# $\mathbf{A}^1$-HOMOTOPY INVARIANCE IN SPECTRAL ALGEBRAIC GEOMETRY

DENIS-CHARLES CISINSKI AND ADEEL A. KHAN

Abstract. We study two different flavours of $\mathbf{A}^1$-homotopy theory in the setting of spectral algebraic geometry, and compare them to classical $\mathbf{A}^1$-homotopy theory. As an application we show that the spectral analogue of Weibel's homotopy invariant K-theory collapses to the classical theory. Along the way we give a new construction of nonconnective algebraic K-theory of stable ∞-categories via a generalization of the Bass–Thomason–Trobaugh construction.

## Contents



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## 1. Introduction

This paper establishes in a systematic way why fundamental invariants from derived geometry, such as Serre's Tor formula, or virtual fundamental classes, have a natural interpretation in homotopy invariant (co)homology theories. In fact, we provide an explicit way to interpret Lurie's spectral geometry into Voevodsky's motivic homotopy theory as follows.

Let $R$ be a commutative ring or connective $\mathcal{E}_\infty$-ring spectrum, and let $R[T]$ denote the polynomial algebra over $R$ in one variable $T$. A peculiarity of the world of $\mathcal{E}_\infty$-ring spectra is that the polynomial algebra $R[T]$ is *free* as an $\mathcal{E}_\infty$-$R$-algebra only when $R$ is of characteristic zero (an $\mathcal{E}_\infty$-$\mathbf{Q}$-algebra). If we write $R\{T\}$ for the free $\mathcal{E}_\infty$-$R$-algebra on one generator $T$ (in degree zero), then in general there is only a comparison homomorphism $R\{T\} \to R[T]$. The $\mathcal{E}_\infty$-ring $R\{T\}$ is *smooth* over $R$ (its cotangent complex $L_{R\{T\}/R}$ is free) but not flat, while the $\mathcal{E}_\infty$-ring $R[T]$ is usually not smooth but instead *fibre-smooth*: that is, it is flat over $R$, and $\pi_0(R[T]) \simeq \pi_0(R)[T]$ is smooth over $\pi_0(R)$ in the sense of ordinary commutative algebra. Let $\mathrm{CAlg}_R^{\mathrm{sm}}$ denote the $\infty$-category of smooth $\mathcal{E}_\infty$-algebras over $R$, and $\mathrm{CAlg}_R^{\mathrm{fibsm}}$ denote the $\infty$-category of fibre-smooth $\mathcal{E}_\infty$-algebras over $R$. Our first main result reads as follows:

**Theorem A.** *Let $R$ be a connective $\mathcal{E}_\infty$-ring. Consider the following $\infty$-categories:*

(i) *The $\infty$-category $\mathbf{H}(R)$ of Nisnevich sheaves of spaces $\mathcal{F} : \mathrm{CAlg}_R^{\mathrm{sm}} \to \mathrm{Spc}$ for which the canonical map $\mathcal{F}(A) \to \mathcal{F}(A\{T\})$ is invertible for every $A \in \mathrm{CAlg}_R^{\mathrm{sm}}$.*

(ii) *The $\infty$-category $\mathbf{H}^\flat(R)$ of Nisnevich sheaves of spaces $\mathcal{F} : \mathrm{CAlg}_R^{\mathrm{fibsm}} \to \mathrm{Spc}$ for which the canonical map $\mathcal{F}(A) \to \mathcal{F}(A[T])$ is invertible for every $A \in \mathrm{CAlg}_R^{\mathrm{fibsm}}$.*

(iii) *The $\infty$-category $\mathbf{H}^{\mathrm{cl}}(\pi_0(R))$ of Nisnevich sheaves of spaces $\mathcal{F} : \mathrm{CAlg}_{\pi_0(R)}^{\mathrm{fibsm}} \to \mathrm{Spc}$ for which the canonical map $\mathcal{F}(A) \to \mathcal{F}(A[T])$ is invertible for every $A \in \mathrm{CAlg}_{\pi_0(R)}^{\mathrm{fibsm}}$.*

*Then (i) and (iii) are equivalent. If $R$ is an $\mathcal{E}_\infty$-$\mathbf{Z}$-algebra, then all three are equivalent.*

See Theorems 2.7.2 and 3.4.1. The result is nontrivial even for ordinary commutative rings $R$ (viewed as discrete $\mathcal{E}_\infty$-rings), in which case it asserts an equivalence $\mathbf{H}(R) \simeq \mathbf{H}^{\mathrm{cl}}(R)$. Note that $\mathrm{CAlg}_R^{\mathrm{fibsm}}$ coincides with the category of commutative rings that are smooth over $R$ in the sense of ordinary commutative algebra, so that $\mathbf{H}^{\mathrm{cl}}(R)$ coincides with the usual $\mathbf{A}^1$-homotopy category considered by Morel and Voevodsky [MV99]. This equivalence was only known in characteristic zero (see Proposition 2.4.6 and Warning 2.4.7 in [Kha19b]). For simplicial commutative rings (regarded as $\mathcal{E}_\infty$-$\mathbf{Z}$-algebras), the equivalence (ii) $\Leftrightarrow$ (iii) was proven in the second author's thesis [Kha16].

Our second subject of discussion is a variant of Weibel's homotopy invariant K-theory for connective $\mathcal{E}_\infty$-rings. Let $\mathrm{KH}^{\mathrm{cl}}$ denote the classical variant [Wei89], defined by starting with nonconnective algebraic K-theory and forcing it to become $\mathbf{A}^1$-homotopy invariant in the sense that the canonical map

$$\mathrm{KH}^{\mathrm{cl}}(R) \to \mathrm{KH}^{\mathrm{cl}}(R[T])$$

is invertible for every commutative ring $R$. We define an analogous construction on the $\infty$-category of connective $\mathcal{E}_\infty$-rings,

$$R \mapsto \mathrm{KH}(R),$$

by forcing $\mathbf{A}^1$-homotopy invariance in the sense that $\mathrm{KH}(R) \to \mathrm{KH}(R\{T\})$ is invertible for any connective $\mathcal{E}_\infty$-ring $R$. We then have the following comparison, a K-theoretic incarnation of Theorem A:

**Theorem B.** *For every connective $\mathcal{E}_\infty$-ring $R$, there is a canonical isomorphism of spectra*

$$\mathrm{KH}(R) \simeq \mathrm{KH}^{\mathrm{cl}}(\pi_0(R)).$$

See Corollary 5.1.9. For connective $\mathcal{E}_\infty$-$\mathbf{Z}$-algebras one may adapt the proof, using the equivalence (i) $\Leftrightarrow$ (iii) in Theorem A, to derive the same result for the variant $\mathrm{KH}^\flat$ constructed by imposing invertibility of the maps $\mathrm{KH}^\flat(R) \to \mathrm{KH}^\flat(R[T])$. This result was communicated to us by B. Antieau and D. Gepner in 2015 in the generality of connective $\mathcal{E}_1$-rings, and has recently been recorded in the case of connective $\mathcal{E}_1$-$\mathbf{Z}$-algebras by Land–Tamme [LT19, Prop. 3.14].

An important ingredient in the proof is the observation that nonconnective K-theory of stable $\infty$-categories, defined as in [BGT13] following Schlichting, can also be described by a variant of the Bass–Thomason–Trobaugh construction [TT90, Sect. 6] defined over the sphere spectrum:

**Theorem C.** *There is an isomorphism*

$$\mathbb{K} \simeq \mathrm{K}^{\mathrm{B}}$$

*of spectrum-valued functors on the $\infty$-category of small stable $\infty$-categories, where* K *is algebraic K-theory,* $\mathbb{K}$ *is nonconnective algebraic K-theory, and* $(-)^{\mathrm{B}}$ *denotes a generalization of the Bass–Thomason–Trobaugh construction (Construction 4.5.5).*

See Example 4.4.4. In fact, we show in Theorem 4.4.3 that the construction $(-)^{\mathrm{B}}$ defines an equivalence between the $\infty$-category of connective spectrum-valued localizing invariants[1] and the $\infty$-category of spectrum-valued localizing invariants. This also implies for example that all operations on connective K-theory deloop to $\mathbb{K}$. The result is inspired by closely related work of Robalo in his framework of noncommutative motivic homotopy theory [Rob15] which gives similar results in the more restrictive setting of dg-categories.

[1] A connective spectrum-valued invariant $E$ is localizing if, for any short exact sequence of stable $\infty$-categories $\mathbf{A}' \to \mathbf{A} \to \mathbf{A}''$, $E(\mathbf{A}')$ is identified with the connective cover of the homotopy fibre of $E(\mathbf{A}) \to E(\mathbf{A}'')$. See Definition 4.4.2.

**Outline.** In the body of the paper, we use the language of spectral algebraic geometry [SAG]. Given a spectral affine[2] scheme $S$, we may define $\mathbf{H}(S)$ as the $\infty$-category of $\mathbf{A}^1$-invariant Nisnevich sheaves on the site $\mathrm{Sm}_{/S}$ of smooth spectral affine schemes over $S$. Then $\mathbf{H}(\mathrm{Spec}(R))$ is equivalent to $\mathbf{H}(R)$ as defined above, and also to the construction given in [Kha19b], see Corollary 2.4.5 of *op. cit.* Similarly we have the variant $\mathbf{H}^\flat(S)$ defined as the $\infty$-category of $\mathbf{A}^{1,\flat}$-invariant Nisnevich sheaves on the site $\mathrm{Sm}^\flat_{/S}$ of fibre-smooth spectral affine schemes over $S$, where $\mathbf{A}^{1,\flat} = \mathrm{Spec}(\mathbf{S}[T])$.

The proof of the equivalence (i) $\Leftrightarrow$ (iii) in Theorem A is given in Sect. 2. The key idea is to reinterpret the derived nil-invariance result of [Kha19b, Thm. A] as a sort of descent statement with respect to the "nil topology" whose coverings are morphisms of the form $X_{\mathrm{cl}} \to X$, where $X_{\mathrm{cl}}$ denotes the classical truncation of the spectral scheme $X$. Since the site $\mathrm{Sm}_{/S}$ is typically not closed under the operation $X \mapsto X_{\mathrm{cl}}$, making sense of this idea requires us to enlargen our site[3]. The first few subsections (2.1, 2.2, 2.3) develop some generalities related to $\mathbf{A}^1$-homotopy theory on various *admissible* sites (Definition 2.1.7) and how variation of site interplays with the basic operations such as inverse/direct image, product, and internal hom. For example, any *narrow* subcategory $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ as in Definition 2.1.10 gives rise to the same $\mathbf{A}^1$-homotopy theory as the smooth site $\mathrm{Sm}_{/S}$ (Example 2.2.4). The key result here is Proposition 2.2.3 which implies that any $\mathbf{A}^1$-invariant Nisnevich sheaf defined on a narrow subcategory $\mathcal{A}_{/S}$ admits a canonical extension $\mathcal{F}^+$ to an $\mathbf{A}^1$-invariant Nisnevich sheaf on any *broad* site $\mathcal{B}_{/S}$ (Definition 2.1.11) that contains $\mathcal{A}_{/S}$. Broad sites are in particular closed under classical truncation.

In Subsect. 2.4 we construct a comparison functor to the classical motivic homotopy category, which we show in Subsect. 2.5 is a left Bousfield localization (the *nil-localization*) if we work over a broad site (Theorem 2.5.3). In Subsect. 2.6 we formulate the nil-descent result alluded to above (Theorem 2.6.2). Finally, we put everything together in Subsect. 2.7 to show that nil-localization gives an equivalence

$$\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \xrightarrow{\sim} \mathbf{H}(S)$$

when we restrict to our narrow subcategory $\mathcal{A}_{/S}$. The last subsection (Subsect. 2.8) extends to the result to presheaves with values in general presentable $\infty$-categories $\mathbf{V}$ (e.g. presheaves of spectra or presheaves of chain complexes).

The second part of Theorem A is proven in Sect. 3. The idea of enlarging sites again plays an important role here. Working on a site large enough that it contains both the spectral affine line $\mathbf{A}^1$ and the flat affine line $\mathbf{A}^{1,\flat}$ allows us to exploit the canonical morphism $\varepsilon : \mathbf{A}^{1,\flat} \to \mathbf{A}^1$ which is a morphism of interval objects (3.3.2). The key input in the comparison is the fact that $\varepsilon$ is a "universal" $\mathbf{A}^{1,\flat}$-equivalence, as long as we work over $\mathrm{Spec}(\mathbf{Z})$ (Lemma 3.3.4). The proof of the comparison is given in Subsect. 3.4.

Sect. 4 is independent of the first three sections and discusses localizing invariants of stable $\infty$-categories in the sense of [BGT13][4]. The main result, Theorem 4.4.3, asserts that every

[2]To simplify the exposition we usually only discuss the affine case. However, all our results extend to spectral schemes and algebraic spaces by descent: see Corollaries 2.7.5, 3.4.6, and 5.1.8.

[3]The same thing happens in classical algebraic geometry with the cdh topology; see [Kha19a].

[4]Note that, unlike [BGT13], we do not require localizing invariants to preserve filtered colimits.

connective spectrum-valued localizing invariant admits a unique delooping to a spectrum-valued localizing invariant. This is proven by generalizing the Bass construction to stable $\infty$-categories over the sphere spectrum (or any connective $\mathcal{E}_\infty$-ring).

Theorem B is proven in Sect. 5 by showing that the equivalence $\mathbf{H}(R) \simeq \mathbf{H}^{\mathrm{cl}}(\pi_0(R))$ of Theorem A or (rather its generalization to sheaves of spectra) sends KH to $\mathrm{KH}^{\mathrm{cl}}$. Unstably this boils down to representability results for the infinite loop spaces, and the stable result is deduced via Bott periodicity.

**Notation and conventions.** We will use the language of $\infty$-categories freely throughout the text. Our main references are [HTT, HA]. The $\infty$-category of spaces and spectra will be denoted by Spc and Spt, respectively, and a morphism in an $\infty$-category will be called an *isomorphism* if it is invertible (= an *equivalence* in the language of [HTT]). We also use the language of spectral algebraic geometry [SAG]. Given a spectral affine scheme $S = \mathrm{Spec}(R)$, we write $\mathrm{Aff}_{/S}$ for the $\infty$-category of spectral affine schemes over $S$, which is equivalent to the opposite of the $\infty$-category of connective $\mathcal{E}_\infty$-$R$-algebras.

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## 2. Comparison with classical motivic homotopy theory

2.1. **Fibred spaces.** For this subsection, we fix an affine spectral scheme $S$ and write $\mathrm{Aff}_{/S}$ for the $\infty$-category of affine spectral schemes over $S$.

**Definition 2.1.1.** A morphism of affine spectral schemes $X \to S$ is called *smooth* (resp. *étale*) if it is of finite presentation and the relative cotangent complex $\mathcal{L}_{X/S}$ is a locally free $\mathcal{O}_X$-module of finite rank (resp. is zero).

**Remark 2.1.2.** From [SAG, Prop. 11.2.2.1] it follows that a morphism of affine spectral schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is smooth if and only if $A \to B$ is differentially smooth in the sense of [SAG, Def. 11.2.2.2].

**Example 2.1.3.** Let $\mathbf{S}$ denote the sphere spectrum. For every integer $n \geqslant 0$, we write $\mathbf{S}\{T_1, \dots, T_n\}$ for the free $\mathcal{E}_\infty$-algebra on $n$ generators $T_i$ (in degree zero). We let $\mathbf{A}^n$ denote the affine spectral scheme $\mathrm{Spec}(\mathbf{S}\{T_1, \dots, T_n\})$ and refer to it as $n$-dimensional *spectral affine space* (over the sphere spectrum). The morphism $\mathbf{A}^n \to \mathrm{Spec}(\mathbf{S})$ is smooth, and we have a canonical isomorphism $(\mathbf{A}^n)_{\mathrm{cl}} \simeq \mathbf{A}^n_{\mathrm{cl}}$, where $\mathbf{A}^n_{\mathrm{cl}}$ denotes the classical affine space over $\mathrm{Spec}(\mathbf{Z})$.

**Remark 2.1.4.** If $X \in \mathrm{Aff}_{/S}$ is smooth over $S$, then Zariski-locally on $X$ there exists an étale $S$-morphism $X \to S \times \mathbf{A}^n$ for some $n \geqslant 0$. This follows from [SAG, Prop. 11.2.2.1].

**Definition 2.1.5** (Nisnevich excision)**.**

(i) A *Nisnevich square* over $X \in \mathrm{Aff}_{/S}$ is a cartesian square of affine spectral schemes

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & X \end{array} \tag{2.1.a}$$

where $j$ is an open immersion, $p$ is étale, and there exists a closed immersion $Z \hookrightarrow X$ complementary to $j$ such that the induced morphism $p^{-1}(Z) \to Z$ is invertible.

(ii) We say that a presheaf of spaces $\mathcal{F}$ on $\mathrm{Aff}_{/S}$ satisfies *Nisnevich excision* if it is reduced, i.e. the space $\Gamma(\varnothing, \mathcal{F})$ is contractible, and for any $X \in \mathrm{Aff}_{/S}$ and any Nisnevich square over $X$ of the form (2.1.a), the induced square of spaces

$$\begin{array}{ccc} \Gamma(X,\mathcal{F}) & \xrightarrow{j^*} & \Gamma(U,\mathcal{F}) \\ \downarrow{\scriptstyle p^*} & & \downarrow \\ \Gamma(V,\mathcal{F}) & \longrightarrow & \Gamma(W,\mathcal{F}) \end{array}$$

is cartesian.

**Definition 2.1.6** ($\mathbf{A}^1$-invariance)**.** Let $\mathcal{F}$ be a presheaf of spaces on $\mathrm{Aff}_{/S}$. We say that $\mathcal{F}$ satisfies $\mathbf{A}^1$-*homotopy invariance* if for every $X \in \mathrm{Aff}_{/S}$, the canonical map of spaces

$$p^* : \Gamma(X, \mathcal{F}) \to \Gamma(X \times \mathbf{A}^1, \mathcal{F})$$

is invertible, where $p : X \times \mathbf{A}^1 \to X$ is the projection of the spectral affine line over $X$.

We will need to consider presheaves defined on smaller subcategories of $\mathrm{Aff}_{/S}$. The following definition identifies the minimal conditions under which Definitions 2.1.5 and 2.1.6 make sense.

**Definition 2.1.7.** We say that a full subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ is *admissible* if it is essentially small and satisfies the following conditions:

(i) The affine spectral scheme $S$ (viewed over $S$ via the identity) belongs to $\mathcal{C}_{/S}$.

(ii) If $X$ belongs to $\mathcal{C}_{/S}$ and $Y$ is étale over $X$, then $Y$ belongs to $\mathcal{C}_{/S}$.

(iii) If $X$ belongs to $\mathcal{C}_{/S}$, then $X \times \mathbf{A}^n$ belongs to $\mathcal{C}_{/S}$ for every $n \geqslant 0$.

**Example 2.1.8.** The full subcategory $\mathrm{Sm}_{/S} \subseteq \mathrm{Aff}_{/S}$ of *smooth* affine spectral $S$-schemes is admissible. This follows from the fact that étale morphisms are smooth, the morphism $\mathbf{A}^n \to \mathrm{Spec}(\mathbf{S})$ is smooth for every $n \geqslant 0$, and the class of smooth morphisms is stable under composition and base change.

**Example 2.1.9.** Let $\mathcal{A}^0_{/S} \subseteq \mathrm{Aff}_{/S}$ denote the full subcategory spanned by $X \in \mathrm{Aff}_{/S}$ which admit an étale morphism

$$X \to S \times \mathbf{A}^n$$

over $S$. Then $\mathcal{A}^0_{/S}$ is admissible. For the third condition, note that if $X$ admits an étale $S$-morphism to $S \times \mathbf{A}^n$, then $X \times \mathbf{A}^m$ admits (for every $m$) an étale $S$-morphism

$$X \times \mathbf{A}^m \to S \times \mathbf{A}^n \times \mathbf{A}^m \xrightarrow{\mathrm{pr}} S \times \mathbf{A}^n,$$

and hence also belongs to $\mathcal{A}^0_{/S}$. Note that $\mathcal{A}^0_{/S}$ is in fact the minimal admissible subcategory of $\mathrm{Aff}_{/S}$.

**Definition 2.1.10.** Let $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. We say that $\mathcal{A}_{/S}$ is *narrow* if it is contained in the full subcategory $\mathrm{Sm}_{/S}$.

**Definition 2.1.11.** Let $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. We say that $\mathcal{A}_{/S}$ is *broad* if it satisfies the following further condition:

(iv) For every $X \in \mathcal{A}_{/S}$, the classical truncation $X_{\mathrm{cl}}$ also belongs to $\mathcal{A}_{/S}$.

Note that there is a minimal broad subcategory of $\mathrm{Aff}_{/S}$, which is the closure of Example 2.1.9 under the operations (ii), (iii), and (iv) (constructed by transfinite iteration).

**Remark 2.1.12.** Note that any broad subcategory contains the minimal narrow subcategory $\mathcal{A}^0_{/S}$ (Example 2.1.9). Note also that, as long as $S$ is not discrete, no admissible subcategory is both narrow and broad, since the morphism $S_{\mathrm{cl}} \to S$ is smooth if and only if it is an isomorphism.

**Definition 2.1.13.** Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be a full subcategory. A $\mathcal{C}$-*fibred space* over $S$ is a presheaf of spaces on $\mathcal{C}_{/S}$.

**Definition 2.1.14.** Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. We say that a $\mathcal{C}$-fibred space $\mathcal{F}$ satisfies *Nisnevich excision* and $\mathbf{A}^1$-*homotopy invariance* if it satisfies the conditions of Definitions 2.1.5 and 2.1.6, respectively (imposed only on objects $X \in \mathcal{C}_{/S}$). We say that $\mathcal{F}$ is a $\mathcal{C}$-*fibred motivic space* over $S$ if it is both Nisnevich excisive and $\mathbf{A}^1$-homotopy invariant.

We denote by

$$\mathrm{Spc}(\mathcal{C}_{/S}) \quad \text{and} \ \mathbf{H}(\mathcal{C}_{/S})$$

the $\infty$-category of $\mathcal{C}$-fibred spaces over $S$ and its full subcategory of motivic objects.

**Example 2.1.15.** In case of the admissible subcategory $\mathrm{Sm}_{/S} \subseteq \mathrm{Aff}_{/S}$ (Example 2.1.8), we write

$$\mathrm{Spc}(S) \coloneqq \mathrm{Spc}(\mathrm{Sm}_{/S}), \quad \mathbf{H}(S) \coloneqq \mathbf{H}(\mathrm{Sm}_{/S}).$$

With this definition we have $\mathbf{H}(\mathrm{Spec}(R)) \simeq \mathbf{H}(R)$ for any connective $\mathcal{E}_\infty$-ring $R$, where the right-hand side is as defined in Theorem A.

