

# EQUIVARIANT HOMOLOGY OF STACKS

ADEEL A. KHAN

ABSTRACT. We use sheaf theory and the six operations to define and study the (equivariant) homology of stacks. The construction makes sense in the algebraic, complex analytic, or even topological categories.

## INTRODUCTION

Let  $X$  be a locally compact and Hausdorff topological space. The singular homology groups  $H_*(X; \mathbf{Z})$  can be described via sheaf theory. Namely, one has the functors  $f_!$  and  $f^!$  of compactly supported direct and inverse image along the projection  $f : X \rightarrow \text{pt}$  on derived categories of sheaves of abelian groups. By Grothendieck–Verdier duality, one has

$$H_*(X; \mathbf{Z}) \simeq H^{-*}(f_! f^! \mathbf{Z}).$$

Similarly, Borel–Moore homology<sup>1</sup> is computed by the complex  $f_* f^! \mathbf{Z}$ . Moreover, the various operations and properties of both variants fall out of the yoga of the six functors right away. We refer to [Gode, KS] for background on sheaves and the six functor formalism<sup>2</sup> and [Bre, Chap. V] for the sheaf-theoretic point of view on homology and Borel–Moore homology.

In this note we are interested in generalizing the above story to define  $G$ -equivariant homology, when  $G$  is a topological group acting on  $X$ . At the same time, we also want to allow  $X$  to be a topological Artin<sup>3</sup> stack. The language of  $\infty$ -categories makes it easy to extend derived categories to stacks, and the six operations are also available in that setting by [Kha5] (see also [LO, LZ, KV], and our review in Sect. 1). The  $G$ -equivariant homology of  $X$  is the *relative* homology of the induced map of quotient stacks  $g : [X/G] \rightarrow BG := [\text{pt}/G]$ , and similarly for the Borel–Moore variant. That is to say,

$$H_*^G(X; \mathbf{Z}) := H^{-*}(BG, g_! g^! \mathbf{Z}), \quad H_*^{\text{BM}, G}(X; \mathbf{Z}) := H^{-*}(BG, g_* g^! \mathbf{Z}).$$

As above, all desired properties of these constructions are derived immediately from the six functor formalism. Of course, they also specialize to homology theories in the complex analytic and algebraic categories.<sup>4</sup>

One motivation to write this note came from work of D. Joyce (see [Joy]), who formulates certain wall-crossing formulas in enumerative geometry in

---

*Date:* 2025-06-04.

<sup>1</sup>a.k.a. locally finite homology

<sup>2</sup>More modern treatments such as [Vol] allow us to treat unbounded derived categories, which is why we do not need to assume finiteness of cohomological dimension above.

<sup>3</sup>or even *higher* Artin

<sup>4</sup>For readers uncomfortable with topological stacks, we note that it is not necessary to pass through underlying topological stacks; see (1.2.2).

terms of vertex algebra structures defined on the homology of certain (algebraic) moduli stacks. This in particular requires a suitable equivariant homology theory as above. See also forthcoming work of A. Bojko [Boj], who uses our definition to refine Joyce’s vertex algebras.<sup>5</sup>

Let us also note that the considerations of this paper make sense in any suitable six functor formalism<sup>6</sup>. For example, one could use the machinery of motivic sheaves (see [Kha1, App. A] or [Kha5, §7]) to define equivariant motivic homology together with a cycle class map to equivariant singular homology. Motivic homology can be thought of as “compactly supported” version of (higher) Chow groups ( $\approx$  motivic Borel–Moore homology) and, unlike the latter (see [Joy, Rem. 2.3(b)]), can be used to define a Chow type lift of Joyce’s vertex algebra structure.

**Acknowledgments.** I am grateful to Arkadij Bojko and Dominic Joyce for that discussions that led to this note as well as feedback on previous drafts. I also thank Charanya Ravi for useful discussions, and James E. Hotchkiss and Anton Mellit for pointing out some errors.

## 1. SHEAVES

### 1.1. Topological stacks.

1.1.1. Let  $\mathbf{Spc}$  denote the category of topological spaces, always implicitly assumed locally compact and Hausdorff. We say that a morphism of topological spaces is *smooth*<sup>7</sup> if it is a topological submersion.

1.1.2. Let  $\mathbf{Stk}$  denote the  $\infty$ -category of topological stacks, i.e., sheaves of  $\infty$ -groupoids on  $\mathbf{Spc}$ . We say that a topological stack  $X$  is *0-Artin* if it is a topological space. We define 1-Artinness as follows.

- (i) A morphism  $f : X \rightarrow Y$  is *0-Artin* (or simply *representable*) if for every  $V \in \mathbf{Spc}$  and every morphism  $V \rightarrow Y$ , the fibred product  $X \times_Y V$  is 0-Artin.
- (ii) A 0-Artin morphism  $f : X \rightarrow Y$  is *smooth* if for every  $Y \in \mathbf{Spc}$  and every morphism  $V \rightarrow Y$ , the base change  $X \times_Y V \rightarrow V$  is smooth.
- (iii) A topological stack  $X$  is *1-Artin* if its diagonal  $\Delta_X : X \rightarrow X \times X$  is 0-Artin, and there exists  $U \in \mathbf{Spc}$  and a morphism<sup>8</sup>  $p : U \twoheadrightarrow X$  which is a smooth surjection.
- (iv) If  $Y$  is a space and  $X$  is 1-Artin, a morphism  $f : X \rightarrow Y$  is *smooth* if there exists a space  $U$  and a smooth surjection  $p : U \twoheadrightarrow X$  such that  $f \circ p : U \rightarrow Y$  is a smooth morphism of spaces.

<sup>5</sup>Combining the discussion in Subsects. 2.4, 2.5 (particularly Remark 2.10), and 3.2, one can see that the definition proposed by Joyce is a quotient of ours.

<sup>6</sup>to be precise: in any topological weave in the sense of [Kha4, Kha5]

<sup>7</sup>We will never use this term to refer to differentiable structures.

<sup>8</sup>which is automatically 0-Artin when  $\Delta_X$  is 0-Artin

Iterating this inductively<sup>9</sup>, we define  $k$ -Artin stacks and (smooth)  $k$ -Artin morphisms for all  $k \geq 0$ . We say  $X$  is *Artin* if it is  $k$ -Artin for some  $k$ , and a morphism is Artin if it is representable by Artin stacks.

