INTRODUCTION TO INTERSECTION THEORY AND MOTIVES

by

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Contents

The main reference for this subject is [**1**]. Part of the arguments exposed here come from this reference.

1. Bézout theorem

Theorem 1.1 **(Bézout)**. — [(1)] *Let P and Q be non-zero homogeneous polynomials in $\mathbf{C}[X,Y,Z]$ of degrees $d>0$ and $e>0$. We let $V_+(P), V_+(Q) \subset \mathbf{P}^2(\mathbf{C})$ be the sets of zeros of P and Q in the projective plane. We assume that P and Q do not have a non-trivial common factor, then the intersection $X = V_+(P) \cap V_+(Q)$ is finite and $\#X \leq de$.*

Moreover, if the curves $V_+(P)$ and $V_+(Q)$ intersects transversally at all intersections points, then $\#X = de$.

$V_+(P)$ and $V_+(Q)$ are two curves in the projective plane which do not have common components. The intersection X must be 0-dimensional and then, X is finite.

We may express P and Q as :

$$P = A_0 Y^d + A_1 Y^{d-1} \cdots + A_d \qquad Q = B_0 Y^e + \cdots + B_e$$

where $A_i, B_i \in \mathbf{C}[X,Z]$ are homogeneous of degree i.

Then, we may consider the resultant $R = \mathrm{Res}_{d,e}(P,Q) \in \mathbf{C}[X,Z]$ of P and Q with respect to the variable Y. It is the determinant of the following matrix of size $d+e$ (it is called the Sylvester matrix of P and

(1) Many French mathematicians wonder whether there is an acute accent in the name of Étienne Bézout. There surely is such as the author of the work *Théorie générale des équations algébriques* printed in Paris in 1779 is « M. Bézout ». The theorem is stated there as « Le degré de l'équation finale résultante d'un nombre quelconque d'équations complettes renfermant un pareil nombre d'inconnues, & de degrés quelconques, est égal au produit des exposans des degrés de ces équations. ». This is a statement which is not limited to dimension 2, i.e., the intersection of two hypersurfaces, which Bézout attributes to Euler.

Q):

$$\begin{pmatrix} A_0 & A_1 & A_2 & \dots & A_d & 0 & \dots & 0 \\ 0 & A_0 & A_1 & \dots & A_{d-1} & A_d & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A_0 & A_1 & \dots & A_{d-1} & A_d \\ B_0 & B_1 & B_2 & \dots & B_e & 0 & \dots & 0 \\ 0 & B_0 & B_1 & \dots & B_{e-1} & B_e & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & B_0 & B_1 & \dots & B_{e-1} & B_e \end{pmatrix}$$

We can prove that R is homogeneous of degree de. To see this, consider a nonzero term in the general formula of determinants. For any line index i, we have to choose a column $\sigma(i)$ such that σ is a permutation and the $(i, \sigma(i))$-coefficient is nonzero. Then, for the first e lines, we have to choose elements $0 \le a_1, \dots, a_e \le d$ and set $\sigma(i) = i + a_i$ and for the last d lines numbers $0 \le b_1, \dots, b_d \le e$ and set $\sigma(e+i) = i + b_i$. The degree of the associated homogeneous term $\varepsilon(\sigma) \prod_{i=1}^{d} A_{a_i} \prod_{i=1}^{e} B_{b_i}$ is $\sum_{i=1}^{e} a_i + \sum_{i=1}^{d} b_i = \sum_{i=1}^{d+e} \sigma(i) - \sum_{i=1}^{d} i - \sum_{i=1}^{e} i$. We see that this does not depend on σ, so that we can compute the degree by looking at the case $\sigma(i) = i$. Then, R is homogeneous of degree de.

For $(x, z) \in \mathbf{C}^2 - \{0\}$, $R(x, z) = 0$ if and only if the polynomials $P(x, Y, z)$ and $Q(x, Y, z)$ have a common root $y \in \mathbf{C}$ or if the leading coefficients $A_0 = P(0, 1, 0) \in \mathbf{C}$ and $B_0 = Q(0, 1, 0) \in \mathbf{C}$ do not vanish both. By doing a generic linear change of variables, we may assume that $(0, 1, 0) \notin X$, so that $R(x, z)$ will vanish if and only if $\exists y \in \mathbf{C}$ such that $[x : y : z] \in X$. By a generic change of variables, we may moreover assume that $[0 : 1 : 0]$ does not belong to any line passing through two different elements u and u' of X. Then, for any $(x, z) \in \mathbf{C}^2 - \{0\}$, $R(x, z)$ vanishes if and only if there exists $y \in \mathbf{C}$ such that $[x : y : z] = 0$ and such y is unique. The cardinality of X is then precisely the number of roots of R in $\mathbf{P}^1(\mathbf{C})$. It is less than or equal to the degree of R, i.e., $\le de$.

A careful analysis shows that if the intersection at a point $u = [x : y : z] \in X$ is transverse, then $[x : z]$ is a single root of R. For simplicity, assume $x = y = 0$ and $z = 1$. Set $\tilde{P}(X, Y) = P(X, Y, 1)$. We have $\tilde{P}(0, 0) = 0$. The smoothness of $V_+(P)$ at this point means that the differential of $\tilde{P}$ at $(0, 0)$ is not zero. We may assume that $\frac{\partial P}{\partial Y}(0, 0) \neq 0$ so that the implicit function theorem says there exists an analytic function $f(X)$ in a neighbourhood of 0 such $f(0) = 0$, $f'(0) \neq 0$ and

$\tilde{P}(X, f(X)) = 0$. Then, we can find $S \in \mathcal{O}[Y]$ where $\mathcal{O}$ is the ring of germs of holomorphic functions in a neighbourhood of 0 (with variable X) such that $\tilde{P} = (Y - f(X))S(X, Y)$. We have $\frac{\partial P}{\partial Y}(0,0) = S(0,0) \neq 0$. Then, $\mathrm{Res}_{d,e|Y}(\tilde{P}, \tilde{Q}) = \tilde{Q}(X, f(X)) \cdot \mathrm{Res}_{d-1,e|Y}(S, \tilde{Q})$.

The reductions made before shows that $\tilde{P}$ and $\tilde{Q}$ do not have common zeros of the form $(0, y)$ other than $(0,0)$. It follows that that $\lambda = \mathrm{Res}_{d-1,e|Y}(S, \tilde{Q})(0) \neq 0$ for $S(0,0) \neq 0$. Finally, we have $\frac{\partial \mathrm{Res}_{d,e|Y}(\tilde{P},\tilde{Q})}{\partial X}(0,0) = \lambda \frac{\partial \tilde{Q}(X, f(X))}{\partial X}(0)$. The fact that the differentials of $\tilde{P}$ and $\tilde{Q}$ are both nonzero and not proportional at $(0,0)$ proves that $\frac{\partial \mathrm{Res}_{d,e|Y}(\tilde{P},\tilde{Q})}{\partial X}(0,0) \neq 0$.

Then, we get that $\mathrm{Res}_{d,e|Y}(P, Q)$ has only single roots, which finishes the proof.

2. Invertible sheaves, divisors

2.1. Invertible sheaves. —

Definition 2.1. — Let X be a scheme. An invertible sheaf $\mathscr{L}$ on X is a vector bundle of rank 1 on X.

On any open subset $U = \operatorname{Spec} A \subset X$, the restriction $\mathscr{L}_{|U}$ corresponds to a finite type A-module L which is projective and of rank 1 (if A is a domain and $K = \mathrm{Frac}(A)$, the latter condition means that $L \otimes_A K$ is a 1-dimensional vector space).

Example 2.2. —
- Let X be a smooth curve. Examples of invertible sheaves are the trivial bundles, tangent bundles, cotangent bundles.
- $X = \mathrm{Spec}(A)$ where A is a Dedekind domain (e.g., $A = \mathscr{O}_K$ where K is a number field), any nonzero ideal of A corresponds to an invertible sheaf. More generally, if $K = \mathrm{Frac}(A)$, a nonzero submodule of K is an A-module which corresponds to an invertible sheaf on X.

Proposition 2.3. — *The tensor product endows the set of isomorphisms classes of invertible sheaves on X with the structure of an abelian group, which is denoted* $\mathrm{Pic}(X)$.

The inverse is given by the dual.

Example 2.4. — If $X = \mathrm{Spec}(\mathscr{O}_K)$, then $\mathrm{Pic}(X)$ is the class group of the number field K.

2.2. Divisors. — We assume that X is noetherian and that its local rings are factorial. These assumptions are satisfied when X is a smooth variety. For simplicity, we assume that X is connected. We denote K the field of meromorphic (or rational) functions on X. A meromorphic function is given by a nonempty subset U of X and a function s on U. Two meromorphic functions given as (s, U) and (t, V) are equal in K if they coincide on $U \cap V$ (i.e., on any given nonempty subset contained in $U \cap V$).

More generally, a rational section of an invertible sheaf $\mathcal{L}$ is given by the data of a nonempty subset U of X and a section s of $\mathcal{L}$ on U. The set of rational sections of $\mathcal{L}$ is obviously a 1-dimensional vector space over K.

Definition 2.5. — A divisor D on X is an element in the free abelian group $\operatorname{Div}(X)$ generated by irreducible closed subsets of X on codimension 1. It can be expressed as $D = \sum_i n_i V_i$ with distinct V_i. We say that D is effective if all n_i are nonnegative.

Example 2.6. — If X is a smooth curve, a divisor on X is a formal linear combination of points of the curve. If $X = \operatorname{Spec} \mathscr{O}_K$, $\operatorname{Div}(X)$ is the free abelian group generated by nonzero prime ideals of $\mathscr{O}_K$.

Proposition 2.7. — *To any invertible sheaf $\mathscr{L}$ on X and a nonzero rational section s of $\mathscr{L}$ is attached a divisor* $\operatorname{div} s \in \operatorname{Div} S$. *It satisfies* $\operatorname{div}(s \otimes t) = \operatorname{div} s + \operatorname{div} t$.

