EQUIVARIANT GENERALIZED COHOMOLOGY VIA STACKS

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ABSTRACT. We prove a general form of the statement that the cohomology of a quotient stack can be computed by the Borel construction. It also applies to the lisse extensions of generalized cohomology theories like motivic cohomology and algebraic cobordism. As a consequence, we deduce the localization property for the equivariant algebraic bordism theory of Deshpande–Krishna–Heller–Malagón-López. We also give a Bernstein–Lunts-type gluing description of the ∞-category of equivariant sheaves on a scheme X, in terms of nonequivariant sheaves on X and sheaves on its Borel construction.

INTRODUCTION

The formalism of Grothendieck's six operations on (derived categories of) étale sheaves can be extended to algebraic stacks (see [LZ, LO]). Specialized to quotient stacks, this affords simple definitions of equivariant (co)homology. For example, if X is a variety with an action of an algebraic group G, we may define the G-equivariant Borel–Moore homology of X (with coefficients in a commutative ring Λ) as the hypercohomology of the complex $f^!(\Lambda_{BG})$, where Λ_{BG} is the constant sheaf on the classifying stack BG and $f : [X/G] \to BG$ is the projection from the quotient stack.

The classical approach to definitions of equivariant cohomology and (Borel–Moore) homology is via algebraic analogues of the Borel construction [Bor]. In this paper, we prove a very general version of a comparison between the two approaches.

On first pass, let's formulate the result for Betti or étale sheaves. Let $\mathbf{D}(X)$ denote the ∞-category of Betti or ℓ-adic sheaves on a locally of finite type

Date: 2022-09-16.

Artin stack X over a field k. Let G be an algebraic group over a field k and let $\mathrm{H}^*_G(X)$ and $\mathrm{H}^{\mathrm{BM},G}_{-*}(X)$ denote the hypercohomology groups of $f_*f^*(\Lambda_{BG})$ and $f_*f^!(\Lambda_{BG})$, respectively, for any X with G-action (where Λ denotes the constant sheaf $\mathbf{Q}$ or $\mathbf{Q}_\ell$, in the Betti or ℓ-adic cases respectively). We regard the Borel construction as a G-equivariant ind-scheme U_∞, e.g. for $G = \mathrm{GL}_n$ this is the infinite Grassmannian $\mathrm{Gr}_{n,\infty}$ of rank n subspaces (see Sect. 3 for details). Then we claim (see Corollaries 5.1 and 5.2):

Theorem A. *For every locally of finite type Artin stack X over k, there are canonical isomorphisms for all $n \in \mathbf{Z}$*

$$\mathrm{H}^n_G(X) \simeq \mathrm{H}^n(X \overset{G}{\times} U_\infty)$$

$$\mathrm{H}^{\mathrm{BM},G}_n(X) \simeq \mathrm{H}^{\mathrm{BM}}_n(X \overset{G}{\times} U_\infty).$$

Here we have written $X \times^G U_\infty$ for the quotient ind-stack $[(X \times U_\infty)/G]$, and we have (essentially by definition)

$$\mathrm{H}^n(X \overset{G}{\times} U_\infty) \simeq \varprojlim_\nu \mathrm{H}^n(X \overset{G}{\times} U_\nu),$$

$$\mathrm{H}^{\mathrm{BM}}_n(X \overset{G}{\times} U_\infty) \simeq \varprojlim_\nu \mathrm{H}^{\mathrm{BM}}_{n+2d_\nu}(X \overset{G}{\times} U_\nu)(-d_\nu),$$

where $d_\nu = \dim(U_\nu/G)$.

For X a quasi-projective k-scheme on which G acts linearly, each $X \times^G U_\nu$ is a scheme (recall that G acts freely on U_ν). The right-hand sides of Theorem A are usually taken as *definitions* of equivariant cohomology and Borel–Moore homology (see e.g. [Lus, §1]). They have also been considered in the case where X itself is a stack, e.g. the moduli stack of objects in an abelian or dg-category (see [Joy, §2.3] or [Dav, 2.3.2]).

Our second main result provides a complete description of the ∞-category of sheaves on a quotient stack. It is inspired by the construction of the equivariant derived category of Bernstein and Lunts (compare [BL, Def. 2.1.3]).

Theorem B. *For every Artin stack X locally of finite type over k with G-action, there is a cartesian square of ∞-categories*

$$\begin{array}{ccc} \mathbf{D}([X/G]) & \longrightarrow & \mathbf{D}(X \times^G U_\infty) \\ \downarrow & & \downarrow \\ \mathbf{D}(X) & \longrightarrow & \mathbf{D}(X \times U_\infty). \end{array}$$

By definition, $\mathbf{D}(X \times^G U_\infty)$ is the limit over ν of $\mathbf{D}(X \times^G U_\nu)$. Thus Theorem B asserts in particular that a sheaf on the quotient stack $[X/G]$ amounts to the data of:

(i) a sheaf $\mathcal{G} \in \mathbf{D}(X)$;

(ii) a collection of sheaves $\mathcal{F}_\nu \in \mathbf{D}(X \times^G U_\nu)$ for every ν, with compatibility isomorphisms $\mathcal{F}_\nu|_{X \times^G U_{\nu+1}} \simeq \mathcal{F}_{\nu+1}$;

(iii) for every ν, an isomorphism $\mathcal{G}|_{X\times U_\nu} \simeq \mathcal{F}_\nu|_{X\times U_\nu}$;

and compatibilities between the isomorphisms of (ii) and (iii). See Corollary 8.5. Note that, in contrast with [BL], we work with unbounded derived categories.

The main result of this paper is in fact an extension of Theorem A to generalized cohomology theories such as motivic cohomology (≈ higher Chow groups), algebraic cobordism, and (variants of) algebraic K-theory. Similarly, we will see that Theorem B applies to the stable motivic homotopy category.

The most relevant distinction between the above setup and the motivic one is the failure of étale descent in the latter setting; for example, algebraic K-theory only satisfies Nisnevich descent.[1] Surprisingly, there are nevertheless well-behaved extensions of these theories to stacks[2] via the mechanism of *lisse extension*, see [KR, Kha4][3]. We may thus use the above approach with quotient stacks to define equivariant versions of generalized cohomology theories:

$$\mathrm{H}^{\mathrm{BM},G}_s(X;E)(-r) = \pi_s R\Gamma([X/G], f^!(E|_{BG})(-r))$$

for a motivic spectrum $E \in \mathbf{SH}(k)$, where $r, s \in \mathbf{Z}$. On the other hand, previous definitions via the Borel construction have long been considered in the literature already by B. Totaro [Tot], Edidin–Graham [EG], Deshpande [Des], Heller–Malagón-López [HML], and A. Krishna [Kri]. The general form of Theorem A will show that these approaches coincide (Corollaries 6.1 and 6.2). For example, we will compare equivariant motivic Borel–Moore homology with the equivariant higher Chow groups of [EG] (Corollary 6.4).

Let $\mathrm{MGL} \in \mathbf{SH}(k)$ denote Voevodsky's algebraic cobordism spectrum. When k is of characteristic zero, a geometric model for "lower" algebraic bordism of quasi-projective k-schemes was given by Levine and Morel [LM]. That this agrees with $\mathrm{H}^{\mathrm{BM}}_{2*}(X;\mathrm{MGL})(-*)$ was proven by Levine [Lev], using the Hoyois–Hopkins–Morel theorem [Hoy2]. We thus get a comparison between $\mathrm{H}^{\mathrm{BM},G}_{2*}(X;\mathrm{MGL})(-*)$ and equivariant algebraic bordism defined using the Borel construction (see Corollary 7.2).

We will use this comparison to deduce the right-exact localization sequence in equivariant algebraic bordism:

Theorem C. *Let k be a field of characteristic zero, G an algebraic group over k, and X a quasi-projective k-scheme with linearized G-action. Then for every G-invariant closed subscheme $Z \subseteq X$ with open complement $U = X \setminus Z$*

[1]With rational coefficients, algebraic K-theory does satisfy étale descent on schemes. However, we warn the reader that there is still a distinction between rational algebraic K-theory of stacks and its lisse-extended version. That is, algebraic K-theory admits a "genuine" extension to stacks, via the theory of perfect complexes on stacks, which does not satisfy étale descent even with rational coefficients (see e.g. [Kha2, §5] and Sect. 4) . In [KR], we also constructed "genuine" counterparts to lisse-extended motivic cohomology and algebraic cobordism which we expect to exhibit similar behaviour.

[2]at least, to quasi-separated Artin stacks with separated diagonal; see Theorem 0.2

[3]This generalizes the étale-local formalism of [Kha1, App. A], which is good enough for oriented B.M. homology theories with rational coefficients, but not with integral coefficients or for non-oriented theories such as the Milnor–Witt version of motivic B.M. homology.

and every $n \in \mathbf{Z}$, there is an exact sequence

$$\Omega_n^G(Z) \to \Omega_n^G(X) \to \Omega_n^G(U) \to 0.$$

For X and Z smooth equidimensional this leads to a Gysin sequence

$$\Omega_G^{n-c}(Z) \to \Omega_G^n(X) \to \Omega_G^n(U) \to 0$$

where $c = \mathrm{codim}(Z, X)$.