1.1.3. Since properness for morphisms of topological spaces is local on the target, we can extend this notion to topological Artin stacks as follows.

A 0-Artin morphism  $f : X \rightarrow Y$  is *proper* if for any morphism  $V \rightarrow Y$  with  $V \in \mathbf{Spc}$ , the base change  $X \times_Y V \rightarrow V$  is a proper morphism of spaces.

A 1-Artin morphism  $f : X \rightarrow Y$  is *proper* if for any morphism  $V \rightarrow Y$  with  $V \in \mathbf{Spc}$ , there exists an (automatically 0-Artin) proper surjection  $Z \twoheadrightarrow X \times_Y V$  with  $Z \in \mathbf{Spc}$  such that  $Z \rightarrow V$  is proper.

1.1.4. The analogous definitions make sense also in the algebraic and complex analytic categories.

The complex analytification functor sends an algebraic space, locally of finite type over the field  $k$  of complex numbers, to its space of  $k$ -points. This extends uniquely to a colimit-preserving functor sending locally of finite type  $k$ -stacks in the algebraic category to stacks in the complex analytic or topological category.

This functor preserves Artin stacks, Artin morphisms, smooth morphisms, vector bundles, and proper representable morphisms.

While our definition of proper morphisms of 1-Artin stacks is not the standard one in the algebraic category (cf. [SP, Tag 0CL4]), it follows from [Ols, Thm. 1.1] and [SP, Tags 0CL3, 06TZ, 06U7, 0DTL] that they agree; hence proper morphisms of 1-Artin stacks are also preserved by complex analytification.

When the discussion does not depend on any particular context, we will use the terms *space* and *stack* agnostically. For example, when read in the algebraic category, *space* will mean “algebraic space locally of finite type over  $k$ ”.

## 1.2. Sheaves.

1.2.1. Fix a commutative ring of coefficients  $\Lambda$ . For a topological space  $X$ , we denote by  $\mathrm{Shv}(X)$  the derived  $\infty$ -category of sheaves of  $\Lambda$ -modules on  $X$ .<sup>10</sup> We extend  $\mathrm{Shv}(-)$  to topological stacks by lisse extension, as in [Kha5, §8]. Thus for a topological stack  $X$ , we have

$$\mathrm{Shv}(X) \simeq \varprojlim_{(S,s)} \mathrm{Shv}(S) \quad (1.1)$$

<sup>9</sup>noting at each point that if  $X$  has  $(n-1)$ -representable diagonal, then any morphism  $U \rightarrow X$  from a space  $U$  is automatically  $(n-1)$ -representable

<sup>10</sup>Equivalently,  $\mathrm{Shv}(X)$  is the  $\infty$ -category of sheaves on  $X$  with values in the derived  $\infty$ -category of  $\Lambda$ -modules. More generally, one can take  $\Lambda$  to be an  $\mathcal{E}_\infty$ -ring spectrum, and  $\mathrm{Shv}(X)$  to be the  $\infty$ -category of sheaves of  $\Lambda$ -module spectra on  $X$ .

where the limit of  $\infty$ -categories is taken over pairs  $(S, s)$  with  $S \in \mathbf{Spc}$  and  $s : S \rightarrow X$  a morphism. If  $X$  is Artin, it is enough to take pairs where  $s$  is *smooth*.

1.2.2. We may also perform the lisse extension within the algebraic or complex analytic categories. For example, if  $X$  is a stack in the algebraic category, we have

$$\mathrm{Shv}(X) := \varprojlim_{(S, s)} \mathrm{Shv}(S)$$

where the limit is taken over pairs  $(S, s)$  with  $S$  a locally of finite type algebraic space (or scheme) and  $s : S \rightarrow X$  a morphism in the algebraic category. For  $X$  Artin, we may again require  $s$  to be smooth. See [Kha5, §3.2].

There is a natural equivalence  $\mathrm{Shv}(X) \simeq \mathrm{Shv}(X^{\mathrm{top}})$ , where  $X^{\mathrm{top}}$  is the underlying topological stack (1.1.4). This follows from the fact that  $X \mapsto X^{\mathrm{top}}$  preserves colimits, and that  $X \mapsto \mathrm{Shv}(X)$  sends colimits of stacks to limits of  $\infty$ -categories (see [Kha5, §3.2]).

1.2.3. The six functor formalism is summed up as follows. By the discussion in (1.1.4) and (1.2.2), it may be read in any of the topological, complex analytic, or algebraic categories according to the reader's preference.

**Theorem 1.2.** *The assignment  $X \mapsto \mathrm{Shv}(X)$  defines a topological weave in the sense of [Kha4, Kha5]. In particular, we have:*

- (i) *Tensor and Hom. For every stack  $X$ , there is a closed symmetric monoidal structure on  $\mathrm{Shv}(X)$ . In particular, there are adjoint bifunctors  $(\otimes, \underline{\mathrm{Hom}})$ . The monoidal unit is the constant sheaf  $\Lambda_X \in \mathrm{Shv}(X)$ .*
- (ii) *\*-Functoriality. For every morphism  $f : X \rightarrow Y$ , there is a pair of adjoint functors*

$$f^* : \mathrm{Shv}(Y) \rightarrow \mathrm{Shv}(X), \quad f_* : \mathrm{Shv}(X) \rightarrow \mathrm{Shv}(Y).$$

*The functor  $f^*$  (resp.  $f_*$ ) is symmetric monoidal (resp. lax symmetric monoidal). The assignments  $f \mapsto f^*$ ,  $f \mapsto f_*$  are compatible with composition up to coherent homotopy.*

- (iii) *!-Functoriality. For every Artin morphism  $f : X \rightarrow Y$ , there is a pair of adjoint functors*

$$f_! : \mathrm{Shv}(X) \rightarrow \mathrm{Shv}(Y), \quad f^! : \mathrm{Shv}(Y) \rightarrow \mathrm{Shv}(X).$$

*The assignments  $f \mapsto f_!$ ,  $f \mapsto f^!$  are compatible with composition up to coherent homotopy.*