We first consider the case $X = \operatorname{Spec}(A)$ is a discrete valuation ring, i.e., a principal domain which has only one nonzero prime ideal. We let x denote the maximal ideal of x. Let $\mathscr{L}$ be an invertible sheaf corresponding to a module L. As A is local, there exists an isomorphism $\varphi \colon L \xrightarrow{\sim} A$.

The set of rational sections of $\mathscr{L}$ identifies to $L \otimes_A K$. We extend φ to an isomorphism $\varphi \colon L \otimes_A K \xrightarrow{\sim} K$. For any $f \in L \otimes_A K$, we set $\operatorname{div} f = v(\varphi(f)) \cdot x$ where $v \colon K^\times \to \mathbf{Z}$ is the valuation. It is obvious that the definition does not depend on φ.

We see that if $L = A$, then $\operatorname{div} s$ is effective if and only if $s \in A$, which means that s is defined at x. If the coefficient is nonnegative, we say that it is the order of vanishing of s at x. If it is negative, we say that there is a pole of order minus this coefficient.

In the general case, for any 1-codimensional closed integral subscheme $Z \subset X$, (i.e., if $X = \mathrm{Spec}(A)$, Z corresponds to a minimal nonzero prime ideal $\mathfrak{p} \subset A$), we can consider the local ring $\mathscr{O}_{X,Z} = A_{(\mathfrak{p})}$ (it defines a scheme $X_{(x)}$) and the localisation of $\mathscr{L}$ and s to this ring $\mathscr{O}_{X,Z}$ which is a discrete valuation ring (for it is factorial). Using the previous case, we get a coefficient $n_Z \in \mathbf{Z}$. Then, we would like to set $\mathrm{div}(s) = \sum_Z n_Z{\cdot}Z$. To do this, we have to prove that the number of such Z with nonzero n_Z is finite. The matter is local on X. We may assume that $X = \mathrm{Spec}(A)$ and that $\mathscr{L} = \mathscr{O}_X$. Then, s is an element in the fraction field of A. Then, we may assume that $s \in A - \{0\}$. All coefficients n_Z will be nonnegative. The Z such that $n_Z > 0$ are precisely those which are contained in the zero locus of s. Actually, they are the irreducible components of the (reduced) zero locus of Z (which are all 1-codimensional by the *Hauptidealsatz*). The ring $A/(s)$ is noetherian and thus only have a finite number of minimal prime ideals.

Theorem 2.8. — *Let X be a noetherian schemes whose local rings are factorial. Let K be the function field of X. We denote $CH^1(X)$ the cokernel of* $\mathrm{div}\colon K^\times \to \mathrm{Div}(X)$. *There is a group morphism* $\mathrm{div}\colon \mathrm{Pic}(X) \to CH^1(X)$ *which is an isomorphism.*

To any invertible sheaf $\mathscr{L}$ and choice of a nonzero rational section s, there is an attached element $\mathrm{div}\, s$ which has a class in $CH^1(X)$. This class in independant of the choice of s. The associated map $\mathrm{div}\colon \mathrm{Pic}(X) \to CH^1(X)$ is a group homomorphism.

It is injective. If s is a rational section of an invertible sheaf $\mathscr{L}$ such that $\mathrm{div}\, s = \mathrm{div}\, f$ for some meromorphic function f, we can replace s by $f^{-1}s$ so as to assume $\mathrm{div}\, s = 0$. Then, I claim that s is a global section of $\mathscr{L}$ which generates $\mathscr{L}$. This claim is local on X, so we may assume that $\mathscr{L}$ is trivial. Then, we only have to do the case where s is a meromorphic function on (an open subset of) X, which can be assumed affine. By localisation, we may even assume $X = \mathrm{Spec}(A)$ where A is local. Then, it is factorial. We get that s can be written as $u \prod_{i=1}^k f_i^{e_i}$, where the f_i belong to a set of representatives of irreductible elements of A. Then, $\mathrm{div}\, s = \sum_i e_i Z_i$ where Z_i is the zero locus of f_i. We assumed that all the coefficients are zero. Then, $s = u$ is invertible.

It it also surjective. Let Z be an integral closed subscheme of X of codimension 1. We have to construct an invertible sheaf $\mathscr{L}$ whose class in $CH^1(X)$ is $[Z]$. We consider the ideal $\mathscr{I}_Z \subset \mathscr{O}_X$ corresponding to the closed subscheme Z. As X is locally factorial, $\mathscr{I}_Z$ is locally generated by a function which is a nonzero divisor. Then, $\mathscr{I}_Z$ is an invertible sheaf. The global section 1 of $\mathscr{O}_X$ is a rational section of $\mathscr{I}_Z$, and its divisor is $-[Z]$. Then, the dual of $\mathscr{I}_Z$ (also denoted $\mathscr{O}(Z)$) corresponds to the class of the divisor $[Z]$.

2.3. Elliptic curves. — Let k be an algebraically closed field of characteristic $\neq 2$. Let $P \in k[X]$ be a polynomial of degree 3 such that $P \wedge P' = 1$. We may consider the projective curve E defined in the plane by the equation $Y^2 Z = P(X/Z)Z^3$. We let $E(k)$ be the set of k-points of E. The only point of E in the line $Z = 0$ (denoted D_∞) at the infinity is $O = [0 : 1 : 0]$ and it is smooth.

This curve is smooth, for we can fix $Z = 1$ to get the affine plane curve of equation $f(X,Y) = 0$ with $f(X,Y) = Y^2 - P(X)$. Assume $f(x,y) = 0$, $\frac{\partial f}{\partial X}(x,y) = -f'(x) = 0$ and $\frac{\partial f}{\partial Y}(x,y) = 2y = 0$. We get $y = 0$, $f(x) = 0$ and $f'(x) = 0$, which contradicts the assumption on P. Then, E is smooth.

We consider the abelian group $\mathrm{Pic}(E)$.

Theorem 2.9. — *The map $E(k) \to \mathrm{Pic}(E)$ which sends P to $[P - O]$ is injective and its image is the kernel $\mathrm{Pic}^0(E)$ of the degree map $\mathrm{Pic}(E) \to$ $\mathbf{Z}$.*

We shall prove later that the degree map $\mathrm{Div}(E) \to \mathbf{Z}$ which maps $\sum_i e_i P_i$ to $\sum_i e_i$ induces a group morphism $\mathrm{Pic}(E) \to \mathbf{Z}$ whose kernel shall be denoted $\mathrm{Pic}^0(E)$. (This is true for all projective and smooth curves.)

For any projective line D in $\mathbf{P}^2$, it makes sense to consider the intersection $D \cap E$. By Bézout's theorem, the cardinality of this intersection is ≤ 3. One can be more precise. *Via* a parametrisation $\mathbf{P}^1 \xrightarrow{\sim} D$ of D, the equation of E restricts to an homogeneous polynomial of degree 3. Then, on D, we have an effective divisor of degree 3. As it lies in $D \cap E$, it also defines an effective divisor of degree 3 on E. Let us denote it $D \cdot E \in \mathrm{Div}(E)$. If we consider one linear equation f of D in $\Gamma(\mathbf{P}^2, \mathscr{O}(1))$, then this divisor $D \cdot E$ is also $\operatorname{div} f_{|E}$ where $f_{|E}$ is the restriction of f, considered as a (regular) section of the invertible sheaf $\mathscr{O}(1)$ on E (it is not completely obvious that the multiplicities are the same as with the previous construction, but this will be a consequence of upcoming formulas).

We say that two divisors are rationally equivalent if they have the same class in $CH^1(E)$. The discussion above shows that the class in $CH^1(E)$ of the divisor $D \cdot E \in \mathrm{Div}(E)$ is independant of the line D. In particular, we can take the line D_∞. In that case, $D_\infty \cdot E = 3O$. This means that if P_1, P_2 and P_3 are three points of $E(k)$ which lies in the same line (in the sense that $P_1+P_2+P_3 = D \cdot E$ for some line D), then $P_1+P_2+P_3 \sim 3O$, i.e., $\varphi(P_1)+\varphi(P_2)+\varphi(P_3) = 0$.

This means that if we take two points P_1 and P_2 of $E(k)$, we can consider the line $D = (P_1P_2)$ (the tangent to E if $P_1 = P_2$). Then, the cycle $D \cdot E$ is of the form $P_1 + P_2 + P_3$ for some point P_3, and it satisfies $\varphi(P_1) + \varphi(P_2) + \varphi(P_3) = 0$ in $\mathrm{Pic}^0(E)$.

For example, if $P = [x : y : z] \in E(k)$, we can consider the vertical line (OP). It intersects E at a third point $Q = [x : -y : z]$ which will be such that $\varphi(Q) = -\varphi(P)$.

The previous constructions shows that the image of $\varphi\colon E(k) \to \mathrm{Pic}^0(E)$ is a group and then, it is surjective. To finish the proof, one has to show that it is injective. Assume that P and P' are such that $\varphi(P) = \varphi(P')$, i.e., $[P] \sim [P']$. We let $R \in E(k)$ be such that O, P and R lie on a line D. It follows that $O + P' + R \sim 3O$. As $3O$ is in the equivalence class associated to the invertible sheaf $\mathscr{O}(1)$ on E, it means that there exists a rational section s of $\mathscr{O}(1)$ such that $\operatorname{div} s = O+P'+R$. As s is effective, this rational s must be a regular section of $\mathscr{O}(1)$. At this stage, we take for granted the non obvious fact that the restriction map

$$\Gamma(\mathbf{P}^2, \mathscr{O}(1)) \to \Gamma(E, \mathscr{O}(1))$$

is surjective (and actually bijective): it is related to the Riemann-Roch theorem for curves. Then, s is the equation of some line D' and $\operatorname{div} s = D' \cdot E = O + P' + R$. As O and R belongs to D' and D, we have $D = D'$. As $D \cdot E = O + P + R$, we get $P = P'$.

Corollary 2.10. — *The set $E(k)$ is naturally equipped with a group structure.*

3. Algebraic cycles

3.1. Definition. — We assume that X is a finite scheme or variety over a field k (sometimes, we will localise these schemes). This assumption shall be implicit until the end of these notes. Points x of the scheme X are in bijection with the set of irreducible closed subsets $Z = \overline{\{x\}}$ of X.