**Remark 2.1.16.** For any admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$, the full subcategories of Nisnevich-excisive, $\mathbf{A}^1$-invariant, and motivic $\mathcal{C}$-fibred spaces are each left Bousfield localizations of the $\infty$-category of $\mathcal{C}$-fibred spaces. The following assertions are proven in the same way as their analogues for Sm-fibred spaces (cf. [Kha19b, Sect. 2]):

(i) The Nisnevich localization functor $\mathcal{F} \mapsto \mathrm{L}_{\mathrm{Nis}}(\mathcal{F})$ is exact (follows from [Kha19b, Thm. 2.2.7]).

(ii) The $\mathbf{A}^1$-localization functor $\mathcal{F} \mapsto \mathrm{L}_{\mathbf{A}^1}(\mathcal{F})$ admits the following description (see [Kha19b, Rem. 2.3.5], [Hoy17, Prop. 3.4]): for every $\mathcal{C}$-fibred space $\mathcal{F}$, the space of sections over any $X \in \mathcal{C}_{/S}$ is computed by a sifted colimit:

$$\Gamma\big(X, \mathrm{L}_{\mathbf{A}^1}(\mathcal{F})\big) \simeq \varinjlim \Gamma(\mathbf{A}^n \times X, \mathcal{F}), \tag{2.1.b}$$

indexed by the opposite of the full subcategory $\mathbf{A}_X \subseteq \mathrm{Aff}_{/X}$ whose objects are spectral affine spaces $X \times \mathbf{A}^n$ $(n \geqslant 0)$.

(iii) The motivic localization functor $\mathcal{F} \mapsto \mathbf{L}(\mathcal{F})$ can be computed as the transfinite composite

$$\mathbf{L}(\mathcal{F}) \simeq \varinjlim_{n \geqslant 0} (\mathrm{L}_{\mathbf{A}^1} \circ \mathrm{L}_{\mathrm{Nis}})^{\circ n}(\mathcal{F}), \tag{2.1.c}$$

for any $\mathcal{F} \in \mathrm{Spc}(\mathcal{C}_{/S})$ (cf. [Kha19b, Rem. 2.4.3]). Moreover, $\mathbf{H}(\mathcal{C}_{/S})$ has universality of colimits.

(iv) The $\infty$-category $\mathbf{H}(\mathcal{C}_{/S})$ of $\mathcal{C}$-fibred motivic spaces is generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_S(X)$, where $\mathrm{h}_S(X)$ is the presheaf on $\mathcal{C}_{/S}$ represented by $X \in \mathcal{C}_{/S}$ (cf. [Kha19b, Prop. 2.4.4]).

2.2. **Extension of fibred spaces.** As in Subsect. 2.1, we fix an affine spectral scheme $S$. We also fix the following data:

**Notation 2.2.1.** Fix an inclusion $\mathcal{C}_{/S} \subseteq \mathcal{D}_{/S}$ of admissible subcategories of $\mathrm{Aff}_{/S}$. We consider the $\infty$-categories

$$\begin{aligned} &\mathrm{Spc}(\mathcal{C}_{/S}) \quad \text{and } \mathbf{H}(\mathcal{C}_{/S}),\\ &\mathrm{Spc}(\mathcal{D}_{/S}) \quad \text{and } \mathbf{H}(\mathcal{D}_{/S}), \end{aligned}$$

as in Definition 2.1.14.

In this subsection we show that there are fully faithful embeddings

$$\mathrm{Spc}(\mathcal{C}_{/S}) \hookrightarrow \mathrm{Spc}(\mathcal{D}_{/S}), \quad \mathbf{H}(\mathcal{C}_{/S}) \hookrightarrow \mathbf{H}(\mathcal{D}_{/S}).$$

**Notation 2.2.2.** Let $\iota : \mathcal{C}_{/S} \hookrightarrow \mathcal{D}_{/S}$ denote the inclusion functor. Restriction along $\iota$ defines a functor $\iota^* : \mathrm{Spc}(\mathcal{D}_{/S}) \to \mathrm{Spc}(\mathcal{C}_{/S})$, whose left adjoint $\iota_! : \mathrm{Spc}(\mathcal{C}_{/S}) \to \mathrm{Spc}(\mathcal{D}_{/S})$ is given by left Kan extension of $\iota$. The latter is uniquely characterized by the property of commutativity with colimits, and the identity $\iota_!\mathrm{h}_S(X) \simeq \mathrm{h}_S(X)$ for $X \in \mathcal{C}_{/S}$. In particular, $\iota_!$ is fully faithful with essential image generated under colimits by objects of the form $\mathrm{h}_S(X)$, with $X \in \mathcal{C}_{/S}$. Similarly, $\iota^*$ also admits a fully faithful right adjoint $\iota_*$ given by right Kan extending $\iota$.

**Proposition 2.2.3.** *In the notation of 2.2.2, the assignment $\mathcal{F} \mapsto \mathbf{L}\iota_!(\mathcal{F})$ induces a fully faithful functor of $\infty$-categories*

$$\mathbf{L}\iota_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{D}_{/S}),$$

*whose essential image is generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_S(X)$, where $X$ belongs to $\mathcal{C}_{/S}$.*

**Example 2.2.4.** Suppose $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ is a *narrow* subcategory, and consider the inclusion $\iota : \mathcal{A}_{/S} \hookrightarrow \mathrm{Sm}_{/S}$. Then the fully faithful embedding

$$\mathbf{L}\iota_! : \mathbf{H}(\mathcal{A}_{/S}) \hookrightarrow \mathbf{H}(\mathrm{Sm}_{/S}) = \mathbf{H}(S)$$

is an *equivalence*. Indeed, by [Kha19b, Prop. 2.4.4], $\mathbf{H}(S)$ is generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_S(X)$, where $X$ belongs to the minimal admissible subcategory $\mathcal{A}^0_{/S}$ (Example 2.1.9), and hence also to $\mathcal{A}_{/S}$.

We will deduce Proposition 2.2.3 from the following lemma:

**Lemma 2.2.5.** *The functors $\iota_! : \mathrm{Spc}(\mathcal{C}_{/S}) \to \mathrm{Spc}(\mathcal{D}_{/S})$ and $\iota^* : \mathrm{Spc}(\mathcal{D}_{/S}) \to \mathrm{Spc}(\mathcal{C}_{/S})$ preserve Nisnevich-local and $\mathbf{A}^1$-local equivalences.*

*Proof.* Since $\iota$ preserves Nisnevich squares and $\mathbf{A}^1$-projections, it follows that $\iota_!$ preserves Nisnevich-local and $\mathbf{A}^1$-local equivalences. The definition of admissibility implies that $\iota$ is also cocontinuous with respect to the Nisnevich topology, so it follows that $\iota^*$ preserves Nisnevich-local equivalences (see [SGA 4, Exp. III, Prop. 2.2] or [Kha19b, Def. 3.1.5]).

For $\mathbf{A}^1$-local equivalences it will suffice to show that, for any $X \in \mathcal{D}_{/S}$, the canonical morphism

$$\iota^*\mathrm{h}_S(X \times \mathbf{A}^1) \to \iota^*\mathrm{h}_S(X)$$

is an $\mathbf{A}^1$-local equivalence of $\mathcal{C}$-fibred spaces. By universality of colimits it suffices to show that, for any $Y \in \mathcal{C}_{/S}$ and any morphism $\varphi : \mathrm{h}_S(Y) \to \iota^*\mathrm{h}_S(X)$ (corresponding to a morphism $Y \to X$ in $\mathcal{D}_{/S}$), the base change

$$\iota^*\mathrm{h}_S(X \times \mathbf{A}^1) \underset{\iota^*\mathrm{h}_S(X)}{\times} \mathrm{h}_S(Y) \to \mathrm{h}_S(Y)$$

is an $\mathbf{A}^1$-local equivalence. Since the morphism $\varphi$ factors as $\mathrm{h}_S(Y) \to \iota^*\mathrm{h}_S(Y) \to \iota^*\mathrm{h}_S(X)$, the morphism in question is a base change of the morphism

$$\iota^*\mathrm{h}_S(X \times \mathbf{A}^1) \underset{\iota^*\mathrm{h}_S(X)}{\times} \iota^*\mathrm{h}_S(Y) \to \iota^*\mathrm{h}_S(Y),$$

which itself is identified with the canonical morphism

$$\mathrm{h}_S(Y \times \mathbf{A}^1) \to \mathrm{h}_S(Y),$$

since $\iota^*$ and $\mathrm{h}_S$ commute with limits and $\iota^*\iota_! = \mathrm{id}$. This is an $\mathbf{A}^1$-local equivalence, so the claim follows. □

*Proof of Proposition 2.2.3.* Since $\iota_!$ preserves motivic equivalences (Lemma 2.2.5), its right adjoint $\iota^*$ preserves motivic spaces and induces a functor $\iota^* : \mathbf{H}(\mathcal{D}_{/S}) \to \mathbf{H}(\mathcal{C}_{/S})$, right adjoint to $\mathbf{L}\iota_!$. Similarly, Lemma 2.2.5 also implies that the right Kan extension functor $\iota_*$ preserves motivic spaces and defines a right adjoint to $\iota^* : \mathbf{H}(\mathcal{D}_{/S}) \to \mathbf{H}(\mathcal{C}_{/S})$. Now the fully faithfulness of $\mathbf{L}\iota_!$, which is equivalent to invertibility of the unit map $\iota^*\mathbf{L}\iota_! \to \mathrm{id}$, follows by passage to left adjoints from the fully faithfulness of $\iota_*$ (which is equivalent to invertibility of the counit map $\iota^*\iota_* \to \mathrm{id}$). The description of the essential image follows from [HTT, Lem. 5.5.8.14]. □

**Corollary 2.2.6.** *There is a canonical invertible natural transformation*

$$\mathrm{L}_{\mathbf{A}^1}\, \iota^* \to \iota^*\, \mathrm{L}_{\mathbf{A}^1}\, .$$

*Proof.* It follows from Lemma 2.2.5 that $\iota^*\,\mathrm{L}_{\mathbf{A}^1}$ takes $\mathbf{A}^1$-invariant values, so the natural transformation in question is induced by the canonical map $\mathrm{id} \to \mathrm{L}_{\mathbf{A}^1}$. The fact that it is invertible follows from the formula (2.1.b), which is valid for both $\mathcal{C}$- and $\mathcal{D}$-fibred spaces. □

2.3. **Functoriality.** We now record the various functorialities of $\mathcal{C}$-fibred motivic spaces as the base varies; this works exactly as in the case $\mathcal{C} = \mathrm{Sm}$ treated in [Kha19b, Subsect. 2.5]. We then discuss the compatibility of these operations, as well as products and internal homs (Remark 2.3.7), under the operation of extension along an inclusion of admissible subcategories (Proposition 2.2.3).

**Notation 2.3.1.** Let $f : T \to S$ be a morphism of affine spectral schemes. Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory, and choose also an admissible subcategory $\mathcal{C}_{/T} \subseteq \mathrm{Aff}_{/T}$ which contains the base changes $X \times_S T$ of every $X \in \mathcal{C}_{/S}$. A minimal such can be constructed as in Definition 2.1.11.

**Construction 2.3.2.** Under the notation of 2.3.1, the base change functor $\mathrm{Aff}_{/S} \to \mathrm{Aff}_{/T}$ restricts to $\mathcal{C}_{/S} \to \mathcal{C}_{/T}$.

(i) The direct image functor $f_*$ on $\mathcal{C}$-fibred spaces is given by restriction along the base change functor $\mathcal{C}_{/S} \to \mathcal{C}_{/T}$. The latter preserves Nisnevich covering families and $\mathbf{A}^1$-projections, so $f_*$ preserves motivic spaces. Its left adjoint $f^*$ on motivic spaces is characterized uniquely by commutativity with colimits and the formula $f^*_{\mathcal{D}}(\mathbf{L}\mathrm{h}_S(X)) \simeq \mathbf{L}\mathrm{h}_T(X \times_S T)$ for $X \in \mathcal{C}_{/S}$.

(ii) Suppose that the morphism $f : T \to S$ exhibits $T$ as an object of the full subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$. Then the base change functor $\mathcal{C}_{/S} \to \mathcal{C}_{/T}$ admits a left adjoint, the forgetful functor

$$(X \to T) \mapsto (X \to T \xrightarrow{f} S)$$

which preserves Nisnevich covering families and $\mathbf{A}^1$-projections. In this case $f^*$ is given by restriction along this forgetful functor, hence preserves motivic spaces and admits a left adjoint $f_\sharp$ characterized uniquely by commutativity with colimits and the formula $f_\sharp^{\mathcal{D}}(\mathbf{L}\mathrm{h}_T(X)) \simeq \mathbf{L}\mathrm{h}_S(X)$ for $X \in \mathcal{C}_{/T}$.

We now discuss the compatibility of these operations under the embedding of Proposition 2.2.3. For this we fix the following notation.

**Notation 2.3.3.** Let $f : T \to S$ be a morphism of affine spectral schemes. Fix an inclusion of admissible subcategories $\mathcal{C}_{/S} \subseteq \mathcal{D}_{/S}$ as in Notation 2.2.1. Fix similarly an inclusion $\mathcal{C}_{/T} \subseteq \mathcal{D}_{/T}$ of admissible subcategories both satisfying the condition of Notation 2.3.1. Write $\mathbf{H}(\mathcal{C}_{/S})$, $\mathbf{H}(\mathcal{D}_{/S})$, $\mathbf{H}(\mathcal{C}_{/T})$, and $\mathbf{H}(\mathcal{D}_{/T})$ for the $\infty$-categories of motivic fibred spaces formed with respect to these choices.

**Remark 2.3.4.** The condition of Notation 2.3.1 guarantees that the base change functor $X \mapsto X \times_S T$ commutes with the inclusions $\iota : \mathcal{C}_{/S} \to \mathcal{D}_{/S}$ and $\iota : \mathcal{C}_{/T} \to \mathcal{D}_{/T}$. From this it follows that the functor $f_*$ commutes with $\iota^*$ and that $f^*$ commutes with $\mathbf{L}\iota_!$. That is, we have commutative squares

$$\begin{array}{ccc} \mathbf{H}(\mathcal{C}_{/S}) & \xrightarrow{\mathbf{L}\iota_!} & \mathbf{H}(\mathcal{D}_{/S}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathbf{H}(\mathcal{C}_{/T}) & \xrightarrow{\mathbf{L}\iota_!} & \mathbf{H}(\mathcal{D}_{/T}), \end{array} \qquad \begin{array}{ccc} \mathbf{H}(\mathcal{D}_{/T}) & \xrightarrow{\iota^*} & \mathbf{H}(\mathcal{C}_{/T}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathbf{H}(\mathcal{D}_{/S}) & \xrightarrow{\iota^*} & \mathbf{H}(\mathcal{C}_{/S}). \end{array}$$

Similarly, if $f$ exhibits $T$ as an object of $\mathcal{C}_{/S}$, then $f_\sharp$ commutes with $\mathbf{L}\iota_!$ and $f^*$ commutes with $\iota^*$.

The following compatibility is less obvious:

**Proposition 2.3.5.** *With notation as in 2.3.3, assume that $\mathcal{C}_{/S}$ is* narrow. *Let $i : Z \hookrightarrow S$ be a closed immersion of affine spectral schemes with affine open complement. Then there is a canonical invertible natural transformation*

$$\mathbf{L}\iota_! \circ i_*^{\mathcal{C}} \to i_*^{\mathcal{D}} \circ \mathbf{L}\iota_! \tag{2.3.a}$$

*of functors $\mathbf{H}(\mathcal{C}_{/Z}) \to \mathbf{H}(\mathcal{D}_{/S})$, where the decorations indicate whether the functor is defined on $\mathcal{C}$-fibred or $\mathcal{D}$-fibred motivic spaces.*

The Sm-fibred localization theorem [Kha19b, Thm. 3.2.2] implies the same for $\mathcal{C}$-fibred motivic spaces (for any narrow $\mathcal{C}$). We will deduce Proposition 2.3.5 by combining this with the following $\mathcal{D}$-fibred variant:

**Theorem 2.3.6** (Localization)**.** *Let the notation be as in Proposition 2.3.5, and let $j : U \to S$ be the open immersion complementary to $i$. Let $\mathcal{F} \in \mathbf{H}(\mathcal{D}_{/S})$ be a $\mathcal{D}$-fibred motivic space over $S$. If $\mathcal{F}$ belongs to the essential image of the functor $\mathbf{L}\iota_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{D}_{/S})$ (Proposition 2.2.3), then there is a cocartesian square*

$$\begin{array}{ccc} j_\sharp j^*(\mathcal{F}) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ j_\sharp(\mathrm{pt}_U) & \longrightarrow & i_* i^*(\mathcal{F}) \end{array}$$

*of $\mathcal{D}$-fibred motivic spaces over $S$.*

*Proof.* From [Kha19b, Prop. 3.1.4] it follows that the functor $i_* : \mathbf{H}(\mathcal{D}_{/Z}) \to \mathbf{H}(\mathcal{D}_{/S})$ commutes with contractible[5] colimits (exactly as in the proof of Thm. 3.1.1 in *loc. cit*). By Proposition 2.2.3 we may assume that $\mathcal{F}$ is the motivic localization of some $X \in \mathcal{C}_{/S}$. Since $\mathcal{C}_{/S}$ is narrow, we can moreover assume by Example 2.2.4 that it belongs to $\mathcal{A}^0_{/S}$ (Example 2.1.9), i.e., that it admits an étale morphism to some $\mathbf{A}^n_S$, $n \geqslant 0$. Then we can proceed exactly as in [Kha19b, Subsect. 4.3]. □

*Proof of Proposition 2.3.5.* The natural transformation (2.3.a) is the composite

$$\mathbf{L}\iota_! \circ i^{\mathcal{C}}_* \xrightarrow{\text{unit}} i^{\mathcal{D}}_* i^*_{\mathcal{D}} \mathbf{L}\iota_! \circ i^{\mathcal{C}}_* \simeq i^{\mathcal{D}}_* \mathbf{L}\iota_! \circ i^*_{\mathcal{C}} i^{\mathcal{C}}_* \xrightarrow{\text{counit}} i^{\mathcal{D}}_* \circ \mathbf{L}\iota_!,$$

where the identification $\mathbf{L}\iota_! \circ i^*_{\mathcal{C}} \simeq i^*_{\mathcal{D}} \circ \mathbf{L}\iota_!$ comes from Remark 2.3.4. By [Kha19b, Prop. 2.4.4] it will suffice to show that the canonical morphism

$$\mathbf{L}\iota_! \circ i^{\mathcal{C}}_*(\mathbf{L}\mathrm{h}_Z(X)) \to i^{\mathcal{D}}_*(\mathbf{L}\mathrm{h}_Z(X))$$

is invertible for all $X \in \mathcal{C}_{/Z}$. Using [Kha19b, Prop. 3.1.4], we may assume that $X$ is of the form $Y \times_S Z$ for some $Y \in \mathcal{C}_{/S}$. Then we conclude by comparing the description of $i^{\mathcal{C}}_* i^*_{\mathcal{C}}(\mathbf{L}\mathrm{h}_S(Y))$ given by [Kha19b, Thm. 3.2.2], and the description of $i^{\mathcal{D}}_* i^*_{\mathcal{D}}(\mathbf{L}\mathrm{h}_S(Y))$ provided by Theorem 2.3.6. □

Finally, we discuss the compatibility of the functors $\mathbf{L}\iota_!$ and $\iota^*$ with products and internal homs. Note that, just as in the $\mathcal{C}$-fibred case [Kha19b, Rem. 2.4.2], the full subcategory $\mathbf{H}(\mathcal{D}_{/S}) \subseteq \mathrm{Spc}(\mathcal{D}_{/S})$ is closed under formation of internal homs.

**Remark 2.3.7.**

(i) Since the functor $\iota^* : \mathbf{H}(\mathcal{D}_{/S}) \to \mathbf{H}(\mathcal{C}_{/S})$ preserves limits (see proof of Proposition 2.2.3), it is symmetric monoidal with respect to the cartesian product.

(ii) The functor $\mathbf{L}\iota_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{D}_{/S})$ is also symmetric monoidal. Indeed, since $\mathbf{L}$ preserves finite products by Remark 2.1.16(iii), it suffices to show the claim for $\iota_! : \mathrm{Spc}(\mathcal{C}_{/S}) \to \mathrm{Spc}(\mathcal{D}_{/S})$. For this we may reduce to representables which is obvious.

(iii) For any $\mathcal{C}$-fibred motivic space $\mathcal{F} \in \mathbf{H}(\mathcal{D}_{/S})$ and $\mathcal{D}$-fibred motivic space $\mathcal{G} \in \mathbf{H}(\mathcal{D}_{/S})$, there is a canonical isomorphism

$$\iota^* \underline{\mathrm{Hom}}(\mathbf{L}\iota_!(\mathcal{F}), \mathcal{G}) \to \underline{\mathrm{Hom}}(\mathcal{F}, \iota^*(\mathcal{G}))$$

of $\mathcal{C}$-fibred motivic spaces, where $\underline{\mathrm{Hom}}$ is taken in $\mathbf{H}(\mathcal{D}_{/S})$ on the left and in $\mathbf{H}(\mathcal{C}_{/S})$ on the right. This follows by adjunction from (ii).

2.4. **Classical fibred spaces.** In this subsection we set up, for a classical affine scheme $S$, a classical variant of the $\infty$-category of $\mathcal{C}$-fibred motivic spaces. For $S$ a spectral affine scheme we then define a pair of adjoint functors

$$\mathbf{L}\upsilon_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad \upsilon^* : \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{C}_{/S}).$$

Later we will focus on understanding these adjunctions when $\mathcal{C}_{/S}$ is broad (Subsect. 2.5) and narrow (Subsect. 2.7).

The following is a classical analogue of Definitions 2.1.7 and 2.1.10.

**Definition 2.4.1.** Let $S$ be an affine scheme. Denote by $\mathrm{AffCl}_{/S}$ the category of classical affine schemes over $S$. We say that a full subcategory $\mathcal{C}_{/S} \subseteq \mathrm{AffCl}_{/S}$ is *admissible* if it is essentially small and satisfies the following conditions:

(i) The affine scheme $S$ (viewed over $S$ via the identity) belongs to $\mathcal{C}_{/S}$.

(ii) If $X$ belongs to $\mathcal{C}_{/S}$ and $Y$ is étale over $X$, then $Y$ belongs to $\mathcal{C}_{/S}$.

(iii) If $X$ belongs to $\mathcal{C}_{/S}$, then $X \times \mathbf{A}^n_{\mathrm{cl}}$ belongs to $\mathcal{C}_{/S}$ for every $n \geqslant 0$.