- (iv) *Base change. For every Artin morphism  $f : X \rightarrow Y$ , the operation  $f_!$  commutes with  $*$ -inverse image. Dually,  $f^!$  commutes with  $*$ -direct image.*
- (v) *Projection formula. Let  $f : X \rightarrow Y$  be an Artin morphism. For every object  $\mathcal{F} \in \mathrm{Shv}(X)$  and every object  $\mathcal{G} \in \mathrm{Shv}(Y)$ , there are canonical*

isomorphisms

$$\begin{aligned} f_!(\mathcal{F}) \otimes \mathcal{G} &\rightarrow f_!(\mathcal{F} \otimes f^*(\mathcal{G})), \\ \underline{\mathrm{Hom}}(f_!(\mathcal{F}), \mathcal{G}) &\rightarrow f_* \underline{\mathrm{Hom}}(\mathcal{F}, f^!(\mathcal{G})), \\ f^! \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) &\rightarrow \underline{\mathrm{Hom}}(f^*(\mathcal{F}), f^!(\mathcal{G})). \end{aligned}$$

- (vi) Forgetting supports. For every 1-Artin morphism  $f : X \rightarrow Y$  with proper diagonal, there is a natural transformation  $\alpha_f : f_! \rightarrow f_*$  which is compatible with composition up to coherent homotopy. Moreover, it is invertible if  $f$  is proper and representable.<sup>11</sup>
- (vii) Localization. Let  $X$  be an Artin stack and let  $i : Z \rightarrow X$  and  $j : U \rightarrow X$  be a complementary pair of closed and open immersions, respectively.<sup>12</sup> Then there are exact triangles of functors

$$\begin{aligned} j_! j^* &\rightarrow \mathrm{id} \rightarrow i_* i^*, \\ i_* i^! &\rightarrow \mathrm{id} \rightarrow j_* j^!. \end{aligned}$$

- (viii) Homotopy invariance. Let  $X$  be a stack and  $\pi : E \rightarrow X$  be the projection of a vector bundle. Then the natural transformations

$$\begin{aligned} \mathrm{id} &\rightarrow \pi_* \pi^*, \\ \pi_! \pi^! &\rightarrow \mathrm{id}, \end{aligned}$$

are invertible.

- (ix) Thom twist. Let  $f : X \rightarrow Y$  be a smooth Artin morphism. Denote by  $\Delta : X \rightarrow X \times_Y X$  and  $\mathrm{pr}_2 : X \times_Y X \rightarrow X$  the diagonal and second projection, respectively. Then the assignment

$$\mathcal{F} \mapsto \mathcal{F} \langle -T_f \rangle := \Delta^! \mathrm{pr}_2^*(\mathcal{F})$$

determines a  $\mathrm{Shv}(X)$ -linear auto-equivalence of  $\mathrm{Shv}(X)$ , whose inverse we denote  $\mathcal{F} \mapsto \mathcal{F} \langle T_f \rangle$ .<sup>13</sup>

- (x) Poincaré duality. Let  $f : X \rightarrow Y$  be an Artin morphism. If  $f$  is smooth, then there is a canonical isomorphism of functors

$$f^!(-) \simeq f^*(-) \langle T_f \rangle.$$

In particular,  $\omega_{X/Y} := f^!(\Lambda_Y) \simeq \Lambda_X \langle T_f \rangle$  is  $\otimes$ -invertible.

See (1.2.7) below for the proof.

<sup>11</sup>See also (1.2.8).

<sup>12</sup>The open complement  $U = X \setminus Z$  is by definition the subfunctor of  $X$  whose  $T$ -points are  $T \rightarrow X$  for which  $T \times_X Z = \emptyset$  (for every  $T \in \mathrm{Spc}$ ). That is to say, it is the largest substack of  $X$  that doesn't intersect  $Z$ . It is easy to see that the morphism  $U \rightarrow X$  is an open immersion.

<sup>13</sup>In the topological category, " $T_f$ " is here just a symbol. The notation is meant to suggest that but one should think of  $\Lambda_X \langle T_f \rangle$  as the *Thom sheaf of the relative tangent microbundle* of  $f$ . In the algebraic or complex analytic (or even  $C^1$ ) categories,  $\Lambda_X \langle T_f \rangle$  can indeed be identified with the Thom sheaf of the relative tangent bundle  $T_f$ , via a deformation to the normal bundle argument.

1.2.4. Let  $f : X \rightarrow Y$  be a smooth morphism of relative dimension  $d$  between Artin stacks. An *orientation* of  $f$  is a trivialization  $\Lambda_X \langle T_f \rangle \simeq \Lambda_X[d]$ . (Note that this depends on the coefficient ring  $\Lambda$ .) Given an orientation, Poincaré duality reads  $f^! \simeq f^* \langle d \rangle$ .

For example, any smooth morphism of Artin stacks in the algebraic or complex analytic categories admits a canonical orientation.<sup>14</sup>

1.2.5. For any cartesian square of stacks

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow p & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

where  $q$  is an Artin morphism, we have in addition to the base change isomorphisms

$$\mathrm{Ex}_!^* : q^* f_! \simeq g_! p^*, \quad \mathrm{Ex}_*^! : q^! f_* \simeq g_* q^!,$$

further exchange transformations

$$\mathrm{Ex}_*^* : q^* f_* \rightarrow g_* p^*, \quad \mathrm{Ex}^{*,!} : g^* q^! \rightarrow p^! f^*,$$

see [Kha5, §A.2], which also become invertible in certain situations:

- (a) Whenever we have the forgetting supports isomorphism  $f_! \simeq f_*$  (and similarly for  $g$ ), one deduces that  $\mathrm{Ex}_*^*$  is invertible. (For example, when  $f$  and  $g$  are proper representable morphisms of 1-Artin stacks.)
- (b) When  $p$  and  $q$  are smooth and hence satisfy Poincaré duality, one deduces that  $\mathrm{Ex}_*^*$  and  $\mathrm{Ex}^{*,!}$  are invertible.