The dimension of the closure Z is the same as the transcendance degree of the field associated to x (the function field of the variety Z). We let $d(x) = \dim \overline{\{x\}}$. We shall call it the dimension of x. We may also define the codimension $c(x)$ of x, it is the dimension of the ring $\mathscr{O}_{X,x}$.

If X is equidimensional (i.e., all irreducible components have the same dimension), then $c(x) = \dim X - d(x)$.

Definition 3.1. — A cycle of dimension d (or a d-cycle) is a formal linear combination of points (or irreducible closed subsets) of dimension d. We let $Z_d(X)$ be the group of d-cycles and $Z_\star(X)$ the corresponding graded abelian group.

Remark 3.2. — $Z_\star(X_{\mathrm{red}}) \xrightarrow{\sim} Z_\star(X)$.

3.2. Cycle associated to a closed subscheme. —

Definition 3.3. — Let $Z \subset X$ be a closed subscheme. We let $C_1, \ldots, C_n$ be its irreducible (integral) components. We let A_i be the localised ring of X at the generic point η_i of C_i. We define $e_i = \lg_{A_i} \mathscr{O}_{Z,\eta_i}$. We set $[Z] = \sum_{i=1}^n e_i[C_i] \in Z_\star(X)$.

If A is a local ring with residue field k and M is an A-module such that there exists a filtration $0 = M_0 \subset M_1 \subset M_2 \subset M_l = M$ by sub-A-modules such that $M_i/M_{i-1} \simeq k$, we say that A is of finite length and $\lg_A M := l$ is independent of the choice of filtrations (Jordan-Hölder).

We may check that the modules considered in the definition are of finite length. For, if A is of the localised ring at generic points of Z, we are reduced to the case $X = \operatorname{Spec} A$ and the set-theoretic closed subset associated to $Z = \operatorname{Spec}(A/I)$ is the closed point $\{\mathfrak{m}\}$, where $\mathfrak{m}$ is the maximal ideal of A which means that $\sqrt{I} = \mathfrak{m}$. Then, by choosing a finite set of generators $x_1, \ldots, x_k$ of the ideal $\mathfrak{m}$, we see that there exists $o \geq 1$ such that $x_i^o \in I$. Then, for a big enough d, any monomial of degree d in the x_i is a multiple of some x_i^o, which proves that $\mathfrak{m}^d \subset I$. Then, A/I is a quotient of $A/\mathfrak{m}^d$ which is easily seen to be an A-module of finite length.

If A is a discrete valuation ring and $I \subset A$ is nonzero ideal, then $\lg_A(A/I)$ is the valuation of a generator of I. Thus, we see that the notation $D \cdot E$ we used for some 0-cycles in the paragraph on elliptic curves correspond to the cycle of the intersection scheme $D \cap E$ (in D, E or $\mathbf{P}^2$).

Example 3.4. — Let $X = \mathbf{A}^2_k$ and $x \in X$ be the closed point associated to the origin. It means we consider $\mathfrak{m} = (X, Y) \subset k[X, Y]$. Nonempty subschemes Z supported by $\{x\}$ corresponds to ideals $I \subset \mathfrak{m}$ which contain some power $\mathfrak{m}^d$. If $d = 1$, $I = \mathfrak{m}$ and the multiplicity $e = \lg_A(A/I)$ is 1. If $d = 2$, we have $\mathfrak{m}^2 \subset I \subset \mathfrak{m}$. We can look at the quotient $A/\mathfrak{m}^2 \simeq k \oplus \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}/\mathfrak{m}^2$ is a 2-dimensional vector space with basis X and Y and whose square is zero. An ideal I as above corresponds to the datum of a subspace V of $\mathfrak{m}/\mathfrak{m}^2$ and then, the length e is $3 - \dim_k V$.

If $I = (Y, X) = \mathfrak{m}$, then $V = \mathfrak{m}/\mathfrak{m}^2$ and $e = 1$.
If $I = (Y, Y - X^2)$, then $V = k \cdot Y$ and $e = 2$.
If $I = (X^2, Y^2, XY) = \mathfrak{m}^2$, then $V = \{0\}$ and $e = 3$.

3.3. Direct image of cycles. — Let $p\colon X \to S$ be a proper morphisms between algebraic varieties. We shall define a graded morphism $Z_\star(X) \to Z_\star(S)$.

Definition 3.5. — Let Z be an integral closed subscheme of X. As p is proper, the image $Z' = p(Z)$ is an (integral) closed subscheme of S. We consider the extension of fields K/K' associated to Z and Z'. If it is finite, we set $p_\star([Z]) = [K : K'] \cdot [Z']$. Otherwise, we set $p_\star([Z]) = 0$. It extends to a group morphism $Z_\star(X) \to Z_\star(S)$.

This construction is obviously functorial in p.

Definition 3.6. — Let X be an integral variety. Let f be a meromorphic function on X. We would like to define the divisor of f as it was previously done in a favourable situation. We consider the normalisation $\nu\colon \tilde{X} \to X$ of X. (If $X = \operatorname{Spec}(A)$, then $\tilde{X} = \operatorname{Spec}(\tilde{A})$ where $\tilde{A}$ is the integral closure of A in its fraction field. The ring $\tilde{A}$ is a finite algebra over A.) We can identify f to a meromorphic function $\nu^\star f$ of $\tilde{X}$. The construction given above actually applies to normal schemes, so that we can consider $\operatorname{div}(\nu^\star f) \in Z_\star(\tilde{X})$ and define $\operatorname{div} f = \nu_\star \operatorname{div}(\nu^\star f) \in Z_\star(X)$.

Proposition 3.7. — *Let $p\colon X \to S$ be a finite and surjective morphism between integral schemes. Let h be a meromorphic function on X, i.e., an element of the function field L of X. We let K denote the function field of S. The extension L/K is finite. Then, $p_\star \operatorname{div} h = \operatorname{div} N_{L/K}(h) \in Z_\star(S)$.*

If we introduce the normalisations of X and S, we immediately see that we may assume that both X and S are normal. As we have to compare some coefficients at some codimension 1 points of S, we may localise S so as to assume that $S = \mathrm{Spec}(A)$ with A a discrete valuation ring with maximal ideal $\mathfrak{p}$. Then, $X = \mathrm{Spec}(B)$ with B a Dedekind domain with only a finite number of nonzero prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_g$.

Lemma 3.8. — *Let A be a discrete valuation ring. Let $v_1, \dots, v_d$ be a basis of a free A-module $M \subset A^d$ of rank d. Then, the valuation of the determinant of the vectors $v_1, \dots, v_d$ is the length of the A-module A^d/M.*

It follows from the structure theory of modules over principal domains.

To prove the proposition, we may assume that $h \in B-\{0\}$ and consider $M = (h) \subset B$. As an A-module, $B \simeq A^d$ where d is the degree of p. The determinant of h is $N_{L/K}(h)$. Using these identifications, we get that the valuation of this determinant, i.e, $v(N_{L/K}(h))$ is the length of the A-module $B/(h)$.

We may introduce the coefficients n_i such that $\mathrm{div}\, h = \sum_{i=1}^{g} n_i \mathfrak{q}_i$. To finish the proof, we have to obtain $\lg_A B/(h) = \sum_{i=1}^{g} n_i f_i$ where f_i is the degree of the extension of residue fields associated to $\mathfrak{q}_i$ and $\mathfrak{p}$. As the statements are multiplicative in h, we may further assume that h is a generator of one of the $\mathfrak{q}_i$, say $\mathfrak{q}_1$, then one has to prove that $\lg_A B/\mathfrak{q}_1 = f_1$. This is obvious as $B/\mathfrak{q}_1$ is the residue field associated to $\mathfrak{q}_1$, the notion of length reduces to that of dimension of vector spaces.

Definition 3.9. — Let X be an (integral) projective variety over a field k. We consider the projection $p\colon X \to \mathrm{Spec}\, k$. It induces a group morphism $\deg\colon Z_0(X) \to Z_0(\mathrm{Spec}\, k) \simeq \mathbf{Z}$.

Proposition 3.10. — *Let C be an (integral) projective curve over a field k. Then, for any nonzero meromorphic function f on C, we have* $\deg(\mathrm{div}\, f) = 0$.

Assume first that $C = \mathbf{P}^1$. Then f is identified to an element $f \in k(T) - \{0\}$. To prove the proposition, we may assume that $f = a_0 + a_1 T + \dots + a_{d-1} T^{d-1} + a_d T^d \in k[T]$ is a monic polynomial. We may even further assume that it is irreducible of some degree d. We let $\mathfrak{p} \subset k[T]$ be the ideal generated by f. The rational function f is regular on the affine line $\mathbf{A}^1 = \mathbf{P}^1 - \{\infty\}$ and this prime ideal $\mathfrak{p}$ is precisely the locus where f vanishes. The contribution of the points of the affine line to the divisor of f shall obviously by $[\mathfrak{p}]$. We also have to consider the point ∞. The complement of the origin $\mathbf{P}^1 - \{0\}$ is identified to $\mathrm{Spec}\, k[U]$ with the identification $TU = 1$ on $\mathbf{P}^1 - \{0, \infty\}$. Thus, we also get

$f = U^{-d}\cdot(1+a_{d-1}U+\cdots+a_0U^d)$ whose U-valuation (i.e., valuation of f at ∞) is $-d$. It follows that $\operatorname{div} f = [\mathfrak{p}]-d[\infty]$. Then, $\deg \operatorname{div} f = d-d = 0$.

In the general case, any "non constant" rational function on the curve C defines a finite surjective morphisn $p\colon C \to \mathbf{P}^1$. Then, $\deg \operatorname{div} f = \deg \operatorname{div} N_{k(C)/k(\mathbf{P}^1)} f = 0$.

Remark 3.11. — The proof is similar to that of the product formula for number fields.

Corollary 3.12. — *If X is a smooth and projective curve, there is a degree morphism* $\operatorname{Pic}(X) \to \mathbf{Z}$ *which maps the isomorphism class of an invertible sheaf $\mathscr{L}$ to the degree of the divisor of a rational section of $\mathscr{L}$.*

3.4. Rational equivalence. —

Definition 3.13. — For any algebraic variety X (irreducible or not), we define a (graded) subgroup $\operatorname{Rat}_\star(X) \subset Z_\star(X)$: it is the subgroup generated by cycles of the form $i_\star \operatorname{div} f$ where $i\colon Z \to X$ is the closed immersion of an integral closed subscheme Z and f is a nonzero meromorphic function of Z.