For example, the full subcategory $\mathrm{SmCl}_{/S} \subseteq \mathrm{AffCl}_{/S}$ of *smooth* affine schemes over $S$ (where smoothness is understood in the sense of classical algebraic geometry) is admissible. We say that an admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{AffCl}_{/S}$ is *narrow* if it is contained in $\mathrm{SmCl}_{/S}$.

**Example 2.4.2.** Let $S$ be a *spectral* affine scheme and $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ an admissible subcategory. Then the full subcategory $\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ spanned by the classical truncations $X_{\mathrm{cl}}$ of all objects $X \in \mathcal{C}_{/S}$ is admissible. The first condition follows from Definition 2.1.7(i), the second follows from Definition 2.1.7(ii) and [HA, Thm. 7.5.0.6], and the third follows from Definition 2.1.7(iii) and the fact that $\mathbf{A}^n_{\mathrm{cl}} \simeq (\mathbf{A}^n)_{\mathrm{cl}}$.

**Definition 2.4.3.** Let $S$ be an affine scheme and let $\mathcal{C}^{\mathrm{cl}}_{/S} \subseteq \mathrm{AffCl}_{/S}$ be an admissible subcategory. A $\mathcal{C}^{\mathrm{cl}}$-*fibred space* over $S$ is a presheaf of spaces on $\mathcal{C}^{\mathrm{cl}}_{/S}$. We say that a $\mathcal{C}^{\mathrm{cl}}$-fibred space $\mathcal{F}$ over $S$ satisfies *Nisnevich excision* if it is reduced, and for any $X \in \mathcal{C}^{\mathrm{cl}}_{/S}$ and any Nisnevich square $Q$ over $X$, the induced square of spaces $\Gamma(Q, \mathcal{F})$ is cartesian. We say that $\mathcal{F}$ satisfies $\mathbf{A}^1_{\mathrm{cl}}$-*homotopy invariance* if for any $X \in \mathcal{C}^{\mathrm{cl}}_{/S}$, the canonical map $\Gamma(X, \mathcal{F}) \to \Gamma(X \times \mathbf{A}^1_{\mathrm{cl}}, \mathcal{F})$ is invertible, where $\mathbf{A}^1_{\mathrm{cl}} = \mathrm{Spec}(\mathbf{Z}[T]) \simeq (\mathbf{A}^1)_{\mathrm{cl}}$ denotes the classical affine line. A *motivic* $\mathcal{C}^{\mathrm{cl}}$-*fibred space* is a $\mathcal{C}^{\mathrm{cl}}$-fibred space that satisfies Nisnevich excision and $\mathbf{A}^1_{\mathrm{cl}}$-homotopy invariance.

**Example 2.4.4.** Let $S = \mathrm{Spec}(R)$ be a spectral affine scheme. Then the $\infty$-category of motivic SmCl-fibred spaces $\mathbf{H}(\mathrm{SmCl}_{/S_{\mathrm{cl}}})$ is equivalent to $\mathbf{H}^{\mathrm{cl}}(\pi_0(R))$ as defined in Theorem A.

For the remainder of this subsection, we fix the following notation:

**Notation 2.4.5.** Let $S$ be a spectral affine scheme. Fix an admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ (Definition 2.1.7). Let $\mathcal{C}^{\mathrm{cl}}_{/S}$ be the induced admissible subcategory of $\mathrm{AffCl}_{/S}$ as in Example 2.4.2. We denote by

$$\mathrm{Spc}(\mathcal{C}_{/S}), \quad \text{resp. } \mathrm{Spc}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}),$$

the $\infty$-category of $\mathcal{C}$-fibred spaces over $S$, resp. of $\mathcal{C}^{\mathrm{cl}}$-fibred spaces over $S_{\mathrm{cl}}$, and by

$$\mathbf{H}(\mathcal{C}_{/S}), \quad \text{resp. } \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}),$$

the full subcategory of motivic objects.

**Construction 2.4.6.** The operation of passing to classical truncations,

$$(X \to S) \mapsto (X_{\mathrm{cl}} \to S_{\mathrm{cl}}),$$

defines a canonical functor $v : \mathrm{Aff}_{/S} \to \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ which restricts to $v : \mathcal{C}_{/S} \to \mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$. Denote by $v^* : \mathrm{Spc}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathrm{Spc}(\mathcal{C}_{/S})$ the functor of restriction along $v$, and by $v_! : \mathrm{Spc}(\mathcal{C}_{/S}) \to \mathrm{Spc}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$ its left adjoint given by left Kan extension of $v$. Recall that $v_!$ is uniquely characterized by commutativity with colimits and the formula $v_!(\mathrm{h}_S(X)) \simeq \mathrm{h}_{S_{\mathrm{cl}}}(X_{\mathrm{cl}})$ for all $X \in \mathcal{C}_{/S}$.

**Lemma 2.4.7.**

(i) *The functor $v_!$ preserves Nisnevich-local equivalences, $\mathbf{A}^1$-local equivalences, and motivic equivalences.*

(ii) *The functor $v^*$ preserves Nisnevich excisive spaces and Nisnevich-local equivalences. In particular, it commutes with $\mathrm{L}_{\mathrm{Nis}}$; that is, there is a canonical invertible natural transformation $\mathrm{L}_{\mathrm{Nis}}\, v^* \to v^*\, \mathrm{L}_{\mathrm{Nis}}$.*

(iii) *The functor $v^*$ sends $\mathbf{A}^1_{\mathrm{cl}}$-invariant $\mathcal{C}^{\mathrm{cl}}$-fibred spaces to $\mathbf{A}^1$-invariant $\mathcal{C}$-fibred spaces. In particular, it sends motivic $\mathcal{C}^{\mathrm{cl}}$-fibred spaces to motivic $\mathcal{C}$-fibred spaces.*

In particular, we find that the functors $v_!$ and $v^*$ descend to a pair of adjoint functors

$$\mathbf{L}v_! : \mathbf{H}(\mathcal{C}_{/S}) \to \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad v^* : \mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{C}_{/S}). \tag{2.4.a}$$

*Proof.* The first claim follows from the fact that $v$ preserves Nisnevich squares and sends $\mathbf{A}^1$ to $\mathbf{A}^1_{\mathrm{cl}}$. By adjunction it follows that $v^*$ preserves Nisnevich-excisive spaces and sends $\mathbf{A}^1_{\mathrm{cl}}$-invariant spaces to $\mathbf{A}^1$-invariant spaces. It remains to show that $v^*$ preserves Nisnevich-local equivalences. For this it is sufficient to check that the functor $v$ is cocontinuous for the Nisnevich topology, i.e., that for all $X \in \mathcal{C}_{/S}$, any Nisnevich covering of $X_{\mathrm{cl}}$ lifts to a Nisnevich covering of $X$ (cf. [Kha19b, Def. 3.1.5]). This follows from [HA, Thm. 7.5.0.6]. □

2.5. **Nil-localization.** In this subsection we study the adjunction (2.4.a) when the admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ is *broad*. In this case, we find that the classical construction $\mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$ is a left Bousfield localization of the spectral variant $\mathbf{H}(\mathcal{C}_{/S})$ (Theorem 2.5.3).

**Notation 2.5.1.** Let $S$ be a spectral affine scheme. Fix a broad subcategory $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$ (Definition 2.1.11), and let $\mathcal{B}^{\mathrm{cl}}_{/S}$ be the induced admissible subcategory of $\mathrm{AffCl}_{/S}$ as in Example 2.4.2. Consider the $\infty$-categories

$$\mathbf{H}(\mathcal{B}_{/S}) \subseteq \mathrm{Spc}(\mathcal{B}_{/S}), \quad \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \subseteq \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}).$$

**Definition 2.5.2.** A $\mathcal{B}$-fibred space $\mathcal{F} \in \mathrm{Spc}(\mathcal{B}_{/S})$ is called *nil-local* if for any $X \in \mathcal{B}_{/S}$, the canonical map of spaces

$$\Gamma(X, \mathcal{F}) \to \Gamma(X_{\mathrm{cl}}, \mathcal{F})$$

is invertible.

**Theorem 2.5.3.** *The functor $v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{B}_{/S})$ is fully faithful, and induces an equivalence*

$$v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}_{\mathrm{nil}}(\mathcal{B}_{/S})$$

*from the $\infty$-category of $\mathcal{B}^{\mathrm{cl}}$-fibred motivic spaces over $S_{\mathrm{cl}}$ to the $\infty$-category $\mathbf{H}_{\mathrm{nil}}(\mathcal{B}_{/S}) \subseteq \mathbf{H}(\mathcal{B}_{/S})$ of nil-local $\mathcal{B}$-fibred motivic spaces over $S$. In particular, the functor $\mathbf{L}v_! : \mathbf{H}(\mathcal{B}_{/S}) \to \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$ is a left Bousfield localization.*

The proof of Theorem 2.5.3 relies on an analysis of the behaviour of the functors

$$v_! : \mathrm{Spc}(\mathcal{B}_{/S}) \to \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad v^* : \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathrm{Spc}(\mathcal{B}_{/S})$$

with respect to $\mathbf{A}^1$-local and Nisnevich-local equivalences, specializing Lemma 2.4.7 to the broad case:

**Proposition 2.5.4.**

(i) *The functor $v_!$ preserves Nisnevich excisive spaces and Nisnevich-local equivalences. In particular, it commutes with $\mathrm{L}_{\mathrm{Nis}}$; that is, there is a canonical invertible natural transformation $\mathrm{L}_{\mathrm{Nis}}\, v_! \to v_!\, \mathrm{L}_{\mathrm{Nis}}$.*

(ii) *The functor $v^*$ preserves Nisnevich excisive spaces and Nisnevich-local equivalences. In particular, it commutes with $\mathrm{L}_{\mathrm{Nis}}$; that is, there is a canonical invertible natural transformation $\mathrm{L}_{\mathrm{Nis}}\, v^* \to v^*\, \mathrm{L}_{\mathrm{Nis}}$.*

(iii) *The functor $v_!$ sends $\mathbf{A}^1$-local equivalences to $\mathbf{A}^1_{\mathrm{cl}}$-local equivalences.*

(iv) *The canonical natural transformations*

$$\mathrm{L}_{\mathbf{A}^1}\, v^* \to v^* \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} \quad \textit{and} \quad v_!\, \mathrm{L}_{\mathbf{A}^1}\, v^* \to \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}$$

*are invertible.*

(v) *The functor $v_!$ preserves motivic equivalences.*

(vi) *The canonical natural transformations*

$$\mathbf{L}v^* \to v^*\mathbf{L} \quad \textit{and} \quad v_!\mathbf{L}v^* \to \mathbf{L}$$

*are invertible.*

The key feature of the broad case is the existence of a left adjoint $u$ to the functor $v : \mathcal{B}_{/S} \to \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$:

**Remark 2.5.5.** Any affine scheme $X_0$ over $S_{\mathrm{cl}}$ can be viewed as a discrete affine spectral scheme over $S$ (by composition with the canonical morphism $S_{\mathrm{cl}} \to S$). This defines a functor $u : \mathrm{AffCl}_{/S_{\mathrm{cl}}} \to \mathrm{Aff}_{/S}$, left adjoint to $v$. Note that $u$ is fully faithful (as the unit map $X_0 \to vu(X_0)$ is always invertible). Since $\mathcal{B}$ is *broad*, our choice of $\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ (Example 2.4.2) guarantees that $u$ restricts to a functor $u : \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \to \mathcal{B}_{/S}$. Restriction along $u$ defines a functor $u^* : \mathrm{Spc}(\mathcal{B}_{/S}) \to \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$, which admits fully faithful left and right adjoints $u_!$ and $u_*$, respectively. By adjunction, we have identifications $v_! \simeq u^*$ and $v^* \simeq u_*$. In particular, it follows that the co-unit

$$v_! v^* \to \mathrm{id}$$

is invertible.

**Remark 2.5.6.** Note that a $\mathcal{B}$-fibred space $\mathcal{F} \in \mathrm{Spc}(\mathcal{B}_{/S})$ is nil-local if and only if it belongs to the essential image of $v^*$, or equivalently if and only if the unit map $\mathcal{F} \to v^* v_!(\mathcal{F}) \simeq v^* u^*(\mathcal{F})$ is invertible. Indeed the counit map $uv(X) \to X$ is the inclusion of the classical truncation, so the map $\Gamma(X, \mathcal{F}) \to \Gamma(X_{\mathrm{cl}}, \mathcal{F})$ is canonically identified with the map

$$\Gamma(X, \mathcal{F}) \to \Gamma(X, v^* u^*(\mathcal{F}))$$

for every $X \in \mathcal{B}_{/S}$.

*Proof of Proposition 2.5.4.* We already know from Lemma 2.4.7 that $v_!$ sends Nisnevich-local equivalences to Nisnevich-local equivalences, $\mathbf{A}^1$-local equivalences to $\mathbf{A}^1_{\mathrm{cl}}$-local equivalences, and motivic equivalences to motivic equivalences. We also know that its right adjoint $v^*$ sends Nisnevich excisive spaces to Nisnevich excisive spaces, $\mathbf{A}^1_{\mathrm{cl}}$-invariant spaces to $\mathbf{A}^1$-invariant spaces, and motivic spaces to motivic spaces.

Let $u : \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \to \mathcal{B}_{/S}$ be as in Remark 2.5.5, so that $v_! \simeq u^*$. Since $u$ preserves Nisnevich squares, the functor $u_!$ preserves Nisnevich-local equivalences. Hence its right adjoint $u^* \simeq v_!$ preserves Nisnevich excisive spaces. This proves claim (i).

Consider claim (iv). Since $v^*$ preserves $\mathbf{A}^1$-invariant spaces, the natural transformation $\mathrm{id} \to \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}$ induces a transformation

$$\mathrm{L}_{\mathbf{A}^1} v^* v_! \to \mathrm{L}_{\mathbf{A}^1} v^* \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} v_! \simeq v^* \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} v_! \tag{2.5.a}$$

which we claim is invertible. For every $X \in \mathcal{B}_{/S}$, let $\mathcal{I}_X \subseteq \mathrm{Aff}_{/X}$ denote the non-full subcategory whose objects are spectral affine spaces $\mathbf{A}^n \times X$ over $X$, and whose morphisms are projections. A variant of the formula (2.1.b) (see [Hoy17, Prop. 3.4]) then yields the functorial isomorphisms

$$\begin{aligned}\Gamma\big(X, \mathrm{L}_{\mathbf{A}^1} v^* u^*(\mathcal{F})\big) &\simeq \varinjlim_{\mathcal{I}_X^{\mathrm{op}}} \Gamma\big(\mathbf{A}^n \times X, v^* u^*(\mathcal{F})\big) \\ &\simeq \varinjlim_{\mathcal{I}_X^{\mathrm{op}}} \Gamma\big(\mathbf{A}^n_{\mathrm{cl}} \times X_{\mathrm{cl}}, \mathcal{F}\big),\end{aligned}$$

by Remark 2.5.6. Similarly, if we write $\mathcal{I}^{\mathrm{cl}}_{X_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/X_{\mathrm{cl}}}$ for the subcategory of classical affine spaces $\mathbf{A}^n_{\mathrm{cl}} \times X_{\mathrm{cl}}$ and projections between them, then [Hoy17, Prop. 3.4] again yields

$$\begin{aligned}\Gamma\big(X, v^*\mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} u^*(\mathcal{F})\big) &\simeq \Gamma\big(X_{\mathrm{cl}}, \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}} u^*(\mathcal{F})\big)\\ &\simeq \varinjlim_{(\mathcal{I}^{\mathrm{cl}}_{X_{\mathrm{cl}}})^{\mathrm{op}}} \Gamma\big(\mathbf{A}^n_{\mathrm{cl}} \times X_{\mathrm{cl}}, u^*(\mathcal{F})\big)\\ &\simeq \varinjlim_{(\mathcal{I}^{\mathrm{cl}}_{X_{\mathrm{cl}}})^{\mathrm{op}}} \Gamma\big(\mathbf{A}^n_{\mathrm{cl}} \times X_{\mathrm{cl}}, \mathcal{F}\big).\end{aligned}$$

Since $v : \mathcal{B}_{/X} \to \mathcal{B}^{\mathrm{cl}}_{/X_{\mathrm{cl}}}$ induces an equivalence $\mathcal{I}_X \simeq \mathcal{I}^{\mathrm{cl}}_{X_{\mathrm{cl}}}$, it follows that (2.5.a) is invertible. Applying $v^*$ on the right, we deduce that the canonical transformation $\mathrm{L}_{\mathbf{A}^1}\, v^* \to v^*\mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}$ is also invertible (since $v_! v^* \simeq \mathrm{id}$ by Remark 2.5.5). Applying $v_!$ on the left, we also obtain the invertible transformation $v_!\, \mathrm{L}_{\mathbf{A}^1}\, v^* \to \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}$.

Claim (vi) follows from claims (ii) and (iv) in view of the formula (2.1.c) (and the analogous formula for $\mathcal{B}^{\mathrm{cl}}$-fibred spaces). □

*Proof of Theorem 2.5.3.* By Remark 2.5.5 we know that the functor

$$v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{B}_{/S})$$

is fully faithful. Its essential image is spanned by objects $\mathcal{F} \in \mathbf{H}(\mathcal{B}_{/S})$ for which the unit map $\mathcal{F} \to v^*\mathbf{L}v_!(\mathcal{F})$ is invertible. By Remark 2.5.6, this condition implies that $\mathcal{F}$ is nil-local. Conversely if $\mathcal{F}$ is nil-local, so that the unit map $\mathcal{F} \to v^*v_!(\mathcal{F})$ is invertible (again by Remark 2.5.6), then using Proposition 2.5.4(vi) we see that the induced map $\mathcal{F} \simeq \mathbf{L}(\mathcal{F}) \to \mathbf{L}v^*v_!(\mathcal{F}) \simeq v^*\mathbf{L}v_!(\mathcal{F})$ is also invertible. □

## 2.6. **Nil descent.**

**Notation 2.6.1.** Let $S$ be a spectral affine scheme. Fix a narrow subcategory $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ and a broad subcategory $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$ containing $\mathcal{A}_{/S}$. We consider the $\infty$-categories

$$\mathbf{H}(\mathcal{A}_{/S}) \subseteq \mathrm{Spc}(\mathcal{A}_{/S}), \quad \mathbf{H}(\mathcal{B}_{/S}) \subseteq \mathrm{Spc}(\mathcal{B}_{/S}).$$

Let $\iota : \mathcal{A}_{/S} \to \mathcal{B}_{/S}$ denote the inclusion, so that we have the fully faithful functor (see Proposition 2.2.3)

$$\mathbf{L}\iota_! : \mathbf{H}(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{B}_{/S}).$$

We are now in a position to state and prove the following result:

**Theorem 2.6.2.** *Let $\mathcal{F} \in \mathbf{H}(\mathcal{A}_{/S})$ be an $\mathcal{A}$-fibred motivic space. Then the $\mathcal{B}$-fibred motivic space $\mathbf{L}\iota_!(\mathcal{F}) \in \mathbf{H}(\mathcal{B}_{/S})$ is nil-local (Definition 2.5.2).*

*Proof.* Set $\mathcal{F}^+ = \mathbf{L}\iota_!(\mathcal{F})$. Let $X \in \mathcal{B}_{/S}$ with structural morphism $f : X \to S$ and choose subcategories $\mathcal{A}_{/X} \subseteq \mathcal{B}_{/X}$ of $\mathrm{Aff}_{/X}$, narrow and broad, respectively, and both satisfying the condition of Notation 2.3.1. By adjunction, there are canonical isomorphisms

$$\begin{aligned}\Gamma(X, \mathcal{F}^+) &\simeq \mathrm{Maps}(\mathrm{pt}_X, f_{\mathcal{B}}^*(\mathcal{F}^+)),\\ \Gamma(X_{\mathrm{cl}}, \mathcal{F}^+) &\simeq \mathrm{Maps}(\mathrm{h}_X(X_{\mathrm{cl}}), f_{\mathcal{B}}^*(\mathcal{F}^+))\\ &\simeq \mathrm{Maps}(i_\sharp^{\mathcal{B}} i_{\mathcal{B}}^*(\mathrm{pt}_X), f_{\mathcal{B}}^*(\mathcal{F}^+))\\ &\simeq \mathrm{Maps}(\mathrm{pt}_X, i_*^{\mathcal{B}} i_{\mathcal{B}}^* f_{\mathcal{B}}^*(\mathcal{F}^+)),\end{aligned}$$

where all the mapping spaces are formed in $\mathbf{H}(\mathcal{B}_{/X})$. Under these identifications the map $\Gamma(X, \mathcal{F}^+) \to \Gamma(X_{\mathrm{cl}}, \mathcal{F}^+)$ is induced by the unit morphism

$$f_{\mathcal{B}}^*(\mathcal{F}^+) \to i_*^{\mathcal{B}} i_{\mathcal{B}}^* f_{\mathcal{B}}^*(\mathcal{F}^+)$$

in $\mathbf{H}(\mathcal{B}_{/X})$. By Remark 2.3.4 and Proposition 2.3.5 this morphism is the image by $\mathbf{L}\iota_!$ of the unit morphism

$$f_{\mathcal{A}}^*(\mathcal{F}) \to i_*^{\mathcal{A}} i_{\mathcal{A}}^* f_{\mathcal{A}}^*(\mathcal{F})$$

in $\mathbf{H}(\mathcal{A}_{/X})$. Since $i$ is a closed immersion with empty complement, this morphism is invertible by the nilpotent invariance property of $\mathbf{H}(\mathcal{A}_{/X})$, see [Kha19b, Cor. 3.2.7] (which applies to any narrow subcategory and not just $\mathrm{Sm}_{/X}$). □

## 2.7. The comparison.

**Notation 2.7.1.** Let $S$ be a spectral affine scheme. We again fix narrow and broad subcategories $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ and $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S_{\mathrm{cl}}}$ as in Notation 2.6.1. We let $\mathcal{A}_{/S}^{\mathrm{cl}}$ and $\mathcal{B}_{/S}^{\mathrm{cl}}$ be the induced admissible subcategories of $\mathrm{AffCl}_{/S}$ as in Example 2.4.2. To simplify notation set

$$\underline{\mathrm{Spc}}(S) \coloneqq \mathrm{Spc}(\mathcal{B}_{/S}), \quad \underline{\mathrm{Spc}}^{\mathrm{cl}}(S_{\mathrm{cl}}) \coloneqq \mathrm{Spc}(\mathcal{B}_{/S_{\mathrm{cl}}}^{\mathrm{cl}}), \quad \underline{\mathbf{H}}(S) \coloneqq \mathbf{H}(\mathcal{B}_{/S}), \quad \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) \coloneqq \mathbf{H}(\mathcal{B}_{/S_{\mathrm{cl}}}^{\mathrm{cl}}),$$

and similarly

$$\mathrm{Spc}(S) \coloneqq \mathrm{Spc}(\mathcal{A}_{/S}), \quad \mathrm{Spc}^{\mathrm{cl}}(S_{\mathrm{cl}}) \coloneqq \mathrm{Spc}(\mathcal{A}_{/S_{\mathrm{cl}}}^{\mathrm{cl}}), \quad \mathbf{H}(S) \coloneqq \mathbf{H}(\mathcal{A}_{/S}), \quad \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \coloneqq \mathbf{H}(\mathcal{A}_{/S_{\mathrm{cl}}}^{\mathrm{cl}}).$$

Recall from Example 2.2.4 that this notation agrees with that of Example 2.1.15, even though $\mathcal{A}_{/S}$ is allowed to be any narrow subcategory.