1.2.6. We also have the following descent statements for the weave  $\mathrm{Shv}(-)$ :

**Proposition 1.3** (Descent). *Let  $p : Y \rightarrow X$  be an Artin morphism and denote by  $Y_\bullet$  its Čech nerve.*

- (i) Smooth descent. *Consider the canonical functor of  $\infty$ -categories*

$$\mathrm{Shv}(X) \rightarrow \mathrm{Tot}(\mathrm{Shv}(Y_\bullet)),$$

*where  $\mathrm{Tot}(-)$  denotes the totalization (homotopy limit), with transition functors given by  $*$ -inverse image (resp. by  $!$ -inverse image). If  $p$  is a smooth surjection, then this is an equivalence.*

- (ii) Proper descent. *Consider the canonical functor of  $\infty$ -categories*

$$\widehat{\mathrm{Shv}}(X) \rightarrow \mathrm{Tot}(\widehat{\mathrm{Shv}}(Y_\bullet)),$$

---

<sup>14</sup>An orientation of  $f$  amounts to a Thom isomorphism for  $T_f$  (in view of Footnote 13). If  $\Lambda$  is an  $\mathcal{E}_\infty$ -ring spectrum equipped with a complex orientation, such as the Eilenberg–MacLane spectrum for an ordinary commutative ring, the complex orientation determines compatible Thom isomorphisms for all complex vector bundles.

where the transition functors are given by  $*$ -inverse image and  $\widehat{\mathrm{Shv}}(-)$  denotes the left completion<sup>15</sup> of  $\mathrm{Shv}(-)$  with respect to the cohomological  $t$ -structure<sup>16</sup>. If  $p$  is a proper representable surjection, then this is an equivalence.

1.2.7. *Proof of Theorem 1.2 and Proposition 1.3.* See [Kha5, §7.1] and [Kha5, §8.3] for the construction of the weave  $\mathrm{Shv}(-)$  in the algebraic and topological categories, respectively (and [Kha5, Cor. 3.4.9] for the abstract statement). In particular, parts (i), (ii), (iii), (iv), (v), (ix), and (x) of Theorem 1.2 are proven there, as well as Proposition 1.3(i). Using Proposition 1.3(i) and (1.2.5), parts (vii) and (viii) of Theorem 1.2 then reduce to the case of spaces. Similarly, Proposition 1.3(ii) reduces<sup>17</sup> to the case of proper morphisms of topological spaces, proven in [Hai, Cor. 2.8].

Theorem 1.2(vi) is only proven for representable morphisms in [Kha5, §8.3]. Let us construct  $\alpha_f : f_! \rightarrow f_*$  for  $f : X \rightarrow Y$  a 1-Artin morphism whose diagonal  $\Delta$  is proper. Since  $\Delta$  is representable, we have the isomorphism  $\alpha_\Delta : \Delta_! \simeq \Delta_*$ . By base change (Theorem 1.2(iv)) for the cartesian square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\mathrm{pr}_2} & X \\ \downarrow \mathrm{pr}_1 & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

we get the natural transformation

$$f^* f_! \simeq \mathrm{pr}_{2,!} \mathrm{pr}_1^* \xrightarrow{\mathrm{unit}} \mathrm{pr}_{2,!} \Delta_! \Delta^* \mathrm{pr}_1^* \simeq \mathrm{id}, \quad (1.4)$$

whence  $\alpha_f : f_! \rightarrow f_*$  by transposition.

Now suppose  $f : X \rightarrow Y$  is a proper morphism of 1-Artin stacks and let us show that  $\alpha_f$  is invertible. Choose a smooth surjection  $v : V \twoheadrightarrow Y$  where  $V \in \mathrm{Spc}$ . By Proposition 1.3(i), it is enough to show that  $v^* f_! \rightarrow v^* f_*$  is invertible. Since  $f_!$  (resp.  $f_*$ ) commutes with arbitrary (resp. smooth)  $*$ -inverse image (1.2.5), we may replace  $Y$  by  $V$  and thereby reduce to the case where  $Y \in \mathrm{Spc}$ . By definition of properness of  $f$ , there exists a proper surjection  $g : Z \twoheadrightarrow X$  where  $Z \in \mathrm{Spc}$  and  $g \circ f : Z \rightarrow Y$  is proper. By Proposition 1.3(ii) we have descent along the Čech nerve  $g_\bullet : Z_\bullet \rightarrow X$  of  $g$ , so it will suffice to show that

$$\alpha_f : f_! g_{n,*} g_n^* \rightarrow f_* g_{n,*} g_n^*$$

is invertible for each  $n$ . This follows from the compatibility of  $\alpha$  with composition, the isomorphisms  $\alpha_{g_n,*} : g_{n,!} \rightarrow g_{n,*}$ , and the isomorphisms  $\alpha_{f \circ g_n,*} : (f \circ g_n)_! \rightarrow (f \circ g_n)_*$ .

<sup>15</sup>So when  $X$  is a topological space,  $\widehat{\mathrm{Shv}}(X)$  is the completed derived  $\infty$ -category of sheaves of abelian groups on  $X$ . See e.g. [Hai, §1.2] or [Lur, §C.5.9]. The canonical functor  $\mathrm{Shv}(X) \rightarrow \widehat{\mathrm{Shv}}(X)$  is an equivalence when  $X$  is a CW complex, and is always an equivalence on eventually coconnective (homologically bounded above) objects.

<sup>16</sup>extended to topological Artin stacks as in [KR, Prop. 4.3], i.e. by imposing that smooth  $*$ -inverse image is  $t$ -exact.

<sup>17</sup>Note that left completion commutes with lisse extension for Artin stacks, since  $*$ -inverse image along smooth morphisms are  $t$ -exact, as in (1.1).

1.2.8. Let  $Y$  be 1-Artin and  $f : X \rightarrow Y$  a proper Deligne–Mumford<sup>18</sup> morphism. Over every point  $y \in Y$ , the fibre  $X \times_Y \{y\}$  has finite discrete stabilizers; we assume their orders are all invertible in the commutative ring  $R$ . In this situation, one can show that the “forget supports” transformation  $\alpha_f : f_! \rightarrow f_*$  is still invertible on cohomologically bounded objects  $\mathcal{F} \in \mathrm{Shv}(X; R)$ .

The basic case is when  $f$  is the projection of the classifying stack  $B_Y(G)$  of a relative group space  $G$  over  $Y$  whose structural morphism  $G \rightarrow Y$  is finite étale<sup>19</sup>, in which case  $\alpha_f$  is invertible on  $\mathrm{Shv}(X; R)$  as soon as the order of  $G$  is invertible in  $R$ .