Definition 3.14. — Two cycles are rationally equivalent if their difference is in $\operatorname{Rat}_\star(X)$. We define $CH_\star(X) = Z_\star(X)/\operatorname{Rat}_\star(X)$.

For example, if X is (irreducible and) locally factorial, we have seen that $CH_{\dim X-1}(X) \simeq \operatorname{Pic}(X)$.

Obviously, for any irreducible X, $CH_{\dim X}(X) \simeq \mathbf{Z}$.

Proposition 3.15. — *For any projective morphism $p\colon X \to S$, the group morphism $p_\star\colon Z_\star(X) \to Z_\star(S)$ induces a graded morphism $p_\star\colon CH_\star(X) \to CH_\star(S)$.*

The general case follows formally from that of a surjective morphism $p\colon X \to S$ between irreducible normal varieties and the case of the cycle $\operatorname{div} f$ of a nonzero meromorphic function f on X. We have to prove that $p_\star \operatorname{div} f \in \operatorname{Rat}_\star(S)$.

First case: $\dim X = \dim S$. I claim that $p_\star \operatorname{div} f = \operatorname{div} N_{L/K}(f)$ where L/K is the finite extension of fields of meromorphic functions on X and S. This formula can be checked by localising S so that we may assume S is a discrete valuation ring. Then, X must be finite over S and we can use the formula of proposition 3.7.

Second case: $\dim X > \dim S$. I claim that $p_\star \operatorname{div} f = 0$. Let Z be an irreducible subvariety of X of dimension $\dim X - 1$. If $p_\star[Z] \neq 0$ in $Z_\star(S)$, then $\dim S \geq \dim p_\star Z = \dim X - 1 \geq \dim S$. Then, we may assume that $\dim X = \dim S + 1$ and we can localise at the generic point of S. Then, we are in the situation $S = \operatorname{Spec} K$ and X is a curve over S. Then, $p_\star \colon Z_0(X) \to Z_0(S)$ identifies to the degree map and we already proved that it vanishes on divisors of meromorphic functions.

Corollary 3.16. — *Let X be a projective variety over a field k. Then the morphism $a\colon X \to \operatorname{Spec} k$ defines a group morphism $a_\star\colon CH_0(X) \to CH_0(\operatorname{Spec} k)$ which identifies to a map* $\deg\colon CH_0(X) \to \mathbf{Z}$.

3.5. Flat inverse image. —

Proposition 3.17. — *Let $f\colon X \to S$ be flat of relative dimension n, i.e., for all $Z \subset S$ irreducitlbe, $f^{-1}(Z)$ is empty or equidimensional of dimension $\dim Z + n$ (this is automatic if S is irreducible and X is equidimensional), then, there exists a unique group morphism $f^\star\colon Z_\star(S) \to Z_\star(X)$ such that:*

(i) *$f^\star$ maps $Z_k(S)$ to $Z_{k+n}(X)$;*

(ii) *for any closed subscheme $Z \subset S$, we have $f^\star([Z]) = [f^{-1}(Z)]$.*

Moreover, $f^\star(\operatorname{Rat}_\star(S)) \subset \operatorname{Rat}_\star(X)$. More precisely, $f^\star \operatorname{div} g = \operatorname{div} f^\star g$ for any nonzero meromorphic function g on S.

We define $f^\star$ so that (ii) applies for all integral closed subschemes. Then, it satisfies (i) by assumption. Then, we have to prove that (ii) applies to all closed subschemes Z of S. By localising at the generic points of the irreducible components of Z, we may assume that $S = \operatorname{Spec}(A)$ is local, with maximal ideal $\mathfrak{m}$ (corresponding closed point x) and that $Z = V(I)$ where $\sqrt{I} = \mathfrak{m}$. Let $e = \lg_A(A/I)$ so that $[Z] = e[x]$. We have to prove that $[f^{-1}(Z)] = e[f^{-1}(x)]$. This computation can be done locally on X, so that we may assume $X = \operatorname{Spec} B$ and even localise at the generic point of an irreducible component of $f^{-1}Z$. Then, $f^{-1}(x) = \operatorname{Spec}(B/\mathfrak{m}B)$ and $f^{-1}(Z) = \operatorname{Spec}(B/IB)$. The B-module $B/\mathfrak{m}B$ is of finite length. We have to prove that $\lg_B(B/IB) = e \lg_B(B/\mathfrak{m}B)$.

We may choose a filtration $I = I_0 \subset I_1 \subset I_e = A$ with $I_k/I_{k-1} \simeq A/\mathfrak{m}$. Then, we get a filtration $J_k = I_k B \subset B$. By flatness, $J_k = I_k \otimes_A B$ and $J_k/J_{k-1} \simeq I_k/I_{k-1} \otimes_A B \simeq (A/\mathfrak{m}) \otimes_A B \simeq B/\mathfrak{m}B$. By dévissage, we get the expected computation of lengths.

The formula $f^\star \operatorname{div} g = \operatorname{div} f^\star g$ can be checked locally on S and on X. Then, we may assume g is a nonzero element of A where $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$. As g can be locally expressed as a quotient, we may assume that $g \in A - \{0\}$. Then, $\operatorname{div} g = [\operatorname{Spec}(A/gA)]$ and $\operatorname{div} f^\star g = [\operatorname{Spec}(B/gB)]$. The expected formula then follows from (ii) applied to the closed subscheme $V(g) \subset S$.

Corollary 3.18. — *If $f\colon X \to S$ is flat, we get a morphism $f^\star\colon CH_\star(S) \to CH_\star(X)$.*

This morphism is not graded. It is reasonable to use a numbering by codimension. If X and S are equidimensional: $CH^c(X) = CH_{\dim X - c}(X)$, $CH^c(S) = CH_{\dim S - c}(S)$. Then, we get a graded morphism $f^\star\colon CH^\star(S) \to CH^\star(X)$.

3.6. Simple computations. —

Proposition 3.19. — *For any X, the morphism $p^\star\colon CH_i(X) \to CH_{i+1}(X \times \mathbf{A}^1)$ is surjective for all $i \in \mathbf{Z}$.*

Lemma 3.20. — *The proposition is true is $X = \operatorname{Spec} K$ where K is a field. More precisely, $CH_1(\mathbf{A}^1_K) \simeq \mathbf{Z}$ and $CH_0(\mathbf{A}^1_K) = 0$.*

We can use the identification $CH_0(\mathbf{A}^1_K) \simeq \operatorname{Pic}(\mathbf{A}^1_K)$. This group is trivial because $K[T]$ is a principal domain.

Lemma 3.21. — *Assume that $Z \subset X$ is a closed subscheme and $U \subset X$ is the open complement. We have the inclusions $i\colon Z \to X$ and $j\colon U \to X$. Then, we have an exact sequence:*

$$CH_\star(Z) \xrightarrow{i_\star} CH_\star(X) \xrightarrow{j^\star} CH_\star(U) \to 0\,.$$

It is obvious. If $z \in Z_\star(X)$ is such that $j^\star z$ is rationally equivalent to zero, we have $j^\star z = \sum_k i_{U_k \to U, \star}[\operatorname{div} f_k]$ where f_k is nonzero rational function on integral subschemes U_k of U. We may consider f_k as meromorphic function on the closure $\overline{U_k}$ in X. Then, we obviously have $z - \sum_k i_{\overline{U_k} \to X, \star}[\operatorname{div} f_k] \in i_\star Z_\star(Z)$.

We can prove the proposition by induction on $\dim X$. Let z be an element of $CH_\star(\mathbf{A}^1_X)$. Let $\eta_1, \dots, \eta_c$ be the generic points of the irreducible components X (which can be assumed to be reduced). We have $\eta_i = \operatorname{Spec} K_i$ where K_i is some field. We proved that $CH_\star(\eta_i) \to CH_\star(\mathbf{A}^1_{\eta_i})$ is surjective (actually bijective). I claim that this surjective map $\prod_i CH_\star(\eta_i) \to \prod_i CH_\star(\mathbf{A}^1_{\eta_i})$ is obtained as the filtering colimit of the maps $CH_\star(U) \to CH_\star(\mathbf{A}^1_U)$ where U varies in the ordered set of dense open subsets of X.

Then, there exists a small enough such U such that the restriction of $z \in CH_\star(\mathbf{A}^1_X)$ to $CH_\star(\mathbf{A}^1_U)$ belongs to the image of $p_U^\star \colon CH_\star(U) \to CH_\star(\mathbf{A}^1_U)$. We choose a class $b \in CH_\star(X)$ such that $p_U^\star b_{|U} = z_{\mathbf{A}^1_U}$. Then, $z - p_X^\star b$ is mapped to zero by the restriction map $CH_\star(\mathbf{A}^1_X) \to CH_\star(\mathbf{A}^1_U)$, then it is of the form $z - p_X^\star b = i_{\mathbf{A}^1_Z \to \mathbf{A}^1_X, \star} z'$ where $z' \in CH_\star(\mathbf{A}^1_Z)$ and $Z = X - U$. By induction, there exists $b' \in CH_\star(Z)$ such that $b' = p_Z^\star(b')$, so that finally $z = p_X^\star(b + i_{Z \to X, \star} b')$.

Corollary 3.22. — *Let $d \geq 0$. We have $CH_d(\mathbf{A}^d_k) \simeq \mathbf{Z}$ and $CH_i(\mathbf{A}^d_k) = 0$ if $i \neq d$.*

Corollary 3.23. — *Let $0 = V_0 \subset V_1 \subset V_2 \subset V_n \subset V_{n+1} = k^{n+1}$ be a complete flag in k^{n+1}, i.e., $\dim V_i = i$. For any $1 \leq i \leq n+1$, V_i defines a subprojective space $\mathbf{P}(V_i) \subset \mathbf{P}^n$. Then, for all $1 \leq i \leq n+1$, $CH_{i-1}(\mathbf{P}^n)$ is generated by the class of $\mathbf{P}(V_i)$* [(2)].