Let $v : \mathcal{B}_{/S} \to \mathcal{B}_{/S_{\mathrm{cl}}}$ and $w : \mathcal{A}_{/S} \to \mathcal{A}_{/S_{\mathrm{cl}}}$ be the classical truncation functors as in Construction 2.4.6. In this subsection we will prove the following result, which gives the equivalence between (i) and (iii) in Theorem A.

**Theorem 2.7.2.** *The adjunction of (2.4.a),*

$$\mathbf{L}w_! : \mathbf{H}(S) \to \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}), \quad w^* : \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \to \mathbf{H}(S),$$

*is an equivalence of ∞-categories.*

The proof will combine the $\mathcal{B}$-fibred nil-localization statement (Theorem 2.5.3) and nil descent for $\mathcal{A}$-fibred spaces (Theorem 2.6.2), as well as Proposition 2.2.3 and the following classical analogue of the latter:

**Proposition 2.7.3.** *Consider the inclusion functor* $\iota : \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \hookrightarrow \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$. *Let* $\iota_! : \mathrm{Spc}^{\mathrm{cl}}(S_{\mathrm{cl}}) \to \underline{\mathrm{Spc}}^{\mathrm{cl}}(S_{\mathrm{cl}})$ *denote the left Kan extension of* $\iota$, *left adjoint to the restriction functor* $\iota^* : \underline{\mathrm{Spc}}^{\mathrm{cl}}(S) \to \mathrm{Spc}^{\mathrm{cl}}(S)$. *Then the assignment* $\mathcal{F} \mapsto \mathbf{L}\iota_!(\mathcal{F})$ *induces a fully faithful functor of* $\infty$*-categories*

$$\mathbf{L}\iota_! : \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \to \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}),$$

*whose essential image is generated under sifted colimits by objects of the form* $\mathbf{L}\mathrm{h}_{S_{\mathrm{cl}}}(X)$, *where* $X \in \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ *admits an étale* $S_{\mathrm{cl}}$*-morphism to a classical affine space* $S_{\mathrm{cl}} \times \mathbf{A}^n_{\mathrm{cl}}$, *for some* $n \geqslant 0$.

*Proof.* Same proof as Proposition 2.2.3. □

**Remark 2.7.4.** Since $w : \mathcal{A}_{/S} \to \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ is the restriction of $v : \mathcal{B}_{/S} \to \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$, we have commutative squares

$$\begin{array}{ccc} \mathbf{H}(S) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \\ \downarrow{\scriptstyle \mathbf{L}\iota_!} & & \downarrow{\scriptstyle \mathbf{L}\iota_!} \\ \underline{\mathbf{H}}(S) & \xrightarrow{\mathbf{L}v_!} & \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}), \end{array} \qquad \begin{array}{ccc} \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{v^*} & \underline{\mathbf{H}}(S) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(S). \end{array} \tag{2.7.a}$$

*Proof of Theorem 2.7.2.* We show that the adjunction $(\mathbf{L}w_!, w^*)$ is an equivalence. For any $\mathcal{A}$-fibred motivic space $\mathcal{F} \in \mathbf{H}(S)$, the $\mathcal{B}$-fibred motivic space $\mathbf{L}\iota_!(\mathcal{F})$ is nil-local by Theorem 2.6.2. Therefore by Theorem 2.5.3 the canonical map

$$\mathbf{L}\iota_!(\mathcal{F}) \to v^*\mathbf{L}v_!\mathbf{L}\iota_!(\mathcal{F})$$

is invertible. Applying $\iota^*$ and using Proposition 2.2.3, we deduce that the canonical map

$$\mathcal{F} \to \iota^*v^*\mathbf{L}v_!\mathbf{L}\iota_!(\mathcal{F})$$

is invertible. This map is identified with the unit $\mathcal{F} \to w^*\mathbf{L}w_!(\mathcal{F})$ under the identifications (Remark 2.7.4 and Proposition 2.7.3)

$$\iota^*v^*\mathbf{L}v_!\mathbf{L}\iota_! \simeq w^*\iota^*\mathbf{L}\iota_!\mathbf{L}w_! \simeq w^*\mathbf{L}w_!.$$

Since $\mathcal{F} \in \mathbf{H}(S)$ was arbitrary, this shows that the unit

$$\mathrm{id} \to w^*\mathbf{L}w_!$$

is invertible, hence $\mathbf{L}w_!$ is fully faithful. It remains to show that $\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})$ is generated under colimits by objects of the form $\mathbf{L}\mathrm{h}_{S_{\mathrm{cl}}}(X_{\mathrm{cl}})$, where $X \in \mathcal{A}_{/S}$. But this follows from the definition of $\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ (Example 2.4.2). □

The next few results are corollaries of Theorems 2.5.3 and 2.7.2.

**Corollary 2.7.5.** *For any quasi-compact quasi-separated spectral algebraic space* $S$, *there are canonical equivalences of* $\infty$*-categories*

$$\mathbf{H}(S) \simeq \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}),$$

*where* $\mathbf{H}(S)$ *is as in* [Kha19b, Def. 2.4.1] *and* $\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})$ *its classical variant.*

*Proof.* Classical truncation defines a functor $w$ from the $\infty$-category of smooth spectral algebraic spaces over $S$ to the category of smooth classical algebraic spaces over $S_{\mathrm{cl}}$. This induces an adjunction

$$\mathbf{L}w_! : \mathbf{H}(S) \to \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}), \quad w^* : \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \to \mathbf{H}(S)$$

which globalizes that of Theorem 2.7.2. To show that the unit and counit maps are invertible, we may use Nisnevich descent [Kha19b, Prop. 2.5.7] to reduce to the affine case proven in Theorem 2.7.2. □

**Corollary 2.7.6.** *Let $S$ be an affine spectral scheme. Then for any $\mathcal{A}$-fibred space $\mathcal{F} \in \mathrm{Spc}(S)$, the canonical map in $\mathbf{H}(S)$*

$$\mathbf{L}(\mathcal{F}) \to w^*\mathbf{L}w_!(\mathcal{F})$$

*is invertible.*

*Proof.* By Theorem A the canonical map $\mathbf{L}(\mathcal{F}) \xrightarrow{\sim} w^*\mathbf{L}w_!(\mathbf{L}(\mathcal{F}))$ is invertible. By Proposition 2.5.4 (v) the canonical map $w_!(\mathcal{F}) \to w_!(\mathbf{L}(\mathcal{F}))$ is a motivic equivalence, whence the claim. □

**Corollary 2.7.7.** *Let $S$ be an affine spectral scheme. Then there is a commutative square*

$$\begin{array}{ccc} \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(S) \\ \downarrow{\scriptstyle \mathbf{L}\iota_!} & & \downarrow{\scriptstyle \mathbf{L}\iota_!} \\ \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{v^*} & \underline{\mathbf{H}}(S). \end{array}$$

*Proof.* The square is obtained by horizontally passing to right adjoints in the left-hand square in (2.7.a), and thus commutes up to a natural transformation which we claim is invertible. Note that both clockwise and counterclockwise composites factor through the full subcategory $\underline{\mathbf{H}}_{\mathrm{nil}}(S)$ of nil-local objects by Theorem 2.6.2 and Remark 2.5.6. By Theorem 2.5.3 it will therefore suffice to show that the natural transformation becomes invertible after post-composition with $\mathbf{L}v_! : \underline{\mathbf{H}}(S) \to \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}})$. This is immediate from the fact that $v^*$ is fully faithful (Theorem 2.5.3) and the commutativity of the right-hand square in (2.7.a). □

**Corollary 2.7.8.** *Let $S$ be an affine spectral scheme. Then there is a commutative square*

$$\begin{array}{ccc} \underline{\mathbf{H}}_{\mathrm{nil}}(S) & \xrightarrow{\mathbf{L}v_!} & \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ \mathbf{H}(S) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}). \end{array}$$

*Proof.* The square is obtained by horizontally passing to left adjoints in the right-hand square in (2.7.a) (and restricting to the full subcategory $\underline{\mathbf{H}}_{\mathrm{nil}}(S) \subseteq \underline{\mathbf{H}}(S)$), and thus commutes up to a natural transformation which we claim is invertible. By Theorem 2.5.3 it will suffice to show this after pre-composition with $v^* : \underline{\mathbf{H}}^{\mathrm{cl}}(S_{\mathrm{cl}}) \to \underline{\mathbf{H}}(S)$. This is immediate from the fact that $v^*$ is fully faithful (Theorem 2.5.3) and the commutativity of the right-hand square in (2.7.a). □

**Remark 2.7.9.** The discussion of Subsect. 2.3 makes sense in the classical setting and provides $\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})$ with the same functorialities as $S$ varies. The equivalence of Theorem A is compatible with all the operations $f_\sharp$, $f^*$, $f_*$, as well as $\times$, and $\underline{\mathrm{Hom}}$:

(i) Let $f : T \to S$ be a morphism of affine spectral schemes. Then we have commutative squares

$$\begin{array}{ccc} \mathbf{H}(S) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*_{\mathrm{cl}}} \\ \mathbf{H}(T) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}), \end{array} \qquad \begin{array}{ccc} \mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(T) \\ \downarrow{\scriptstyle f_{\mathrm{cl},*}} & & \downarrow{\scriptstyle f_*} \\ \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(S). \end{array}$$

Indeed, the left-hand square is induced by the commutative square

$$\begin{array}{ccc} \mathcal{A}_{/S} & \xrightarrow{w} & \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \\ \downarrow & & \downarrow \\ \mathcal{A}_{/T} & \xrightarrow{w} & \mathcal{A}^{\mathrm{cl}}_{/T_{\mathrm{cl}}}, \end{array}$$

where the upper horizontal arrow is (derived) base change along $f$, and the lower horizontal arrow is classical base change along $f_{\mathrm{cl}}$. The right-hand square is obtained by passage to right adjoints.

(ii) Since the horizontal arrows in the squares above are equivalences (Theorem A), the squares are horizontally right- and left-adjointable, respectively. In other words, they give rise to further commutative squares

$$\begin{array}{ccc} \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(S) \\ \downarrow{\scriptstyle f^*_{\mathrm{cl}}} & & \downarrow{\scriptstyle f^*} \\ \mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(T), \end{array} \qquad \begin{array}{ccc} \mathbf{H}(T) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{\mathrm{cl},*}} \\ \mathbf{H}(S) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}). \end{array}$$

(iii) Similarly $f$ is smooth, then we have commutative squares

$$\begin{array}{ccc} \mathbf{H}(T) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) \\ \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_{\mathrm{cl},\sharp}} \\ \mathbf{H}(S) & \xrightarrow{\mathbf{L}w_!} & \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}), \end{array} \qquad \begin{array}{ccc} \mathbf{H}^{\mathrm{cl}}(T_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(T) \\ \downarrow{\scriptstyle f_{\mathrm{cl},\sharp}} & & \downarrow{\scriptstyle f_\sharp} \\ \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}) & \xrightarrow{w^*} & \mathbf{H}(S). \end{array}$$

Here the left-hand square is induced by the commutative square

$$\begin{array}{ccc} \mathcal{A}_{/T} & \xrightarrow{v} & \mathcal{A}^{\mathrm{cl}}_{/T_{\mathrm{cl}}} \\ \downarrow & & \downarrow \\ \mathcal{A}_{/S} & \xrightarrow{v} & \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}. \end{array}$$

The right-hand square comes from the horizontal right-adjointability of the left-hand one.

(iv) Finally, consider the operations $\times$ and $\underline{\mathrm{Hom}}$. Note that $w^*$ is cartesian monoidal since it is a right adjoint. Its left adjoint $\mathbf{L}w_!$ is also monoidal, since $w_!$ is clearly monoidal (as can be checked on representables) and $\mathbf{L}$ preserves finite products by Remark 2.1.16(iii). Then by adjunction we have a canonical isomorphism

$$w^*\big(\underline{\mathrm{Hom}}(\mathbf{L}w_!(\mathcal{F}), \mathcal{G})\big) \to \underline{\mathrm{Hom}}(\mathcal{F}, v^*(\mathcal{G}))$$

in $\mathbf{H}(S)$, for any $\mathcal{F} \in \mathbf{H}(S)$ and $\mathcal{G} \in \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})$. This induces in turn for every $\mathcal{F}, \mathcal{G} \in \mathbf{H}(S)$ isomorphisms

$$\begin{aligned}\underline{\mathrm{Hom}}(\mathbf{L}w_!(\mathcal{F}), \mathbf{L}w_!(\mathcal{G})) &\simeq \mathbf{L}w_!w^*\big(\underline{\mathrm{Hom}}(\mathbf{L}w_!(\mathcal{F}), \mathbf{L}w_!(\mathcal{G}))\big)\\ &\simeq \mathbf{L}w_!\big(\underline{\mathrm{Hom}}(\mathcal{F}, v^*\mathbf{L}w_!(\mathcal{G}))\big)\\ &\simeq \mathbf{L}w_!\big(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})\big)\end{aligned}$$

where the first and third isomorphisms come from Theorem A.

2.8. **V-linear motivic objects.** Let $S$ be an affine spectral scheme and let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. By replacing the $\infty$-category of spaces in Definition 2.1.14 with any given presentable $\infty$-category $\mathbf{V}$, we can define a $\mathbf{V}$-linear variant of the construction $\mathbf{H}(\mathcal{C}_{/S})$:

**Definition 2.8.1.** A *$\mathcal{C}$-fibred motivic $\mathbf{V}$-object* is a $\mathbf{V}$-valued presheaf $(\mathcal{C}_{/S})^{\mathrm{op}} \to \mathbf{V}$ satisfying $\mathbf{A}^1$-homotopy invariance and Nisnevich excision. We write $\mathbf{H}(\mathcal{C}_{/S})_{\mathbf{V}}$ for the $\infty$-category of motivic $\mathcal{C}$-fibred $\mathbf{V}$-objects.

Similarly, given an admissible subcategory $\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S_{\mathrm{cl}}}$, we may consider the $\infty$-category $\mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})_{\mathbf{V}}$ of $\mathcal{C}^{\mathrm{cl}}$-fibred motivic $\mathbf{V}$-objects. We have $\mathbf{V}$-linear analogues of each of the categories defined in *loc. cit.*:

$$\mathrm{Spc}(\mathcal{C}_{/S})_{\mathbf{V}}, \quad \mathrm{Spc}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})_{\mathbf{V}}, \quad \mathbf{H}(\mathcal{C}_{/S})_{\mathbf{V}}, \quad \mathbf{H}^{\mathrm{cl}}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})_{\mathbf{V}}.$$

**Example 2.8.2.** Taking $\mathbf{V}$ to be the stable presentable $\infty$-category Spt of spectra, we obtain $\infty$-categories of fibred motivic spectra. We will refer to these as fibred *motivic $S^1$-spectra*, to distinguish them from the notion of motivic spectra with respect to the Thom space of the trivial line bundle.

**Remark 2.8.3.** If $\mathbf{V}$ is *stable*, then so is $\mathbf{H}(\mathcal{C}_{/S})_{\mathbf{V}}$ for any admissible subcategory $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$.

**Remark 2.8.4.** The $\infty$-category $\mathbf{H}(\mathcal{C}_{/S})_{\mathbf{V}}$ can also be described as the tensor product of presentable $\infty$-categories $\mathbf{H}(\mathcal{C}_{/S}) \otimes \mathbf{V}$ in the sense of [HA, Sect. 4.8]. An analogous description holds for the classical variant $\mathbf{H}(\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$, for $\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ admissible. It follows that when $\mathbf{V}$ is presentably symmetric monoidal, these categories also inherit presentably symmetric monoidal structures. Remark 2.3.7 then carries over to the $\mathbf{V}$-linear setting.

The description of Remark 2.8.4 immediately gives the following generalization of the comparison result of Theorem A (or rather the more precise statement proven in Subsect. 2.7):

**Theorem 2.8.5.** *Let $S$ be a spectral affine scheme and let $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ be a* narrow *subcategory. Let $\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S}$ be as in Example 2.4.2 and let $w : \mathcal{A}_{/S} \to \mathcal{A}_{/S_{\mathrm{cl}}}$ denote the classical truncation functor (Construction 2.4.6). Then for any presentable $\infty$-category $\mathbf{V}$, the adjunction*

$$\mathbf{L}w_! : \mathbf{H}(S)_{\mathbf{V}} \to \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})_{\mathbf{V}}, \quad w^* : \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}})_{\mathbf{V}} \to \mathbf{H}(S)_{\mathbf{V}}$$

*is an equivalence of $\infty$-categories. Moreover, this equivalence is compatible with the operations $f^*$, $f_*$ for any morphism $f : T \to S$, with $f_\sharp$ when $f$ exhibits $T$ as an object of $\mathcal{A}_{/S}$, with products and with internal homs.*

Finally, let us note the following two properties which are specific to the stable case.

**Proposition 2.8.6.** *Let $\mathbf{V}$ be a* stable *presentable $\infty$-category. Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. Then on the $\infty$-category of $\mathcal{C}$-fibred $\mathbf{V}$-objects, the $\mathbf{A}^1$-localization functor $\mathrm{L}_{\mathbf{A}^1}$ preserves the property of Nisnevich excision. In particular, the motivic localization functor $\mathbf{L}$ can be computed by the formula*

$$\mathbf{L} \simeq \mathrm{L}_{\mathbf{A}^1} \, \mathrm{L}_{\mathrm{Nis}} \, .$$

*Proof.* Let $\mathcal{F}$ be a $\mathcal{C}$-fibred $\mathbf{V}$-object. If $\mathcal{F}$ is Nisnevich-excisive, then its $\mathbf{A}^1$-localization $\mathrm{L}_{\mathbf{A}^1}(\mathcal{F})$ is still Nisnevich-excisive, in view of the formula (2.1.b) and the fact that colimits commute with finite limits in stable $\infty$-categories. Therefore the claim follows from Remark 2.1.16 (iii). □

**Corollary 2.8.7.** *Let $\mathbf{V}$ be a* stable *presentable $\infty$-category. Let $\mathcal{F} \in \mathbf{H}(S)_{\mathbf{V}}$ be an $\mathcal{A}$-fibred $\mathbf{V}$-object over $S$ (where $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ is narrow). If $\mathcal{F}$ is Nisnevich-excisive, then the canonical map*

$$\mathrm{L}_{\mathbf{A}^1}(\mathcal{F}) \to w^* \, \mathrm{L}_{\mathbf{A}^1} \, w_!(\mathcal{F})$$

*is invertible.*

*Proof.* It follows from Theorem 2.8.5 (cf. Corollary 2.7.6) that the canonical map $\mathbf{L}(\mathcal{F}) \to w^*\mathbf{L}w_!(\mathcal{F})$ is invertible. Since $\mathcal{F}$ and hence $w_!(\mathcal{F})$ are Nisnevich-excisive, we conclude by Proposition 2.8.6. □

## 3. Comparison with $\mathbf{A}^{1,\flat}$-motivic homotopy theory

### 3.1. Flat affine spaces.

**Notation 3.1.1.** Let $R$ be an $\mathcal{E}_\infty$-ring. Denote by $R[T_1, \ldots, T_n]$ denote the polynomial $R$-algebra in $n$ variables $T_1, \ldots, T_n$ (in degree zero). This is by definition the monoid algebra $R[\mathbf{N}^n] = R \otimes \Sigma^\infty_+(\mathbf{N}^n)$, where $\mathbf{N}$ is the set of natural numbers, viewed as a discrete (additive) $\mathcal{E}_\infty$-monoid space. Note that we have canonical isomorphisms $\pi_*(R[T_1, \ldots, T_n]) \simeq \pi_*(R) \otimes_{\pi_0(R)} \pi_0(R)[T_1, \ldots, T_n]$, so that $R[T_1, \ldots, T_n]$ is flat over $R$.

**Definition 3.1.2.** For every $n \geqslant 0$, let $\mathbf{A}^{n,\flat}$ denote the affine spectral scheme $\mathrm{Spec}(\mathbf{S}[T_1, \dots, T_n])$, where $\mathbf{S}$ is the sphere spectrum. Note that we have a canonical isomorphism $(\mathbf{A}^{n,\flat})_{\mathrm{cl}} \simeq \mathbf{A}^n_{\mathrm{cl}}$. We refer to $\mathbf{A}^{n,\flat}$ as the *flat affine space* (over the sphere spectrum). If $S$ is classical, then $S \times \mathbf{A}^{n,\flat} = S \times \mathbf{A}^n_{\mathrm{cl}}$.

**Remark 3.1.3.** The affine spectral schemes $\mathbf{A}^{n,\flat}$ are equipped with the following additional structure:

(i) The flat affine line $\mathbf{A}^{1,\flat}$ has the structure of a commutative monoid under the operation of multiplication. This is induced by the cocommutative comonoid structure on the commutative monoid $\mathbf{N}$. For example, the multiplication morphism $\mathbf{A}^{1,\flat} \times \mathbf{A}^{1,\flat} \to \mathbf{A}^{1,\flat}$ corresponds to the diagonal $\Sigma^\infty_+(\mathbf{N}) \to \Sigma^\infty_+(\mathbf{N} \times \mathbf{N}) \simeq \Sigma^\infty_+(\mathbf{N}) \otimes \Sigma^\infty_+(\mathbf{N})$. Similarly the counit of $\mathbf{N}$ induces an $\mathcal{E}_\infty$-ring homomorphism $\mathbf{S}[T] \simeq \Sigma^\infty_+(\mathbf{N}) \to \Sigma^\infty_+(\mathrm{pt}) \simeq \mathbf{S}$ which corresponds to the unit section $s_1 : \mathrm{Spec}(\mathbf{S}) \to \mathbf{A}^{1,\flat}$.