We sketch a proof in the general case. It is enough to work in the topological category. We begin by showing that  $f_!$  is left t-exact on the left completion  $\widehat{\mathrm{Shv}}(X; R)$ . We may assume that  $Y$  is a space. Consider the underlying space  $|X|$  of  $X$ , i.e. the set  $\pi_0(X(\mathrm{pt}))$  with opens  $|U|$  for every open substack  $U \subseteq X$ . Then  $f$  factors through the projection  $\pi : X \rightarrow |X|$  and the morphism of spaces  $|f| : |X| \rightarrow |Y| \simeq Y$ . One can check that, when  $f$  is proper,  $\pi$  and  $|f|$  are proper. We know that  $|f|_! \simeq |f|_*$  is left t-exact, so by functoriality it will suffice to consider  $\pi_!$ . On  $\widehat{\mathrm{Shv}}$  the functors  $x^*$  are t-exact and jointly conservative as  $x$  ranges among points of  $|X|$  (see e.g. [Hai, Obs. 1.30]), so by the base change formula it is enough to show that  $\pi_{x,!}$  is left t-exact for each fibre  $\pi_x : X_x \rightarrow \{x\}$ . By [Noo1, Prop. 10.2(v)], this fibre is the projection  $BG \rightarrow \{x\}$  where  $G$  is the stabilizer group space of  $X$  at  $x$ . Since the order of  $G$  is invertible in  $R$  by assumption,  $\pi_{x,!} \simeq \pi_{x,*}$  is indeed left t-exact by above.

Now let us show that  $\alpha_f$  is invertible on a cohomologically bounded object  $\mathcal{F} \in \mathrm{Shv}(X; R)$ . It is enough to consider the case of  $\mathcal{F} \in \mathrm{Shv}(X; R)^\vee$  discrete. By smooth base change we may assume  $Y$  is a space. By definition of properness of  $f$ , there exists a proper surjection  $g : Z \twoheadrightarrow X$  where  $Z \in \mathrm{Spc}$  and  $g \circ f : Z \rightarrow Y$  is proper. By Proposition 1.3(ii) we have descent along the Čech nerve  $g_\bullet : Z_\bullet \rightarrow X$  of  $g$ , i.e.,  $\mathcal{F} \simeq \mathrm{Tot}(g_{\bullet,*} g_\bullet^* \mathcal{F})$ . Since totalizations in the 1-categories  $\mathrm{Shv}(X; R)^\vee$  and  $\mathrm{Shv}(Y; R)^\vee$  can be computed as finite limits (equalizers), and  $f_! : \mathrm{Shv}(X; R)^\vee \rightarrow \mathrm{Shv}(Y; R)^\vee$  is exact, we find that  $\alpha_f : f_!(\mathcal{F}) \rightarrow f_*(\mathcal{F})$  is the limit over  $[n] \in \Delta$  of the maps

$$\alpha_f : f_! g_{n,*} g_n^*(\mathcal{F}) \rightarrow f_* g_{n,*} g_n^*(\mathcal{F}).$$

But these are all invertible, as follows from the compatibility of  $\alpha$  with composition, the isomorphisms  $\alpha_{g_n,*} : g_{n,!} \rightarrow g_{n,*}$ , and the isomorphisms  $\alpha_{f \circ g_n,*} : (f \circ g_n)_! \rightarrow (f \circ g_n)_*$ .

## 2. HOMOLOGY

**2.1. Definition.** As before, we fix a commutative ring<sup>20</sup> of coefficients  $\Lambda$ , and work in either the topological, complex analytic, or algebraic category. Let  $f : X \rightarrow S$  be an Artin morphism of stacks and  $\Lambda_S \in \mathrm{Shv}(S)$  the constant sheaf.

<sup>18</sup>i.e., representable by Deligne–Mumford stacks. See [Noo1, §14] for the definition of Deligne–Mumford stacks in the topological category.

<sup>19</sup>in the topological category, read: a proper local homeomorphism

<sup>20</sup>and also as before, one can take more generally an  $\mathcal{E}_\infty$ -ring spectrum



We define the following objects of  $\mathrm{Shv}(S)$ :

- (i) *Cochains*:  $\mathbf{C}^\bullet(X/S; \Lambda) := f_* f^*(\Lambda_S)$ .
- (ii) *Compactly supported cochains*:  $\mathbf{C}_c^\bullet(X/S; \Lambda) := f_! f^*(\Lambda_S)$ .
- (iii) *Borel–Moore chains*:  $\mathbf{C}_\bullet^{\mathrm{BM}}(X/S; \Lambda) := f_* f^!(\Lambda_S)$ .
- (iv) *Chains*:  $\mathbf{C}_\bullet(X/S; \Lambda) := f_! f^!(\Lambda_S)$ .

We denote their respective complexes of (derived) global sections by

$$\mathbf{C}^\bullet(X/S; \Lambda), \quad \mathbf{C}_c^\bullet(X/S; \Lambda), \quad \mathbf{C}_\bullet^{\mathrm{BM}}(X/S; \Lambda), \quad \mathbf{C}_\bullet(X/S; \Lambda),$$

respectively, and their respective cohomology groups by

$$\mathrm{H}^*(X/S; \Lambda), \quad \mathrm{H}_c^*(X/S; \Lambda), \quad \mathrm{H}_{-*}^{\mathrm{BM}}(X/S; \Lambda), \quad \mathrm{H}_{-*}(X/S; \Lambda).$$

We leave the coefficient  $\Lambda$  implicit when there is no risk of ambiguity. When  $S = \mathrm{pt}$ , we omit the decoration  $/S$ . By adjunction, we may write

$$\begin{aligned} \mathbf{C}^\bullet(X/S; \Lambda) &\simeq \Gamma(X, \Lambda_X) \simeq \mathbf{C}^\bullet(X; \Lambda), \\ \mathbf{C}_\bullet^{\mathrm{BM}}(X/S; \Lambda) &\simeq \Gamma(X, \omega_{X/S}) \end{aligned}$$

where  $\Lambda_X \simeq f^*(\Lambda_S)$  and  $\omega_{X/S} := f^!(\Lambda_S)$ .

**2.2. Operations.** The various compatibilities between the six operations give rise to the expected operations on (Borel–Moore) homology. We record these for the sheaves  $\mathbf{C}_\bullet(X/S)$  and  $\mathbf{C}_\bullet^{\mathrm{BM}}(X/S)$ , leaving the ring of coefficients  $\Lambda$  implicit in the notation.

**2.2.1. Forgetting supports.** If  $f : X \rightarrow S$  is a morphism of 1-Artin stacks with proper diagonal, then  $\alpha_f : f_! \rightarrow f_*$  induces a map

$$\mathbf{C}_\bullet(X/S) \rightarrow \mathbf{C}_\bullet^{\mathrm{BM}}(X/S)$$

which is invertible if  $X \rightarrow S$  is proper and Deligne–Mumford.