We can do induction on n. We use the exact sequence

$$CH_\star(\mathbf{P}(V_n)) \to CH_\star(\mathbf{P}^n) \to CH_\star(\mathbf{P}^n - \mathbf{P}(V_n)) \to 0$$

The variety $\mathbf{P}^n - \mathbf{P}(V_n)$ is isomorphic to $\mathbf{A}^n$, so that its Chow group, except in dimension n are zero. Then, the maps $CH_{i-1}(\mathbf{P}(V_n)) \to CH_{i-1}(\mathbf{P}^n)$ for $i \neq n$ are surjective, so that the induction shows $CH_{i-1}(\mathbf{P}^n)$ is generated by $[\mathbf{P}(V_i)]$. In top dimension, $CH_n(\mathbf{P}^n)$ is obviously generated by $[\mathbf{P}(V_{n+1})]$.

4. Products

4.1. First Chern class of a line bundle. —

Definition 4.1. — Let X be a finite type scheme over a field k. Let $\mathscr{L}$ be a line bundle over X. We define a group morphism

$$c_1(\mathscr{L}) \cap - \colon Z_\star(X) \to CH_{\star-1}(X)$$

in the following way. Let Z be a integral closed subscheme of X. We denote $i_Z \colon Z \to X$ the corresponding closed immersion. We define

[(2)] We do not have the tools yet to prove that $CH_{i-1}(\mathbf{P}^n)$ is actually isomorphic to $\mathbf{Z}$.

$c_1(\mathscr{L}) \cap [Z] = i_{Z\star}([\operatorname{div} s])$ where s is a nonzero meromorphic section of the restriction $\mathscr{L}_{|Z}$.

Proposition 4.2. — *The morphism $c_1(\mathscr{L}) \cap -$ factors through rational equivalence to give a pairing* $\operatorname{Pic}(X) \times CH_\star(X) \to CH_{\star-1}(X)$.

One is reduced to the following proposition:

Proposition 4.3. — *If X is an integral scheme, then on 1-codimensional cycles, the previous construction induces a commutative pairing* $\operatorname{Pic}(X) \times \operatorname{Pic}(X) \to CH^2(X)$.

We shall do only the favourable case where X is locally factorial (the general case is [**1**, Theorem 2.4]). We already have a bilinear pairing $b\colon Z^1(X) \times Z^1(X) \to CH^2(X)$. Let D and D' be two integral closed subschemes on X of codimension 1. We can define the cycles $[D]$, $[D']$ and the invertible sheaves $\mathscr{O}(D)$ and $\mathscr{O}(D')$. The pairing of $[D]$ and $[D']$ is the divisor class associated to $\mathscr{O}(D)_{|D'}$ considered as a cycle on D' and pushed through $D' \to X$.

We should prove $b([D],[D']) = b([D'],[D])$. It is obviously true if $D = D'$. Now, assume $D \neq D'$. We consider the invertible sheaf $\mathscr{O}(D)$ associated to D. The unit of $\mathscr{O}_X$ defines a global section of $\mathscr{O}(D)$ which is invertible on $X - D$. The divisor of this global section is precisely $[D]$. Then, $\mathscr{O}(D)_{|D'}$ is equipped with the restriction of that section and it is invertible on $D' - D \cap D'$. Then, it is a nonzero global section. Its divisor represents $b([D],[D'])$ and is supported on $D \cap D'$.

Then, we see that we can localise X at points of codimension 2 lying in the intersection of D and D'. Thus, we may assume $X = \operatorname{Spec} A$ is local (2-dimensional) with closed point x. As we assumed X is factorial, we can assume that D is the hypersurface defined by $f \in A$ and D' by $g \in A$. We may identify $\mathscr{O}(D)$ to $f^{-1}A \subset K = \operatorname{Frac}(A)$, equipped with the section $1 \in A$. This identifies to $A \subset K$ equipped with the section $f \in A$. Then, $b([D],[D'])$ identifies to the divisor of the image of f in the ring $A/(g)$ of functions on D'. By definition, $b([D],[D'])$ is the cycle class of the subscheme $A/(f,g)$. This is not changed when we invert D and D'.

Proposition 4.4. — *If $\mathscr{L}$ is a line bundle on X equipped with a trivialisation $t\colon \mathscr{O}_{X-Z} \xrightarrow{\sim} \mathscr{L}_{|X-Z}$ outside a closed subscheme Z, then $c_1(\mathscr{L}) \cap -\colon CH_\star(X) \to CH_{\star-1}(X)$ refines to a morphism $c_1(\mathscr{L},t) \cap -\colon CH_\star(X) \to CH_{\star-1}(Z)$.*

4.2. Projective bundles, vector bundles. —

Proposition 4.5. — *Let $n \geq 0$. We consider the invertible sheaf $\mathscr{O}(1)$ on $\mathbf{P}^n$ (it is the dual of the universal line $\mathscr{O}(1) \subset \mathscr{O}_{\mathbf{P}^n}^{n+1}$). For any $0 \leq i \leq n$, we define u^i the image of $[\mathbf{P}^n] \in CH^0(\mathbf{P}^n)$ by the ith power of $c_1(\mathscr{O}(1)) \cap -$ acting on $CH_\star(\mathbf{P}^n)$. Then, $CH^i(\mathbf{P}^n)$ is a free abelian group generated by u^i.*

If $V \subset k^{n+1}$ is a subspace of codimension i, expressing V as the intersection of i hyperplanes, one can see that $u^i = [\mathbf{P}(V)]$. We already know that u^i generates $CH^i(\mathbf{P}^n)$. To finish the proof, it suffices to construct a morphism $\varphi_i \colon CH^i(\mathbf{P}^n) \to \mathbf{Z}$ which maps u^i to 1. To do this, we may consider the $(n-i)$th power $c_1(\mathscr{O}(1))^{\circ(n-i)} \cap - \colon CH^i(\mathbf{P}^n) \to CH^n(\mathbf{P}^n)$ of $c_1(\mathscr{O}(1)) \cap -$ followed by the degree map $CH_0(\mathbf{P}^n) \to \mathbf{Z}$.

Proposition 4.6. — *Let X be a scheme. Let $\mathscr{E}$ be a vector bundle of rank $n+1$ on X. Then, we have a canonical isomorphism $\prod_{i=0}^n CH_{\star+i-n}(X) \to CH_\star(\mathbf{P}(\mathscr{E}))$* [(3)] *which maps a family of classes $x_0, \dots, x_n$ to $\sum_{i=0}^n c_1(\mathscr{O}(1))^i \cap \pi^\star x_i$ where $\pi \colon \mathbf{P}(\mathscr{E}) \to X$ denotes the projection from the projective bundle.*

The proof uses variants of arguments seen above.

Corollary 4.7. — *Let X be a scheme. Let $\mathscr{E}$ be a vector bundle of rank n on X. We identify $\mathscr{E}$ to a scheme above X with a projection $\pi \colon \mathscr{E} \to X$. Then, $\pi^\star \colon CH_{\star-n}(X) \to CH_\star(\mathscr{E})$ is an isomorphism.*

We may embed $\mathscr{E}$ as an open subscheme of $\mathbf{P}(\mathscr{E}^\vee \oplus \mathscr{O}_X)$ where a vector v is mapped to the point of homogeneous coordinates $[v:1]$. The complement of $\mathscr{E}$ is the closed subscheme which identifies to $\mathbf{P}(\mathscr{E}^\vee) \subset \mathbf{P}(\mathscr{E}^\vee \oplus \mathscr{O}_X)$. Then, we have an exact sequence:

$$CH_\star(\mathbf{P}(\mathscr{E}^\vee)) \xrightarrow{i_\star} CH_\star(\mathbf{P}(\mathscr{E}^\vee \oplus \mathscr{O}_X)) \to CH_\star(\mathscr{E}) \to 0\ .$$

The Chow groups of these two projective bundles are computed in terms of $CH_\star(X)$ and we can compute the map between the sames. We let $\pi \colon CH_\star(\mathbf{P}(\mathscr{E}^\vee)) \to X$ and $\pi' \colon CH_\star(\mathbf{P}(\mathscr{E}^\vee) \oplus \mathscr{O}_X) \to X$ denote the two projections and $i \colon \mathbf{P}(\mathscr{E}^\vee) \to \mathbf{P}(\mathscr{E}^\vee \oplus \mathscr{O}_X)$ denote the inclusion. Then, we see that for $x \in CH_\star(X)$, $i_\star(c_1(\mathscr{O}(1))^k \cap \pi^\star x) = c_1(\mathscr{O}(1))^{k+1} \cap \pi'^\star x$: this follows from the fact that $\mathscr{O}(1)$ on $\mathbf{P}(\mathscr{E}^\vee \oplus \mathscr{O}_X)$ is trivialised outside

(3) From this stage, we use the Grothendieck $\mathbf{P}$, so that $\mathbf{P}(\mathscr{E})$ does not parametrise lines in $\mathscr{E}$ but hyperplanes in it. There are good reasons to do this...

$\mathbf{P}(\mathscr{E}^\vee)$ so that $c_1(\mathscr{O}(1)) \cap \pi'^\star x$ is represented by $i_\star(\pi_\star x)$ which is the expected formula for $k = 0$ and the general case follows by applying $c_1(\mathscr{O}(1))^k \cap -$ on both sides. Then, writting $CH_\star(\mathscr{E})$ as the cokernel of the shifting map we computed:

$$\bigoplus_{k=0}^{n-1} CH_{\star+k+1-n}(X) \to \bigoplus_{k=0}^{n} CH_{\star+k-n}(X) ,$$

we obtain that the cokernel identifies to $CH_{\star-n}(X)$.

4.3. Blow-ups. —

Definition 4.8. — Let $i\colon Y \to X$ be a closed immersion defined by an ideal $\mathscr{I} \subset \mathscr{O}_X$ of finite type. Then, $\mathrm{Bl}_Y X = \mathrm{Proj}\,(\oplus_{n\geq 0}\mathscr{I}^n T^n)$ where $\oplus_{n\geq 0}\mathscr{I}^n T^n$ is the obvious subalgebra of $\mathscr{O}_X[T]$.