(ii) The flat affine line $\mathbf{A}^{1,\flat}$ also admits a zero section $s_0 : \mathrm{Spec}(\mathbf{S}) \to \mathbf{A}^{1,\flat}$, which can be constructed as follows. Identify the discrete pointed $\mathcal{E}_\infty$-monoid space $\mathrm{pt}_+$ with the set $\{0, 1\}$, viewed as a multiplicative monoid with base point 0 and identity element 1. Since $\mathbf{N}$ is freely generated as a (discrete) commutative monoid by the element $1 \in \mathbf{N}$, either choice of element $i \in \{0, 1\}$ gives rise to a unique homomorphism $\sigma'_i : \mathbf{N}_+ \to \mathrm{pt}_+$ of pointed commutative monoids such that $\sigma'_i(1) = i$. Regarding $\sigma'_i$ as a homomorphism of discrete pointed $\mathcal{E}_\infty$-monoid spaces, application of the symmetric monoidal functor $\Sigma^\infty$ produces $\mathcal{E}_\infty$-ring homomorphisms

$$\sigma_i : \mathbf{S}[T] \simeq \Sigma^\infty(\mathbf{N}_+) \to \Sigma^\infty(\mathrm{pt}_+) \simeq \mathbf{S}$$

for each $i \in \{0, 1\}$. For $i = 1$ this is the same homomorphism defining the unit section $s_1$, and we let $s_0$ denote the section $\mathrm{Spec}(\mathbf{S}) \to \mathrm{Spec}(\mathbf{S}[T]) = \mathbf{A}^{1,\flat}$ corresponding to $\sigma_0$.

(iii) For every $n \geqslant 0$, the flat affine space $\mathbf{A}^{n,\flat}$ admits the structure of a module over $\mathbf{A}^{1,\flat}$. This is induced by the canonical comodule structure on the commutative monoid $\mathbf{N}^n$ over the comonoid $\mathbf{N}$, where the coaction homomorphism sends $\mathbf{N}^n \to \mathbf{N} \times \mathbf{N}^n$

$$(k_1, k_2, \dots, k_n) \mapsto (k_1 + \cdots + k_n, k_1, k_2, \dots, k_n).$$

In particular, there is a "scalar multiplication" morphism

$$\mathbf{A}^{1,\flat} \times \mathbf{A}^{n,\flat} \to \mathbf{A}^{n,\flat}.$$

(iv) After base change along $\mathrm{Spec}(\mathbf{Z}) \to \mathrm{Spec}(\mathbf{S})$, the flat affine spaces $\mathrm{Spec}(\mathbf{Z}) \times \mathbf{A}^{n,\flat} \simeq \mathbf{A}^n_{\mathrm{cl}}$ become abelian groups under addition.

**Remark 3.1.4.** The zero section of $\mathbf{A}^{1,\flat}$ is compatible with the multiplicative structure, in the sense that $\mathbf{A}^{1,\flat}$ defines an *interval object* in the sense of Morel–Voevodsky [MV99, Sect. 2.3].

3.2. $\mathrm{Sm}^\flat$**-fibred motivic spaces.** Unlike $S \times \mathbf{A}^n$, the flat affine spaces $S \times \mathbf{A}^{n,\flat}$ are not smooth over $S$ in the sense of Definition 2.1.1 (except under the conditions of Remark 3.3.1). However, they are smooth in the following sense:

**Definition 3.2.1.** A morphism of affine spectral schemes $X \to S$ is called *fibre-smooth* if it is almost of finite presentation, flat, and on classical truncations induces a morphism $X_{\mathrm{cl}} \to S_{\mathrm{cl}}$ that is smooth in the sense of classical algebraic geometry.

**Remark 3.2.2.** From [SAG, Cor. 11.2.4.2] and [EGA $\mathrm{IV}_4$, § 17.3] it follows that a morphism of affine spectral schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is fibre-smooth if and only if $A \to B$ is fibre-smooth in the sense of [SAG, Def. 11.2.3.1].

**Remark 3.2.3.** Let $f : X \to S$ be a fibre-smooth morphism. Then Zariski-locally on $X$, there exists an étale $S$-morphism $X \to S \times \mathbf{A}^{n,\flat}$ for some $n \geqslant 0$. This follows from Remark 11.2.3.5 and Proposition 11.2.4.1 of [SAG], combined with [HA, Thm. 7.5.0.6]. Contrast with Remark 2.1.4.

The following is the same as Definition 2.1.7 except for the last condition.

**Definition 3.2.4.** We say that a full subcategory $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ is $\flat$*-admissible* if it is essentially small and satisfies the following conditions:

(i) The affine spectral scheme $S$ (viewed over $S$ via the identity) belongs to $\mathcal{C}^\flat_{/S}$.

(ii) If $X$ belongs to $\mathcal{C}^\flat_{/S}$ and $Y$ is étale over $X$, then $Y$ belongs to $\mathcal{C}^\flat_{/S}$.

(iii) If $X$ belongs to $\mathcal{C}^\flat_{/S}$, then $X \times \mathbf{A}^{n,\flat}$ belongs to $\mathcal{C}^\flat_{/S}$ for every $n \geqslant 0$.

A $\flat$*-narrow* subcategory $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ is a $\flat$-admissible subcategory which is contained in the full subcategory $\mathrm{Sm}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ of fibre-smooth spectral affine schemes over $S$.

**Example 3.2.5.** Let $\mathcal{A}^{\flat,0}_{/S} \subseteq \mathrm{Aff}_{/S}$ denote the full subcategory spanned by $X \in \mathrm{Aff}_{/S}$ for which the structural morphism $X \to S$ factors through an étale morphism

$$X \to S \times \mathbf{A}^{n,\flat}$$

over $S$. Then $\mathcal{A}^{\flat,0}_{/S}$ is the minimal $\flat$-admissible subcategory of $\mathrm{Aff}_{/S}$ (same proof as Example 2.1.9).

**Remark 3.2.6.** Let $S$ be a spectral affine scheme. Let $\mathcal{C}^\flat \subseteq \mathrm{Aff}_{/S}$ be a $\flat$-admissible subcategory. As in Example 2.4.2, the full subcategory $\mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ spanned by classical truncations of objects in $\mathcal{C}^\flat$ is admissible.

**Definition 3.2.7.** Let $S$ be a spectral affine scheme and $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ a $\flat$-admissible subcategory. Let $\mathcal{F}$ be a $\mathcal{C}^\flat$-fibred space (Definition 2.1.13), i.e., a presheaf of spaces on $\mathcal{C}^\flat_{/S}$. We say that $\mathcal{F}$ is Nisnevich excisive if it is reduced and sends Nisnevich squares $Q$ to cartesian squares $\Gamma(Q, \mathcal{F})$ (Definition 2.1.5). We say that $\mathcal{F}$ satisfies $\mathbf{A}^{1,\flat}$*-homotopy invariance* if for any $X \in \mathcal{C}^\flat_{/S}$, the canonical map $\Gamma(X, \mathcal{F}) \to \Gamma(X \times \mathbf{A}^{1,\flat}, \mathcal{F})$ is invertible, where $\mathbf{A}^{1,\flat}$ denotes the flat affine line (Definition 3.1.2). We say that $\mathcal{F}$ is *motivic* if it is Nisnevich excisive and $\mathbf{A}^{1,\flat}$-homotopy invariant. We denote by $\mathbf{H}(\mathcal{C}^\flat_{/S}) \subseteq \mathrm{Spc}(\mathcal{C}^\flat_{/S})$ the full subcategory of motivic $\mathcal{C}^\flat$-fibred spaces.

**Notation 3.2.8.** If $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ is both admissible and $\flat$-admissible, then there is possible ambiguity in the terminology "$\mathcal{C}$-fibred motivic space" and in the notation $\mathbf{H}(\mathcal{C}_{/S})$. To maintain the distinction we introduce the following convention: we write $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ for the same full subcategory when it is to be regarded as a $\flat$-admissible subcategory. Thus a *motivic $\mathcal{C}$-fibred space* is an $\mathbf{A}^1$-invariant Nisnevich-excisive $\mathcal{C}$-fibred space, while a *motivic $\mathcal{C}^\flat$-fibred space* is an $\mathbf{A}^{1,\flat}$-invariant Nisnevich-excisive $\mathcal{C}$-fibred space. In particular, $\mathbf{H}(\mathcal{C}_{/S})$ and $\mathbf{H}(\mathcal{C}^\flat_{/S})$ are two distinct full subcategories of $\mathrm{Spc}(\mathcal{C}_{/S}) = \mathrm{Spc}(\mathcal{C}^\flat_{/S})$.

**Remark 3.2.9.** For any $\flat$-admissible subcategory $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$, the full subcategories of Nisnevich-excisive, $\mathbf{A}^{1,\flat}$-homotopy invariant, and motivic $\mathcal{C}^\flat$-fibred spaces are each left Bousfield localizations of the $\infty$-category of $\mathcal{C}^\flat$-fibred spaces. We write $\mathrm{L}_{\mathrm{Nis}}$, $\mathrm{L}_{\mathbf{A}^{1,\flat}}$, and $\mathbf{L}^\flat$ for the respective localization functors. The remarks in 2.1.16 remain valid *mutatis mutandis* up to replacing the spectral affine spaces $\mathbf{A}^n$ by the flat affine spaces $\mathbf{A}^{n,\flat}$ in item (ii).

**Proposition 3.2.10.** *Let $S$ be a spectral affine scheme. Fix an inclusion $\mathcal{C}^\flat_{/S} \subseteq \mathcal{D}^\flat_{/S}$ of $\flat$-admissible subcategories of $\mathrm{Aff}_{/S}$. Denote by $\iota : \mathcal{C}^\flat_{/S} \hookrightarrow \mathcal{D}^\flat_{/S}$ the inclusion functor and by $\iota_! : \mathrm{Spc}(\mathcal{C}^\flat_{/S}) \to \mathrm{Spc}(\mathcal{D}^\flat_{/S})$ its left Kan extension. Then the assignment $\mathcal{F} \mapsto \mathbf{L}^\flat \iota_!(\mathcal{F})$ induces a fully faithful functor of $\infty$-categories*

$$\mathbf{L}^\flat \iota_! : \mathbf{H}(\mathcal{C}^\flat_{/S}) \to \mathbf{H}(\mathcal{D}^\flat_{/S}),$$

*whose essential image is generated under sifted colimits by objects of the form $\mathbf{L}^\flat(\mathrm{h}_S(X))$, where $X$ belongs to $\mathcal{C}_{/S}$.*

*Proof.* Same proof as Proposition 2.2.3. □

**Remark 3.2.11.** Let $\mathcal{C}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ be $\flat$-admissible. Using the interval structure on $\mathbf{A}^{1,\flat}$ (Remarks 3.1.3 and 3.1.4), we can make sense of $\mathbf{A}^{1,\flat}$-*homotopies* between morphisms of $\mathcal{C}^\flat$-fibred spaces, and therefore of *strict* $\mathbf{A}^{1,\flat}$-*homotopy equivalences* (see [Kha19b, Def. 2.3.6]).

## 3.3. $\mathbf{A}^1$-homotopies vs. $\mathbf{A}^{1,\flat}$-homotopies.

**Remark 3.3.1.** For any connective $\mathcal{E}_\infty$-ring $R$ and integer $n \geqslant 0$, there is a canonical homomorphism of $\mathcal{E}_\infty$-$R$-algebras $R\{T_1, \ldots, T_n\} \to R[T_1, \ldots, T_n]$ determined by the assignment $T_i \mapsto T_i$. This gives rise to canonical morphisms

$$\varepsilon_S : S \times \mathbf{A}^{n,\flat} \to S \times \mathbf{A}^n$$

for every affine spectral scheme $S$, which are invertible if and only if either $n = 0$ or $S$ is of characteristic zero (i.e., $S = \mathrm{Spec}(R)$ with $R$ an $\mathcal{E}_\infty$-$\mathbf{Q}$-algebra).

**Remark 3.3.2.** The map $\varepsilon : \mathbf{A}^{1,\flat} \to \mathbf{A}^1$ is compatible with interval structures. That is, it preserves the zero and unit sections, and is compatible with the multiplicative structures in the sense that the diagram

$$\begin{array}{ccc} \mathbf{A}^{1,\flat} \times \mathbf{A}^{1,\flat} & \longrightarrow & \mathbf{A}^{1,\flat} \\ \downarrow & & \downarrow \\ \mathbf{A}^1 \times \mathbf{A}^1 & \longrightarrow & \mathbf{A}^1 \end{array}$$

commutes. By restriction of scalars, we may therefore regard $\mathbf{A}^n$ as a module over $\mathbf{A}^{1,\flat}$. Note that for every $n \geq 0$, the morphism $\varepsilon : \mathbf{A}^{n,\flat} \to \mathbf{A}^n$ is then $\mathbf{A}^{1,\flat}$-linear.

Note also that over $\operatorname{Spec}(\mathbf{Z})$, $\varepsilon$ defines a group homomorphism

$$\mathbf{A}^n_{\mathrm{cl}} \simeq \operatorname{Spec}(\mathbf{Z}) \times \mathbf{A}^{n,\flat} \to \operatorname{Spec}(\mathbf{Z}) \times \mathbf{A}^n$$

with respect to the additive structures (Remark 3.1.3(iv)).

**Lemma 3.3.3.** *Let $S$ be an affine spectral scheme. Let $\mathcal{C}_{/S}$ be an admissible and $\flat$-admissible subcategory of* $\mathrm{Aff}_{/S}$. *Then we have:*

(i) *Every $\mathbf{A}^1$-local equivalence between $\mathcal{C}$-fibred spaces over $S$ is an $\mathbf{A}^{1,\flat}$-local equivalence.*

(ii) *Every $\mathbf{A}^{1,\flat}$-homotopy invariant $\mathcal{C}$-fibred space over $S$ is $\mathbf{A}^1$-homotopy invariant.*

*In particular, there is an inclusion*

$$\mathbf{H}(\mathcal{C}^\flat_{/S}) \subseteq \mathbf{H}(\mathcal{C}_{/S})$$

*of subcategories of* $\mathrm{Spc}(\mathcal{C}_{/S})$.

*Proof.* By adjunction, the two statements are equivalent. To prove the first it will suffice to show that for every $X \in \mathcal{C}_{/S}$, the morphism $\mathrm{h}_S(X \times \mathbf{A}^1) \to \mathrm{h}_S(X)$ is an $\mathbf{A}^{1,\flat}$-local equivalence. In fact, we claim that every strict $\mathbf{A}^1$-homotopy equivalence is a strict $\mathbf{A}^{1,\flat}$-homotopy equivalence (hence *a fortiori* an $\mathbf{A}^{1,\flat}$-local equivalence). Indeed, the canonical map $\varepsilon_S : S \times \mathbf{A}^{1,\flat} \to S \times \mathbf{A}^1$ (Remark 3.3.1) is a morphism of interval objects (Remark 3.3.2), so composition with $\varepsilon_S$ sends elementary $\mathbf{A}^1$-homotopies to elementary $\mathbf{A}^{1,\flat}$-homotopies. □

The following key lemma shows that the comparison morphism $\varepsilon : \mathbf{A}^{n,\flat} \to \mathbf{A}^n$ is a "universal" $\mathbf{A}^{1,\flat}$-local equivalence, at least over $\operatorname{Spec}(\mathbf{Z})$. The reason for this restriction is that the proof uses the additive structure on the flat affine spaces (Remark 3.1.3(iv)).

**Lemma 3.3.4.** *Let $S$ be a spectral affine scheme defined over* $\operatorname{Spec}(\mathbf{Z})$. *Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible and $\flat$-admissible subcategory. Given $X \in \mathcal{C}_{/S}$ and an $S$-morphism $f : X \to \mathbf{A}^n_S$, consider the cartesian square*

$$\begin{array}{ccc} X^\flat & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ S \times \mathbf{A}^{n,\flat} & \xrightarrow{\varepsilon_S} & S \times \mathbf{A}^n. \end{array}$$

*Then the morphism* $\mathrm{h}_S(X^\flat) \to \mathrm{h}_S(X)$ *is an* $\mathbf{A}^{1,\flat}$*-local equivalence of* $\mathcal{C}$*-fibred spaces.*

*Proof.* Note that $f$ induces a section $s : X \to (S \times \mathbf{A}^n) \times_S X = X \times \mathbf{A}^n$ of the projection $X \times \mathbf{A}^n \to X$, and similarly $g$ induces a section $t : X \to X \times \mathbf{A}^{n,\flat}$. These fit into a factorization of the given square:

$$\begin{array}{ccc} X^\flat & \longrightarrow & X \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ X \times \mathbf{A}^{n,\flat} & \xrightarrow{\varepsilon_X} & X \times \mathbf{A}^n \\ \downarrow & & \downarrow \\ S \times \mathbf{A}^{n,\flat} & \xrightarrow{\varepsilon_S} & S \times \mathbf{A}^n \end{array}$$

Since the lower square is cartesian, so is the upper square. Therefore we may as well assume that $X = S$, in which case the claim becomes that $\mathrm{h}_X(X^\flat)$ is $\mathbf{A}^{1,\flat}$-contractible. Since $S$ is defined over $\mathrm{Spec}(\mathbf{Z})$, $\varepsilon$ is a group homomorphism with respect to the additive structures (Remark 3.3.2). Therefore it gives rise to an isomorphism between $X^\flat$ and the fibre of $\varepsilon$ over the zero section. Thus we may also assume that $s$ is the zero section.

Since $s : X \to \mathbf{A}^n_X$ lifts to the zero section $s : X \to \mathbf{A}^n_{\flat,X}$, it induces an $X$-morphism $X \to X^\flat$ and hence a base point of the $\mathcal{C}$-fibred space $\mathrm{h}_X(X^\flat)$. It will suffice to exhibit an $\mathbf{A}^{1,\flat}$-homotopy contracting $\mathrm{h}_X(X^\flat)$ to this base point. Recall that $\mathbf{A}^{1,\flat}$ acts compatibly on $\mathbf{A}^n_{\flat,X}$ and $\mathbf{A}^n_X$ (Remarks 3.1.3 and 3.3.2). The action on the latter restricts along the zero section $s : X \to \mathbf{A}^n_X$ to the trivial action on $X$. The induced $\mathbf{A}^{1,\flat}$-action on $X^\flat$ is a morphism

$$\mathbf{A}^{1,\flat} \times X^\flat \to X^\flat,$$

which induces the $\mathbf{A}^{1,\flat}$-homotopy desired. □

**Corollary 3.3.5.** *Let $S$ be a spectral affine scheme over* $\mathrm{Spec}(\mathbf{Z})$. *Let* $\mathcal{A}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ *be a* $\flat$*-narrow subcategory and let* $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ *be an admissible and* $\flat$*-admissible subcategory containing* $\mathcal{A}^\flat_{/S}$. *Let* $\iota : \mathcal{A}^\flat_{/S} \hookrightarrow \mathcal{C}_{/S}$ *denote the inclusion. Then the essential image of the fully faithful functor (Proposition 3.2.10)*

$$\mathbf{L}^\flat \iota_! : \mathbf{H}(\mathcal{A}^\flat_{/S}) \to \mathbf{H}(\mathcal{C}^\flat_{/S})$$

*is generated under sifted colimits by objects of the form* $\mathbf{L}^\flat \mathrm{h}_S(X)$, *where* $X \in \mathrm{Sm}_{/S}$ *is affine and admits an étale* $S$*-morphism to the* spectral *affine space* $\mathbf{A}^n_S$, *for some* $n \geqslant 0$.

*Proof.* Let $\mathcal{E}$ and $\mathcal{E}_\flat$ denote the full subcategories of $\mathbf{H}(\mathcal{C}^\flat_{/S})$ generated under sifted colimits by objects of the form $\mathbf{L}^\flat \mathrm{h}_S(X)$, where $X$ admits an étale $S$-morphism to $S \times \mathbf{A}^n$, respectively to $S \times \mathbf{A}^{n,\flat}$, for some $n \geqslant 0$. We may as well assume $\mathcal{A}^\flat_{/S}$ is the minimal $\flat$-admissible subcategory (Example 3.2.5), spanned by $X \in \mathrm{Aff}_{/S}$ which admit an étale morphism $X \to S \times \mathbf{A}^{n,\flat}$ for some $n \geqslant 0$. Then by Proposition 3.2.10, $\mathcal{E}_\flat$ is identified with the essential image of the functor in question, so it will suffice to show that $\mathcal{E} = \mathcal{E}_\flat$.

Let $X \in \mathrm{Aff}_{/S}$. Suppose $X$ admits an étale $S$-morphism $f : X \to S \times \mathbf{A}^n$ for some $n \geqslant 0$. The base change of $f$ along $\varepsilon_S : S \times \mathbf{A}^{n,\flat} \to S \times \mathbf{A}^n$ is an étale morphism $X^\flat \to S \times \mathbf{A}^{n,\flat}$. By

Lemma 3.3.4, the canonical morphism $X^\flat \to X$ induces an isomorphism $\mathbf{L}^\flat \mathrm{h}_S(X^\flat) \simeq \mathbf{L}^\flat \mathrm{h}_S(X)$, hence in particular $\mathbf{L}^\flat \mathrm{h}_S(X) \in \mathcal{E}_\flat$. This shows $\mathcal{E} \subseteq \mathcal{E}_\flat$.

For the other direction, suppose $X \in \mathrm{Aff}_{/S}$ admits an étale $S$-morphism $g : X \to S \times \mathbf{A}^{n,\flat}$ for some $n \geqslant 0$. Since $\varepsilon_S$ is an isomorphism on classical truncations, it follows from [HA, Thm. 7.5.0.6] that there exists $Y \in \mathrm{Aff}_{/S}$, an étale $S$-morphism $f : Y \to S \times \mathbf{A}^n$, and a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ S \times \mathbf{A}^{n,\flat} & \xrightarrow{\varepsilon_S} & S \times \mathbf{A}^n. \end{array}$$

By Lemma 3.3.4, there is an isomorphism $\mathbf{L}^\flat \mathrm{h}_S(X) \simeq \mathbf{L}^\flat \mathrm{h}_S(Y)$, hence in particular $\mathbf{L}^\flat \mathrm{h}_S(X) \in \mathcal{E}$. This shows $\mathcal{E}_\flat \subseteq \mathcal{E}$. □

3.4. **The comparison.** In this subsection we prove the following statement, which in particular yields the equivalence between (ii) and (iii) in Theorem A.

**Theorem 3.4.1.** *Let $S$ be a spectral affine scheme defined over* $\mathrm{Spec}(\mathbf{Z})$. *Let $\mathcal{A}^\flat_{/S}$ be any $\flat$-narrow subcategory of* $\mathrm{Aff}_{/S}$ *and let $w^\flat : \mathcal{A}^\flat_{/S} \to \mathcal{A}^{\mathrm{cl}}_{/S}$ be the restriction of the classical truncation functor $v : \mathcal{B}_{/S} \to \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$. Then the adjunction*

$$\mathbf{L}w^\flat_! : \mathbf{H}(\mathcal{A}^\flat_{/S}) \to \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S}) \quad w^* : \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S}) \to \mathbf{H}(\mathcal{A}^\flat_{/S})$$

*is an equivalence.*

We fix the following notation for this subsection.