**2.2.2. Products.** Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow S$  be Artin morphisms. We have the operations of *composition product*:

$$\circ : g_* \mathbf{C}_\bullet(X/Y) \otimes \mathbf{C}_\bullet(Y/S) \rightarrow \mathbf{C}_\bullet(X/S), \quad (2.1)$$

$$\circ : g_* \mathbf{C}_\bullet^{\mathrm{BM}}(X/Y) \otimes \mathbf{C}_\bullet^{\mathrm{BM}}(Y/S) \rightarrow \mathbf{C}_\bullet^{\mathrm{BM}}(X/S). \quad (2.2)$$

Taking  $f = g = \mathrm{id}$ , these become the *cup product* on cochains:

$$\cup : \mathbf{C}^\bullet(Y) \otimes \mathbf{C}^\bullet(Y) \rightarrow \mathbf{C}^\bullet(Y), \quad (2.3)$$

via which  $\mathbf{C}^\bullet(Y)$  becomes an  $\mathcal{E}_\infty$ -algebra in  $\mathrm{Shv}(Y)$ . Taking  $f = \mathrm{id}$ , they become the *cap products*:

$$\cap : g_* \mathbf{C}^\bullet(Y) \otimes \mathbf{C}_\bullet^{\mathrm{BM}}(Y/S) \rightarrow \mathbf{C}_\bullet^{\mathrm{BM}}(Y/S), \quad (2.4)$$

$$\cap : g_* \mathbf{C}^\bullet(Y) \otimes \mathbf{C}_\bullet(Y/S) \rightarrow \mathbf{C}_\bullet(Y/S), \quad (2.5)$$

via which  $\mathbf{C}_\bullet^{\mathrm{BM}}(Y/S)$  and  $\mathbf{C}_\bullet(Y/S)$  are modules over  $g_* \mathbf{C}^\bullet(Y)$  for any Artin morphism  $g : Y \rightarrow S$ .

We recall the construction of (2.1); that of (2.2) is similar. It comes from a natural transformation

$$g_* f_! f^! g^*(-) \otimes g_! g^!(-) \rightarrow (g \circ f)_!(g \circ f)^!(-) \quad (2.6)$$

defined as the composite

$$\begin{aligned} g_* f_! f^! g^*(-) \otimes g_! g^!(-) &\simeq g_!(g^* g_* f_! f^! g^*(-) \otimes g^!(-)) \\ &\rightarrow g_!(g^!(-) \otimes f_! f^! g^*(-)) \\ &\simeq g_! f_!(f^* g^!(-) \otimes f^! g^*(-)) \\ &\rightarrow g_! f_!(f^!(g^*(-) \otimes g^!(-))) \\ &\rightarrow g_! f_!(f^! g^!(-)) \\ &\simeq (g \circ f)_!(g \circ f)^!(-), \end{aligned}$$

where the (iso)morphisms are: the projection formula for  $g_!$ ; the unit  $\text{id} \rightarrow g^* g_*$ ; the projection formula for  $f_!$ ; the exchange transformation  $\text{Ex}_{\otimes}^{*,!} : f^*(-) \otimes f^!(-) \rightarrow f^!(- \otimes -)$ ; and the exchange transformation  $\text{Ex}_{\otimes}^{*,1} : g^*(-) \otimes g^!(-) \rightarrow g^!(- \otimes -)$ . (See [Kha5, A.1.2] for the definition of the latter two.)

**2.2.3. Functoriality: change of base.** Let  $p : S' \rightarrow S$  be a morphism of stacks. For any Artin morphism  $X \rightarrow S$ , the unit  $\text{id} \rightarrow p_* p^*$  induces maps

$$\begin{aligned} \mathbf{C}_{\bullet}(X/S) &\rightarrow \mathbf{C}_{\bullet}(X_{S'}/S'), \\ \mathbf{C}_{\bullet}^{\text{BM}}(X/S) &\rightarrow \mathbf{C}_{\bullet}^{\text{BM}}(X_{S'}/S'), \end{aligned}$$

where  $X_{S'} = X \times_S S'$ .

**2.2.4. Functoriality: direct image.** Let  $f : X \rightarrow Y$  be a morphism of relatively Artin stacks over a stack  $S$ . The counit  $f_! f^! \rightarrow \text{id}$  induces a map

$$f_* : \mathbf{C}_{\bullet}(X/S) \rightarrow \mathbf{C}_{\bullet}(Y/S).$$

If  $f$  is a *proper* Deligne–Mumford morphism of 1-Artin stacks, so that  $f_* \simeq f_!$ , then  $f_* f^! \simeq f_! f^! \rightarrow \text{id}$  induces a map

$$f_* : \mathbf{C}_{\bullet}^{\text{BM}}(X/S) \rightarrow \mathbf{C}_{\bullet}^{\text{BM}}(Y/S).$$

**2.2.5. Functoriality: Gysin.** Let  $f : X \rightarrow Y$  be a smooth Artin morphism of relative dimension  $d$  over a stack  $S$ . The Poincaré duality isomorphism  $f^* \langle T_f \rangle \simeq f^!$  yields a natural transformation  $\text{id} \rightarrow f_* f^* \simeq f_* f^! \langle -T_f \rangle$ , whence a map

$$f^! : \mathbf{C}_{\bullet}^{\text{BM}}(Y/S) \rightarrow \mathbf{C}_{\bullet}^{\text{BM}}(X/S) \langle -T_f \rangle,$$

where by abuse of notation we write, with  $p : X \rightarrow S$  the structural morphism,

$$\mathbf{C}_{\bullet}^{\text{BM}}(X/S) \langle -T_f \rangle := p_* \langle -T_f \rangle p^! (\Lambda_S).$$

Recall that we may trivialize  $\langle -T_f \rangle \simeq [-d]$  given a suitable orientation of  $f$ .