Locally, $\mathscr{I} = (f_0, f_1, \dots, f_n)$, then, we have a morphism $X - Y \to \mathbf{P}^n \times X$ given by the homogeneous coordinates $[f_0 : \cdots : f_n]$. Then, the blow-up $\mathrm{Bl}_Y X$ is the schematic closure of $X - Y$ in $\mathbf{P}^n \times X$.

There is a map $\pi\colon \mathrm{Bl}_Y X \to X$ which is an isomorphism above $X - Y$ such that $\pi^{-1}(Y) = \mathrm{Proj}(\oplus_{n\geq 0}\mathscr{I}^n/\mathscr{I}^{n+1})$ which is the projectivisation of the normal cone of i.

Remark 4.9. — If $V \subset X$ is a closed subscheme and $W = Y \cap Y$, then the blow-up $\mathrm{Bl}_W V$ identifies to a closed subscheme of $\mathrm{Bl}_Y X$: it is called the strict transform of V. Note: it is contained in $\pi^{-1}(V)$, but the inclusion is not an equality in general.

4.4. Deformation to the normal cone. — We fix a closed immersion $i\colon Y \to X$. The goal of this paragraph is to define a morphism $i^\star\colon CH_\star(X) \to CH_{\star-c}(Y)$ under some circumstances. We assume that i is a regular immersion of codimension $c > 0$. Locally on Y (with $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(A/I)$, this means that I is generated by c elements $f_1, \dots, f_c$ which form a regular sequence: f_1 is not a zero divisor in A, $[f_2]$ is not a zero divisor in $A/(f_1)$... and $[f_c]$ is not a zero divisor in $A/(f_1, \dots, f_{c-1})$. If $\mathscr{I}$ is the ideal defining Y (locally, it corresponds to I), the conormal sheaf of i is the quotient $\mathscr{I}/\mathscr{I}^2$ considered as a $\mathscr{O}_Y$-Module. It is a locally free sheaf of ranf c on Y.

Example 4.10. — Any hypersurface H in $\mathbf{P}^n$ (or $\mathbf{A}^n$) defines a regular immersion $H \to \mathbf{P}^n$ of codimension 1.

If X and Y are smooth varieties, any closed immersion $i\colon Y \to X$ is a regular immersion. If TY and TX are the tangent bundles to X and Y, then the conormal sheaf on Y is the dual of the normal bundle $N_{X/Y}$ which is the quotient $(i^\star TX)/TY$.

Definition 4.11. — We define $B = \mathrm{Bl}_{Y\times\{\infty\}}(X \times \mathbf{P}^1)$: it is the blow-up of $Y \times \infty$ in $X \times \mathbf{P}^1$. We have a projection $\pi\colon B \to X \times \mathbf{P}^1$ which is an isomorphism above the open subset $X \times \mathbf{P}^1 - Y \times \{\infty\}$ (it is birational) and $\pi^{-1}(Y \times \{\infty\}) \simeq \mathbf{P}(\mathscr{N}_{X\times\mathbf{P}^1/Y\times\{\infty\}})$ where $\mathscr{N}_{X\times\mathbf{P}^1/Y\times\{\infty\}}$ is the conormal sheaf of the immersion $i'\colon Y = Y \times \{\infty\} \to X \times \mathbf{P}^1$. Obviously, $\mathscr{N}_{X\times\mathbf{P}^1/Y\times\{\infty\}} = \mathscr{N}_{X/Y} \oplus \mathscr{O}_Y$. Then, geometrically $\pi^{-1}(Y \times \{\infty\})$ identifies to the projective bundle of the direct sum of the normal bundle to i and of the trivial vector bundle of rank 1 on Y.

Definition 4.12. — Above ∞, the blow-up B contains the blow-up $\mathrm{Bl}_Y X$ of Y in X as a closed subscheme. We remove it to get the deformation space $D = B - \mathrm{Bl}_Y X$. This $\mathrm{Bl}_Y X$ meets $\mathbf{P}(\mathscr{N}_{X/Y} \oplus \mathscr{O}_Y)$ as the closed subscheme given by the hyperplane $\mathbf{P}(\mathscr{N}_{X/Y}) \subset \mathbf{P}(\mathscr{N}_{X/Y} \oplus \mathscr{O}_Y)$. Then, if we let $p\colon D \to \mathbf{P}^1$ denote the projection and identify $\mathbf{A}^1$ to $\mathbf{P}^1 - \{\infty\}$, then $p^{-1}(\mathbf{A}^1) \simeq \mathbf{A}^1 \times X$ and $p^{-1}(\infty) \simeq N_{X/Y}$ is the normal bundle of i. There also exists a canonical closed immersion $Y \times \mathbf{P}^1 \to D$ which can be considered as a 1-parameter family of immersions of Y in the fibers of D. Outside ∞, this is the given immersion $i\colon Y \to X$ but at ∞, it identifies to the zero section $Y \to N_{X/Y}$ of the normal bundle. This is the reason why this construction is the "deformation to the normal cone" [(4)].

Definition 4.13. — The morphism $i^\star\colon CH_\star(X) \to CH_{\star-c}(Y)$ is defined as follows:

$$\begin{tikzcd}
CH_\star(X) \arrow[r, "\sim"] & CH_{\star+1}(\mathbf{A}^1 \times X) & \\
& CH_{\star+1}(D) \arrow[u, two heads] \arrow[r, "c_1(p^\star\mathscr{O}(1),t=\infty)\cap -"] & CH_\star(N_{X/Y}) \\
& CH_{\star+1}(N_{X/Y}) \arrow[u] \arrow[ur, "0"] & CH_{\star-c}(Y) \arrow[u, "\sim"]
\end{tikzcd}$$

(4) If i was not assumed regular, instead of the normal bundle, we would only have a cone.

A priori, a choice appears in this construction when we lift an element of $CH_{\star+1}(\mathbf{A}^1 \times X)$ to an element of $CH_{\star+1}(D)$. The lifting is defined up to the image of the pushforward $CH_{\star+1}(N_{X/Y}) \to CH_{\star+1}(D)$. But the composition with $c_1(p^\star\mathscr{O}(1), t = \infty) \cap -$ gives the zero map $CH_{\star+1}(N_{X/Y}) \to CH_\star(N_{X/Y})$ because the restriction of the invertible sheaf $p^\star\mathscr{O}(1)$ to $N_{X/Y}$ is trivial.

4.5. Variations, examples. —

4.5.1. Normal cone. — Let $i\colon Y \to X$ be a regular closed immersion of codimension c. Let V be an integral closed subscheme of dimension k in X. We denote $W = V \cap Y$. We consider W as a closed subscheme of V. Let$\mathscr{I}$ be the ideal of W in V. There is a cone $C_W V = \operatorname{Spec} \bigoplus_{n\geq 0} \mathscr{I}/\mathscr{I}^2$ over W: it is the normal cone of $W \to V$. This cone identifies to a subcone of the restriction of the vector bundle $N_{X/Y}$ from Y to W. Then, it has a class in $CH_k(N_{X/Y_{|W}}) \xleftarrow{\sim} CH_{k-c}(W)$ which can be pushed by $CH_{k-c}(W) \to CH_{k-c}(Y)$. The class obtained in $CH_{k-c}(Y)$ is precisely $i^\star[V]$.

Remark 4.14. — Assume Y is integral, that $Y$and V intersects properly, i.e., the integral components $W_1, \ldots, W_k$ of W are of dimension $k-c$. Then, the class constructed (which we shall denote here $i^\star_{\mathrm{ref}}V$) above in $CH_{k-c}(W)$ decomposes uniquely as $\sum_{i=1}^k m_i[W_i]$. The number m_i is the multiplicity of W_i in "the product $Y \cdot V$".

4.5.2. Curves on surfaces. — Assume C and D are two smooth curves on a surface X and that they intersects transversally at some points $x_1, \ldots, x_n$. We consider $i\colon C \to X$ which is a regular immersion of codimension 1. The canonical closed immersion $C_W D \to N_{X/C_{|W}}$ where $W = \{x_1, \ldots, x_n\}$ is then an isomorphism. Then, $i^\star_{\mathrm{ref}}D = [W]$.

The situation is orthogonal to this one when $C = D$. Then, $W = C \cap C = C$, and the normal cone $C_C C$ is the zero section of the normal bundle $N_{X/C}$. By the following lemma, we see that $i^\star_{\mathrm{ref}}C = c_1(N_{X/C}) in CH_0(C)$.

Lemma 4.15. — *Let X be an integral scheme. Let $\mathscr{L}$ be an invertible sheaf over X, which we also consider as a scheme over X. We let $p\colon \mathscr{L} \to X$ be the projection and $s\colon X \to \mathscr{L}$ be the zero section. Then, $s_\star[X] = p^\star c_1(\mathscr{L}) \in CH^1(\mathscr{L})$ where $c_1(\mathscr{L})$ where $\mathscr{L}$ is line bundle on an integral scheme U is a notation for $c_1(\mathscr{L}) \cap [U] \in CH^1(U)$.*

We consider $\mathscr{L}$ as an open subscheme of $\mathbf{P}(\mathscr{L}^\vee \oplus \mathscr{O}_X)$. We have a canonical surjection $p^\star \mathscr{L}^\vee \oplus p^\star \mathscr{O}_X \to \mathscr{O}(1)$. The second component $p^\star \mathscr{O}_X \to \mathscr{O}(1)$ is surjective on the open subset $\mathscr{L}$, so that $\mathscr{O}(1)_{|\mathscr{L}}$ is trivial.

By tensoring with $\mathscr{L}$, the first component $p^\star \mathscr{L}^\vee \to \mathscr{O}(1)$ defines a section of $p^\star \mathscr{L} \otimes \mathscr{O}(1)$ whose divisor is precisely the class $s_\star[X]$ of the zero section. Then, $p^\star c_1(\mathscr{L}) + c_1(\mathscr{O}(1)) = c_1(p^\star \mathscr{L} \otimes \mathscr{O}(1)) = s_\star[X] \in CH^1(\mathbf{P}(\mathscr{L}^\vee \oplus \mathscr{O}_X))$. The result follows by taking the restriction of the open subset $\mathscr{L}$.

