VIRTUAL EXCESS INTERSECTION THEORY

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To my little brother, on the occasion of his birthday.

Abstract. We prove a K-theoretic excess intersection formula for derived Artin stacks. When restricted to classical schemes, it gives a refinement and new proof of R. Thomason's formula.

Introduction

Suppose given a commutative square of derived Artin stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y} \end{array}$$

where f and f' are quasi-smooth closed immersions. We call this an *excess intersection square* if the following conditions hold:

(i) The square is cartesian on underlying classical stacks, i.e., the canonical morphism $\mathcal{X}' \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ induces an isomorphism $\mathcal{X}'_{\mathrm{cl}} \simeq (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')_{\mathrm{cl}}$.

(ii) The canonical morphism of (shifted) relative cotangent complexes

$$p^* \mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1] \to \mathcal{L}_{\mathcal{X}'/\mathcal{Y}'}[-1]$$

is surjective (on π_0). In particular, its fibre Δ is locally free of finite rank.

For a derived stack $\mathcal{X}$, let $\mathrm{K}(\mathcal{X})$ denote the algebraic K-theory space of perfect complexes on $\mathcal{X}$. In this note we prove the following theorem:

Theorem 0.1. *For any excess intersection square as above, there is a canonical homotopy*

$$q^* f_*(-) \simeq f'_*(p^*(-) \cup e(\Delta))$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{Y}')$*, where* $e(\Delta) \in \mathrm{K}(\mathcal{X}')$ *is the Euler class of the excess sheaf.*

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See [KR, Sect. 2] for background on quasi-smoothness. A closed immersion of classical stacks is quasi-smooth if and only if it is regular (a.k.a. a local complete intersection) in the sense of [SGA6, Exp. VII, Déf. 1.4], so we have:

Corollary 0.2. *For any cartesian square of Artin stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array}$$

where f and f' are regular closed immersions, there is a canonical homotopy

$$q^* f_*(-) \simeq f'_*(p^*(-) \cup e(\Delta))$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{Y}')$, *where Δ is the excess sheaf.*

For quasi-compact quasi-separated schemes, this refines Thomason's formula [Th, Thm. 3.1], in that the proof provides an explicit chain of homotopies between the two sides.

For a homotopy cartesian[1] square, the excess sheaf Δ vanishes, so this recovers the base change formula. More interesting are the following two special cases:

Corollary 0.3 (Self-intersections)**.** *For any quasi-smooth closed immersion $f : \mathcal{X} \hookrightarrow \mathcal{Y}$ of derived Artin stacks, there is a canonical homotopy*

$$f^* f_* \simeq (-) \cup e(\mathcal{N}_{\mathcal{X}/\mathcal{Y}})$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{X})$.

This is the result of applying Theorem 0.1 to the self-intersection square

$$\begin{array}{ccc} \mathcal{X} & = & \mathcal{X} \\ \| & & \downarrow{\scriptstyle f} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array}$$

where the excess sheaf is the conormal sheaf $\mathcal{N}_{\mathcal{X}/\mathcal{Y}} = \mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1]$. Similarly, we get a generalization of the "formule clef" (key formula) of [SGA6, Exp. VII]:

Corollary 0.4 (Blow-ups)**.** *For any quasi-smooth closed immersion $f : \mathcal{X} \hookrightarrow \mathcal{Y}$ of derived Artin stacks, consider the blow-up square*

$$\begin{array}{ccc} \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) & \xrightarrow{i} & \mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

Then there is a canonical homotopy

$$q^* f_* \simeq i_*(p^*(-) \cup e(\Delta))$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}))$.

[1] We remind the reader that a commutative square of *classical* stacks is homotopy cartesian (in the ∞-category of derived stacks) if and only if it is cartesian and Tor-independent.

The blow-up square is the universal excess square over $f : \mathcal{X} \to \mathcal{Y}$ such that the upper morphism i is a virtual Cartier divisor; see [KR].

