

$$\begin{tikzcd} C \times C & \underline{C}(2_+) \arrow[l, "\simeq"'] \arrow[r] & \underline{C}(1_+) \cong C \arrow[from=1-1, to=1-3, dashed, bend right, "\otimes"'] \end{tikzcd}$$

$x \in ob\, C$ rigid if there exists $x^\vee \in ob\, C$

as well as $1 \xrightarrow{\eta} x \otimes x^\vee$ and $x^\vee \otimes x \xrightarrow{\varepsilon} 1$

such that, for any $y, z \in ob\, C$

$$\begin{tikzcd} Hom_C(x\otimes y, z) \arrow[r] \arrow[drr, "\simeq"] & Hom_C(x^\vee \otimes x \otimes y, x^\vee \otimes z) \arrow[r] & Hom_C(x \otimes x^\vee \otimes y, x^\vee \otimes z) \arrow[d] \\ & & Hom_C(1 \otimes y, x^\vee \otimes z) \\ & & Hom_C(y, x^\vee \otimes z) \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

If $\mathcal{C}$ is stable, $x \in \mathrm{ob}\,\mathcal{C}$ is rigid iff it is rigid in $\mathrm{ho}(\mathcal{C})$.

Example:

X quasi-compact quasi-separated scheme: any perfect complex of X is rigid in $D_{\mathrm{QCoh}}(X)$.

In fact: rigid = perfect in this context.