4.5.3. Proper intersections. — Assume X is regular and Y and V are integral closed subschemes in X that intersects properly. Let Z be an integral component of $W = Y \cap V$. We have defined a multiplicity m of Z in the product Y and V (actually only if Y embeds regularly in X).

***Theorem 4.16* (Serre).** — *Let A be the local ring of X at W. Let I and J be the ideals of I corresponding to Y and V. Then, $m = \sum_{i=0}^{\dim A} (-1)^i \lg_A \operatorname{Tor}_i^A(A/I, A/J)$.*

The term corresponding to i gives a contribution which corresponds to the naive definition we may imagine, which would be taking the cycle $[W]$ of the intersection W. We give an example where the two are different, a phenomenon which may happen only if $\dim A \geq 4$ (for in other cases, we have intersection with divisors for which the naive definition is enough):

***Example 4.17* ([**1**, 7.1.5]).** — We consider $A = k[X, Y, Z, T]$ and the ideals $I = (XT - YZ, X^2Z - Y^3, YT^2 - Z^3, Y^2T - Z^2X)$ and $J = (X, T)$. Then, I and J are two primes ideals of codimension 2 which intersects properly and such that $\lg_A(A/(I+J)) = 5$ but the intersection multiplicity is 4.

We shall compute $\operatorname{Tor}_i^A(A/I, A/J)$. First, $A/I = k[M']$ where M' is the monoid obtained from $M = X^{\mathbf{N}} \cdot Y^{\mathbf{N}} \cdot Z^{\mathbf{N}} \cdot T^{\mathbf{N}}$ by taking the quotient by the relations $XT \sim YZ$, $X^2Z \sim Y^3$, $YT^2 \sim Z^3$ and $Y^2T \sim Z^2X$. There is a map of monoids $M' \xrightarrow{\varphi} \mathbf{N}^2$ such that $\varphi(X) = (4, 0)$, $\varphi(Y) = (3, 1)$, $\varphi(Z) = (1, 3)$ and $\varphi(T) = (0, 4)$.

The image of φ is $\{(a, b) \in \mathbf{N}^2, 4|a + b\} - \{(2, 2)\}$ and φ is injective (hint: construct an inverse map and check it is a morphism).

Then, we have an injective map $A/I = k[M'] \to k[\mathbf{N}^2]$, where the latter is a polynomial algebra with two variables, so that I is a prime ideal such that $\dim(A/I) = 2$.

We have an equality of ideals $I+J=(X,T,YZ,Y^3,Z^3)$, whose radical is (X,Y,Z,T). The quotient $A/(I+J)$ is isomorphic to $k[Y,Z]/(Y^3,Z^3)$ divided by the ideal generated by YZ which is the vector space with basis YZ, Y^2Z, YZ^2, Y^2Z^2, which shows that $\lg_A A/(I+J)=5$.

We compute the higher Tor in two steps. As T is not a zero divisor in A, the derived tensor product $A/I\otimes^{\mathbf{L}}_A A/(T)$ is represented by the complex $[A/I \xrightarrow{T} A/I]$ (in homological degrees 1 and 0). Obviously, $T\notin I$ and A/I is a domain, so that the multiplication with T is injective. Then, there are no higher Tor at this stage: $A/I\otimes^{\mathbf{L}}_A A/(T)=A/(I+(T))=B$ with $B=k[X,Y,Z]/(YZ,X^2Z-Y^3,Z^3,Z^2X)$ which is easily seen to be isomorphic, as a $k[X]$-module, to

$$k[X]1\oplus k[X]Y\oplus k[X]Y^2\oplus k[X]Z\oplus k[X]/(X)Z^2\ .$$

Then, $B\otimes^{\mathbf{L}}_{A/T} A/(T,X)$ is represented by $[B\xrightarrow{X} B]$. The cokernel was already computed and the kernel is the line in B spanned by Z^2, so that $\dim \mathrm{Tor}_i^A(A/I,A/J)$ is 5 if $i=0$, 1 if $i=1$ and 0 if $i\geq 2$, which proves the claim.

4.6. Products. —

Definition 4.18. — Let X be a smooth variety over a field k. We consider the diagonal morphism $\Delta\colon X\to X\times X$. The external product of cycles defines a bilinear morphism $CH^a(X)\times CH^b(X)\to CH^{a+b}(X\times X)$. By composing with $\Delta^\star\colon CH^\star(X\times X)\to CH^\star(X)$, we get a bilinear morphism $CH^a(X)\times CH^b(X)\to CH^{a+b}(X)$.

The construction of the preceding paragraph enjoys many good properties, especially compability with composition of closed regular immersions. Then, this construction gives a graded ring structure on $CH^\star(X)$. The multiplication is commutative (and not up to signs as it happens with some cohomology theories: actually one should think of classes of cycles as cohomology classes in even degrees).

5. Chow motives

We fix a base field k.

5.1. Effective Chow motives. —

Definition 5.1. — Let X and Y be smooth and projective varieties over k. We let d_X and d_Y be the dimensions of these varieties. If Y is equidimensional, we set $\mathrm{Corr}(X,Y) = CH_{d_Y}(Y \times_k X)$. If Y has connected components Y_i of dimensions d_i, then $\mathrm{Corr}(X,Y) = \prod_i \mathrm{Corr}(X,Y_i)$. The group $\mathrm{Corr}(X,Y)$ is the group of Chow correspondences.

Definition 5.2. — We define a category $\mathbf{PreCHM}^{\mathrm{eff}}(k)$. Its objects are the projective and smooth varieties X over k. The object in $\mathbf{PreCHM}^{\mathrm{eff}}(k)$ corresponding to X is denoted $h(X)$. We define the group of morphisms $h(X) \to h(Y)$ in $\mathbf{PreCHM}^{\mathrm{eff}}(k)$ by the formula $\mathrm{Hom}_{\mathbf{PreCHM}^{\mathrm{eff}}(k)}(h(X),h(Y)) = \mathrm{Corr}(X,Y)$. Assume X,Y,Z are three (connected) projective and smooth varieties over k, that $\alpha \in \mathrm{Hom}_{\mathbf{PreCHM}^{\mathrm{eff}}(k)}(h(X),h(Y)) = CH_{d_Y}(Y \times X)$ and $\beta \in \mathrm{Hom}_{\mathbf{PreCHM}^{\mathrm{eff}}(k)}(h(Y),h(Z)) = CH_{d_Z}(Z \times Y)$. We define $\beta \circ \alpha \in \mathrm{Hom}_{\mathbf{PreCHM}^{\mathrm{eff}}(k)}(h(X),h(Z)) = CH_{d_Z}(Z \times X)$ by the formula:

$$\beta \circ \alpha = p_{13,\star}(p_{12}^\star \beta \cdot p_{23}^\star \alpha) ,$$

where the product is taken in $CH^\star(Z \times Y \times X)$ and the p_{ij} denote projections from $Z \times Y \times X$ to some factors.

This defines an additive category $\mathbf{PreCHM}^{\mathrm{eff}}(k)$.

Proposition 5.3. — *There is a functor* $\mathbf{SmProj}(k)^{\mathrm{opp}} \to \mathbf{PreCHM}^{\mathrm{eff}}(k)$ *that maps X to $h(X)$ and a morphism $f\colon Y \to X$ to the class of the graph $\Gamma_f \subset Y \times X$ considered as an element in $f^\star \in \mathrm{Corr}(X,Y)$.*

As a result, the identity of $h(X)$ is given by the class of the diagonal in $CH_{d_X}(X \times X)$.

Proposition 5.4. — *The group* $\mathrm{End}_{\mathbf{PreCHM}^{\mathrm{eff}}(k)}(h(\mathbf{P}^1)) = CH^1(\mathbf{P}^1 \times \mathbf{P}^1)$ *is isomorphic to $\mathbf{Z}^2$ and generated by the classes $p = [\infty \times \mathbf{P}^1]$ and $q = [\mathbf{P}^1 \times \infty]$. Then, $p \in \mathrm{End}_{\mathbf{PreCHM}^{\mathrm{eff}}(k)}(\mathbf{P}^1)$ is $f^\star$ for the constant morphism $f\colon \mathbf{P}^1 \to \mathbf{P}^1$ with value ∞. We have $p \circ p = p$, $q \circ q = q$, $p \circ q = q \circ p = 0$ and $p + q = \mathrm{id}_{h(X)}$.*

The fact that p and q are a basis of $CH^1(\mathbf{P}^1 \times \mathbf{P}^1)$ follows from the general computation of Chow groups of projective bundles. We define the two projections $\pi_1, \pi_2 \colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. Then, we see that $\pi_{1,\star}p = 0$, $\pi_{2,\star}p = 1$, $\pi_{1,\star}q = 1$ and $\pi_{2,\star}q = 0$. For a given class $z \in CH^1(\mathbf{P}^1 \times \mathbf{P}^1)$, this enables to find an expression for z as a linear combination of p and

q. For example, the diagonal $\Delta \in \mathbf{P}^1 \times \mathbf{P}^1$ is such that $\pi_{1,\star}[\Delta] = 1$ and $\pi_{2,\star}[\Delta] = 1$ so that $[\Delta] = p + q$.

Furthermore, $p \circ p = p$ because $p \circ p$ corresponds to $(f \circ f)^\star = f^\star$. Then, p is a projector of $h(X)$. As usual, we have the supplementary projector $\mathrm{id}_{h(X)} - p = [\Delta] - p = q$ which satisfies $p \circ q = q \circ p = 0$ and $q \circ q = q$.

We have a morphism $h(\operatorname{Spec} k) \to h(\mathbf{P}^1)$ corresponding to the projection $\mathbf{P}^1 \to \operatorname{Spec} k$ and a morphism in the other way $h(\mathbf{P}^1) \to h(\operatorname{Spec} k)$ given by the inclusion of $\infty \colon \operatorname{Spec} k \to \mathbf{P}^1$. The composition $h(\operatorname{Spec} k) \to h(\operatorname{Spec} k) \to h(\operatorname{Spec} k)$ is the identity.