It is not clear whether Theorem C was expected. In equivariant Chow homology the analogous property is obvious (see [EG, Prop. 5]) since any homogeneous component of $\mathrm{A}_*^G(X)$ can be computed using a single approximation U_ν/G for large enough ν (as opposed to the entire tower U_∞/G). In equivariant algebraic bordism, A. Krishna showed exactness at the end (i.e., surjectivity of restriction to an open) and explained why he did not believe exactness in the middle should hold (see Prop. 5.3 in [Kri] and the discussion just before). In a paper that appeared on arXiv around the same time, J. Heller and J. Malagón-López claimed[4] that the localization sequence holds as in Theorem C (see [HML, Thm. 20]), but their proof also only shows exactness on the right (this mistake is well-known in the area).

We also consider the lisse-extension of algebraic K-theory and its variants. In this example there are already "genuine" extensions of K-theory and G-theory to stacks, defined via perfect and coherent complexes, respectively, which do *not* agree with the lisse extension. Instead, we find e.g. that the lisse-extended G-theory of a quotient stack $[X/G]$ can be computed by the Borel construction (see Corollary 4.4). The latter can often be identified with the derived completion of the G-theory spectrum (with respect to the augmention ideal in the representation ring of G), see [CJ, TVdB]. Therefore, lisse-extended G-theory may be regarded as a "global" version of completed G-theory.

0.1. **Outline.** We begin in Sect. 1 by recalling the lisse extension construction from [KR, §12]. The key technical result (Proposition 2.2) is proven in Sect. 2: given a lisse-extended $\mathbf{A}^1$-invariant Nisnevich sheaf F, a stack S, and an ind-stack U_∞ over S satisfying appropriate hypotheses, it states that $F(S)$ can be approximated by the pro-system $F(U_\infty)$. After reductions, this eventually boils down to an argument extracted from [MV].

In Sect. 3 we specialize to the setting considered in the introduction. The Borel construction satisfies the conditions of Proposition 2.2, so we find that for a lisse-extended $\mathbf{A}^1$-invariant sheaf of spectra F, its value on a quotient stack $[X/G]$ can be computed as the homotopy limit

$$F([X/G]) \simeq \varprojlim_{\nu} F(X \overset{G}{\times} U_\nu).$$

[4]They use a different definition of cobordism than [Kri] (which is based on [Des]), but show in [HML, Rem. 14] that the two are isomorphic.

Note that a priori there is no reason that this should yield isomorphisms

$$\pi_i F([X/G]) \simeq \varprojlim_{\nu} \pi_i F(X \overset{G}{\times} U_\nu).$$

That this is the case was surprising to us and follows from the fact that the above formula for $F([X/G])$ can be refined to an isomorphism of pro-spectra.

We then apply the result to various cohomology theories of interest. In Sect. 4 we consider lisse-extended algebraic K-theory and variants. Sect. 5 deals with the Betti/étale sheaf theories and proves Theorem A. We then consider generalized cohomology theories, i.e., cohomology theories represented by motivic spectra, in Sect. 6. Our results on algebraic (co)bordism are in Sect. 7.

Finally, we prove Theorem B in Sect. 8.

0.2. **Conventions on stacks.**

0.2.1. *The Nisnevich topology.* Recall that the Nisnevich topology is generated by families of étale morphisms $(Y_\alpha \to X)_\alpha$ such that the morphism $\coprod_\alpha Y_\alpha \to X$ is surjective on field-valued points (see e.g. [BH, App. A]).

A smooth morphism of schemes admits étale-local sections if and only if it is surjective (see [EGA, Cor. 17.16.3(ii)]). Here is the Nisnevich analogue:

Lemma 0.1. *Let $f : X \to Y$ be a smooth morphism of schemes. Then the following conditions are equivalent:*

(i) *The morphism f is surjective on field-valued points.*

(ii) *There exists a Nisnevich cover $Y' \twoheadrightarrow Y$ such that the base change $X \times_Y Y' \to Y'$ admits a section.*

Moreover, if Y is quasi-compact and quasi-separated, then Y' in (ii) can also be taken to be affine.

Proof. *(ii) $\implies$ (i):* We will show that for every field-valued point $y : \mathrm{Spec}(\kappa) \to Y$, the base change $X_{\{y\}} = X \times_Y \mathrm{Spec}(\kappa) \to \mathrm{Spec}(\kappa)$ admits a section. By base change, the condition implies that there is a Nisnevich cover $S \twoheadrightarrow \mathrm{Spec}(\kappa)$ such that $X_S \simeq X_{\{y\}} \times_{\mathrm{Spec}(\kappa)} S \to S$ admits a section. Since $S \twoheadrightarrow \mathrm{Spec}(\kappa)$ is surjective on field-valued points by definition, it admits a section. The composition of the section $\mathrm{Spec}(\kappa) \to S$, the section $S \to X_S$, and the morphism $X_S \to X_\kappa$ is then a section of $X_{\{y\}} \to \mathrm{Spec}(\kappa)$ as desired.

(i) $\implies$ (ii): Let $y : \mathrm{Spec}(\kappa) \to Y$ be a point and $x : \mathrm{Spec}(\kappa) \to X$ a lift. Since $f : X \to Y$ is smooth, there exists by [EGA, IV_4, 18.6.6(i), 18.5.17] a morphism $\tilde{x} : S \to X$ extending x, where S is the henselization of Y at y. Recall that S can be identified with the cofiltered limit of elementary étale neighbourhoods[5] of (Y, y). Since X is locally of finite presentation over Y, it follows that there exists an étale neighbourhood $Y'_y \to Y$ over y such that the Y-morphism $\tilde{x} : S \to X$ factors through $Y'_y \to X$. Then the disjoint union

[5]Here an elementary étale neighbourhood is an étale morphism $Y' \to Y$ along which y lifts to $y' : \mathrm{Spec}(\kappa) \to Y'$.

$Y' = \coprod_y Y'_y \twoheadrightarrow Y$ over all field-valued point y is an étale morphism which is surjective on field-valued points, i.e., a Nisnevich cover, with the desired property. If Y is quasi-compact, then there is a finite subcover refining Y' (see e.g. [EHIK, Lem. 2.1.2]), so in particular we may take Y' quasi-compact. We may then further replace Y' by a Zariski cover by an affine scheme. □

0.2.2. *Stacks.* We work with *higher* stacks throughout the paper. Thus a *stack* is a presheaf of ∞-groupoids on the site of k-schemes that satisfies hyperdescent with respect to the étale topology.

Let $\tau \in \{\text{ét}, \text{Nis}\}$ stand for either the étale or Nisnevich topology. We say a morphism of stacks $f : X \to Y$ *admits τ-local sections* if, for any scheme T and any morphism $t : T \to Y$, there exists a τ-cover $T' \twoheadrightarrow T$ such that the base change $X \times_Y T' \to T'$ admits a section.

A morphism $f : X \to Y$ is *schematic* if for every scheme V and every morphism $V \to Y$, the fibred product $X \times_Y V$ is a scheme. A stack X is *$(\tau, 0)$-Artin* if it has schematic (-1)-truncated diagonal and there exists a scheme U and an étale morphism $U \to X$ with τ-local sections. For $\tau = \text{ét}$, these are the algebraic spaces; for $\tau = \text{Nis}$, these are the quasi-separated algebraic spaces by [Knu, Chap. 2, Thm. 6.4].

For $n > 0$, a morphism $f : X \to Y$ is *$(\tau, n-1)$-representable* if for every scheme V and every morphism $V \to Y$, the fibred product $X \times_Y V$ is $(\tau, n-1)$-Artin. A stack X is *(τ, n)-Artin* if it has $(\tau, n-1)$-representable diagonal and there exists a scheme U and a smooth morphism $U \to X$ with τ-local sections.

A stack is *τ-Artin* if it is (τ, n)-Artin for some n.

The $(\text{ét}, 1)$-Artin stacks are Artin stacks (or algebraic stacks) as defined e.g. in [SP, Tag 026O]. More generally, the $(\text{ét}, n)$-Artin stacks and ét-Artin stacks are n-Artin stacks and higher Artin stacks as defined in [Gai, §4.2]. We will usually drop the "ét" from the notation.

A 1-Artin stack is $(\text{Nis}, 1)$-Artin if and only if it is quasi-separated with separated diagonal:

Theorem 0.2. *Let X be a quasi-separated 1-Artin stack with separated diagonal. Then there exists a scheme U and a smooth morphism $U \to X$ with Nisnevich-local sections. In particular, X is $(\text{Nis}, 1)$-Artin.*

Proof. For every field-valued point $x : \text{Spec}(\kappa) \to X$, there exists by [LMB, Thm. 6.3] an affine scheme $U(x)$ and a smooth morphism $U(x) \to X$ along which x lifts. Then the disjoint union $U = \coprod_x U_x \to X$ is a smooth morphism which is surjective on field-valued points. By the assumptions, $U \to X$ is a quasi-separated 0-representable morphism. To show it admits Nisnevich-local sections, let $T \to X$ be a morphism with T a scheme. The base change $U \times_X T$ is a quasi-separated algebraic space and thus admits by [Knu, Chap. 2, Thm. 6.4] an étale morphism $U' \twoheadrightarrow U \times_X T$ which is surjective on field-valued points. By Lemma 0.1, $U' \to T$ admits Nisnevich-local sections, hence so does $U \times_X T \to T$. □

0.3. **Notation.** We denote by Ani the ∞-category of anima, a.k.a. homotopy types or ∞-groupoids. We work over a fixed commutative ring k, which we leave implicit in the notation. We write ${}^{\tau}\mathrm{Stk}$ for the ∞-category of locally of finite type τ-Artin stacks over k, and Sch for the full subcategory of schemes locally of finite type over k. Given $S \in {}^{\tau}\mathrm{Stk}$ we write ${}^{\tau}\mathrm{Stk}_S$ for the ∞-category of locally of finite type τ-Artin stacks over S, and Sch_S for the full subcategory of schemes locally of finite type over S.