The proof of Theorem 0.1 is inspired by Fulton's proof of the Grothendieck–Riemann–Roch formula [Fu, Chap. 15], and uses a similar argument involving the deformation space $\mathrm{Bl}_{\mathcal{X}\times\{\infty\}}(\mathcal{Y}\times\mathbf{P}^1)$ to reduce to the case where f and f' are zero sections of (projective completions of) vector bundles. In contrast, the original proof of Thomason made use of a rather opaque method called "spectral sequences".

Theorem 0.1 was used in [BK, Lect. 10] to generalize the Grothendieck–Riemann–Roch theorem of [SGA6] to projective quasi-smooth morphisms of derived schemes. If X is a (possibly derived) scheme, then for any quasi-smooth closed immersion $i : \mathrm{Z} \to \mathrm{X}$, the class

$$i_*(1) = [i_*(\mathcal{O}_{\mathrm{Z}})] \in \mathrm{K}(\mathrm{X})$$

is the K-theoretic version of the cohomological virtual fundamental class $[\mathrm{Z}]$ [Kh2, (3.21)]. The GRR formula implies in particular that this class lives in the expected degree of the γ-filtration (determined by the relative virtual dimension).

1. Derived symmetric powers

We start by briefly recalling some background on quasi-coherent sheaves, perfect complexes, and K-theory of derived stacks; see [Kh1] for details. Let $\mathcal{X}$ be a derived Artin stack. We denote by $\mathrm{Qcoh}(\mathcal{X})$ the stable presentable ∞-category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules, and by $\mathrm{Qcoh}(\mathcal{X})_{\geqslant 0}$ the full subcategory of connective objects. Recall that $\mathrm{Qcoh}(\mathcal{X})$ is the limit

$$\varprojlim_{u:\mathrm{X}\to\mathcal{X}} \mathrm{Qcoh}(\mathrm{X})$$

taken over the ∞-category $\mathrm{Lis}(\mathcal{X})$ of smooth morphisms $u : \mathrm{X} \to \mathcal{X}$, where X is an affine derived scheme. In other words, the sheaf $\mathcal{X} \mapsto \mathrm{Qcoh}(\mathcal{X})$ is right Kan extended from affines. For an affine $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$, $\mathrm{Qcoh}(\mathrm{X})$ is equivalent to the stable ∞-category of (nonconnective) modules over the simplicial commutative ring R, in the sense of [SAG, Not. 25.2.1.1]. Similarly for the connective and perfect subcategories $\mathrm{Qcoh}(-)_{\geqslant 0}$ and $\mathrm{Perf}(-)$, respectively. Recall that $\mathrm{K}(\mathcal{X})$ is defined as the algebraic K-theory space of the stable ∞-category $\mathrm{Perf}(\mathcal{X})$.

By $\mathrm{QcohAlg}(\mathcal{X})$ we denote the presentable ∞-category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebras. This admits a similar description as above, and for $\mathrm{Spec}(\mathrm{R})$ is equivalent to the ∞-category of R-algebras.

The *derived symmetric algebra* functor

$$\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}} : \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0} \to \mathrm{QcohAlg}(\mathcal{X})$$

is left adjoint to the forgetful functor. For any morphism $f : \mathcal{X}' \to \mathcal{X}$ there are natural isomorphisms

$$f^*(\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F})) \simeq \mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}'}}(f^*\mathcal{F})$$

for every $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0}$ (since the forgetful functors commute with f_*).

The derived symmetric algebra $\mathrm{Sym}^*_{\mathcal{O}_\mathcal{X}}(\mathcal{F})$ can be described in terms of the derived symmetric powers $\mathrm{Sym}^n_{\mathcal{O}_\mathcal{X}}(\mathcal{F})$. These were constructed in the affine case in [SAG, Sect. 25.2.2], and extend to stacks by descent.