**Notation 3.4.2.** Let $S$ be a spectral affine scheme. Let $\mathcal{A}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$ be a $\flat$-narrow subcategory and $\mathcal{B} \subseteq \mathrm{Aff}_{/S}$ be a $\flat$-admissible and broad subcategory containing $\mathcal{A}^\flat_{/S}$. Let $\iota : \mathcal{A}^\flat_{/S} \hookrightarrow \mathcal{B}_{/S}$ denote the inclusion. Let $\mathcal{B}^\flat_{/S}$ be as in Remark 3.2.8 and let $\mathcal{B}^{\mathrm{cl}}_{/S} \subseteq \mathrm{AffCl}_{/S}$ be as in Example 2.4.2. We write

$$\mathbf{H}(\mathcal{B}_{/S}), \quad \mathbf{H}(\mathcal{B}^\flat_{/S}), \quad \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$$

for the $\infty$-categories of motivic spaces formed respectively out of the admissible subcategory $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$, the $\flat$-admissible subcategory $\mathcal{B}^\flat_{/S} \subseteq \mathrm{Aff}_{/S}$, and the admissible subcategory $\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \subseteq \mathrm{AffCl}_{/S}$.

**Remark 3.4.3.** By construction the classical truncation functor $v : \mathrm{Aff}_{/S} \to \mathrm{AffCl}_{/S_{\mathrm{cl}}}$ (Construction 2.4.6) restricts to a functor $v : \mathcal{B}_{/S} \to \mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$. Since the latter preserves Nisnevich squares and sends $\mathbf{A}^{1,\flat}$ to $\mathbf{A}^1_{\mathrm{cl}}$, one sees as in Lemma 2.4.7 that $v$ induces an adjunction

$$\mathbf{L}v_! : \mathbf{H}(\mathcal{B}^\flat_{/S}) \to \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{B}^\flat_{/S}).$$

Proposition 2.5.4 also holds *mutatis mutandis* for the functors

$$v_! : \mathrm{Spc}(\mathcal{B}^\flat_{/S}) \to \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}), \quad v^* : \mathrm{Spc}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathrm{Spc}(\mathcal{B}^\flat_{/S}).$$

In particular, $v^*$ commutes with $\mathbf{L}^\flat$. By Remark 2.5.6 this implies that $\mathbf{L}^\flat$ preserves nil-local objects.

**Corollary 3.4.4.** *The functor*

$$v^* : \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \to \mathbf{H}(\mathcal{B}^{\flat}_{/S})$$

*is fully faithful, with essential image spanned by the nil-local objects of* $\mathbf{H}(\mathcal{B}^{\flat}_{/S})$.

*Proof.* Follows immediately by combining Theorem 2.5.3 with Lemma 3.3.3. □

**Corollary 3.4.5.** *If $S$ is defined over* $\mathrm{Spec}(\mathbf{Z})$, *then for every $\mathcal{F} \in \mathbf{H}(\mathcal{A}^{\flat}_{/S})$, the $\mathcal{B}^{\flat}$-fibred motivic space $\mathbf{L}^{\flat}\iota_!(\mathcal{F}) \in \mathbf{H}(\mathcal{B}^{\flat}_{/S})$ is nil-local.*

*Proof.* By Theorem 2.6.2, every object in the essential image of

$$\mathbf{L}\iota_! : \mathbf{H}(\mathcal{A}^{0}_{/S}) \to \mathbf{H}(\mathcal{B}_{/S})$$

is nil-local, where $\mathcal{A}^0_{/S}$ is the minimal admissible subcategory of $\mathrm{Aff}_{/S}$ (Example 2.1.9). It follows from Remark 3.4.3 that the same holds for objects in the essential image of

$$\mathbf{L}^{\flat}\mathbf{L}\iota_! \simeq \mathbf{L}^{\flat}\iota_! : \mathbf{H}(\mathcal{A}^{0}_{/S}) \to \mathbf{H}(\mathcal{B}^{\flat}_{/S}),$$

where the isomorphism is due to Lemma 3.3.3. By Corollary 3.3.5 this coincides with the essential image of the functor in question. □

*Proof of Theorem 3.4.1.* We are free to choose any $\flat$-admissible and broad subcategory $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$ containing $\mathcal{A}^{\flat}_{/S}$ as in Notation 3.4.2. By Proposition 3.2.10 we have the commutative diagram

$$\begin{array}{ccc}
\mathbf{H}(\mathcal{A}^{\flat}_{/S}) & \xrightarrow{\mathbf{L}w^{\flat}_!} & \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}) \\
{\scriptstyle \mathbf{L}^{\flat}\iota_!}\downarrow & & \downarrow{\scriptstyle \mathbf{L}\iota_!} \\
\mathbf{H}(\mathcal{B}^{\flat}_{/S}) & \xrightarrow{\mathbf{L}v_!} & \mathbf{H}(\mathcal{B}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})
\end{array}$$

where the vertical arrows are fully faithful. Write $\langle\mathcal{A}^{\flat}_{/S}\rangle$ for the essential image of the left-hand vertical arrow, and $\langle\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}\rangle$ for that of the right-hand vertical arrow. It will suffice to show that $\mathbf{L}v_!$ restricts to an equivalence $\langle\mathcal{A}^{\flat}_{/S}\rangle \simeq \langle\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}\rangle$. By Corollary 3.4.5, every object in the source is nil-local. Thus the claim follows from Corollary 3.4.4. □

**Corollary 3.4.6.** *For any quasi-compact quasi-separated spectral algebraic space $S$ over* $\mathrm{Spec}(\mathbf{Z})$, *there are canonical equivalences of $\infty$-categories*

$$\mathbf{H}^{\flat}(S) \simeq \mathbf{H}^{\mathrm{cl}}(S_{\mathrm{cl}}).$$

*Here $\mathbf{H}^{\flat}(S)$ is the $\infty$-category of $\mathbf{A}^{1,\flat}$-invariant Nisnevich sheaves on the site of (quasi-compact quasi-separated) fibre-smooth spectral algebraic spaces over $S$ (where fibre-smoothness is defined as in* [SAG, Def. 11.2.5.5]*).*

*Proof.* Follows from Theorem 3.4.1 by Nisnevich descent as in Corollary 2.7.5. □

## 4. The Bass construction

4.1. **Localizing invariants.** We briefly establish our notations and conventions for localizing invariants. We work with $R$-linear stable $\infty$-categories, over a fixed connective $\mathcal{E}_\infty$-ring $R$, although the discussion makes sense in the greater generality of $\mathrm{Perf}(S)$-linear $\infty$-categories, for any spectral algebraic space $S$.

**Notation 4.1.1.** Let Stab denote the $\infty$-category of small stable $\infty$-categories. For any connective $\mathcal{E}_\infty$-ring $R$, let $\mathrm{Stab}_R$ denote the $\infty$-category of small stable $R$-linear $\infty$-categories.

**Notation 4.1.2.** Let $R$ be a connective $\mathcal{E}_\infty$-ring and $E : \mathrm{Stab}_R \to \mathrm{Spt}$ a functor. For any $R$-algebra $A$, resp. spectral algebraic space $X$ over $\mathrm{Spec}(R)$, we set

$$E(A) := E(\mathrm{Perf}_A), \quad \text{resp. } E(X) := E(\mathrm{Perf}(X)),$$

where $\mathrm{Perf}_A$ is the stable $\infty$-category of $A$-modules and $\mathrm{Perf}(X)$ is the stable $\infty$-category of perfect complexes on $X$.

**Remark 4.1.3.** Any $R$-linear stable $\infty$-category $\mathbf{A}$ corepresents a functor $h_{\mathbf{A}} : \mathrm{Stab}_R \to \mathrm{Spt}$ given by the assignment

$$h_{\mathbf{A}}(\mathbf{A}') = \mathrm{Maps}_R(\mathbf{A}, \mathbf{A}'),$$

where $\mathrm{Maps}_R$ here denotes the mapping spectrum in the $\infty$-category of functors $\mathrm{Stab}_R \to \mathrm{Spt}$.

**Remark 4.1.4.** The Day convolution product endows the $\infty$-category of functors $\mathrm{Stab}_R \to \mathrm{Spt}$ with a closed symmetric monoidal structure for which the Yoneda embedding is symmetric monoidal. Given a functor $E : \mathrm{Stab}_R \to \mathrm{Spt}$, we write $E^{\mathbf{A}}$ for the internal hom object $\underline{\mathrm{Hom}}_R(h_{\mathbf{A}}, E)$, for any $\mathbf{A} \in \mathrm{Stab}_R$. Note that the assignment $\mathbf{A} \mapsto h_{\mathbf{A}}$ is *contravariant*, while $\mathbf{A} \mapsto E^{\mathbf{A}}$ is *covariant*. By the Yoneda lemma, the functor $E^{\mathbf{A}} : \mathrm{Stab}_R \to \mathrm{Spt}$ is given by

$$E^{\mathbf{A}}(\mathbf{A}') = E(\mathbf{A} \otimes \mathbf{A}')$$

for every $\mathbf{A}' \in \mathrm{Stab}_R$.

**Definition 4.1.5.** We say that $E$ is *additive* if it sends split exact sequences of stable $\infty$-categories [BGT13, Def. 5.18] to split exact triangles of spectra. We say $E$ is *localizing* if it sends short exact sequences of stable $\infty$-categories [BGT13, Def. 5.12] to exact triangles of spectra. Note that, unlike [BGT13], we do not require that $E$ commutes with filtered colimits.

**Remark 4.1.6.** Note that if $E$ is additive (resp. localizing), then the same holds for $E^{\mathbf{A}}$ for any $\mathbf{A} \in \mathrm{Stab}_R$.

**Example 4.1.7.** Let $\mathrm{K} : \mathrm{Stab} \to \mathrm{Spt}$ denote the algebraic K-theory functor. Recall that this is defined using the Waldhausen $S_\bullet$-construction (see [HA, Rmk. 1.2.2.5], [BGT13, Def. 7.1], or [Bar16, Sect. 10]) and takes values in connective spectra. Then K is additive by Waldhausen's additivity theorem ([BGT13, Prop. 7.10]).

Let $\mathbb{K} : \mathrm{Stab} \to \mathrm{Spt}$ denote the nonconnective algebraic K-theory functor, defined e.g. as in [BGT13, Def. 9.6] following Schlichting. This is a localizing invariant such that $\mathbb{K}_{\geq 0} \simeq \mathrm{K}$.

4.2. **The projective bundle formula.** In this subsection we prove a projective bundle formula computing $E(\mathbf{P}^{1,\flat}_R)$ for the *flat* projective line over any connective $\mathcal{E}_\infty$-ring $R$, and any $R$-linear additive invariant $E$. This essentially follows from a result of Lurie [SAG, Thm. 7.2.2.1].

4.2.1. Consider the following subsets of $\mathbf{Z} \times \mathbf{Z}$:

- $M^+$ consists of pairs $(m,n)$ with $m+n=0$ and $m \geqslant 0$.
- $M^-$ consists of pairs $(m,n)$ with $m+n=0$ and $n \geqslant 0$.
- $M^\pm$ consists of pairs $(m,n)$ with $m+n=0$.

We view each of these as (additive) discrete commutative monoids. For any connective $\mathcal{E}_\infty$-ring $R$, we write $R[T]$, $R[T^{-1}]$ and $R[T^\pm]$ for the monoid algebras $R \otimes \Sigma^\infty_+(M^+)$, $R \otimes \Sigma^\infty_+(M^-)$, and $R \otimes \Sigma^\infty_+(M^\pm)$, respectively.

4.2.2. We write $p_+ : \mathrm{Spec}(R[T]) \to \mathrm{Spec}(R)$, $p_- : \mathrm{Spec}(R[T^{-1}]) \to \mathrm{Spec}(R)$, and $p_\pm : \mathrm{Spec}(R[T^\pm]) \to \mathrm{Spec}(R)$ for the respective projections. Note that under the obvious isomorphisms $M^+ \simeq \mathbf{N} \simeq M^-$, both $\mathrm{Spec}(R[T])$ and $\mathrm{Spec}(R[T^{-1}])$ are canonically identified with the *flat affine line* $\mathbf{A}^{1,\flat}_R$ (Definition 3.1.2). Similarly, under the isomorphism $M^\pm \simeq \mathbf{Z}$, $R[T^\pm]$ is identified with the monoid algebra $R \otimes \Sigma^\infty_+(\mathbf{Z})$. It can also be identified with the localization of $R[T]$ away from $T \in \pi_0(R[T]) \simeq \pi_0(R)[T]$, or the localization of $R[T^{-1}]$ away from $T^{-1}$. In particular, the projections $p_+$, $p_-$ and $p_\pm$ are fibre-smooth in the sense of [SAG, Def. 11.2.3.1], and the affine spectral scheme $\mathbf{G}^\flat_{m,R} = \mathrm{Spec}(R[T^\pm])$ is equipped with open immersions

$$j_+ : \mathrm{Spec}(R[T^\pm]) \to \mathrm{Spec}(R[T]), \quad j_- : \mathrm{Spec}(R[T^-]) \to \mathrm{Spec}(R[T]).$$

The construction of the zero and unit sections of $\mathbf{A}^{1,\flat}$ described in Remark 3.1.3 can be adapted as follows. We recall the constructions of the zero and unit sections of $\mathrm{Spec}(R[T])$. Consider the set $\{0,1\}$, viewed as a pointed multiplicative monoid with base point 0 and identity element 1. Since $M^+$ is freely generated as a (discrete) commutative monoid by the element $(1,0) \in M^+$, either choice of element $i \in \{0,1\}$ determines a unique homomorphism $M^+ \to \{0,1\}$ sending $(1,0) \mapsto i$. Each of these gives rise to $\mathcal{E}_\infty$-ring homomorphisms

$$\sigma_i : R[T] \to R \otimes \Sigma^\infty(\{0,1\}) \simeq R,$$

where we identify $\{0,1\}$ with the pointed 0-sphere $S^0$. We let $s_i$ denote the induced morphisms

$$s_i : \mathrm{Spec}(R) \to \mathrm{Spec}(R[T])$$

for each $i \in \{0,1\}$. The obvious analogous construction gives sections $s_i : \mathrm{Spec}(R) \to \mathrm{Spec}(R[T^{-1}])$. Similarly it is clear that the unit section $s_1$ factors through a morphism $s_1 : \mathrm{Spec}(R) \to \mathrm{Spec}(R[T^\pm])$.

4.2.3. We denote by $\mathbf{P}^{1,\flat}_R$ the *flat projective line* over $R$, see [SAG, Constr. 5.4.1.3] (where it is denoted $\mathbf{P}^1_R$). This is equipped with a canonical morphism $q : \mathbf{P}^{1,\flat}_R \to \mathrm{Spec}(R)$ which is fibre-smooth. Moreover there is a cartesian and cocartesian square of spectral schemes

$$\begin{array}{ccc} \mathrm{Spec}(R[T^{\pm}]) & \xrightarrow{j_-} & \mathrm{Spec}(R[T^{-1}]) \\ \downarrow{\scriptstyle j_+} & & \downarrow{\scriptstyle k_-} \\ \mathrm{Spec}(R[T]) & \xrightarrow{k_+} & \mathbf{P}^{1,\flat}_R \end{array} \tag{4.2.a}$$

where every arrow is an open immersion. In particular, this is a Nisnevich square.

**Notation 4.2.4.** For a functor $E : \mathrm{Stab}_R \to \mathrm{Spt}$, we write:

$$E^+ := E^{\mathrm{Perf}(R[T])}, \quad E^- := E^{\mathrm{Perf}(R[T^{-1}])}, \quad E^{\pm} := E^{\mathrm{Perf}(R[T^{\pm}])}, \quad E^{\boxtimes} := E^{\mathrm{Perf}(\mathbf{P}^{1,\flat}_R)}.$$

Note that $E^+(\mathbf{A}) = E(\mathbf{A} \otimes \mathrm{Perf}(R[T]))$ for $\mathbf{A} \in \mathrm{Stab}_R$, and similarly for $E^-$, $E^{\pm}$ and $E^{\boxplus}$.

**Theorem 4.2.5.** *Let $E$ be an additive invariant of $R$-linear stable $\infty$-categories. Then the two functors* $\mathrm{Perf}(\mathrm{Spec}(R)) \to \mathrm{Perf}(\mathbf{P}^{1,\flat}_R)$*, given by $\mathcal{F} \mapsto q^*(\mathcal{F})$ and $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(-1)$, induce a canonical isomorphism*

$$q^* \oplus (q^* \otimes \mathcal{O}(-1)) : E \oplus E \xrightarrow{\sim} E^{\boxplus}.$$

*Proof.* By Yoneda, it will suffice to show that the maps

$$E(\mathbf{A}) \oplus E(\mathbf{A}) \to E(\mathbf{A} \otimes \mathrm{Perf}(\mathbf{P}^{1,\flat}_R))$$

are invertible for every $\mathbf{A} \in \mathrm{Stab}_R$. For this it will suffice to show that the map

$$E(R) \oplus E(R) \to E(\mathbf{P}^{1,\flat}_R)$$

is invertible for every additive invariant $E$ (as we can then apply this to every $E^{\mathbf{A}}$).

By [SAG, Thm. 7.2.2.1] there is a semi-orthogonal decomposition on $\mathrm{Qcoh}(\mathbf{P}^{1,\flat}_R)$ into two full subcategories both canonically equivalent to $\mathrm{Qcoh}(\mathrm{Spec}(R)) \simeq \mathrm{Mod}_R$. Moreover, an inspection of the proof of *loc. cit.* shows that this restricts to a semi-orthogonal decomposition on $\mathrm{Perf}(\mathbf{P}^{1,\flat}_R)$ by two full subcategories both equivalent to $\mathrm{Perf}(\mathrm{Spec}(R)) \simeq \mathrm{Perf}_R$. Indeed, both functors $q^*$ and $q_*$ preserve perfect complexes (the latter by [SAG, Thm. 6.1.3.2]). The claim then follows by definition of additive invariants. □

**Remark 4.2.6.** Let $\alpha$ denote the composite morphism

$$\alpha : E \oplus E \xrightarrow{\mu} E \oplus E \xrightarrow{q^* \oplus (q^* \otimes \mathcal{O}(-1))} E^{\boxplus},$$

where $\mu$ is the isomorphism induced by the invertible matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

By construction, $\alpha$ fits in the commutative diagram

$$\begin{array}{ccccc} & & E \oplus E & & \\ & \swarrow^{p_-^* \oplus 0} & \downarrow \alpha & \searrow^{p_+^* \oplus 0} & \\ E^- & \xleftarrow[k_-^*]{} & E^{\boxtimes} & \xrightarrow[k_+^*]{} & E^+. \end{array} \tag{4.2.b}$$

4.3. **The Bass fundamental sequence.**

**Theorem 4.3.1.** *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a localizing invariant. Then for any $R$-linear stable $\infty$-category $\mathbf{A}$, there is a natural exact sequence of abelian groups*

$$0 \to E_n^{\mathbf{A}}(R) \xrightarrow{(p_+^*, -p_-^*)} E_n^{\mathbf{A}}(R[T]) \oplus E_n^{\mathbf{A}}(R[T^{-1}]) \xrightarrow{(j_+^*, j_-^*)} E_n^{\mathbf{A}}(R[T^{\pm}]) \xrightarrow{\partial} E_{n-1}^{\mathbf{A}}(R) \to 0$$

*for every integer $n$, where we write $E_n = \pi_n E$.*

*Proof.* By replacing $E$ with $E^{\mathbf{A}}$, we may assume that $\mathbf{A} = \mathrm{Perf}_R$. Since $E$ satisfies Nisnevich descent (e.g. [CMNN, App. A]), the Nisnevich square (4.2.a) gives rise to a cartesian square

$$\begin{array}{ccc} E(\mathbf{P}_R^{1,\flat}) & \xrightarrow{k_-^*} & E(R[T^{-1}]) \\ \downarrow k_+^* & & \downarrow j_-^* \\ E(R[T]) & \xrightarrow{j_+^*} & E(R[T, T^{-1}]) \end{array}$$

and hence to a Mayer–Vietoris long exact sequence

$$\begin{aligned} \cdots \to E_{n+1}(R[T, T^{-1}]) \xrightarrow{\partial} E_n(\mathbf{P}_R^{1,\flat}) \xrightarrow{(k_+^*, -k_-^*)} \\ E_n(R[T]) \oplus E_n(R[T^{-1}]) \xrightarrow{j_+^* \oplus j_-^*} E_n(R[T, T^{-1}]) \xrightarrow{\partial} \cdots \end{aligned} \tag{4.3.a}$$

By the projective bundle formula (Theorem 4.2.5) and Remark 4.2.6, there is a canonical isomorphism $E(R) \oplus E(R) \simeq E(\mathbf{P}_R^{1,\flat})$ under which the map $(k_+^*, -k_-^*)$ in (4.3.a) is $(p_+^*, p_-^*)$ on the first copy of $E_n(R)$ and $(0,0)$ on the second. Here $p_+$ and $p_-$ denote the respective projections $p_+ : \mathrm{Spec}(R[T]) \to \mathrm{Spec}(R)$ and $p_- : \mathrm{Spec}(R[T^{-1}]) \to \mathrm{Spec}(R)$. Since the boundary map is then surjective onto the second copy of $E_n(R)$, and the zero section induces canonical retractions of both $p_+^*$ and $p_-^*$, we see that the long exact sequence (4.3.a) splits up into short exact sequences as in the claim. □

**Remark 4.3.2.** In the case where the localizing invariant $E$ is nonconnective algebraic K-theory $\mathbb{K}$ (Example 4.1.7), the map $\partial : \mathbb{K}_n(R[T^{\pm}]) \to \mathbb{K}_{n-1}(R)$ in the Bass fundamental sequence admits a natural *splitting*, up to an automorphism of $\mathbb{K}_{n-1}(R)$. Indeed, consider the automorphism of $R[T, T^{-1}]$ given by multiplication by $T$. This induces a point $b \in \mathbb{K}(R[T, T^{-1}])[-1]$ which we call the *Bott class*. Now cup product with $b$ induces a canonical map

$$\mathbb{K}(R) \xrightarrow{p_{\pm}^*} \mathbb{K}(R[T^{\pm}]) \xrightarrow{b \cup} \mathbb{K}(R[T^{\pm}])[-1] \xrightarrow{\partial} \mathbb{K}(R)$$

which we claim is invertible. By $\mathbb{K}(R)$-linearity, this is equivalent to the assertion that $\partial$ sends $b \in \mathbb{K}_1(R[T^{\pm}]) \simeq \mathrm{K}_1(R[T^{\pm}])$ to a unit in $\mathbb{K}_0(R) \simeq \mathrm{K}_0(R)$. Since the 1-truncation

$\tau_{\leqslant 1}(\mathrm{K})$ is insensitive to positive homotopy groups [Lur14, Lect. 20, Cor. 4], we may replace $R$ by $\pi_0(R)$. Now the claim is classical, see e.g. the proof of [TT90, Thm. 6.1(b)].