0.4. **Acknowledgments.** The first-named author spoke about a preliminary version of these results at the INI workshop "Algebraic K-theory, motivic cohomology and motivic homotopy theory" in June 2022. At that point, he was able to prove the homology-level comparison in Corollary 6.1 only for coefficients that are *eventually coconnective* in the homotopy t-structure; in particular, this did not apply to algebraic cobordism (so that it was not possible to deduce Theorem C). After writing up these results, the authors realized that it was possible to prove the much more general result in Theorem 3.2 via a completely different argument. We would like to thank Jens Hornbostel, Henry July, and Marc Levine for their interest in and questions related to Theorem C.

We acknowledge support from MOST grant 110-2115-M-001-016-MY3 (A.K.) and EPSRC grant no EP/R014604/1 (A.K. and C.R.). We would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme KAH2 where work on this paper was undertaken.

1. Lisse extension

Given $S \in {}^{\tau}\mathrm{Stk}$, we denote by Lis_S (resp. ${}^{\tau}\mathrm{LisStk}_S$) the ∞-category of pairs (T,t) where $T \in \mathrm{Sch}$ (resp. $T \in {}^{\tau}\mathrm{Stk}$) and $t : T \to S$ is a smooth morphism. Let $F : \mathrm{Lis}_S^{\mathrm{op}} \to \mathcal{V}$ be a presheaf, where $\mathcal{V}$ is an ∞-category.

Definition 1.1. The *lisse extension* of F is the presheaf

$$F^{\triangleleft} : {}^{\tau}\mathrm{LisStk}_S^{\mathrm{op}} \to \mathcal{V}$$

defined as the right Kan extension of F along the fully faithful functor $\mathrm{Lis}_S \hookrightarrow {}^{\tau}\mathrm{LisStk}_S$. In particular, we have

$$F^{\triangleleft}(S) \simeq \varprojlim_{(T,t)} F(T)$$

where the limit is taken over $(T,t) \in \mathrm{Lis}_S$.

Given a presheaf $F : \mathrm{Sch}^{\mathrm{op}} \to \mathcal{V}$, we may restrict along the forgetful functor $\mathrm{Lis}_S \to \mathrm{Sch}$ and form the lisse extension of the resulting presheaf F_S on Lis_S. On the other hand, we may also describe $F_S^{\triangleleft}$ more directly in terms of F. Moreover, this will work for presheaves with restricted functoriality (e.g. only for smooth or lci morphisms).

Definition 1.2. Let ${}^{\tau}\mathrm{Stk}^{?} \subseteq {}^{\tau}\mathrm{Stk}$ be a wide subcategory containing all smooth morphisms, and denote by $\mathrm{Sch}^{?} \subseteq \mathrm{Sch}$ the intersection $\mathrm{Sch} \cap {}^{\tau}\mathrm{Stk}^{?}$.

Let $F : \mathrm{Sch}^{?,\mathrm{op}} \to \mathcal{V}$ be a presheaf. Its *lisse extension* $F^{\lhd}$ is its right Kan extension $F : {}^{\tau}\mathrm{Stk}^{?,\mathrm{op}} \to \mathcal{V}$ along $\mathrm{Sch}^{?} \hookrightarrow {}^{\tau}\mathrm{Stk}^{?}$.

Remark 1.3. Let $\mathrm{Sch}^{?}_{/S}$ denote the full subcategory of the slice ${}^{\tau}\mathrm{Stk}^{?}_{/S}$ spanned by pairs $(T, t : T \to X)$ where T is a scheme (and t is a morphism in ${}^{\tau}\mathrm{Stk}^{?}$). Note that morphisms $(T,t) \to (T',t')$ are morphisms $T \to T'$ in $\mathrm{Sch}^{?}$ which are compatible with t and t'. Then we have

$$F^{\lhd}(S) \simeq \varprojlim_{(T,t)\in \mathrm{Sch}^{?}_{/S}} F(T).$$

Proposition 1.4. *Let $F : \mathrm{Sch}^{?,\mathrm{op}} \to \mathcal{V}$ be a presheaf. Assume that $\mathrm{Sch}^{?}$ contains all lci morphisms and that its morphisms are stable under smooth base change*[6]*. If F satisfies τ-descent, then for every $S \in {}^{\tau}\mathrm{Stk}$ there is a canonical isomorphism*

$$F^{\lhd}|_{{}^{\tau}\mathrm{LisStk}_S} \to (F|_{\mathrm{Lis}_S})^{\lhd}$$

of presheaves on ${}^{\tau}\mathrm{LisStk}_S$.

For example, we will apply Proposition 1.4 to presheaves on Sch and the subcategories

$$\mathrm{Sch}^{\mathrm{sm}},\ \mathrm{Sch}^{\mathrm{lci}} \quad \subseteq \quad \mathrm{Sch}$$

containing only the smooth and lci morphisms, respectively.

Lemma 1.5. *Let $F : \mathrm{Sch}^{?,\mathrm{op}} \to \mathcal{V}$ be a presheaf where $\mathrm{Sch}^{?}$ satisfies the conditions of Proposition 1.4. If F satisfies τ-descent, then for every $S \in {}^{\tau}\mathrm{Stk}$ and every smooth morphism $p : U \to S$ admitting τ-local sections, the canonical map*

$$F^{\lhd}(S) \to \mathrm{Tot}(F(U_\bullet)) \tag{1.6}$$

is invertible, where $U_\bullet$ is the Čech nerve of p, "Tot" denotes the totalization of a cosimplicial object.

Proof. Assume first that S is a scheme, so that the terms of $U_\bullet$ are algebraic spaces. By assumption, there exists a scheme S' and a τ-cover $S' \twoheadrightarrow S$ over which p admits a section. Since F satisfies descent for the Čech nerve of $S' \twoheadrightarrow S$, we may replace S by S' and thereby assume that p admits a section. This section (which is lci hence determines a morphism in $\mathrm{Sch}^{?}$) gives rise to a splitting of the augmented simplicial object $U_\bullet \to S$, so that the map (1.6) is invertible by [Lur, Lem. 6.1.3.16].

Now we consider the general case. For every pair (T,t) where $T \in \mathrm{Sch}$ and $t : T \to S$ is a morphism in ${}^{\tau}\mathrm{Stk}^{?}$, denote by $U_T \twoheadrightarrow T$ the base change of p and by $U_{T,\bullet}$ its Čech nerve. The canonical map

$$F(T) \to \mathrm{Tot}(F(U_{T,\bullet})) \tag{1.7}$$

is invertible by above.

[6] i.e., if $X \to Y$ is a morphism in $\mathrm{Sch}^{?}$ and $Y' \to Y$ is a smooth morphism in Sch, then the base change $X \times_Y Y' \to Y'$ belongs to $\mathrm{Sch}^{?}$

Note that for every smooth S-scheme V, the base change functor $\mathrm{Sch}^{?,\mathrm{op}}_{/S} \to \mathrm{Sch}^{?,\mathrm{op}}_{/V}$ is cofinal. Indeed, given $(T,t) \in \mathrm{Sch}^{?}_{/V}$ we have a section $s : T \to T \times_S V$ over V of the projection $T \times_S V \to T$. Since the latter is smooth, s is lci and determines a morphism in $\mathrm{Sch}^{?}_{/V}$ whose target lies in the essential image of the functor in question. Thus, the morphism (1.6) is the limit over $(T,t) \in \mathrm{Sch}^{?}_{/S}$ of the isomorphisms (1.7). □

Proof of Proposition 1.4. It will suffice to show that for every $X \in {}^{\tau}\mathrm{LisStk}_S$, the projection map

$$F^{\lhd}(X) \simeq \varprojlim_{(T,t)\in \mathrm{Sch}^{?}_{/X}} F(T) \to \varprojlim_{(T,t)\in (\mathrm{Lis}_S)_{/X}} F(T) \tag{1.8}$$

is invertible. Here $(\mathrm{Lis}_S)_{/X}$ is the ∞-category of pairs (T,t) where $T \in \mathrm{Lis}_S$ and $t : T \to X$ is a morphism in ${}^{\tau}\mathrm{LisStk}_S$.

Let $p : U \twoheadrightarrow X$ be a smooth morphism admitting τ-local sections where U is a scheme. Denote by $U_\bullet$ the Čech nerve of p, so that there is an equivalence $\mathrm{Tot}(F(U_\bullet)) \simeq F^{\lhd}(X)$ by Lemma 1.5. This defines a diagram $\mathbf{\Delta}^{\mathrm{op}} \to (\mathrm{Lis}_S)_{/X}$, so by projection there is a canonical map

$$\varprojlim_{(T,t)\in (\mathrm{Lis}_S)_{/X}} F(T) \to \varprojlim_{[n]\in \mathbf{\Delta}} F(U_n) \simeq F^{\lhd}(X).$$

One verifies that this is inverse to (1.8). □

2. A PRO-APPROXIMATION LEMMA

In this section we will show the key technical result of the paper (Proposition 2.2), which is adapted from an argument of Morel and Voevodsky. The following definition is inspired by [MV, §4, Def. 2.1] (see also Remark 2.4 below).

Notation 2.1. Let $S \in {}^{\tau}\mathrm{Stk}$ and $\{U_\nu\}_\nu$ a filtered diagram in ${}^{\tau}\mathrm{Stk}_S$ with monomorphisms as transition maps. Suppose that for every index ν there is a vector bundle V_ν over S, an open immersion $U_\nu \hookrightarrow V_\nu$ over S, and a closed substack $W_\nu \subseteq V_\nu$ complementary to U_ν containing the zero section, such that the following conditions hold:

(i) For every affine scheme T and every morphism $T \to S$, there exists an index ν_0 such that the morphism $U_{\nu_0} \times_S T \to T$ admits τ-local sections.