Construction 1.1. Let $q : (\mathrm{SCRMod})_{\geqslant 0} \to \mathrm{SCRing}$ denote the cocartesian fibration associated to the presheaf of ∞-categories $\mathrm{R} \mapsto (\mathrm{Mod}_\mathrm{R})_{\geqslant 0}$; its objects are pairs (R, M), where $\mathrm{R} \in \mathrm{SCRing}$ is a simplicial commutative ring and $\mathrm{M} \in (\mathrm{Mod}_\mathrm{R})_{\geqslant 0}$ is a connective R-module. By [SAG, Constr. 25.2.2.1], there is for each integer $n \geqslant 0$ a functor $(\mathrm{SCRMod})_{\geqslant 0} \to (\mathrm{SCRMod})_{\geqslant 0}$ given informally by the assignment

$$(\mathrm{R}, \mathrm{M}) \mapsto (\mathrm{R}, \mathrm{Sym}^n_\mathrm{R}(\mathrm{M})).$$

This functor preserves q-cocartesian morphisms [SAG, Prop. 25.2.3.1] and therefore induces functors

$$\mathrm{Sym}^n_\mathrm{R} : (\mathrm{Mod}_\mathrm{R})_{\geqslant 0} \to (\mathrm{Mod}_\mathrm{R})_{\geqslant 0},$$

which define a natural transformation of functors as R varies. By right Kan extension, this extends to a natural transformation on the ∞-category of derived Artin stacks. In other words for every derived Artin stack $\mathcal{X}$ we have functors

$$\mathrm{Sym}^n_{\mathcal{O}_\mathcal{X}} : \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0} \to \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0},$$

which commute with f^*.

Lemma 1.2. *Let $\mathcal{X}$ be a derived Artin stack. Then for every connective quasi-coherent sheaf $\mathcal{E} \in \mathrm{Qcoh}(\mathcal{X})_{\geqslant 0}$, there is a canonical isomorphism*

$$\mathrm{Sym}^*_{\mathcal{O}_\mathcal{X}}(\mathcal{E}) \simeq \bigoplus_{n \geqslant 0} \mathrm{Sym}^n_{\mathcal{O}_\mathcal{X}}(\mathcal{E})$$

in $\mathrm{Qcoh}(\mathcal{X})$.

Proof. As $\mathcal{X}$ varies, both $\mathrm{Sym}^*_{\mathcal{O}_\mathcal{X}}(-)$ and $\bigoplus_n \mathrm{Sym}^n_{\mathcal{O}_\mathcal{X}}(-)$ define natural transformations $\mathrm{Qcoh}(-)_{\geqslant 0} \to \mathrm{Qcoh}(-)_{\geqslant 0}$ of presheaves on the ∞-category of derived Artin stacks. On the restrictions to affines, the two are canonically equivalent by [SAG, Constr. 25.2.2.6]. Since $\mathrm{Qcoh}(-)_{\geqslant 0}$ is right Kan extended from affines, the claim follows. □

Lemma 1.3. *Let $\mathcal{X}$ be a derived Artin stack and $\mathcal{E}' \to \mathcal{E} \to \mathcal{E}''$ a cofibre sequence of connective perfect complexes. Then there are canonical equivalences*

$$[\mathrm{Sym}^n_{\mathcal{O}_\mathcal{X}}(\mathcal{E})] \simeq \bigoplus_{i+j=n} [\mathrm{Sym}^i_{\mathcal{O}_\mathcal{X}}(\mathcal{E}') \otimes_{\mathcal{O}_\mathcal{X}} \mathrm{Sym}^j_{\mathcal{O}_\mathcal{X}}(\mathcal{E}'')]$$

for every $n \geqslant 0$. If $\mathrm{Sym}^n_{\mathcal{O}_\mathcal{X}}(\mathcal{E}) = 0$ for all sufficiently large $n \gg 0$, and similarly for $\mathcal{E}'$ and $\mathcal{E}''$, then also

$$[\mathrm{Sym}^*_{\mathcal{O}_\mathcal{X}}(\mathcal{E})] \simeq [\mathrm{Sym}^*_{\mathcal{O}_\mathcal{X}}(\mathcal{E}') \otimes_{\mathcal{O}_\mathcal{X}} \mathrm{Sym}^*_{\mathcal{O}_\mathcal{X}}(\mathcal{E}'')]$$

in $\mathrm{K}(\mathcal{X})$.