### 4.4. **Delooping localizing invariants of connective spectra.**

**Notation 4.4.1.** We let $\mathrm{Spt}_{\geqslant 0}$ denote the full subcategory of Spt spanned by connective spectra. A *connective fibre sequence* of spectra is a diagram

$$F \to X \to Y,$$

together with a null-homotopy of the composite $F \to Y$, such that the induced map $F \to \mathrm{Fib}(X \to Y)$ induces an isomorphism

$$\tau_{\geqslant 0}(F) \simeq \tau_{\geqslant 0}(\mathrm{Fib}(X \to Y))$$

of connective spectra.

**Definition 4.4.2.** We say that a functor $\mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ is *localizing* if it sends short exact sequences to connective fibre sequences.

**Theorem 4.4.3.** *Let $R$ be a connective $\mathcal{E}_\infty$-ring. The assignment $E \mapsto \tau_{\geqslant 0}(E)$ determines an equivalence*

$$\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}) \to \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}_{\geqslant 0})$$

*from the $\infty$-category of localizing invariants $\mathrm{Stab}_R \to \mathrm{Spt}$, to the $\infty$-category of localizing invariants $\mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$.*

**Example 4.4.4.** Let $\mathrm{K} : \mathrm{Stab} \to \mathrm{Spt}_{\geqslant 0}$ and $\mathbb{K} : \mathrm{Stab} \to \mathrm{Spt}$ be as in Example 4.1.7. Since $\mathbb{K}$ is localizing and satisfies $\mathbb{K}_{\geq 0} \simeq \mathrm{K}$, it follows from Theorem 4.4.3 that there is a canonical isomorphism of localizing invariants

$$\mathbb{K} \simeq \mathrm{K}^{\mathrm{B}}.$$

### 4.5. **The Bass construction.** Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be an arbitrary functor.

4.5.1. Define $V(E)$ and $W(E)$ such that there are cocartesian squares

$$\begin{array}{ccccccc} E & \xrightarrow{\mathrm{incl}_1} & E \oplus E & \xrightarrow{\alpha} & E^{\boxplus} & \xrightarrow{(k_+^*, -k_-^*)} & E^+ \oplus E^- \\ \downarrow & & \downarrow \mathrm{pr}_2 & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E & \longrightarrow & W(E) & \longrightarrow & V(E) \end{array} \tag{4.5.a}$$

in $\mathrm{Fun}(\mathrm{Stab}_R, \mathrm{Spt})$. Here $\alpha$ is as in Remark 4.2.6. We denote by $\psi_E$ the composite $E \to W(E) \to V(E)$.

**Remark 4.5.2.** The commutative diagram (4.2.b) provides a null-homotopy of the composite

$$E \xrightarrow{\mathrm{incl}_2} E \oplus E \xrightarrow{\alpha} E^{\boxplus} \xrightarrow{(k_+^*, -k_-^*)} E^+ \oplus E^-.$$

Since the composite $\mathrm{pr}_2 \circ \mathrm{incl}_2$ is the identity, combining this with the commutative diagram (4.5.a) yields a canonical null-homotopy of the morphism $\psi_E : E \to V(E)$.

**Remark 4.5.3.** The commutative diagram (4.2.b) identifies the upper horizontal composite with the morphism $(p_+^*, -p_-^*) : E \to E^+ \oplus E^-$. It follows that $V(E)$ fits in an exact triangle

$$E \xrightarrow{(p_+^*,-p_-^*)} E^+ \oplus E^- \to V(E).$$

Moreover, since the morphisms $p_+^*$ and $p_-^*$ admit splittings induced by the homomorphisms $R[T] \to R$ and $R[T^{-1}] \to R$, respectively, it follows that the associated long exact sequence splits into short exact sequences

$$0 \to \pi_n(E) \xrightarrow{(p_+^*,-p_-^*)} \pi_n(E^+) \oplus \pi_n(E^-) \to \pi_n(V(E)) \to 0 \tag{4.5.b}$$

for every $n$.

**Remark 4.5.4.** Note that we have commutative squares

$$\begin{array}{ccc} E^{\boxplus} & \xrightarrow{k_+^*} & E^+ \\ \downarrow{\scriptstyle k_-^*} & & \downarrow{\scriptstyle j_+^*} \\ E^- & \xrightarrow{j_-^*} & E^{\pm}, \end{array} \qquad \begin{array}{ccc} E^{\boxplus} & \xrightarrow{(k_+^*,-k_-^*)} & E^+ \oplus E^- \\ \downarrow & & \downarrow{\scriptstyle j_+^* \oplus j_-^*} \\ 0 & \longrightarrow & E^{\pm}, \end{array}$$

where the right-hand square is induced from the left-hand one. This gives rise to a canonical morphism $\theta_E : V(E) \to E^{\pm}$ fitting into the commutative diagram:

$$\begin{array}{ccc} E^{\boxplus} & \xrightarrow{(k_+^*,-k_-^*)} & E^+ \oplus E^- \\ \downarrow & & \downarrow \\ W(E) & \longrightarrow & V(E) \\ \downarrow & & \downarrow{\scriptstyle \theta_E} \\ 0 & \longrightarrow & E^{\pm} \end{array} \qquad (E^+ \oplus E^- \xrightarrow{j_+^* \oplus j_-^*} E^{\pm})$$

**Construction 4.5.5.** Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a functor. Denote by $U(E)$ the fibre of the morphism $\theta_E : V(E) \to E^{\pm}$, so that there is a fibre sequence

$$U(E) \to V(E) \xrightarrow{\theta_E} E^{\pm}.$$

The canonical null-homotopy of the composite $W(E) \to V(E) \xrightarrow{\theta_E} E^{\pm}$ defined above gives rise to a canonical morphism $W(E) \to U(E)$. In particular, we get a canonical morphism

$$\phi_E : E \to W(E) \to U(E).$$

This gives rise to a tower

$$E \xrightarrow{\phi_E} U(E) \xrightarrow{U(\phi_E)} U^2(E) \xrightarrow{U^2(\phi_E)} \cdots$$

whose colimit we denote $E^{\mathrm{B}}$, and call the *Bass construction* on $E$.

**Remark 4.5.6.** Since the functors $V$ and $(-)^{\pm}$ commute with colimits and with $(-)^{\mathbf{A}}$ for $\mathbf{A} \in \mathrm{Stab}_R$, the same holds for $U$. In particular, it follows that there are canonical identifications $U^k(\phi_E) \simeq \phi_{U^k(E)}$ for each $k \geqslant 0$.

**Theorem 4.5.7.** *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ be a functor. The Bass construction $E^{\mathrm{B}}$ satisfies the following properties:*

(i) *If $E$ satisfies the projective bundle formula and is $Q_\pm$-excisive, then the natural morphism $E \to E^{\mathrm{B}}$ induces an isomorphism $E \simeq \tau_{\geqslant 0}(E^{\mathrm{B}})$.*

(ii) *If $E : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ is a localizing invariant, then the functor $E^{\mathrm{B}} : \mathrm{Stab}_R \to \mathrm{Spt}$ is a localizing invariant.*

## 4.6. **Proof of Theorem 4.5.7.**

**Lemma 4.6.1.** *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ be a functor. If $E$ satisfies the projective bundle formula and is $Q_\pm$-excisive, then we have:*

(i) *There is a connective fibre sequence*

$$E \xrightarrow{\psi_E} V(E) \xrightarrow{\theta_E} E^\pm$$

*which is natural in $E$.*

(ii) *There exist canonical morphisms*

$$\tau_{\geqslant 0}(\Omega(E^\pm)) \to E, \quad \sigma_E : E \to \tau_{\geqslant 0}(\Omega(E^\pm))$$

*which exhibit $E$ as a retract of $\tau_{\geqslant 0}(\Omega(E^\pm))$, and are natural in $E$.*

*Proof.* We first show the following weaker version of (i):

($*$) There is a connective fibre sequence

$$\tau_{\geqslant 0}(\Omega(E^\pm)) \to E \xrightarrow{\psi_E} V(E).$$

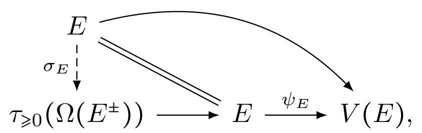
Since $E$ satisfies the projective bundle formula, $\alpha : E \oplus E \to E^{\boxplus}$ is invertible. Considering the diagram (4.5.a), it follows that $E \to W(E)$ is also invertible and that it will suffice to show that the diagram

$$\tau_{\geqslant 0}(\Omega(E^\pm)) \to W(E) \to V(E)$$

is a connective fibre sequence. Since the $\infty$-category $\mathrm{Spt}_{\geqslant 0}$ is prestable, the right-hand square in (4.5.a) is also cartesian [SAG, Cor. C.1.2.6], and in particular induces an isomorphism on fibres. The fact that $E$ is $Q_\pm$-excisive implies that the fibre[6] of the upper arrow $(k_+^*, -k_-^*) : E^{\boxplus} \to E^+ \oplus E^-$ is $\tau_{\geqslant 0}(\Omega(E^\pm))$. This shows claim ($*$).

[6]Note that this fibre, computed in $\mathrm{Spt}_{\geqslant 0}$, is the same as the connective cover of the fibre computed in Spt.

For part (ii), the canonical null-homotopy of the morphism $\psi_E : E \to V(E)$ (Remark 4.5.2) gives rise to a morphism $\sigma_E$ fitting in the commutative diagram




since the lower row is a fibre sequence by claim $(*)$.

Finally we prove (i). By (ii), the diagram $E \to V(E) \to E^{\pm}$ is a retract of the diagram

$$\tau_{\geqslant 0}(\Omega(E^{\pm})) \to \tau_{\geqslant 0}(\Omega(V(E)^{\pm})) \to \tau_{\geqslant 0}(\Omega((E^{\pm})^{\pm})).$$

It will suffice to show that the latter is a fibre sequence. Since the functor $E \mapsto E^{\pm}$ is left-exact and commutes with $\tau_{\geqslant 0}$, this follows from the fact that the diagram

$$\tau_{\geqslant 0}(\Omega(E)) \to \tau_{\geqslant 0}(\Omega(V(E))) \to \tau_{\geqslant 0}(\Omega(E^{\pm}))$$

is a fibre sequence, by $(*)$. □

**Corollary 4.6.2.** *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant 0}$ be a localizing invariant. Then the morphism $\phi_E : E \to U(E)$ induces an isomorphism $E \simeq \tau_{\geqslant 0}(U(E))$.*

*Proof.* The claim is equivalent to the assertion that the diagram

$$E \to V(E) \to E^{\pm}$$

is a connective fibre sequence. This is Lemma 4.6.1(i), which applies since $E$ is localizing. □

**Lemma 4.6.3.** *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a functor. If $E$ is localizing, then we have:*

(i) *There is an exact triangle*

$$E \xrightarrow{\psi_E} V(E) \xrightarrow{\theta_E} E^{\pm}$$

*which is natural in $E$. In other words, the morphism $\phi_E : E \to U(E)$ is invertible.*

(ii) *The canonical morphism $E \to E^{\mathrm{B}}$ is invertible.*

(iii) *There exist canonical morphisms*

$$\Omega(E^{\pm}) \to E, \quad \sigma_E : E \to \Omega(E^{\pm})$$

*which exhibit $E$ as a retract of $\Omega(E^{\pm})$, and are natural in $E$.*

*Proof.* Parts (i) and (iii) follow by the same argument as in the proof of Lemma 4.6.1. The second follows from (i). □

**Lemma 4.6.4.** *The functor $E \mapsto U(E)$ preserves $\tau_{\geqslant 0}$-equivalences. That is, let $E \to E'$ be a morphism in $\mathrm{Fun}(\mathrm{Stab}_R, \mathrm{Spt})$ which induces an isomorphism $\tau_{\geqslant 0}(E) \simeq \tau_{\geqslant 0}(E')$. Then the induced map*

$$\tau_{\geqslant 0}(U(E)) \to \tau_{\geqslant 0}(U(E'))$$

*is invertible.*

*Proof.* Note that the analogous property holds for the functors $(-)^{\pm}$ and $V(-)$: in fact, they are even t-exact (the latter in view of the exact sequences (4.5.b)). As $U(-)$ is the fibre of the morphism $V(-) \to (-)^{\pm}$, the claim follows. □

**Lemma 4.6.5.** *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a functor. We have:*

(i) *There is an exact triangle*

$$E^{\mathrm{B}} \xrightarrow{\psi_{E^{\mathrm{B}}}} V(E^{\mathrm{B}}) \xrightarrow{\theta_{E^{\mathrm{B}}}} (E^{\mathrm{B}})^{\pm}$$

*which is natural in $E$.*

(ii) *There exist canonical morphisms*

$$\Omega((E^{\mathrm{B}})^{\pm}) \to E^{\mathrm{B}}, \quad \sigma_{E^{\mathrm{B}}} : E^{\mathrm{B}} \to \Omega((E^{\mathrm{B}})^{\pm})$$

*which exhibit $E^{\mathrm{B}}$ as a retract of $\Omega((E^{\mathrm{B}})^{\pm})$, and are natural in $E$.*

*Proof.* Part (ii) will follow from (i) as in the proof of Lemma 4.6.1. For (i) it will suffice to show that the morphism $\phi_{E^{\mathrm{B}}} : E^{\mathrm{B}} \to U(E^{\mathrm{B}})$ is invertible. Since the functors $U$, $V$ and $W$ each commute with colimits, it is clear from the construction that this morphism is the colimit of the morphisms $\phi_{U^k(E)} \simeq U^k(\phi_E) : U^k(E) \to U^{k+1}(E)$, $k \geqslant 0$ (Remark 4.5.6). That is, $\phi_{E^{\mathrm{B}}}$ is identified with the canonical morphism fitting in the diagram

$$\begin{array}{ccccccccc}
E & \xrightarrow{\phi_E} & U(E) & \xrightarrow{U(\phi_E)} & U^2(E) & \xrightarrow{U^2(\phi_E)} & \cdots & \longrightarrow & E^{\mathrm{B}} \\
\downarrow{\scriptstyle \phi_E} & & \downarrow{\scriptstyle U(\phi_E)} & & \downarrow{\scriptstyle U^2(\phi_E)} & & & & \downarrow{\scriptstyle \phi_{E^{\mathrm{B}}}} \\
U(E) & \xrightarrow[U(\phi_E)]{} & U^2(E) & \xrightarrow[U^2(\phi_E)]{} & U^3(E) & \xrightarrow[U^3(\phi_E)]{} & \cdots & \longrightarrow & U(E^{\mathrm{B}}),
\end{array}$$

and is clearly invertible. □

**Corollary 4.6.6.** *Let $E : \mathrm{Stab}_R \to \mathrm{Spt}$ be a functor. For every integer $n \geqslant 0$, denote by $(E^{\mathrm{B}})^{\pm n} : \mathrm{Stab}_R \to \mathrm{Spt}$ the functor*

$$((((E^{\mathrm{B}})^{\pm})^{\pm})^{\cdots})^{\pm}$$

*obtained from the Bass construction $E^{\mathrm{B}}$ by an $n$-fold iteration of the functor $(-)^{\pm}$ (e.g. $(E^{\mathrm{B}})^{\pm 0} = E^{\mathrm{B}}$). Then for each $n \geqslant 0$, the functor*

$$\Sigma^n(E^{\mathrm{B}}) : \mathrm{Stab}_R \to \mathrm{Spt}$$

*is a retract of $(E^{\mathrm{B}})^{\pm n} : \mathrm{Stab}_R \to \mathrm{Spt}$.*

*Proof.* By induction, it will suffice to consider $n = 1$ and, by adjunction, to show that $E^{\mathrm{B}}$ is a retract of $\Omega(E^{\mathrm{B}})^{\pm}$. This is Lemma 4.6.5(ii). □

4.6.7. *Proof of Theorem 4.5.7(i).* By Corollary 4.6.2, $\phi_E : E \to U(E)$ is a $\tau_{\geqslant 0}$-equivalence. It follows then from Lemma 4.6.4 that $U^k(\phi_E) : U^k(E) \to U^{k+1}(E)$ is a $\tau_{\geqslant 0}$-equivalence for every $k \geqslant 0$. It follows that the transfinite composite $E \to E^{\mathrm{B}}$ is also a $\tau_{\geqslant 0}$-equivalence.

4.6.8. *Proof of Theorem 4.5.7(ii).* Let $\mathbf{A}' \to \mathbf{A} \to \mathbf{A}''$ be an exact sequence of small stable $\infty$-categories and consider the induced diagram of spectra

$$E^{\mathrm{B}}(\mathbf{A}') \to E^{\mathrm{B}}(\mathbf{A}) \to E^{\mathrm{B}}(\mathbf{A}'').$$

To show that this is an exact triangle, it will suffice to show that each of the induced diagrams

$$\tau_{\geqslant n}(E^{\mathrm{B}}(\mathbf{A}')) \to \tau_{\geqslant n}(E^{\mathrm{B}}(\mathbf{A})) \to \tau_{\geqslant n}(E^{\mathrm{B}}(\mathbf{A}''))$$

is a fibre sequence in $\mathrm{Spt}_{\geqslant n}$, for every $n \leqslant 0$. In other words, it will suffice to show that each functor $\tau_{\geqslant n}(E^{\mathrm{B}}) : \mathrm{Stab}_R \to \mathrm{Spt}_{\geqslant n}$ is a localizing invariant.

By Corollary 4.6.6 we know that $\tau_{\geqslant n}(E^{\mathrm{B}})$ is a retract of $\tau_{\geqslant 0}((E^{\mathrm{B}})^{\pm n})$. Since $(-)^{\pm}$ is left-exact, the latter is isomorphic to $\tau_{\geqslant 0}(E^{\mathrm{B}})^{\pm n} \simeq E^{\pm n}$ (Theorem 4.5.7(i)), which is localizing because $E$ is.

4.7. **Proof of Theorem 4.4.3.** We are now ready to prove Theorem 4.4.3, which asserts that the canonical functor

$$\tau_{\geqslant 0} : \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}) \to \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}_{\geqslant 0}),$$

given by the assignment $E \mapsto \tau_{\geqslant 0}(E)$, is an equivalence.

4.7.1. Note that the Bass construction (Construction 4.5.5) defines a functor $E \mapsto E^{\mathrm{B}}$ from $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}_{\geqslant 0})$ to $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt})$. We claim first that this is a fully faithful left adjoint to $\tau_{\geqslant 0}$. For $E \in \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt}_{\geqslant 0})$, the unit map

$$\eta_E : E \xrightarrow{\sim} \tau_{\geqslant 0}(E^{\mathrm{B}})$$

comes from the natural isomorphisms of Theorem 4.5.7(i). For $E \in \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt})$, the co-unit map

$$\varepsilon_E : (\tau_{\geqslant 0}(E))^{\mathrm{B}} \to E^{\mathrm{B}}$$

is induced from $\tau_{\geqslant 0}(E) \to E$ in view of the fact that the natural map $E \to E^{\mathrm{B}}$ is invertible (Lemma 4.6.3(ii)). One easily verifies the triangle identities.

4.7.2. In order to conclude that the functor $\tau_{\geqslant 0}$ is an equivalence, it will suffice to show that it is conservative (so that the co-unit maps are necessarily isomorphisms). Let $E \to E'$ be a morphism of localizing invariants in $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Stab}_R, \mathrm{Spt})$, and suppose that the induced morphism $\tau_{\geqslant 0}(E) \to \tau_{\geqslant 0}(E')$ is invertible. We argue by decreasing induction on $n$ that the map $\pi_n(E(\mathbf{A})) \to \pi_n(E'(\mathbf{A}))$ is invertible for every $n \leqslant 0$ and every $\mathbf{A} \in \mathrm{Stab}_R$. For $n = 0$ this holds by assumption. The induction step follows from the Bass fundamental sequence (Theorem 4.3.1). The claim follows.

## 5. Homotopy invariant K-theory

### 5.1. Homotopy invariant K-theory.

**Notation 5.1.1.** Given a spectral scheme $S$, let $\mathrm{Perf}(S)$ denote the stable $\infty$-category of perfect complexes on $S$. Denote by $\mathrm{K}(S)$ and $\mathrm{K}^{\mathrm{B}}(S)$ the spectra

$$\mathrm{K}(\mathrm{Perf}(S)), \quad \mathrm{K}^{\mathrm{B}}(\mathrm{Perf}(S)),$$

respectively. Here K is algebraic K-theory (Example 4.1.7) and $\mathrm{K}^{\mathrm{B}}$ is the Bass construction, equivalent to nonconnective algebraic K-theory $\mathbb{K}$ (Example 4.4.4).

**Construction 5.1.2.** For any spectral scheme $S$, consider the spectrum

$$\mathrm{KH}(S) := \varinjlim \mathrm{K}(S \times \mathbf{A}^n),$$

where $\mathbf{A}^n$ is the $n$-dimensional spectral affine space (Example 2.1.3) and the colimit is indexed by the opposite of the (cosifted) full subcategory $\mathbf{A}_S \subseteq \mathrm{Aff}_{/S}$ whose objects are spectral affine spaces $S \times \mathbf{A}^n$ $(n \geqslant 0)$. We also write $\mathrm{KH}(R) = \mathrm{KH}(\mathrm{Spec}(R))$ for any connective $\mathcal{E}_\infty$-ring $R$.