(ii) For every index ν, there exists $\beta > \nu$ such that the transition map $U_\nu \to U_\beta$ factors as follows:

$$\begin{array}{ccc} V_\nu \smallsetminus W_\nu & \xrightarrow{(0,\mathrm{id})} & V_\nu \times_S V_\nu \smallsetminus W_\nu \times_S W_\nu \\ \| & & \downarrow \\ U_\nu & \hookrightarrow & U_\beta. \end{array}$$

Let $\mathcal{V}$ be an ∞-category. A presheaf $F : \mathrm{Lis}^{\mathrm{op}}_S \to \mathcal{V}$ is $\mathbf{A}^1$-*invariant* if for every $X \in \mathrm{Lis}_S$, the canonical map $F(X) \to F(X \times \mathbf{A}^1)$ is invertible.

Proposition 2.2. *Let $S \in {}^{\tau}\mathrm{Stk}$ and $\{U_\nu\}_\nu$ as in Notation 2.1. Then for every $\mathbf{A}^1$-invariant τ-sheaf $F : \mathrm{Lis}_S^{\mathrm{op}} \to \mathcal{V}$, the canonical morphism in* $\mathrm{Pro}(\mathcal{V})$

$$F^{\triangleleft}(S) \to \{F^{\triangleleft}(U_\nu)\}_\nu.$$

is invertible. In particular, the pro-object $\{F^{\triangleleft}(U_\nu)\}_\nu$ is essentially constant.

Lemma 2.3. *Let S and $\{U_\nu\}_\nu$ be as in Notation 2.1. Suppose that S is an affine scheme and that there exists an index ν_0 such that $U_{\nu_0} \to S$ admits a section. Then the presheaf $U_\infty := \varinjlim_\nu U_\nu$ (where the colimit is taken in presheaves) is $\mathbf{A}^1$-contractible on smooth affine S-schemes.*

Proof. The following argument is extracted from the proof of [MV, §4, Prop. 2.3]. The claim is that the animum $R\Gamma(X, \mathrm{L}_{\mathbf{A}^1} U_\infty)$ is contractible for every affine $X \in \mathrm{Lis}_S$, where $\mathrm{L}_{\mathbf{A}^1}$ denotes the $\mathbf{A}^1$-localization functor (see e.g. [Hoy1, Proof of Prop. C.6]), i.e. that the simplicial set

$$\mathrm{Maps}_{\mathrm{Fun}(\mathrm{Lis}_S^{\mathrm{op}}, \mathrm{Ani})}(X \times \mathbf{A}^\bullet, U_\infty) \simeq \varinjlim_\nu \mathrm{Maps}_{\mathrm{Lis}_S}(X \times \mathbf{A}^\bullet, U_\nu)$$

is a contractible Kan complex. By [EHKSY, Lem. A.2.6] and closed gluing for the presheaf U_∞, it is enough to show that for every $n \geqslant 0$ and every affine $X \in \mathrm{Lis}_S$, the restriction map

$$\mathrm{Maps}(X \times \mathbf{A}^n, U_\infty) \to \mathrm{Maps}(X \times \partial\mathbf{A}^n, U_\infty)$$

is surjective on π_0, where $\partial\mathbf{A}^n$ is as in *loc. cit.*

Let $\nu \geqslant \nu_0$ be an index. Denote by $s : S \to U_{\nu_0} \to U_\nu$ the induced section and by $t : X \to S \to U_\nu$ its composite with the structural morphism. The existence of t shows the surjectivity for $n = 0$.

Let $n > 0$ and $f : X \times \partial\mathbf{A}^n \to U_\nu$ a morphism over S. We claim that this extends to a morphism $g : X \times \mathbf{A}^n \to U_\beta$ for some index $\beta \geqslant \nu$. Since X and S are affine and V_ν is a vector bundle over S, there exists an S-morphism $g' : X \times \mathbf{A}^n \to V_\nu$ which restricts to f on $X \times \partial\mathbf{A}^n$. Since $X \times \partial\mathbf{A}^n$ and $g'^{-1}(W_\nu)$ are disjoint as closed subschemes of $X \times \mathbf{A}^n$, there exists for the same reason an S-morphism $g'' : X \times \mathbf{A}^n \to V_\nu$ which restricts to 0 on $X \times \partial\mathbf{A}^n$ and to

$$g'^{-1}(W_\nu) \to S \xrightarrow{s} U_\nu \subseteq V_\nu$$

on $g'^{-1}(W_\nu)$. By construction, the induced S-morphism

$$(g'', g') : X \times \mathbf{A}^n \to V_\nu \underset{S}{\times} V_\nu$$

restricts to $(f, 0) : X \times \partial\mathbf{A}^n \to V_\nu \times_S V_\nu$, and factors through the complement of $W_\nu \times_S W_\nu$. Let $\beta > \nu$ and the morphism $V_\nu \times_S V_\nu \smallsetminus W_\nu \times_S W_\nu \to U_\beta$ be as in assumption (ii). Then the composite

$$g : X \times \mathbf{A}^n \xrightarrow{(g'', g')} V_\nu \underset{S}{\times} V_\nu \smallsetminus W_\nu \underset{S}{\times} W_\nu \to U_\beta$$

fits into the commutative diagram

$$\begin{array}{ccc} X \times \partial\mathbf{A}^n & \xrightarrow{f} & U_\beta \\ \big\downarrow & \overset{g}{\nearrow} & \big\downarrow \\ X \times \mathbf{A}^n & \longrightarrow & S \end{array}$$

as desired. ☐

Proof of Proposition 2.2. Consider the unique colimit-preserving functor

$$F_! : \mathrm{Fun}(\mathrm{Lis}_S^{\mathrm{op}}, \mathrm{Ani}) \to \mathrm{Fun}(\mathcal{V}, \mathrm{Ani})$$

which sends the presheaf represented by $X \in \mathrm{Lis}_S$ to the functor $\mathcal{V} \to \mathrm{Ani}$ sending $V \mapsto \mathrm{Maps}(F(X), V)$. Recall that a morphism $A \to B$ in $\mathrm{Fun}(\mathrm{Lis}_S^{\mathrm{op}}, \mathrm{Ani})$ is a *local equivalence* if for every $\mathbf{A}^1$-invariant τ-sheaf $F \in \mathrm{Fun}(\mathrm{Lis}_S^{\mathrm{op}}, \mathrm{Ani})$, the map

$$\mathrm{Maps}(B, F) \to \mathrm{Maps}(A, F)$$

is invertible. Since F satisfies $\mathbf{A}^1$-invariance and τ-descent, $F_!$ inverts local equivalences. Using the fully faithful embedding of $\mathrm{Pro}(\mathcal{V})$ in $\mathrm{Fun}(\mathcal{V}, \mathrm{Ani})^{\mathrm{op}}$, it will thus suffice to show that $\varinjlim_\nu U_\nu \to S$ is a local equivalence in $\mathrm{Fun}(\mathrm{Lis}_S^{\mathrm{op}}, \mathrm{Ani})$.

By universality of colimits it is enough to show that for every $T \in \mathrm{Lis}_S$, the base change

$$\varinjlim_\nu U_\nu \underset{S}{\times} T \to T$$

is a local equivalence. Since local equivalences are preserved by the colimit-preserving functor $\mathrm{Fun}(\mathrm{Lis}_T^{\mathrm{op}}, \mathrm{Ani}) \to \mathrm{Fun}(\mathrm{Lis}_S^{\mathrm{op}}, \mathrm{Ani})$ sending $U \in \mathrm{Lis}_T$ to $U \in \mathrm{Lis}_S$, and because the data and assumptions in Notation 2.1 are stable under base change, we may thus replace S by T to assume that it is a scheme. We can moreover assume that it is affine, arguing similarly using the fact that S can be written as a colimit of affines up to Zariski-local equivalence. Finally, up to τ-local equivalence we may assume by condition (i) that $U_{\nu_0} \to S$ admits a section for some index ν_0. Now the claim follows from Lemma 2.3. ☐

Remark 2.4. In Notation 2.1, a sufficient condition for (i) is that for every field κ (which can be assumed algebraically closed if τ = ét) and every κ-valued point $s : \mathrm{Spec}(\kappa) \to S$, there exists an index ν_s and a lift $\mathrm{Spec}(\kappa) \to U_{\nu_s}$. Indeed, let us show that if S is affine then there exists an index ν_0 such that $U_\nu \to S$ admits τ-local sections. The assumption implies that the disjoint union $\coprod_\nu U_\nu \to S$ is a smooth morphism (not necessarily of finite type) which is surjective on field-valued points. By Lemma 0.1 (and its étale analogue [EGA, Cor. 17.16.3(ii)]), there exists an affine scheme S' and a τ-cover $S' \twoheadrightarrow S$ along which the base change $\coprod_\nu U_\nu \times_S S' \to S'$ admits a section. Since S' is quasi-compact, there is a finite subset I of indices through which the section factors. Any section of U_ν gives rise to a section of U_β for any $\beta > \nu$ (by composition with the transition map), so we may assume that I consists of a single index ν_0.

3. The Borel construction

Notation 3.1.

(i) Let S be an algebraic space and G an fppf group scheme over S which is *embeddable*, i.e., admits an embedding as a closed subgroup scheme of $\mathrm{GL}_S(\mathcal{E})$ for some finite locally free sheaf $\mathcal{E}$ on S.