Proof. For every such cofibre sequence and every integer $n \geqslant 0$, there is a canonical filtration

$$\mathrm{Sym}^n_{\mathcal{O}_\mathcal{X}}(\mathcal{E}') = \mathrm{F}^{0,n} \to \mathrm{F}^{1,n} \to \cdots \to \mathrm{F}^{n,n} = \mathrm{Sym}^n_{\mathcal{O}_\mathcal{X}}(\mathcal{E})$$

together with cofibre sequences

$$\mathrm{F}^{i-1,n} \to \mathrm{F}^{i,n} \to \mathrm{Sym}^{n-i}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}') \otimes_{\mathcal{O}_{\mathcal{X}}} \mathrm{Sym}^{i}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}'')$$

for each $0 < i \leqslant n$. This is constructed in [SAG, Constr. 25.2.5.4] for affines and the construction extends to stacks by a similar right Kan extension procedure. Since $\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$ is perfect for all n [SAG, Prop. 25.2.5.3] (and similarly for $\mathcal{E}'$ and $\mathcal{E}''$), these cofibre sequences give rise to the desired equivalances in $\mathrm{K}(\mathcal{X})$. The second claim follows from Lemma 1.2, since the assumption guarantees $\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})$ is also perfect (and similarly for $\mathcal{E}'$ and $\mathcal{E}''$). □

Definition 1.4. *The Euler class of a finite locally free sheaf $\mathcal{E}$ is defined by*

$$e(\mathcal{E}) = [\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1])] = \sum_{n\geqslant 0} (-1)^n \cdot [\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1])] \in \mathrm{K}(\mathcal{X}).$$

There are canonical isomorphisms $\mathrm{Sym}^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1]) \simeq \Lambda^n_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E})[n]$ (cf. [SAG, Prop. 25.2.4.2]), where $\Lambda^n_{\mathcal{O}_{\mathcal{X}}}(-)$ denotes the derived exterior power, so this agrees with the usual definition of the Euler class (often denoted $\lambda_{-1}(\mathcal{E})$).

2. Projective bundles

Given a connective perfect complex $\mathcal{F}$ on a derived Artin stack $\mathcal{X}$, there is an associated "generalized vector bundle"

$$\mathbf{V}_{\mathcal{X}}(\mathcal{F}) = \mathrm{Spec}_{\mathcal{X}}(\mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F})),$$

defined as the relative spectrum of its derived symmetric algebra. It is the moduli stack of cosections $\mathcal{F} \to \mathcal{O}_{\mathcal{X}}$.

This construction can exhibit some surprising behaviour. For example, if $\mathcal{E}$ is a finite locally free sheaf, consider the morphism $p : \mathbf{V}_{\mathcal{X}}(\mathcal{E}[1]) \to \mathcal{X}$. This is a quasi-smooth closed immersion that fits in the homotopy cartesian square

$$\begin{array}{ccc} \mathbf{V}_{\mathcal{X}}(\mathcal{E}[1]) & \xrightarrow{p} & \mathcal{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s} \\ \mathcal{X} & \xrightarrow{s} & \mathbf{V}_{\mathcal{X}}(\mathcal{E}), \end{array}$$

where s is the zero section. By the base change formula we obtain a canonical isomorphism

$$s^* s_* \simeq p_* p^* \simeq (-) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1]),$$

where the second isomorphism follows from $p_*(\mathcal{O}) = \mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1])$ and the projection formula. We have just proven the following lemma in the special case where $t = s$:

Lemma 2.1. *Let $\mathcal{X}$ be a derived Artin stack. Given a finite locally free sheaf $\mathcal{E}$ and a cosection $t : \mathcal{E} \to \mathcal{O}_{\mathcal{X}}$, we let $i : \mathcal{Z} \to \mathcal{X}$ denote its derived zero locus,* *so that there is a homotopy cartesian square*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{i} & \mathcal{X} \\ \downarrow & & \downarrow t \\ \mathcal{X} & \xrightarrow{s} & \mathbf{V}_{\mathcal{X}}(\mathcal{E}). \end{array}$$

Then there is an essentially unique homotopy

$$i_* i^* \simeq (-) \cup e(\mathcal{E})$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{X})$.