Then, in any reasonable sense, the morphism p which is the composition of the morphism given above $h(\mathbf{P}^1) \to h(\operatorname{Spec} k) \to h(\mathbf{P}^1)$ should be an endomorphism of $h(\mathbf{P}^1)$ which is a projector on the direct factor $h(\operatorname{Spec} k)$. However, there does not exist any object M in $\mathbf{PreCHM}^{\mathrm{eff}}(k)$ such that $h(\mathbf{P}^1) \simeq h(\operatorname{Spec} k) \oplus M$.

Definition 5.5. — The category of effective Chow motives $\mathbf{CHM}^{\mathrm{eff}}(k)$ is the Karoubian (or pseudo-abelian) envelope of $\mathbf{PreCHM}^{\mathrm{eff}}(k)$. An object of $\mathbf{CHM}^{\mathrm{eff}}(k)$ is an object M of $\mathbf{PreCHM}^{\mathrm{eff}}(k)$ equipped with a projector $p \in \operatorname{End}_{\mathbf{PreCHM}^{\mathrm{eff}}}(M)$. Such a tuple (M, p) should be considered as the (formal) image of p. Then, we define

$$\operatorname{Hom}_{\mathbf{CHM}^{\mathrm{eff}}(k)}((M,p),(N,q)) = q \circ \operatorname{Hom}_{\mathbf{PreCHM}^{\mathrm{eff}}(k)}(M,N) \circ p \,,$$

considered as a subgroup of $\operatorname{Hom}_{\mathbf{PreCHM}^{\mathrm{eff}}(k)}(M,N)$.

This defines an additive category $\mathbf{CHM}^{\mathrm{eff}}(k)$, with a fully faithful functor $\mathbf{PreCHM}^{\mathrm{eff}}(k) \to \mathbf{CHM}^{\mathrm{eff}}(k)$ which maps M to (M, id_M). If $M \in \mathbf{PreCHM}^{\mathrm{eff}}(k)$ and $p \in \operatorname{End}_{\mathbf{PreCHM}^{\mathrm{eff}}(k)}(M)$ such that $p \circ p = p$, we have a canonical decomposition in $\mathbf{CHM}^{\mathrm{eff}}(k)$: $M \simeq (M,p) \oplus (M, \mathrm{id}_M - p)$.

Definition 5.6. — The Lefschetz motive $\mathbf{L}$ is the motive $(h(\mathbf{P}^1), q)$ with q as defined above. We denote $\mathbf{1}$ the unit motive $h(\operatorname{Spec} k)$. Then, we have a canonical decomposition $h(\mathbf{P}^1) = \mathbf{1} \oplus \mathbf{L}$.

5.2. Tensor products. —

Definition 5.7. — We define a functor $\otimes \colon \mathbf{PreCHM}^{\mathrm{eff}}(k) \times \mathbf{PreCHM}^{\mathrm{eff}}(k) \to \mathbf{PreCHM}^{\mathrm{eff}}(k)$ by setting $h(X) \otimes h(Y) = h(X \times_k Y)$. On morphisms, it is given corresponds to the external product of cycles, if

$\alpha\colon h(X) \to h(X')$ is given by an element in $CH_{d_{X'}}(X' \times X)$ and $\beta\colon h(Y) \to h(Y')$ is given by an element in $CH_{d_{Y'}}(Y' \times Y)$, then $\alpha \otimes \beta\colon h(X \times Y) \to h(X' \times Y')$ is given by the external product of α and β in $CH_{d_{X'}+d_{Y'}}(X' \times X \times Y' \times Y) \simeq CH_{d_{X'}+d_{Y'}}(X' \times Y' \times X \times Y)$.

This definition extends to a functor $\otimes\colon \mathbf{CHM}^{\mathrm{eff}}(k) \times \mathbf{CHM}^{\mathrm{eff}}(k) \to \mathbf{CHM}^{\mathrm{eff}}(k)$.

Proposition 5.8. — *The functor* $-\otimes \mathbf{L}\colon \mathbf{CHM}^{\mathrm{eff}}(k) \to \mathbf{CHM}^{\mathrm{eff}}(k)$ *is a fully faithful functor: for any* M *and* M'*, the obvious map* $\mathrm{Hom}_{\mathbf{CHM}^{\mathrm{eff}}(k)}(M, M') \to \mathrm{Hom}_{\mathbf{CHM}^{\mathrm{eff}}(k)}(M \otimes \mathbf{L}, M' \otimes \mathbf{L})$ *is a bijection.*

Definition 5.9. — A Chow motive is a tuple (M, n) with $M \in \mathbf{CHM}^{\mathrm{eff}}(k)$ and $n \in \mathbf{Z}$. A morphism $(M, n) \to (M', n')$ is an element in the inductive limit $\lim_{k \geq \max(-n,-n')} \mathrm{Hom}_{\mathbf{CHM}^{\mathrm{eff}}(k)}(M \otimes \mathbf{L}^{\otimes n+k}, M' \otimes \mathbf{L}^{\otimes n'+k})$. This constitutes a category $\mathbf{CHM}(k)$. We have an obvious fully faithful functor $\mathbf{CHM}^{\mathrm{eff}}(k) \to \mathbf{CHM}(k)$ which maps M to $(M, 0)$.

Proposition 5.10. — *Let* X *and* Y *be two (connected) smooth and projective varieties over* k*. Then, for all* n *and* n'*,* $\mathrm{Hom}_{\mathbf{CHM}(k)}(h(X) \otimes \mathbf{L}^{\otimes n}, h(Y) \otimes \mathbf{L}^{\otimes n'}) \simeq CH_{d_Y - n + n'}(Y \times X)$*.*

Definition 5.11. — Let $M \in \mathbf{CHM}(k)$. For any $n \in \mathbf{Z}$, we set $CH^n(M) = \mathrm{Hom}(\mathbf{L}^{\otimes n}, M)$ and $CH_n(M) = \mathrm{Hom}(M, \mathbf{L}^{\otimes b})$.

Remark 5.12. — If $f\colon X \to Y$ is a morphism in $\mathbf{SmProj}(k)$, then the morphism $f^\star\colon h(Y) \to h(X)$ constructed using the graph of f induces a morphism : $CH_\star(X) \to CH_\star(Y)$ which is the graded pushforward $f_\star$ associated to the proper morphism f and there is also an induced map $f^\star\colon CH^\star(Y) \to CH^\star(X)$ which corresponds to the previous constructions if f is flat or a closed immersion.

***Proposition 5.13* (Manin's identity principle)**

If $M \xrightarrow{\varphi} M'$ *is a morphism in* $\mathbf{CHM}(k)$*, then* φ *is an isomorphism if and only if* $\varphi^\star\colon CH^\star(M \otimes N) \to CH^\star(M' \otimes N)$ *is a bijection for all Chow motives* N*.*

This is a good exercise.

Proposition 5.14. — *Let* X *be a smooth and projective variety over* k*. Let* $\mathscr{E}$ *be a vector bundle of rank* $n+1$ *over* X*. Then, there is an* *isomorphism*

$$h(\mathbf{P}(\mathscr{E}) \simeq \bigoplus_{i=0}^{n} h(X) \otimes \mathbf{L}^{\otimes i} \,.$$

This follows from the computation of the Chow groups of projective bundles and Manin's identity principle.

5.3. Realization functors. — Assume the base field k is embedded into $\mathbf{C}$ *via* $\iota\colon k \to \mathbf{C}$. Then, the singular cohomology defines a functor $H_B\colon \mathbf{SmProj}(k)^{\mathrm{opp}} \to \mathrm{GrVec}_{\mathbf{Q}}$. This extends to a (covariant) functor $r_B\colon \mathbf{CHM}(k) \to \mathrm{GrVec}_{\mathbf{Q}}$ which maps the motive $h(X)$ to $H^\star(X(\mathbf{C}), \mathbf{Q})$. The key inputs for this construction are the Poincaré duality isomorphism and the cycle class map $CH^n(X) \to H^{2n}(X(\mathbf{C}), \mathbf{Q}) \otimes \mathbf{Q}(n)$ where $\mathbf{Q}(n) = \mathbf{Q}(1)^{\otimes n}$ and $\mathbf{Q}(1)$ is the dual of $H^2(\mathbf{P}^1, \mathbf{Q})$. This r_B is the Betti realization functor. Similar functors exist for other Weil cohomology theories: De Rham cohomology, ℓ-adic étale cohomology...

Then, if two motives M and N are isomorphic, then $r_B(M) \simeq r_B(N)$, $r_\ell(M) \simeq r_\ell(N)$, $r_{DR}(M) \simeq r_{DR}(N)$. Sometimes, it is easy to construct such isomorphisms in each cohomology theory, but Grothendieck's idea of motives was to obtain such isomorphism (or some morphisms) as induced by algebraic cycles and isomorphisms of motives. Then, such (iso)morphisms of motives can have different incarnations in each cohomology theories. It is expected that all reasonable isomorphisms in cohomology of algebraic varieties can be explained by algebraic cycles...

Theorem 5.15. — *Let X be a smooth and projective variety. Let Y be a closed subvariety of X of codimension c. Then, there is an isomorphism of Chow groups:*

$$CH^\star(X) \bigoplus \oplus_{i=1}^{c-1} CH^{\star - i}(Y) \xrightarrow{\sim} CH^\star(\mathrm{Bl}_Y X) \,.$$

If follows from an exact sequence:

$$0 \to CH^{\star - c}(Y) \to CH^{\star - 1}(\mathbf{P}(\mathscr{N}_{X/Y})) \oplus CH^\star(X) \overset{i_\star \oplus p^\star}{\to} CH^\star(\mathrm{Bl}_Y X) \to 0$$

where $i\colon \mathbf{P}(\mathscr{N}_{X/Y}) \to \mathrm{Bl}_Y X$ and $p\colon \mathrm{Bl}_Y X \to X$ are the obvious morphisms.

Corollary 5.16. — *Let X be a smooth and projective variety. Let Y be a closed subvariety of X of codimension c. Then, there is an isomorphism of motives:*

$$h(\mathrm{Bl}_Y X) \simeq h(X) \bigoplus \oplus_{i=1}^{c-1} h(Y) \otimes \mathbf{L}^{\otimes i} \,.$$

As a corollary of this isomorphism, we get a computation of the cohomology of a blow-up in all Weil cohomology theories.

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