(ii) Fix an embedding $G \subseteq \mathrm{GL}_S(\mathcal{E})$ and let U_ν, for every integer $\nu > 0$, be the open subspace of $\mathbf{V}_S(\mathcal{E})^{\oplus\nu}$ where the diagonal action of G is free, so that the quotients $[U_\nu/G] = U_\nu/G$ are algebraic spaces. The G-equivariant closed immersions $U_\nu \to U_{\nu+1}$ determine a filtered diagram $\{U_\nu\}_\nu$ of G-equivariant algebraic spaces of finite type over S, whence a filtered diagram $\{U_\nu/G\}_\nu$ of Artin stacks of finite type over the classifying stack $BG = [S/G]$. Compare [MV, §4.2].

(iii) Given $X \in {}^\tau\mathrm{Stk}_S$ with G-action, let $X \times^G_S U_\nu := [X/G] \times_{BG} (U_\nu/G)$.

Theorem 3.2. *Let $F : \mathrm{Sch}^{\mathrm{lci,op}} \to \mathrm{Spt}$ be an $\mathbf{A}^1$-invariant τ-sheaf of spectra. Let S, G, and $\{U_\nu\}_\nu$ be as in Notation 3.1. Then for every $X \in {}^\tau\mathrm{Stk}_S$ with G-action, there is a canonical isomorphism*

$$R\Gamma([X/G], F^\triangleleft) \to \{R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, F^\triangleleft)\}_\nu$$

of pro-spectra, where $F^\triangleleft : {}^\tau\mathrm{Stk}^{\mathrm{lci,op}} \to \mathrm{Spt}$ denotes the lisse extension (Definition 1.2). Moreover, the homomorphisms

$$\mathrm{H}^i([X/G], F^\triangleleft) \to \{\mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_\nu, F^\triangleleft)\}_\nu$$

are bijective for all $i \in \mathbf{Z}$.

Proof. By Proposition 1.4, we may replace F by its restriction $\mathrm{Lis}^{\mathrm{op}}_{[X/G]} \to \mathrm{Spt}$.

The filtered diagram $\{U_\nu\}_\nu$ over S satisfies the assumptions of Notation 2.1 (compare [MV, §4, Ex. 2.2] and Remark 2.4). Moreover, the transition maps and the maps $U_\nu \hookrightarrow V_\nu = \mathbf{V}_S(\mathcal{E})^{\oplus\nu}$ are all G-equivariant, so the same holds for the quotient $\{U_\nu/G\}_\nu$ over $BG = [S/G]$. By base change it also holds for $\{X \times^G_S U_\nu\}_\nu$ over $[X/G]$. Thus the first statement follows from Proposition 2.2.

The second will follow from the first if we show that the canonical surjections

$$\mathrm{H}^i([X/G], F^\triangleleft) \simeq \pi_{-i}\left(\varprojlim_\nu R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, F)\right) \twoheadrightarrow \varprojlim_\nu \mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_\nu, F)$$

are injective, or equivalently (by the Milnor exact sequence) that the $\varprojlim^1_\nu$'s of the abelian groups

$$\pi_{-i+1}\left(R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, F)\right)$$

vanish for all i. For this it suffices to check the Mittag–Leffler condition for the pro-abelian group $\{\pi_{-i+1} R\Gamma(X \times^G_S U_\nu, F)\}_\nu$. But by the first statement, it is isomorphic to a constant pro-system. □

Corollary 3.3. *Let the notation be as in Theorem 3.2. Then there is a canonical isomorphism of spectra*

$$R\Gamma([X/G], F^\triangleleft) \to \varprojlim_\nu R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, F)$$

and of abelian groups

$$\mathrm{H}^i([X/G], F^\triangleleft) \to \varprojlim_\nu \mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_\nu, F).$$

for all $i \in \mathbf{Z}$.

4. Algebraic K-theory

Let $\mathrm{Sch}^{\mathrm{ft}} \subseteq \mathrm{Sch}$ denote the full subcategory of schemes of finite type (over the base k). Consider the Nisnevich sheaf of spectra $\mathrm{K} : \mathrm{Sch}^{\mathrm{ft,op}} \to \mathrm{Spt}$ which sends $X \in \mathrm{Sch}^{\mathrm{ft}}$ to its Bass–Thomason–Trobaugh K-theory spectrum (see e.g. [Kha2, Def. 2.6, Rem. 2.15]). By right Kan extension, it defines a Nisnevich sheaf $\mathrm{K} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Spt}$.

Write $\mathrm{KH} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Spt}$ for its $\mathbf{A}^1$-invariant version, so that

$$\mathrm{KH}(X) \coloneqq \varinjlim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{K}(X \times \mathbf{A}^n)$$

for every $X \in \mathrm{Sch}$, see e.g. [Kha2, §4.2]. The canonical map $\mathrm{K}(X) \to \mathrm{KH}(X)$ is invertible when X is regular.

We let S, G, and $\{U_\nu\}_\nu$ be as in Theorem 3.2.

Corollary 4.1. *For every* $X \in \mathrm{Sch}_S$ *with* G*-action, there are canonical isomorphisms*

$$\mathrm{KH}^{\triangleleft}([X/G]) \simeq \varprojlim_{\nu} \mathrm{KH}(X \overset{G}{\underset{S}{\times}} U_\nu)$$

and

$$\mathrm{KH}_i^{\triangleleft}([X/G]) \simeq \varprojlim_{\nu} \mathrm{KH}_i(X \overset{G}{\underset{S}{\times}} U_\nu)$$

for every $i \in \mathbf{Z}$.

Remark 4.2. More generally, we get the same computation for any localizing invariant of stable ∞-categories which satisfies $\mathbf{A}^1$-invariance. For example, this also applies to topological K-theory (over the complex numbers) and periodic cyclic homology in characteristic zero.

Corollary 4.3. *If* X *is regular, then moreover*

$$\mathrm{K}^{\triangleleft}([X/G]) \simeq \varprojlim_{\nu} \mathrm{K}(X \overset{G}{\underset{S}{\times}} U_\nu)$$

and

$$\mathrm{K}_i^{\triangleleft}([X/G]) \simeq \varprojlim_{\nu} \mathrm{K}_i(X \overset{G}{\underset{S}{\times}} U_\nu).$$

Suppose the base ring k is noetherian. Since lci morphisms are of finite Tor-amplitude, G-theory (= algebraic K-theory of coherent sheaves) defines a presheaf of spectra $\mathrm{G} : \mathrm{Sch}^{\mathrm{lci,op}} \to \mathrm{Spt}$, see e.g. [Kha2, §3].

Corollary 4.4. *For every* $X \in \mathrm{Sch}_S$ *with* G*-action, there are canonical isomorphisms*

$$\mathrm{G}^{\triangleleft}([X/G]) \simeq \varprojlim_{\nu} \mathrm{G}(X \overset{G}{\underset{S}{\times}} U_\nu)$$

and

$$\mathrm{G}_i^{\triangleleft}([X/G]) \simeq \varprojlim_{\nu} \mathrm{G}_i(X \overset{G}{\underset{S}{\times}} U_\nu)$$

for all $i \in \mathbf{Z}$.

Remark 4.5. Consider the presheaf $\mathrm{K}(-)_{\mathbf{Q}} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Spt}$ sending $X \in \mathrm{Sch}$ to its rationalized K-theory spectrum $\mathrm{K}(X) \otimes \mathbf{Q}$. This satisfies étale descent (see e.g. [Kha2, Thm. 5.1]). Thus $\mathrm{K}(-)_{\mathbf{Q}}^{\triangleleft}$ is the unique extension of $\mathrm{K}_{\mathbf{Q}}$ to an étale sheaf on Artin stacks; in particular, it coincides with the construction $\mathrm{K}^{\text{ét}}(-)_{\mathbf{Q}}$ in [Kha2, §5.2].

5. Betti and étale (co)homology

Let $\mathbf{D}$ be the one of the following sheaf theories:

(i) *Betti:* Suppose $k = \mathbf{C}$. For every locally of finite type k-scheme S, let $\mathbf{D}(S) \coloneqq \mathrm{D}(S(\mathbf{C}), \Lambda)$ denote the derived ∞-category of sheaves of Λ-modules on the topological space $S(\mathbf{C})$, for some commutative ring Λ.

(ii) *Étale (torsion coefficients):* For every locally of finite type k-scheme S, let $\mathbf{D}(S) \coloneqq \mathrm{D}_{\text{ét}}(S, \Lambda)$ denote the derived ∞-category of sheaves of Λ-modules on the small étale site of S, where Λ is a commutative ring of positive characteristic n, with n invertible in k.

(iii) *Étale (adic coefficients):* For every locally of finite type k-scheme S, let $\mathbf{D}(S)$ denote the limit of ∞-categories

$$\varprojlim_{n>0} \mathrm{D}_{\text{ét}}(S, \Lambda/\mathfrak{m}^n)$$

where Λ is a discrete valuation ring whose residue characteristic is invertible in k.

These all satisfy étale descent, so one may take $\tau = \text{ét}$ in this section, and the lisse extension $\mathbf{D}^{\triangleleft}$ is the unique étale sheaf on ${}^{\tau}\mathrm{Stk}$ which restricts to $\mathbf{D}$ on Sch. In particular, it coincides with the extension to stacks considered in [LZ].