This definition may appear ad-hoc. In the language of $\mathcal{C}$-fibred $S^1$-spectra (Example 2.8.2), we can give a more systematic definition:

**Construction 5.1.3.** Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be an admissible subcategory. The assignments $X \mapsto \mathrm{K}(X)$ and $X \mapsto \mathrm{K}^{\mathrm{B}}(X)$ define presheaves of spectra

$$(\mathcal{C}_{/S})^{\mathrm{op}} \to \mathrm{Spt},$$

which we shall denote by $\mathrm{K}|_{\mathcal{C}}$ and $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}$ and which we view as $\mathcal{C}$-fibred $S^1$-spectra over $S$. Then $\mathrm{KH}|_{\mathcal{C}}$ is the $\mathcal{C}$-fibred $S^1$-spectrum defined as the $\mathbf{A}^1$-localization of $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}$:

$$\mathrm{KH}|_{\mathcal{C}} := \mathrm{L}_{\mathbf{A}^1}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}).$$

The formula (2.1.b) shows that the spectrum of global sections recovers $\mathrm{KH}(S)$ as defined above:

$$\Gamma(S, \mathrm{KH}|_{\mathcal{C}}) \simeq \mathrm{KH}(S).$$

**Proposition 5.1.4.** *The $\mathcal{C}$-fibred $S^1$-spectrum $\mathrm{KH}|_{\mathcal{C}}$ satisfies Nisnevich excision and $\mathbf{A}^1$-homotopy invariance; that is, it is motivic. Moreover, the canonical morphism of motivic $\mathcal{C}$-fibred $S^1$-spectra*

$$\mathbf{L}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}) \to \mathrm{KH}|_{\mathcal{C}}$$

*is invertible.*

*Proof.* Recall that any localizing invariant satisfies Nisnevich excision, see e.g. [CMNN, Prop. A.13], so $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}$ satisfies Nisnevich excision. Thus by Proposition 2.8.6 we have

$$\mathbf{L}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}) \simeq \mathrm{L}_{\mathbf{A}^1}\,\mathrm{L}_{\mathrm{Nis}}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}) \simeq \mathrm{L}_{\mathbf{A}^1}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}),$$

where the latter is $\mathrm{KH}|_{\mathcal{C}}$ by definition. □

**Definition 5.1.5.** We may repeat Construction 5.1.3 in the setting of $\mathcal{C}^{\mathrm{cl}}$-fibred spectra over $S_{\mathrm{cl}}$ (notation as in Example 2.4.2). The restriction of $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}$ along $u : \mathcal{C}^{\mathrm{cl}}_{/S_{\mathrm{cl}}} \to \mathcal{C}_{/S}$ (Remark 2.5.5) is the $\mathcal{C}^{\mathrm{cl}}$-fibred $S^1$-spectrum $u^*(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}}) \simeq v_!(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}})$ which we denote simply by $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}^{\mathrm{cl}}}$. We define the $\mathcal{C}^{\mathrm{cl}}$-fibred $S^1$-spectrum $\mathrm{KH}^{\mathrm{cl}}|_{\mathcal{C}^{\mathrm{cl}}}$ as the $\mathbf{A}^1_{\mathrm{cl}}$-localization of $\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}^{\mathrm{cl}}}$:

$$\mathrm{KH}^{\mathrm{cl}}|_{\mathcal{C}^{\mathrm{cl}}} := \mathrm{L}_{\mathbf{A}^1_{\mathrm{cl}}}(\mathrm{K}^{\mathrm{B}}|_{\mathcal{C}^{\mathrm{cl}}}).$$

We define $\mathrm{KH}^{\mathrm{cl}}(S_{\mathrm{cl}})$ as the spectrum of global sections $\Gamma(S_{\mathrm{cl}}, \mathrm{KH}^{\mathrm{cl}}|_{\mathcal{C}^{\mathrm{cl}}})$. This is nothing else than Weibel's homotopy invariant K-theory spectrum (see [Cis13] for this point of view).

To formulate the main result of this subsection, we introduce the following notation:

**Notation 5.1.6.** Let $\mathcal{A}_{/S} \subseteq \mathrm{Aff}_{/S}$ be a narrow subcategory (e.g. $\mathcal{A}_{/S} = \mathrm{Sm}_{/S}$), let $\mathcal{A}^{\mathrm{cl}}_{/S} \subseteq \mathrm{AffCl}_{/S}$ be as in Example 2.4.2, and let $w : \mathcal{A}_{/S} \to \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ be the classical truncation functor (Construction 2.4.6). Recall the equivalence

$$\mathbf{L}w_! : \mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}} \to \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})_{\mathrm{Spt}}, \quad w^* : \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})_{\mathrm{Spt}} \to \mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}}$$

from Theorem 2.8.5.

Then we have:

**Theorem 5.1.7.** *Let the notation be as in 5.1.6. Then there are canonical isomorphisms*

$$\mathbf{L}w_!(\mathrm{KH}|_{\mathcal{A}}) \simeq \mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}},$$
$$\mathrm{KH}|_{\mathcal{A}} \simeq w^*(\mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}})$$

*of motivic $\mathcal{A}^{\mathrm{cl}}$-fibred $S^1$-spectra over $S_{\mathrm{cl}}$, resp. of motivic $\mathcal{A}$-fibred $S^1$-spectra over $S$.*

See Subsect. 5.4 for the proof. Note that this immediately implies Theorem B.

**Corollary 5.1.8.** *For every quasi-compact quasi-separated spectral algebraic space $S$, there is a canonical isomorphism of spectra $\mathrm{KH}(S) \simeq \mathrm{KH}^{\mathrm{cl}}(S_{\mathrm{cl}})$, functorial in $S$.*

*Proof.* By Nisnevich descent we may assume that $S$ is affine. Passing to global sections in Theorem 5.1.7, we get an isomorphism of spectra

$$\mathrm{KH}(S) = \Gamma(S, \mathrm{KH}|_{\mathcal{A}}) \xrightarrow{\sim} \Gamma(S, w^*(\mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}})) \simeq \Gamma(S_{\mathrm{cl}}, \mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}}) = \mathrm{KH}^{\mathrm{cl}}(S_{\mathrm{cl}})$$

as claimed. □

**Corollary 5.1.9.** *For any connective $\mathcal{E}_\infty$-ring $R$, there is a canonical isomorphism of spectra $\mathrm{KH}(R) \simeq \mathrm{KH}^{\mathrm{cl}}(\pi_0(R))$, functorial in $R$.*

5.2. **Connective comparison.** In this subsection our goal is to prove the following two statements:

**Proposition 5.2.1.** *Let the notation be as in 5.1.6. There is a canonical isomorphism of $\mathcal{A}^{\mathrm{cl}}$-fibred $S^1$-spectra*

$$\mathbf{L}w_!(\mathrm{K}|_{\mathcal{A}}) \to \mathbf{L}(\mathrm{K}|_{\mathcal{A}^{\mathrm{cl}}}).$$

**Proposition 5.2.2.** *Let the notation be as in 5.1.6. Let $\mathcal{B}_{/S} \subseteq \mathrm{Aff}_{/S}$ be a broad subcategory containing $\mathcal{A}_{/S}$ and let $\iota : \mathcal{A}_{/S} \hookrightarrow \mathcal{B}_{/S}$ denote the inclusion. Then the canonical morphisms of $\mathcal{B}$-fibred $S^1$-spectra*

$$\mathrm{L}_{\mathrm{Nis}}\,\iota_!(\mathrm{K}|_{\mathcal{A}}) \to \mathrm{K}|_{\mathcal{B}}$$
$$\mathbf{L}\iota_!(\mathrm{K}|_{\mathcal{A}}) \to \mathbf{L}(\mathrm{K}|_{\mathcal{B}})$$

*are invertible.*

**Corollary 5.2.3.** *The motivic $\mathcal{B}$-fibred $S^1$-spectrum $\mathbf{L}(\mathrm{K}|_{\mathcal{B}})$ is nil-local (Definition 2.5.2).*

*Proof.* By Proposition 5.2.2, $\mathbf{L}(\mathrm{K}|_{\mathcal{B}})$ belongs to the essential image of $\mathbf{L}\iota_! : \mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}} \to \mathbf{H}(\mathcal{B}_{/S})_{\mathrm{Spt}}$. Hence it is nil-local by Theorem 2.6.2. □

We will deduce Propositions 5.2.1 and 5.2.2 from a representability statement, Proposition 5.2.5 below.

**Construction 5.2.4.** Let $R$ be a connective $\mathcal{E}_\infty$-ring. Denote by $\mathrm{Mod}_R^{\mathrm{proj}}$ the $\infty$-category of finitely generated projective $R$-modules, and by $\mathrm{Mod}_R^{\mathrm{free}}$ the full subcategory of free $R$-modules of finite rank. Let $X(R)$ denote the underlying $\infty$-groupoid $(\mathrm{Mod}_R^{\mathrm{free}})^{\simeq}$, obtained by discarding non-invertible morphisms. Note that $X(R)$ is nothing else than the coproduct of the classifying spaces $\mathrm{BGL}_n(R)$ over $n \geqslant 0$, where $\mathrm{GL}_n(R)$ is the space of automorphisms of the free $R$-module $R^{\oplus n}$. Note also that the symmetric monoidal structure on $\mathrm{Mod}_R^{\mathrm{proj}}$ induces a structure of grouplike $\mathcal{E}_\infty$-monoid on $X(R)$. Moreover, formation of $X(R)$ is functorial and we may regard the assignment $\mathrm{Spec}(R) \mapsto X(R)$ as a presheaf of grouplike $\mathcal{E}_\infty$-spaces on the site of affine spectral schemes.

**Proposition 5.2.5.** *Let $\mathcal{C}_{/S} \subseteq \mathrm{Aff}_{/S}$ be any admissible subcategory. Denote by $X_S$ the presheaf on $\mathcal{C}_{/S}$ given by the assignment $\mathrm{Spec}(R) \mapsto X(R)$, and by $(X_S)^{\mathrm{gp}}$ its group completion. Then there is a canonical morphism of $\mathcal{C}$-fibred $\mathcal{E}_\infty$-groups*

$$(X_S)^{\mathrm{gp}} \to \Omega^\infty(\mathrm{K}|_{\mathcal{C}})$$

*which induces an isomorphism $\mathrm{L}_{\mathrm{Zar}}(X_S)^{\mathrm{gp}} \simeq \Omega^\infty(\mathrm{K}|_{\mathcal{C}})$.*

*Proof.* Let $X'_S$ denote the presheaf $\mathrm{Spec}(R) \mapsto (\mathrm{Mod}_R^{\mathrm{proj}})^{\simeq}$. Then by [Lur14, Lect. 19, Thm. 5] there is a canonical isomorphism $(X'_S)^{\mathrm{gp}} \simeq \Omega^\infty(\mathrm{K}|_{\mathcal{C}})$. Therefore it will suffice to show that the monomorphism $X_S \hookrightarrow X'_S$ induces an effective epimorphism of Zariski sheaves $\mathrm{L}_{\mathrm{Zar}}(X_S) \to X'_S$ (see [HTT, Ex. 5.2.8.16]). This is clear since every finitely generated projective $R$-module is Zariski-locally free. □

**Remark 5.2.6.** Note that the functors $\mathrm{L}_{\mathrm{Nis}}\,\iota_!$ and $\mathbf{L}\iota_!$ preserve connective objects. Indeed, the essential images of the fully faithful functors

$$\mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}_{\geq 0}} \hookrightarrow \mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}},$$
$$\mathbf{H}(\mathcal{B}_{/S})_{\mathrm{Spt}_{\geq 0}} \hookrightarrow \mathbf{H}(\mathcal{B}_{/S})_{\mathrm{Spt}},$$

are generated under colimits by objects of the form $\Sigma^\infty_+(X)$ with $X \in \mathcal{A}_{/S}$, resp. $X \in \mathcal{B}_{/S}$.

*Proof of Proposition 5.2.1.* Since both source and target are connective, it will suffice to show the claim for the underlying $\mathcal{A}$-fibred $\mathcal{E}_\infty$-group $\Omega^\infty(\mathrm{K}|_\mathcal{A})$. Since each of the functors $\mathrm{L}_{\mathrm{Nis}}$, $\iota_!$ and $\iota^*$ commutes with colimits and finite products, and hence with group completion (see e.g. [Hoy, Lem. 5.5]), we reduce using Proposition 5.2.5 to showing the analogous claim for the $\mathcal{A}$-fibred $\mathcal{E}_\infty$-group $X_S$, i.e., that the canonical morphism

$$\mathbf{L}w_!(X_S) \simeq X^{\mathrm{cl}}_{S_{\mathrm{cl}}}$$

is invertible, where the right-hand side is the construction analogous to Construction 5.2.4 in classical algebraic geometry. The claim now follows from the fact that $w : \mathcal{A}_{/S} \to \mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}}$ sends the spectral affine schemes[7] $\mathrm{GL}_{n,S}$ to their classical counterparts. □

*Proof of Proposition 5.2.2.* As above, we reduce to the analogous claim for the $\mathcal{B}$-fibred $\mathcal{E}_\infty$-group $X_S$. This follows from the fact that the classifying spaces $\mathrm{BGL}_{n,S}$ are colimits of finite products of the smooth spectral schemes $\mathrm{GL}_{n,S}$. □

5.3. **Bott periodicity.** We keep the notation of 5.1.6. In this subsection we will show that $\mathrm{KH}|_\mathcal{B}$ can be described as the Bott periodization of the motivic localization of $\mathrm{K}|_\mathcal{B}$, and similarly for $\mathrm{KH}|_\mathcal{A}$. This will allow us to prove a nonconnective analogue of Proposition 5.2.2, see Corollary 5.3.7.

**Construction 5.3.1.** Let $R$ be a connective $\mathcal{E}_\infty$-ring. Denote by $R\{T\}$ the free $\mathcal{E}_\infty$-$R$-algebra on one generator $T$ and by $R\{T, T^{-1}\}$ the localization away from $T \in \pi_0(R\{T\}) \simeq \pi_0(R)[T]$. Just as in Remark 4.3.2, the automorphism of $R\{T, T^{-1}\}$ given by multiplication by $T$ induces a canonical element $b \in \mathrm{K}_1(R\{T, T^{-1}\})$ which we also call the Bott class. Again by [Lur14, Lect. 20, Cor. 4] there is a canonical isomorphism $\mathrm{K}_1(R\{T, T^{-1}\}) \simeq \mathrm{K}_1(\pi_0(R)[T, T^{-1}])$ under which $b$ corresponds to the usual Bott class. In particular, the canonical bijection $\mathrm{K}_1(R\{T, T^{-1}\}) \to \mathrm{K}_1(R[T, T^{-1}])$ (induced by $\varepsilon$, see Remark 3.3.1) sends $b$ to $b$.

**Remark 5.3.2.** The Bott class may be regarded as a morphism of $\mathcal{B}$-fibred $S^1$-spectra

$$b : \Sigma^\infty(\mathbf{G}_{m,S}, 1)[1] \to \mathrm{K}|_\mathcal{B}.$$

To simplify notation, we set $\mathbf{T}_S := \Sigma^\infty(\mathbf{G}_{m,S}, 1)[1]$.

**Definition 5.3.3.** Recall that the $\mathcal{B}$-fibred $S^1$-spectrum $\mathrm{K}|_\mathcal{B}$ admits an $\mathcal{E}_\infty$-ring structure, induced by the symmetric monoidal structure on perfect complexes. We say that a $\mathrm{K}|_\mathcal{B}$-module $\mathcal{F}$ is *Bott-periodic* if the canonical morphism

$$b : \mathcal{F} \to \underline{\mathrm{Hom}}(\mathbf{T}_S, \mathcal{F})$$

induced by the Bott class (via the action of $\mathrm{K}|_\mathcal{B}$ on $\mathcal{F}$) is invertible. We define Bott-periodic $\mathrm{K}|_\mathcal{A}$-modules similarly. Note that the full subcategory spanned by Bott-periodic $\mathrm{K}|_\mathcal{B}$-modules (resp. $\mathrm{K}|_\mathcal{A}$-modules) is a left localization; we refer to the left adjoint Q as *Bott periodization.*

[7] Since $\mathrm{GL}_{n,S}$ are Zariski-open inside spectral affine spaces, they belong not only to $\mathrm{Sm}_{/S}$ but even to $\mathcal{A}^0_{/S}$ (Example 2.1.9) and hence to any narrow $\mathcal{A}_{/S}$.

**Theorem 5.3.4.** *The canonical morphisms*

$$\mathrm{K}|_{\mathcal{B}} \to \mathrm{KH}|_{\mathcal{B}}$$
$$\mathrm{K}|_{\mathcal{A}} \to \mathrm{KH}|_{\mathcal{A}}$$

*induce isomorphisms*

$$\mathrm{Q}\left(\mathbf{L}(\mathrm{K}|_{\mathcal{B}})\right) \simeq \mathrm{KH}|_{\mathcal{B}},$$
$$\mathrm{Q}\left(\mathbf{L}(\mathrm{K}|_{\mathcal{A}})\right) \simeq \mathrm{KH}|_{\mathcal{A}}$$

*of Bott-periodic motivic fibred $S^1$-spectra.*

The following description of Bott periodization will be used in the proof of Theorem 5.3.4.

**Remark 5.3.5.** Explicitly, the Bott periodization of a *motivic* $\mathrm{K}|_{\mathcal{B}}$-module $\mathcal{F}$ can be computed as the colimit of the tower

$$\mathcal{F} \xrightarrow{b} \underline{\mathrm{Hom}}(\mathbf{T}_S, \mathcal{F}) \xrightarrow{b} \underline{\mathrm{Hom}}(\mathbf{T}_S^{\otimes 2}, \mathcal{F}) \xrightarrow{b} \cdots$$

according to Theorem 3.8 and the proof of Lemma 4.9 of [Hoy]. Moreover, if $\mathcal{F}$ is nil-local, then we may replace $b$ by the Bott class in $\mathbf{G}^{\flat}_{m,S}$ (Remark 4.3.2).

**Remark 5.3.6.** Since the functors $\mathbf{L}\iota_!$ and $\iota^*$ are symmetric monoidal (Remarks 2.3.7 and 2.8.4), they extend to an adjunction

$$\mathbf{L}\iota_! : \mathrm{Mod}_{\mathbf{L}(\mathrm{K}|_{\mathcal{A}})}(\mathbf{H}(\mathcal{A}_{/S})_{\mathrm{Spt}}) \rightleftarrows \mathrm{Mod}_{\mathbf{L}(\mathrm{K}|_{\mathcal{B}})}(\mathbf{H}(\mathcal{B}_{/S})_{\mathrm{Spt}}) : \iota^*.$$

Since $\mathbf{T}_S$ belongs to the essential image of $\mathbf{L}\iota_!$ (as $\mathbf{G}_{m,S}$ belongs to $\mathcal{A}_{/S}$), it follows from Remark 2.3.7(iii) and the fact that $\iota^*$ preserves colimits that $\iota^*$ preserves Bott-periodic objects and commutes with the Bott periodization functor Q. Its left adjoint on Bott-periodic objects is given by $\mathcal{F} \mapsto \mathrm{Q}(\mathbf{L}\iota_!(\mathcal{F}))$ and preserves Q-equivalences.

*Proof of Theorem 5.3.4.* Since $\iota^*$ commutes with Q (Remark 5.3.5), it will suffice to consider the first map. Just as in the classical setting [Cis13, Prop. 2.10], one observes that up to $\mathbf{A}^{1,\flat}$-localization, hence also up to $\mathbf{A}^1$-localization by Lemma 3.3.3, the Bass construction (Construction 4.5.5) simplifies to give the formula

$$\mathrm{KH}|_{\mathcal{B}} \simeq \varinjlim \left( \mathbf{L}(\mathrm{K}|_{\mathcal{B}}) \xrightarrow{b} \underline{\mathrm{Hom}}\left(\Sigma^\infty(\mathbf{G}^{\flat}_{m,S}, 1)[1], \mathbf{L}(\mathrm{K}|_{\mathcal{B}})\right) \xrightarrow{b} \cdots \right),$$

where the maps are induced by the Bott class $b \in \mathrm{K}_1(\mathbf{G}^{\flat}_{m,S})$ (Remark 4.3.2). But by Remark 5.3.5 and Corollary 5.2.3, this is isomorphic to the Bott periodization $\mathrm{Q}(\mathbf{L}(\mathrm{K}|_{\mathcal{B}}))$ with respect to $b \in \mathrm{K}_1(\mathbf{G}_{m,S})$. □

Using Theorem 5.3.4 we may deduce the following $S^1$-stable analogue of Proposition 5.2.2, which holds up to $\mathbf{A}^1$-homotopy and Bott periodization:

**Corollary 5.3.7.** *The canonical morphism*

$$\mathbf{L}\iota_!(\mathrm{KH}|_{\mathcal{A}}) \simeq \mathbf{L}\iota_!\iota^*(\mathrm{KH}|_{\mathcal{B}}) \to \mathrm{KH}|_{\mathcal{B}}$$

*induces an isomorphism*

$$\mathrm{Q}\left(\mathbf{L}\iota_!(\mathrm{KH}|_{\mathcal{A}})\right) \simeq \mathrm{KH}|_{\mathcal{B}}$$

*of Bott-periodic motivic $\mathcal{B}$-fibred $S^1$-spectra.*

*Proof.* Follows immediately from Theorem 5.3.4 and Remark 5.3.6. □

**Remark 5.3.8.** In the statement of Corollary 5.3.7, we can replace the source with $\mathbf{L}\iota_!(\mathrm{K}^{\mathrm{B}}|_{\mathcal{A}})$. That is, the canonical map

$$\mathrm{Q}(\mathbf{L}\iota_!(\mathrm{K}^{\mathrm{B}}|_{\mathcal{A}})) \to \mathrm{KH}|_{\mathcal{B}}$$

is also invertible. This follows from Proposition 5.1.4 and the fact that $\iota_!$ preserves motivic equivalences (Lemma 2.2.5).

5.4. **Proof of Theorem 5.1.7.** The only remaining ingredient is the behaviour of Bott periodization under the equivalence

$$\mathbf{L}w_! : \mathbf{H}(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$$

of Theorem A. But the fact that it commutes with internal homs (Remark 2.7.9(iv)) immediately implies that we have

$$\mathbf{L}w_!(\mathrm{Q}(\mathcal{F})) \simeq \mathrm{Q}^{\mathrm{cl}}(\mathbf{L}w_!(\mathcal{F}))$$

in $\mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$ for every $\mathbf{L}(\mathrm{K})$-module $\mathcal{F}$ in $\mathbf{H}(\mathcal{A}_{/S})$. Here $\mathrm{Q}^{\mathrm{cl}}$ denotes Bott periodization of a SmCl-fibred motivic space with the classical Bott element.

Now consider the commutative diagram in $\mathbf{H}(\mathcal{A}^{\mathrm{cl}}_{/S_{\mathrm{cl}}})$

$$\begin{array}{ccc}
\mathbf{L}w_!\,(\mathrm{Q}(\mathbf{L}(\mathrm{K}|_{\mathcal{A}}))) & \longrightarrow & \mathbf{L}w_!(\mathrm{KH}|_{\mathcal{A}}) \\
\downarrow & & \downarrow \\
\mathrm{Q}(\mathbf{L}(\mathrm{K}|_{\mathcal{A}^{\mathrm{cl}}})) & \longrightarrow & \mathrm{KH}^{\mathrm{cl}}|_{\mathcal{A}^{\mathrm{cl}}}.
\end{array}$$

The assertion of Theorem 5.1.7 is that the right-hand vertical arrow is invertible. Since the horizontal arrows are isomorphisms by Theorem 5.3.4 (and its classical analogue), it will suffice to demonstrate the invertibility of the left-hand vertical arrow. Since $\mathbf{L}w_!$ commutes with Q, this is identified with the Bott periodization of the canonical morphism

$$\mathbf{L}w_!(\mathrm{K}|_{\mathcal{A}}) \simeq \mathbf{L}w_!(\mathbf{L}(\mathrm{K}|_{\mathcal{A}})) \to \mathbf{L}(\mathrm{K}|_{\mathcal{A}^{\mathrm{cl}}}),$$

which is invertible by Proposition 5.2.1.

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Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

*Email address*: `denis-charles.cisinski@mathematik.uni-regensburg.de`

*URL*: `http://www.mathematik.uni-regensburg.de/cisinski/`

Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France

*Email address*: `khan@ihes.fr`

*URL*: `https://www.preschema.com`