Combining this with the isomorphism  $\alpha_f : f_! \rightarrow f_*$  when  $f$  is moreover proper and Deligne–Mumford with  $Y$  1-Artin, we have the natural transformation  $\text{id} \rightarrow f_* f^* \simeq f_! f^! \langle -T_f \rangle$  which yields a map

$$f^! : \mathbf{C}_{\bullet}(Y/S) \rightarrow \mathbf{C}_{\bullet}(X/S) \langle -T_f \rangle.$$

2.2.6. *Fundamental class.* Let  $f : X \rightarrow S$  be a smooth Artin morphism. The *relative fundamental class*

$$[X/S] \in \mathbf{C}_{\bullet}^{\mathrm{BM}}(X/S)\langle -T_f \rangle$$

is the image of  $1 \in \mathbf{C}^{\bullet}(S) \simeq \mathbf{C}_{\bullet}^{\mathrm{BM}}(S/S)$  by the Gysin map  $f^! : \mathbf{C}_{\bullet}^{\mathrm{BM}}(S/S) \rightarrow \mathbf{C}_{\bullet}^{\mathrm{BM}}(X/S)\langle -T_f \rangle$ .<sup>21</sup>

2.2.7. *Virtual functoriality.* Let us work in the  $C^{\infty}$ -category; we refer to [Ste] for background on (derived)  $C^{\infty}$ -stacks. Given a derived  $C^{\infty}$ -stack  $X$  and  $\pi : E \rightarrow X$  a  $C^{\infty}$ -vector bundle, we consider the construction<sup>22</sup>

$$\mathcal{F}\langle -E \rangle := 0^! \pi^*(\Lambda_X)$$

as an endofunctor in  $\mathcal{F} \in \mathrm{Shv}(X)$ , where  $0 : X \rightarrow E$  denotes the zero section. This is an auto-equivalence (as can be checked locally) and we denote its inverse by  $\mathcal{F} \mapsto \mathcal{F}\langle E \rangle$ . As in [Kha4, §2.6], compatibility with exact sequences of vector bundles implies that this extends to a canonical homomorphism

$$K_0(\mathrm{Perf}(X)) \rightarrow \mathrm{Shv}(X)$$

from the Grothendieck group of perfect complexes on  $X$  to the Picard group of  $\otimes$ -invertible objects of  $\mathrm{Shv}(X)$ .

Now let  $f : X \rightarrow Y$  be a *quasi-smooth morphism of derived  $C^{\infty}$ -stacks* (see again [Ste] for a reference for quasi-smoothness in this context). The relative cotangent complex is perfect and gives rise to a Thom twist  $\langle T_f^{\mathrm{vir}} \rangle$  on  $\mathrm{Shv}(X)$ . One can define in this situation a canonical natural transformation, the *virtual Gysin transformation*

$$\mathrm{gys}_f : f^*(-)\langle T_f^{\mathrm{vir}} \rangle \rightarrow f^!(-),$$

which reduces to the Poincaré duality isomorphism  $f^! \simeq f^*(-)\langle T_f \rangle$  when  $f$  is smooth. See [Kha1, §3.1] for the construction of  $\mathrm{gys}_f$  in the algebraic category; the same works word-for-word in the complex analytic and  $C^{\infty}$ -categories, see e.g. [PY2, §4].<sup>23</sup>

Now, as in (2.2.5),  $\mathrm{gys}_f$  gives rise to virtual Gysin maps:

$$f_{\mathrm{vir}}^! : \mathbf{C}_{\bullet}^{\mathrm{BM}}(Y/S) \rightarrow \mathbf{C}_{\bullet}^{\mathrm{BM}}(X/S)\langle -T_f^{\mathrm{vir}} \rangle$$

when  $X$  and  $Y$  are defined over some base  $S$ . In particular, taking  $Y = S$  we get the (relative) *virtual fundamental class*

$$[X/S]^{\mathrm{vir}} := f_{\mathrm{vir}}^!(1) \in \mathbf{C}_{\bullet}^{\mathrm{BM}}(X/S)\langle -T_{X/S}^{\mathrm{vir}} \rangle.$$

<sup>21</sup>Here  $x \in \mathbf{C}_{\bullet}^{\mathrm{BM}}(X)\langle -T_f \rangle$  is shorthand for any of the following equivalent things: a morphism  $x : \Lambda_X \rightarrow \mathbf{C}_{\bullet}^{\mathrm{BM}}(X)\langle -T_f \rangle$  in  $\mathrm{Shv}(X)$ ; a morphism  $x : \Lambda \rightarrow \mathbf{C}_{\bullet}^{\mathrm{BM}}(X)\langle -T_f \rangle$  in the derived  $\infty$ -category of  $\Lambda$ -modules; an object of the  $\infty$ -groupoid underlying the complex  $\mathbf{C}_{\bullet}^{\mathrm{BM}}(X)\langle -T_f \rangle$ .

<sup>22</sup>where  $\mathrm{Shv}(-)$  and the various operations are defined via underlying topological stacks

<sup>23</sup>In fact, the construction does not require the full data of a derived structure on  $f : X \rightarrow Y$ . The latter determines in particular a closed immersion of the intrinsic normal cone  $\mathfrak{C}_{X/Y}$  into the virtual normal bundle  $N_{X/Y}^{\mathrm{vir}}$ , and this linear shadow of the derived structure is enough: see [Kha2, Varnt. 2.4] or [Kha1, §3.3].

If  $f$  is moreover proper and Deligne–Mumford with  $Y$  1-Artin, we also have

$$f_{\text{vir}}^! : \mathbf{C}_\bullet(Y/S) \rightarrow \mathbf{C}_\bullet(X/S)\langle -T_f^{\text{vir}} \rangle.$$

For example, the above discussion applies to any quasi-smooth Artin morphism of derived stacks in the algebraic or complex analytic categories (see e.g. [Kha3] or [PY1] for some background on derived stacks in the respective contexts). In that case, the canonical orientation of  $f$  yields a canonical trivialization  $\langle T_f^{\text{vir}} \rangle \simeq [2d]$  where  $d$  is the relative complex dimension.<sup>24</sup>

**2.3. Properties.** The following statements immediately follow from various parts of Theorem 1.2.

**Theorem 2.7.** *Let  $S$  be a stack and  $f : X \rightarrow S$  an Artin morphism.*

- (i) *Localization. For any closed immersion  $i : Z \hookrightarrow X$  with complementary open immersion  $j : U \hookrightarrow X$ , there are exact triangles in  $\text{Shv}(S)$ .*

$$\begin{aligned} \mathbf{C}_\bullet^{\text{BM}}(Z/S) &\xrightarrow{i_*} \mathbf{C}_\bullet^{\text{BM}}(X/S) \xrightarrow{j^!} \mathbf{C}_\bullet^{\text{BM}}(U/S), \\ \mathbf{C}_\bullet(U/S) &\xrightarrow{j_*} \mathbf{C}_\bullet(X/S) \rightarrow \mathbf{C}_\bullet((X, U)_S), \end{aligned}$$

where  $\mathbf{C}_\bullet((X, U)_S) = f_! i^* f^! (\Lambda_S)$  is the complex of relative chains on  $(X, U)$  over  $S$ .