Proof. The result of composing such a square with the open immersion $\mathbf{V}_{\mathcal{X}}(\mathcal{E}) \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ is still homotopy cartesian. Let $\bar{s} : \mathcal{X} \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ and $\bar{t} : \mathcal{X} \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$, respectively, denote the induced morphisms. By the base change formula we have

$$i_* i^* \simeq \bar{t}^* \bar{s}_*.$$

In the case $t = s$, we have as above $\bar{s}^* \bar{s}_* \simeq (-) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathrm{Sym}^*_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}[1])$, so in particular $\bar{s}^* \bar{s}_* \simeq (-) \cup e(\mathcal{E})$ in K-theory. Thus it will suffice to exhibit an (essentially unique) homotopy between the two maps $\mathrm{K}(\mathbf{P}(\mathcal{E} \oplus \mathcal{O})) \to \mathrm{K}(\mathcal{X})$ induced by $\bar{s}^*$ and $\bar{t}^*$. But from the projective bundle formula [Kh1, Cor. 3.4.1] it follows that there is an exact triangle

$$\mathrm{K}(\mathbf{P}_{\mathcal{X}}(\mathcal{E})) \xrightarrow{\infty_*} \mathrm{K}(\mathbf{P}_{\mathcal{X}}(\mathcal{E} \oplus \mathcal{O})) \xrightarrow{\bar{u}^*} \mathrm{K}(\mathcal{X}),$$

for *any* $\bar{u} : \mathcal{X} \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ induced by a section $u : \mathcal{X} \to \mathbf{V}_{\mathcal{X}}(\mathcal{E})$. The claim follows. □

Let $\pi : \mathbf{P}(\mathcal{E} \oplus \mathcal{O}) \to \mathcal{X}$ denote the projection. We have on $\mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ the canonical exact triangle of locally free sheaves

$$\mathcal{Q} \to \pi^*(\mathcal{E}) \oplus \mathcal{O} \to \mathcal{O}(1).$$

Recall that the zero section $\bar{s} : \mathcal{X} \to \mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ can be written as the derived zero locus of the canonical cosection

$$\mathcal{Q} \to \pi^*(\mathcal{E}) \oplus \mathcal{O} \to \mathcal{O}.$$

Thus we get:

Corollary 2.2. *There is a canonical homotopy*

$$\bar{s}_*(-) \simeq e(\mathcal{Q}) \cup \pi^*(-)$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathbf{P}(\mathcal{E} \oplus \mathcal{O}))$.

Proof. By Lemma 2.1, $\bar{s}_*(\mathcal{O}) \simeq e(\mathcal{Q})$. By the projection formula, $\bar{s}_*(-) \simeq \bar{s}_*(\mathcal{O}) \cup \pi^*(-) \simeq e(\mathcal{Q}) \cup \pi^*(-)$. □

We are now ready to prove a special case of Theorem 0.1. Let $p : \mathcal{X}' \to \mathcal{X}$ be a morphism of derived Artin stacks. Let $\mathcal{E}$ and $\mathcal{E}'$ be finite locally free sheaves on $\mathcal{X}$ and $\mathcal{X}'$, respectively, together with a surjection

$$p^*(\mathcal{E}) \twoheadrightarrow \mathcal{E}'$$

whose fibre we denote Δ. This induces an excess intersection square

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\bar{s}'} & \mathbf{P}(\mathcal{E}'\oplus\mathcal{O}) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{\bar{s}} & \mathbf{P}(\mathcal{E}\oplus\mathcal{O}), \end{array}$$

where $\bar{s}$ and $\bar{s}'$ are the zero sections.

Claim 2.3. *The excess intersection formula*

$$q^*\bar{s}_* \simeq \bar{s}'_*(p^*(-)\cup e(\Delta))$$

holds for the above square.