For any $X \in {}^{\tau}\mathrm{Stk}$ and any sheaf $\mathcal{F} \in \mathbf{D}^{\triangleleft}(X)$, form the presheaf of spectra

$$(T, t : T \to X) \mapsto R\Gamma(T, \mathcal{F}) \coloneqq \mathrm{Maps}_{\mathbf{D}(T)}(\Lambda_T, t^*(\mathcal{F}))$$

on ${}^{\tau}\mathrm{LisStk}_X$, which by construction is lisse-extended from its restriction to Lis_X. Moreover, the latter is an $\mathbf{A}^1$-invariant étale sheaf.

Let S, G, and $\{U_\nu\}_\nu$ be as in Notation 3.1. We deduce from Theorem 3.2:

Corollary 5.1. *Let* $X \in {}^{\tau}\mathrm{Stk}_S$ *with* G*-action. Then for every* $\mathcal{F} \in \mathbf{D}^{\triangleleft}([X/G])$*, there are canonical isomorphisms*

$$R\Gamma([X/G], \mathcal{F}) \simeq \varprojlim_{\nu} R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F})$$

and

$$\mathrm{H}^i([X/G],\mathcal{F}) \simeq \varprojlim_{\nu} \mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F})$$

for every $i \in \mathbf{Z}$.

Let $\Lambda_{BG} \in \mathbf{D}^{\triangleleft}(BG)$ be the constant sheaf with coefficients in the commutative ring Λ. For $X \in {}^{\tau}\mathrm{Stk}$ with G-action, define the equivariant Borel–Moore homology spectrum

$$\mathrm{C}^{\mathrm{BM},G}_{\bullet}(X;\Lambda) = R\Gamma([X/G], f^!(\Lambda_{BG}))$$

where $f : [X/G] \to [\mathrm{Spec}(k)/G] = BG$ is the projection. We denote its homotopy groups by

$$\mathrm{H}^{\mathrm{BM},G}_s(X;\Lambda) = \pi_s R\Gamma([X/G], f^!(\Lambda_{BG})) \simeq \mathrm{H}^{-s}([X/G], f^!(\Lambda_{BG}))$$

for $s \in \mathbf{Z}$. See [LZ] for the construction of the functor $f^!$.

Consider the presheaf

$$\begin{aligned}(T, t : T \to [X/G]) \mapsto R\Gamma(T, t^* f^!(\Lambda_{BG})) &\simeq R\Gamma(T, t^! f^!(\Lambda_{BG})(-d_t)[-2d_t] \\ &\simeq \mathrm{C}^{\mathrm{BM}}_{\bullet}(T;\Lambda)(-d_t)[-2d_t]\end{aligned}$$

on ${}^{\tau}\mathrm{LisStk}_X$, where d_t is the relative dimension of t. Since $a : BG \to S$ is smooth, we have $\Lambda_{BG} \simeq a^!(\Lambda)(g)[2g]$ where g is the relative dimension of $G \to S$.

Corollary 5.2. *For every* $X \in {}^{\tau}\mathrm{Stk}_S$ *with* G*-action, we have*

$$\mathrm{C}^{\mathrm{BM},G}_{\bullet}(X;\Lambda) \simeq \varprojlim_{\nu} \mathrm{C}^{\mathrm{BM}}_{\bullet}(X \overset{G}{\underset{S}{\times}} U_\nu;\Lambda)(-d_\nu + g)[-2d_\nu + 2g],$$

where d_ν *is the relative dimension of* $U_\nu \to S$*, and*

$$\mathrm{H}^{\mathrm{BM},G}_s(X;\Lambda) \simeq \varprojlim_{\nu} \mathrm{H}^{\mathrm{BM}}_{s+2d_\nu-2g}(X \overset{G}{\underset{S}{\times}} U_\nu;\Lambda)(-d_\nu + g)$$

for every $s \in \mathbf{Z}$.

6. Generalized cohomology theories

In this section we simply observe that the discussion of Sect. 5 can be generalized to $\mathbf{D}$ any constructible ∞-category in the sense of [Kha3]. For concreteness, we take the universal case $\mathbf{D} = \mathbf{SH}$ and we let $\tau = \mathrm{Nis}$.

Given a scheme S, let $\mathbf{SH}(S)$ denote the stable ∞-category of motivic spectra over S (see e.g. [Hoy1, App. C]), and consider the lisse extension $\mathbf{SH}^{\triangleleft}(-)$ with respect to $*$-inverse image. For any $X \in {}^{\mathrm{Nis}}\mathrm{Stk}$ and any motivic spectrum $\mathcal{F} \in \mathbf{SH}^{\triangleleft}(X)$, the presheaf of spectra

$$(T, t : T \to X) \mapsto R\Gamma(T, \mathcal{F}) := \mathrm{Maps}_{\mathbf{SH}^{\triangleleft}(T)}(\mathbf{1}_T, t^*(\mathcal{F}))$$

on ${}^{\tau}\mathrm{LisStk}_X$ is lisse-extended from its restriction to Lis_X. The latter is an $\mathbf{A}^1$-invariant Nisnevich sheaf (by definition of $\mathbf{SH}(-)$).

Let S, G, and $\{U_\nu\}_\nu$ be as in Notation 3.1. The first isomorphism below is a vast generalization of [KR, Thm. 12.9].

Corollary 6.1. *Let $X \in {}^{\tau}\mathrm{Stk}_S$ with G-action. Then for every $\mathcal{F} \in \mathbf{SH}^{\triangleleft}([X/G])$, there are canonical isomorphisms*

$$R\Gamma([X/G], \mathcal{F}) \simeq \varprojlim_{\nu} R\Gamma(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F})$$

and

$$\mathrm{H}^i([X/G], \mathcal{F}) \simeq \varprojlim_{\nu} \mathrm{H}^i(X \overset{G}{\underset{S}{\times}} U_\nu, \mathcal{F})$$

for every $i \in \mathbf{Z}$.

The six functor formalism on $\mathbf{SH}(-)$ persists to the lisse extension $\mathbf{SH}^{\triangleleft}(-)$ (see [KR, Kha4][7]). In particular, for any locally of finite type morphism f in ${}^{\tau}\mathrm{Stk}$ one has the adjoint pair of functors $(f_!, f^!)$.

Given a motivic spectrum $E \in \mathbf{SH}(\mathrm{Spec}(k))$, let $E_{BG} = E|_{BG}$ denote its $*$-inverse image in $\mathbf{SH}^{\triangleleft}(BG)$. For $X \in {}^{\tau}\mathrm{Stk}$ with G-action, define the equivariant Borel–Moore homology spectrum

$$\mathrm{C}^{\mathrm{BM},G}_{\bullet}(X; E) = R\Gamma([X/G], f^!(E_{BG}))$$

where $f : [X/G] \to [\mathrm{Spec}(k)/G] = BG$ is the projection. We denote its homotopy groups by

$$\mathrm{H}^{\mathrm{BM},G}_s(X; E) = \pi_s R\Gamma([X/G], f^!(E_{BG})) \simeq \mathrm{H}^{-s}([X/G], f^!(E_{BG}))$$

for $s \in \mathbf{Z}$.

We have the presheaf

$$\begin{aligned}(T, t : T \to [X/G]) \mapsto R\Gamma(T, t^* f^!(E_{BG})) &\simeq R\Gamma(T, t^! f^!(E_{BG})\langle -\Omega_t\rangle)\\ &\simeq \mathrm{C}^{\mathrm{BM}}_{\bullet}(T; E)\langle -\Omega_t\rangle\end{aligned}$$

on ${}^{\tau}\mathrm{LisStk}_X$. Here $\langle -\Omega_t\rangle$ is the inverse of the operation of Thom twist by the relative cotangent sheaf of u (so that an orientation of E determines a Thom isomorphism $E\langle -\Omega_u\rangle \simeq E(-d_t)[-2d_t]$, where d_t is the relative dimension of t). Since $a : BG \to S$ is smooth, we have $E_{BG} \simeq a^!(E)\langle \Omega_{G/S}\rangle$.

Corollary 6.2. *For every $X \in {}^{\tau}\mathrm{Stk}_S$ with G-action, there are canonical isomorphisms*

$$\mathrm{C}^{\mathrm{BM},G}_{\bullet}(X; E) \simeq \varprojlim_{\nu} \mathrm{C}^{\mathrm{BM}}_{\bullet}(X \overset{G}{\underset{S}{\times}} U_\nu; E)\langle -\Omega_{U_\nu/S} + \Omega_{G/S}\rangle$$

and

$$\mathrm{H}^{\mathrm{BM},G}_s(X; E) \simeq \varprojlim_{\nu} \mathrm{H}^{\mathrm{BM}}_s(X \overset{G}{\underset{S}{\times}} U_\nu; E)\langle -\Omega_{U_\nu/S} + \Omega_{G/S}\rangle$$

for every $s \in \mathbf{Z}$.

---

[7] See also [Cho], which constructs the !-operations for representable morphisms. This suffices for our discussion of equivariant B.M. homology below in case $X \in {}^{\tau}\mathrm{Stk}$ is a scheme or algebraic space with G-action (so that $f : [X/G] \to BG$ is representable), hence in particular for our applications in Sect. 7.

Corollary 6.3. *Suppose E is oriented. Then for every $X \in {}^{\tau}\mathrm{Stk}_S$ with G-action, we have*

$$\mathrm{C}_{\bullet}^{\mathrm{BM},G}(X;E) \simeq \varprojlim_{\nu} \mathrm{C}_{\bullet}^{\mathrm{BM}}(X \overset{G}{\underset{S}{\times}} U_\nu; E)(-d_\nu + g)[-2d_\nu + 2g],$$

where d_ν (resp. g) is the relative dimension of $U_\nu \to S$ (resp. $G \to S$), and

$$\mathrm{H}_s^{\mathrm{BM},G}(X;E) \simeq \varprojlim_{\nu} \mathrm{H}_{s+2d_\nu-2g}^{\mathrm{BM}}(X \overset{G}{\underset{S}{\times}} U_\nu; E)(-d_\nu + g)$$

for every $s \in \mathbf{Z}$.