Proof. Let $\pi : \mathbf{P}(\mathcal{E}\oplus\mathcal{O}) \to \mathcal{X}$ and $\pi' : \mathbf{P}(\mathcal{E}'\oplus\mathcal{O}) \to \mathcal{X}'$ denote the respective projections. Let $\mathcal{Q}$ and $\mathcal{Q}'$ denote the respective universal hyperplane sheaves on $\mathbf{P}(\mathcal{E}\oplus\mathcal{O})$ and $\mathbf{P}(\mathcal{E}'\oplus\mathcal{O})$. The surjection $p^*(\mathcal{E}) \to \mathcal{E}'$ gives rise to a canonical morphism $q^*\mathcal{Q} \to \mathcal{Q}'$, whose fibre is $(\pi')^*(\Delta)$. Thus Lemma 1.3 provides a canonical homotopy

$$e(q^*\mathcal{Q}) \simeq e((\pi')^*\Delta)\cup e(\mathcal{Q}')$$

in $\mathrm{K}(\mathbf{P}(\mathcal{E}'\oplus\mathcal{O}))$. Now two applications of Corollary 2.2 give:

$$\begin{aligned} q^*\bar{s}_* &\simeq q^*(\pi^*(-)\cup e(\mathcal{Q})) \\ &\simeq (\pi')^*p^*(-)\cup e(q^*\mathcal{Q}) \\ &\simeq (\pi')^*p^*(-)\cup e((\pi')^*\Delta)\cup e(\mathcal{Q}') \\ &\simeq (\pi')^*(p^*(-)\cup e(\Delta))\cup e(\mathcal{Q}') \\ &\simeq \bar{s}'_*(p^*(-)\cup e(\Delta)), \end{aligned}$$

as desired. ∎

3. Deformation space

Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth closed immersion of derived Artin stacks. Write M for the blow-up $\mathrm{Bl}_{\mathcal{X}\times\{\infty\}}(\mathcal{Y}\times\mathbf{P}^1)$ as in [KR]. It fits in a commutative diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{s_0} & \mathcal{X}\times\mathbf{P}^1 & \xleftarrow{s_\infty} & \mathcal{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle f_\infty} \\ \mathcal{Y} & \xrightarrow{\sigma_0} & \mathrm{M} & \xleftarrow{\sigma_\infty} & \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}\oplus\mathcal{O}) \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow{\scriptstyle \pi_\infty} \\ \{0\} & \longrightarrow & \mathbf{P}^1 & \longleftarrow & \{\infty\} \end{array}$$

The two left-hand squares and upper right-hand square are homotopy cartesian. The morphism $\hat{f}$ is

$$\mathcal{X}\times\mathbf{P}^1 = \mathrm{Bl}_{\mathcal{X}\times\{\infty\}}(\mathcal{X}\times\mathbf{P}^1) \to \mathrm{Bl}_{\mathcal{X}\times\{\infty\}}(\mathcal{Y}\times\mathbf{P}^1),$$

induced by $f\times\mathrm{id} : \mathcal{X}\times\mathbf{P}^1 \to \mathcal{Y}\times\mathbf{P}^1$, and the morphism f_∞ is the zero section. There are canonical morphisms $r : \mathcal{X}\times\mathbf{P}^1 \to \mathcal{X}$, $\rho : \mathrm{M}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{Y}$, retractions of s_0 and σ_0, respectively.

Denote by $\mathrm{M}_\infty := \mathrm{M} \times^{\mathbf{R}}_{\mathbf{P}^1} \{\infty\}$ the special fibre, and by $i_\infty : \mathrm{M}_\infty \to \mathrm{M}$ the inclusion. Then we have a canonical homotopy

$$(\sigma_0)_*(\sigma_0)^* \simeq (i_\infty)_*(i_\infty)^* \tag{3.1}$$

of maps $\mathrm{K}(\mathrm{M}) \to \mathrm{K}(\mathrm{M})$. Indeed, we have $0_*(\mathcal{O}) \simeq \infty_*(\mathcal{O})$ in $\mathrm{K}(\mathbf{P}^1)$, so by the base change formula there is a canonical identification

$$\begin{aligned}(\sigma_0)_*(\mathcal{O}) \simeq (\sigma_0)_*(\pi_0)^*(\mathcal{O}) \simeq \widehat{\pi}^* 0_*(\mathcal{O}) \\ \simeq \widehat{\pi}^* \infty_*(\mathcal{O}) \simeq (i_\infty)_*(\pi_\infty)^*(\mathcal{O}) \simeq (i_\infty)_*(\mathcal{O})\end{aligned}$$

in $\mathrm{K}(\mathrm{M})$. Thus the claim follows from the projection formula.

The fibre M_∞ fits in a homotopy cartesian and cocartesian square

$$\begin{array}{ccc} \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) & \longrightarrow & \mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}} \oplus \mathcal{O}) & \longrightarrow & \mathrm{M}_\infty. \end{array}$$

That is, M_∞ is the sum of the two virtual Cartier divisors $\mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}} \oplus \mathcal{O})$ and $\mathrm{Bl}_{\mathcal{X}}(\mathcal{Y})$ on M. We denote by $i_\infty : \mathrm{M}_\infty \to \mathrm{M}$ the inclusion, and by $b : \mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \to \mathrm{M}$, and $c : \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) \to \mathrm{M}$ the composites with i_∞. We have a canonical homotopy

$$(i_\infty)_*(i_\infty)^* \simeq (\sigma_\infty)_*(\sigma_\infty)^* + b_*b^* - c_*c^* \tag{3.2}$$

of maps $\mathrm{K}(\mathrm{M}) \to \mathrm{K}(\mathrm{M})$, by the following lemma.

Lemma 3.3. *Let* $\mathrm{D} \hookrightarrow \mathcal{X}$ *and* $\mathrm{D}' \hookrightarrow \mathcal{X}$ *be virtual Cartier divisors on a derived Artin stack* $\mathcal{X}$. *Denote by* $\mathrm{D} \cap \mathrm{D}' = \mathrm{D} \times^{\mathbf{R}}_{\mathcal{X}} \mathrm{D}'$ *their intersection and by*

$$\mathrm{D} + \mathrm{D}' = \mathrm{D} \underset{\mathrm{D}\cap\mathrm{D}'}{\sqcup} \mathrm{D}'$$

their sum. Then we have a canonical homotopy

$$(i_{\mathrm{D}+\mathrm{D}'})_*(i_{\mathrm{D}+\mathrm{D}'})^* \simeq (i_{\mathrm{D}})_*(i_{\mathrm{D}})^* + (i_{\mathrm{D}'})_*(i_{\mathrm{D}'})^* - (i_{\mathrm{D}\cap\mathrm{D}'})_*(i_{\mathrm{D}\cap\mathrm{D}'})^*$$

of maps $\mathrm{K}(\mathcal{X}) \to \mathrm{K}(\mathcal{X})$.

Proof. By definition of $\mathrm{D} + \mathrm{D}'$ we have

$$(i_{\mathrm{D}+\mathrm{D}'})_*(\mathcal{O}_{\mathrm{D}+\mathrm{D}'}) \simeq (i_{\mathrm{D}})_*(\mathcal{O}_{\mathrm{D}}) \underset{(i_{\mathrm{D}\cap\mathrm{D}'})_*(\mathcal{O}_{\mathrm{D}\cap\mathrm{D}'})}{\times} (i_{\mathrm{D}'})_*(\mathcal{O}_{\mathrm{D}'})$$

in $\mathrm{Perf}(\mathcal{X})$. This induces in $\mathrm{K}(\mathcal{X})$ a canonical homotopy

$$(i_{\mathrm{D}+\mathrm{D}'})_*(\mathcal{O}_{\mathrm{D}+\mathrm{D}'}) \simeq (i_{\mathrm{D}})_*(\mathcal{O}_{\mathrm{D}}) + (i_{\mathrm{D}'})_*(\mathcal{O}_{\mathrm{D}'}) - (i_{\mathrm{D}\cap\mathrm{D}'})_*(\mathcal{O}_{\mathrm{D}\cap\mathrm{D}'}).$$