Suppose k is a field, $S = \mathrm{Spec}(k)$, Λ is a commutative ring in which the characteristic exponent of k is invertible, and $E = \Lambda^{\mathrm{mot}} \in \mathbf{SH}(k)$ is the Λ-linear motivic cohomology spectrum. Combining Corollary 6.3 with the comparison between motivic Borel–Moore homology of schemes and higher Chow groups (see [MVW, Prop. 19.18] and [CD, Cor. 8.12]), we get:

Corollary 6.4. *For every quasi-separated algebraic space X of finite type over k with G-action, there are canonical isomorphisms*

$$\mathrm{H}_{s+2n}^{\mathrm{BM},G}(X;\Lambda^{\mathrm{mot}})(-n) \simeq \mathrm{A}_n^G(X,s) \otimes \Lambda$$

for all $n, s \in \mathbf{Z}$, where on the right-hand side are the G-equivariant higher Chow groups of X [EG, §2.7].

Similarly, one has

$$\mathrm{H}_{s+2n}^{\mathrm{BM}}([X/G];\Lambda^{\mathrm{mot}})(-n) \simeq \mathrm{A}_\nu([X/G],s) \otimes \Lambda,$$

where the right-hand side is defined in [EG, §5.3] or [Kre]. One expects this comparison to generalize to any Artin stack of finite type over k with affine stabilizers, see also [BP].

Remark 6.5. The right-hand side of Corollary 6.2 indicates how one may define G-equivariant (higher) Chow–Witt groups in terms of the Borel construction, in such a way that the resulting theory identifies with the generalized Borel–Moore homology theory associated with the Λ-linear Milnor–Witt motivic cohomology spectrum.

7. Algebraic bordism

Let $S = \mathrm{Spec}(k)$ and let G and $\{U_\nu\}_\nu$ be as in Theorem 3.2.

Corollary 6.3 applied to the algebraic cobordism spectrum $\mathrm{MGL} \in \mathbf{SH}(k)$ (and its Tate twists) shows that the associated equivariant Borel–Moore homology theory can be computed by the Borel construction, for any $X \in \mathrm{Sch}$ with G-action.

Corollary 7.1. *For every $r, s \in \mathbf{Z}$ there are canonical isomorphisms*

$$\mathrm{H}_s^{\mathrm{BM},G}(X;\mathrm{MGL})(-r) \simeq \varprojlim_{\nu} \mathrm{H}_{s+2d_\nu-2g}^{\mathrm{BM}}(X \overset{G}{\times} U_\nu; \mathrm{MGL})(-r - d_\nu + g)$$

where $d_\nu = \dim(U_\nu)$ and $g = \dim(G)$.

When k is a field of characteristic zero, Levine and Morel defined a (lower) algebraic bordism theory $\Omega_*(-)$ on quasi-projective k-schemes (see [LM]). By [Lev], this theory is identified with $\mathrm{H}^{\mathrm{BM}}_{2*}(-;\mathrm{MGL})(-*)$, in a way compatible with proper push-forwards and Gysin pull-backs along open immersions. From these compatibilities one can further deduce compatibility with Gysin pull-backs along lci morphisms, from a direct comparison of the constructions of [LM, §6.5.2] and [DJK, §3.2].

Building on this theory, Heller and Malagón-López defined in [HML] a (lower) G-equivariant algebraic bordism theory $\Omega^G_*(-)$ by the formula

$$\Omega^G_n(X) := \varprojlim_{\nu} \Omega_{n+d_\nu}(X \overset{G}{\times} U_\nu)$$

for X with linearized G-action. We have thus proven:

Corollary 7.2. *For every quasi-projective k-scheme X with linearized G-action, we have canonical isomorphisms*

$$\mathrm{H}^{\mathrm{BM},G}_{2n}(X;\mathrm{MGL})(-n) \simeq \Omega^G_n(X)$$

for all $n \in \mathbf{Z}$. Furthermore, these isomorphisms are compatible with direct image along G-equivariant proper morphisms and Gysin pull-back along G-equivariant lci morphisms.

Moreover, we deduce the right-exact localization property for $\Omega^G_*(-)$:

Theorem 7.3. *Let X be a quasi-projective k-scheme with linearized G-action. Then for every G-equivariant closed immersion $i : Z \to X$ with complementary open immersion $j : U \to X$, the localization sequence*

$$\Omega^G_*(Z) \xrightarrow{i_*} \Omega^G_*(X) \xrightarrow{j^*} \Omega^G_*(U) \to 0$$

is exact.

The proof will require following vanishing statement:

Corollary 7.4. *Let k be a perfect field, G an algebraic group over k, and X a quasi-separated algebraic space of finite type over k with G-action. Then for every integer $n \in \mathbf{Z}$, the spectrum $\mathrm{C}^{\mathrm{BM},G}_\bullet(X;\mathrm{MGL})\langle -n\rangle$ is connective. That is, we have*

$$\mathrm{H}^{\mathrm{BM},G}_s(X;\mathrm{MGL})(-r) = 0$$

for all $r, s \in \mathbf{Z}$ with $s < 2r$.

Proof. By Corollary 7.1, we may assume that the algebraic group G is trivial. If X is a smooth scheme, then by Poincaré duality the claim is equivalent to the vanishing of $\mathrm{H}^q(X;\mathrm{MGL}(p))$ for $q > 2p$, see e.g. [BKWX, Thm. B.1]. In general, the schematic locus defines a dense open $U \subseteq X$ (see [SP, Tag 06NH]). Since the field k is perfect, the smooth locus V of U is a further dense open. Using the localization sequence

$$\mathrm{H}^{\mathrm{BM}}_s(Z;\mathrm{MGL})(-r) \to \mathrm{H}^{\mathrm{BM}}_s(X;\mathrm{MGL})(-r) \to \mathrm{H}^{\mathrm{BM}}_s(V;\mathrm{MGL})(-r),$$

where $Z \subseteq X$ is the reduced closed complement, we conclude by noetherian induction. □

Proof of Theorem 7.3. The localization exact triangle

$$\mathrm{C}_\bullet^{\mathrm{BM},G}(Z;\mathrm{MGL}) \to \mathrm{C}_\bullet^{\mathrm{BM},G}(X;\mathrm{MGL}) \to \mathrm{C}_\bullet^{\mathrm{BM},G}(U;\mathrm{MGL})$$

induces a long exact sequence in $\mathrm{H}_*^{\mathrm{BM},G}(-;\mathrm{MGL})(-n)$ for every $n \in \mathbf{Z}$. Since $\mathrm{H}_{2n-1}^{\mathrm{BM},G}(Z;\mathrm{MGL})(-n) = 0$ (Corollary 7.4), this gives the exact sequence

$$\mathrm{H}_{2n}^{\mathrm{BM},G}(Z;\mathrm{MGL})(-n) \xrightarrow{i_*} \mathrm{H}_{2n}^{\mathrm{BM},G}(X)(-n) \xrightarrow{j^*} \mathrm{H}_{2n}^{\mathrm{BM},G}(U)(-n) \to 0.$$

Finally, we conclude via the identifications of Corollary 7.2. The statement about functoriality is clear from the base change formula (for proper direct image and Gysin pull-backs) and the non-equivariant version of the same statement. □

8. Categorification

Let $\mathbf{D} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Pres}$ be a τ-sheaf with values in the ∞-category Pres of presentable ∞-categories and left adjoint functors. Denote by $\mathbf{D}^{\triangleleft} : {}^{\tau}\mathrm{Stk}^{\mathrm{op}} \to \infty\text{-Cat}$ its lisse extension as in Definition 1.2. Given a morphism $f : X \to Y$ we denote the induced functor by $f^* = \mathbf{D}(f) : \mathbf{D}(Y) \to \mathbf{D}(X)$, and by $f_* : \mathbf{D}(X) \to \mathbf{D}(Y)$ its right adjoint.

We will assume that $\mathbf{D}$ satisfies the following two properties:

(i) $\mathbf{A}^1$-*invariance*: for every $X \in \mathrm{Sch}$, the unit morphism

$$\mathrm{id} \to \pi_*\pi^*$$

is fully faithful, where $\pi : X \times \mathbf{A}^1 \to X$ is the projection. In other words, $\pi^* : \mathbf{D}(X) \to \mathbf{D}(X \times \mathbf{A}^1)$ is fully faithful.

(ii) *Smooth base change formula*: for every cartesian square in Sch

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y, \end{array}$$

the base change transformation

$$v^*f_* \xrightarrow{\text{unit}} g_*g^*v^*f_* \simeq g_*u^*f^*f_* \xrightarrow{\text{counit}} g_*u^*$$

is invertible.

For example, we may take $\mathbf{D} = \mathbf{SH}$ (with $\tau = \mathrm{Nis}$, see Sect. 6) or more generally any constructible ∞-category in the sense of [Kha3].

Fix S, G, and $\{U_n\}_n$ as in Notation 3.1. For every $X \in {}^{\tau}\mathrm{Stk}_S$ with G-action we consider the square

$$\begin{array}{ccc} X \times_S U_n & \xrightarrow{p_n} & X \\ \downarrow{\scriptstyle v_n} & & \downarrow{\scriptstyle u} \\ X \times_S^G U_n & \xrightarrow{q_n} & [X/G] \end{array} \tag{8.1}$$

where p_n and q_n are the projections, and u and v_n are the quotient maps.

Theorem 8.2. *For every $\mathcal{F} \in \mathbf{D}^{\triangleleft}([X/G])$, the unit maps induce a canonical isomorphism*

$$\mathcal{F} \to \varprojlim_n q_{n,*} q_n^*(\mathcal{F})$$

in $\mathbf{D}^{\triangleleft}([X/G])$.

Proof. For a fixed $\mathcal{F} \in \mathbf{D}^{\triangleleft}([X/G])$, the presheaf

$$F : {}^{\tau}\mathrm{LisStk}_{[X/G]}^{\mathrm{op}} \to \mathbf{D}^{\triangleleft}([X/G])$$

sending $(T, t : T \to [X/G]) \mapsto t_* t^*(\mathcal{F})$ is, by construction, lisse-extended from its restriction to $\mathrm{Lis}_{[X/G]}$. Thus the claim follows from Proposition 2.2 applied to $F|_{\mathrm{Lis}_{[X/G]}}$. □

Consider the right Kan extension of $\mathbf{D}^{\triangleleft}$ to ind-objects, so that

$$\mathbf{D}(X \underset{S}{\times} \{U_n\}_n) \simeq \varprojlim_n \mathbf{D}(X \underset{S}{\times} U_n), \quad \mathbf{D}(X \overset{G}{\underset{S}{\times}} \{U_n\}_n) \simeq \varprojlim_n \mathbf{D}(X \overset{G}{\underset{S}{\times}} U_n)$$

where the transition functors are $*$-inverse image. We have the induced functors

$$\begin{aligned} p^* = (p_n^*)_n : \mathbf{D}^{\triangleleft}(X) &\to \mathbf{D}^{\triangleleft}(X \underset{S}{\times} \{U_n\}_n), \\ q^* = (q_n^*)_n : \mathbf{D}^{\triangleleft}([X/G]) &\to \mathbf{D}^{\triangleleft}(X \overset{G}{\underset{S}{\times}} \{U_n\}_n). \end{aligned} \tag{8.3}$$

Corollary 8.4. *The functor (8.3) is fully faithful.*

Proof. The functor q^* admits as right adjoint $(\mathcal{F}_n) \mapsto \varprojlim_n q_{n,*}\mathcal{F}_n$, so fully faithfulness amounts to invertibility of the unit map

$$\mathcal{F} \to \varprojlim_n q_{n,*} q_n^*(\mathcal{F})$$

for all $\mathcal{F} \in \mathbf{D}([X/G])$. This follows by passage to the limit from the pro-isomorphism of Theorem 8.2. □

We say that the group scheme G is *Nisnevich-special* if the quotient morphism $S \twoheadrightarrow [S/G] = BG$ admits Nisnevich-local sections, i.e., if every étale G-torsor is Nisnevich-locally trivial. For example, this includes special group schemes in the sense of Serre such as $\mathrm{GL}_{n,S}$.

Corollary 8.5. *If τ = ét or G is Nisnevich-special, then the squares (8.1) induce a cartesian square of ∞-categories*

$$\begin{array}{ccc} \mathbf{D}^{\triangleleft}([X/G]) & \xrightarrow{q^*} & \mathbf{D}^{\triangleleft}(X \times_S^G \{U_n\}_n) \\ \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle v^*} \\ \mathbf{D}^{\triangleleft}(X) & \xrightarrow{p^*} & \mathbf{D}^{\triangleleft}(X \times_S \{U_n\}_n). \end{array}$$

Remark 8.6. If $\tau = \mathrm{Nis}$ and G is not Nisnevich-special, then one still has a cartesian square of ∞-categories

$$\begin{array}{ccc} \mathbf{D}^{\triangleleft}([X/G]) & \xrightarrow{q^*} & \mathbf{D}^{\triangleleft}(X \times_S^G \{U_n\}_n) \\ \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle v^*} \\ \mathbf{D}^{\triangleleft}(Y) & \xrightarrow{p^*} & \mathbf{D}^{\triangleleft}(Y' \times_S^G \{U_n\}_n), \end{array}$$

where $u : Y \twoheadrightarrow [X/G]$ is any smooth morphism with Nisnevich-local sections and $Y' \to Y$ is the G-torsor classified by $Y \to [X/G] \to BG$, since u^* is conservative in this case.

Lemma 8.7. *With notation as above, suppose that X is a scheme. Then the cartesian square*

$$\begin{array}{ccc} X \times_S \{U_n\}_n & \xrightarrow{p} & X \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ X \times_S^G \{U_n\}_n & \xrightarrow{q} & [X/G] \end{array}$$

satisfies the smooth base change formula. That is, the natural transformation

$$u^* q_* \xrightarrow{\text{unit}} p_* p^* u^* q_* \simeq p_* v^* q^* q_* \xrightarrow{\text{counit}} p_* v^*$$

is invertible.

Proof. This is the limit over n of the natural transformations

$$u^* q_{n,*} \to p_{n,*} v_n^*,$$

associated to the squares (8.1). Using descent for the Čech nerve of $u : X \twoheadrightarrow [X/G]$ and its base change v_n, which we denote $X_\bullet$ and $Y_\bullet$ respectively, [GR, Vol. I, Pt. I, Chap. 1, 2.6.4] implies that this is map is in turn the limit of the corresponding natural transformations for all the squares

$$\begin{array}{ccc} Y_{m+1} & \xrightarrow{p_m} & X_{m+1} \\ \downarrow{\scriptstyle d^i} & & \downarrow{\scriptstyle d^i} \\ Y_m & \xrightarrow{q_m} & X_m \end{array}$$

where the horizontal arrows are base changed from q and p and the vertical arrows d^i are the face maps (for $0 \leqslant i \leqslant m$). By the smooth base change formula for schemes (ii), these are invertible for all m and all i. □

Lemma 8.8. *Suppose given a commutative square of ∞-categories*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{q^*} & \mathcal{C}' \\ \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle v^*} \\ \mathcal{D} & \xrightarrow{p^*} & \mathcal{D}'. \end{array}$$

where p^* and q^* are fully faithful with respective right adjoints q_* and p_*, and the base change transformation

$$u^* q_* \xrightarrow{\text{unit}} p_* p^* u^* q_* \simeq p_* v^* q^* q_* \xrightarrow{\text{counit}} p_* v^*$$

is invertible, and that v^* is conservative. Then the essential image of q^* is spanned by objects $c' \in \mathcal{C}'$ for which $v^*(c')$ belongs to the essential image of p^*.

Proof. Note that an object $c' \in \mathcal{C}'$ belongs to the essential image of q^* if and only if the counit $q^*q_*(c') \to c'$ is invertible. Indeed, the condition is clearly sufficient. Conversely, suppose $c' \simeq q^*(c)$ for an object $c \in \mathcal{C}$. By the adjunction identities, the composite

$$q^*(c) \xrightarrow{\text{unit}} q^*q_*q^*(c) \xrightarrow{\text{counit}} q^*(c)$$

is the identity. Since q^* is fully faithful, the first arrow is invertible. It follows that the second arrow is also invertible.

Now since v^* is conservative, invertibility of the counit $q^*q_*(c') \to c'$ is equivalent to invertibility of

$$\text{counit} : p^*p_*v^*(c') \simeq p^*u^*q_*(c') \simeq v^*q^*q_*(c') \xrightarrow{\text{counit}} v^*(c')$$

where we have used the base change isomorphism. As in the first paragraph, since p^* is fully faithful this is equivalent to the condition that $v^*(c')$ belongs to the essential image of p^*. □

Proof of Corollary 8.5. Given $(T,t) \in \mathrm{Lis}_{[X/G]}$, we may form the base change of the squares (8.1) along $t : T \to [X/G]$ to get

$$\begin{array}{ccc} T' \times_S \{U_n\}_n & \xrightarrow{p_T} & T' \\ \downarrow{\scriptstyle v_T} & & \downarrow{\scriptstyle u_T} \\ T \times_{BG} \{U_n/G\}_n & \xrightarrow{q_T} & T \end{array}$$

where $T' \to T \simeq [T'/G]$ is the G-torsor classified by $T \to [X/G] \to BG$. By definition of $\mathbf{D}^{\lhd}$, the square in question is the limit over (T,t) of the squares

$$\begin{array}{ccc} \mathbf{D}^{\lhd}(T) & \xrightarrow{q_T^*} & \mathbf{D}^{\lhd}(T \times_{BG} \{U_n/G\}_n) \\ \downarrow{\scriptstyle u_T^*} & & \downarrow{\scriptstyle v^*} \\ \mathbf{D}^{\lhd}(T') & \xrightarrow{p^*} & \mathbf{D}^{\lhd}(T' \times_S \{U_n\}_n). \end{array}$$

We may therefore replace X by T' and thereby assume that X is a scheme.

By Corollary 8.4 the upper horizontal arrow is fully faithful. The same holds for the lower horizontal arrow (note that $\{U_n\}_n$ also serves as a Borel construction for the trivial group). This implies that the square is cartesian on mapping spaces. Essential surjectivity of the functor

$$\mathbf{D}([X/G]) \to \mathbf{D}(X) \underset{\mathbf{D}(X \times_S \{U_n\}_n)}{\times} \mathbf{D}(X \overset{G}{\underset{S}{\times}} \{U_n\}_n)$$

then follows from Lemma 8.8 in view of the base change formula $u^*q_* \simeq p_*v^*$ (Lemma 8.7) and the conservativity of v^* (since u and hence v admits τ-local sections). □

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