We conclude using the projection formula. □

Since the intersection

$$\mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \underset{\mathrm{M}}{\times} (\mathcal{X} \times \mathbf{P}^1) = \mathrm{Bl}_{\mathcal{X}}(\mathcal{Y}) \underset{\mathrm{M}_\infty}{\times} \mathcal{X} = \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) \underset{\mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}\oplus\mathcal{O}})}{\times} \mathcal{X}$$

is empty, we have $b^*\hat{f}_* = 0$ and $c^*\hat{f}_* = 0$ by the base change formula. Thus (3.1) and (3.2) induce the homotopy

$$(\sigma_0)_*(\sigma_0)^*\hat{f}_* \simeq (i_\infty)_*(i_\infty)^*\hat{f}_* \simeq (\sigma_\infty)_*(\sigma_\infty)^*\hat{f}_* \tag{3.4}$$

of maps $\mathrm{K}(\mathcal{X} \times \mathbf{P}^1) \to \mathrm{K}(\mathrm{M})$.

4. Proof

Consider an excess intersection square of the form

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y}. \end{array}$$

We keep the notation of the previous section, so $\mathrm{M} = \mathrm{Bl}_{\mathcal{X}\times\{\infty\}}(\mathcal{Y}\times\mathbf{P}^1)$, etc. We consider all the same constructions for $f' : \mathcal{X}' \to \mathcal{Y}'$, with notation decorated by primes: $\mathrm{M}' = \mathrm{Bl}_{\mathcal{X}'\times\{\infty\}}(\mathcal{Y}'\times\mathbf{P}^1)$, and so on. We have morphisms of excess intersection squares

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y}. \end{array} \quad \hookrightarrow \quad \begin{array}{ccc} \mathcal{X}'\times\mathbf{P}^1 & \xrightarrow{\hat{f}'} & \mathrm{M}' \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle \hat{q}} \\ \mathcal{X}\times\mathbf{P}^1 & \xrightarrow{\hat{f}} & \mathrm{M} \end{array} \quad \hookleftarrow \quad \begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'_\infty} & \mathbf{P}(\mathcal{N}_{\mathcal{X}'/\mathcal{Y}'}\oplus\mathcal{O}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q_\infty} \\ \mathcal{X} & \xrightarrow[f_\infty]{} & \mathbf{P}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}\oplus\mathcal{O}). \end{array}$$

Using $r \circ s_0 = \mathrm{id}$ and the base change formula, we have

$$\begin{aligned} q^* f_* &\simeq q^* f_* (s_0)^* r^* \\ &\simeq q^* (\sigma_0)^* \hat{f}_* r^* \\ &\simeq (\sigma'_0)^* \hat{q}^* \hat{f}_* r^* \end{aligned}$$

and similarly

$$\begin{aligned} (q_\infty)^*(f_\infty)_* &\simeq (q_\infty)^*(f_\infty)_*(s_\infty)^* r^* \\ &\simeq (q_\infty)^*(\sigma_\infty)^* \hat{f}_* r^* \\ &\simeq (\sigma'_\infty)^* \hat{q}^* \hat{f}_* r^*. \end{aligned}$$

Thus (3.4) induces an equivalence

$$(\sigma'_0)_* q^* f_* \simeq (\sigma'_0)_*(\sigma'_0)^* \hat{q}^* \hat{f}_* r^* \simeq (\sigma'_\infty)_*(\sigma'_\infty)^* \hat{q}^* \hat{f}_* r^* \simeq (\sigma'_\infty)_*(q_\infty)^*(f_\infty)_*.$$

Applying ρ'_* gives

$$q^* f_* \simeq \rho'_*(\sigma'_\infty)_*(q_\infty)^*(f_\infty)_*$$

since $\rho' \circ \sigma'_0 = \mathrm{id}$. Finally, Claim 2.3 yields the desired equivalence

$$q^* f_* \simeq \rho'_*(\sigma'_\infty)_*(f'_\infty)_*(p^*(-) \cup e(\Delta)) \simeq f'_*(p^*(-) \cup e(\Delta)).$$

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