- (ii) *Homotopy invariance. For any vector bundle  $\pi : E \rightarrow X$  of rank  $r$ , the maps in  $\text{Shv}(S)$*

$$\begin{aligned} \pi^! : \mathbf{C}_\bullet^{\text{BM}}(X/S) &\rightarrow \mathbf{C}_\bullet^{\text{BM}}(E/S)\langle -T_\pi \rangle, \\ \pi_* : \mathbf{C}_\bullet(E/S) &\rightarrow \mathbf{C}_\bullet(X/S) \end{aligned}$$

*are invertible.*

- (iii) *Poincaré duality. If  $f : X \rightarrow S$  is smooth, cap product with the fundamental class induces a canonical isomorphism in  $\text{Shv}(S)$ .*

$$(-) \cap [X/S] : \mathbf{C}^\bullet(X/S) \rightarrow \mathbf{C}_\bullet^{\text{BM}}(X/S)\langle -T_{X/S} \rangle.$$

- (iv) *Smooth descent. For any smooth surjective Artin morphism  $p : Y \rightarrow X$  with Čech nerve  $Y_\bullet$ , there are canonical isomorphisms in  $\text{Shv}(S)$*

$$\begin{aligned} \mathbf{C}_\bullet^{\text{BM}}(X/S) &\simeq \varprojlim_{[n] \in \Delta} \mathbf{C}_\bullet^{\text{BM}}(Y_n/S)\langle -T_{Y_n/X} \rangle, \\ \mathbf{C}_\bullet(X/S) &\simeq \varinjlim_{[n] \in \Delta^{\text{op}}} \mathbf{C}_\bullet(Y_n/S). \end{aligned}$$

- (v) *Proper descent. Suppose that the relative dualizing complex  $\omega_{X/S} = f^! (\Lambda_S)$  is eventually coconnective (e.g.,  $X \rightarrow S$  is a relative CW complex locally on the source and target). For any proper surjective morphism*

<sup>24</sup>If  $\Lambda$  is an  $\mathcal{E}_\infty$ -ring spectrum, as opposed to an ordinary commutative ring, we need a complex orientation on  $\Lambda$  here.

$p : Y \rightarrow X$  with Čech nerve  $Y_\bullet$ , there are canonical isomorphisms

$$\begin{aligned} C_\bullet^{\text{BM}}(X/S) &\simeq \varinjlim_{[n] \in \Delta^{\text{op}}} C_\bullet^{\text{BM}}(Y_n/S), \\ C_\bullet(X/S) &\simeq \varinjlim_{[n] \in \Delta^{\text{op}}} C_\bullet(Y_n/S) \end{aligned}$$

in  $\text{Shv}(S)$ .

**2.4. Homology and homotopy types.** We define the *homotopy type* of a topological stack, cf. [Sim, Bla, Noo2]. Denote by  $\mathbf{H}$  the  $\infty$ -category of homotopy types (a.k.a.  $\infty$ -groupoids). We have the tautological functor

$$\text{Spc} \rightarrow \mathbf{H}, \quad X \mapsto |X|,$$

which exhibits its target as the  $\infty$ -categorical localization at weak homotopy equivalences. By left Kan extension, this extends uniquely to a colimit-preserving functor

$$\text{Spc} \rightarrow \mathbf{H}, \quad X \mapsto |X|.$$

In particular,  $|X|$  is canonically presented as the homotopy colimit

$$|X| \simeq \varinjlim_{(S,s)} |S|$$

taken over pairs  $(S, s)$  with  $S \in \text{Spc}$  and  $s : S \rightarrow X$  a morphism.

For a topological stack  $X$ , the complex of chains  $C_\bullet(X)$  can be described in terms of its homotopy type  $|X| \in \mathbf{H}$ . Recall first of all that on topological spaces, the functor  $X \mapsto C_\bullet(X)$  descends to a functor

$$C_\bullet^{\mathbf{H}}(-) : \mathbf{H} \rightarrow D(\Lambda)$$

valued in the derived  $\infty$ -category of  $\Lambda$ -modules (where  $\Lambda$  is the ring of coefficients), since it inverts weak homotopy equivalences.

The following is a tautological reformulation of smooth codescent for  $C_\bullet(-) : \text{Stk} \rightarrow D(\Lambda)$  (Theorem 2.7(iv)).

**Proposition 2.8.** *For every topological stack  $X$ , there is a canonical natural isomorphism*

$$C_\bullet(X) \simeq C_\bullet^{\mathbf{H}}(|X|)$$

in  $D(\Lambda)$ .

*Proof.* By construction, the left-hand triangle and outer composite in the following diagram commute:

$$\begin{array}{ccccc} \text{Spc} & \hookrightarrow & \text{Stk} & \xrightarrow{C_\bullet} & D(\Lambda) \\ & \searrow & \downarrow |{-}| & \nearrow C_\bullet^{\mathbf{H}} & \\ & & \mathbf{H} & & \end{array}$$

Theorem 2.7(iv) implies that  $C_\bullet(-) : \text{Stk} \rightarrow D(\Lambda)$  is also left Kan extended from  $\text{Spc}$ . It follows that the right-hand triangle commutes if and only if the outer one does.  $\square$

**2.5. Relative homology.** If  $S$  is a compact topological manifold, then may describe the complex of relative chains  $C_\bullet(X/S) = a_{S,*}f_!f^!(\Lambda_S)$  in more “absolute” terms:

$$C_\bullet(X/S) \simeq a_{S,!}f_!f^!(\omega_S\langle -T_S \rangle) \simeq C_\bullet(X)\langle -T_S \rangle.$$

More generally, using chains on pairs as in Theorem 2.7(i), we have:

**Lemma 2.9.** *Let  $S$  be a topological manifold of dimension  $d$  and  $f : X \rightarrow S$  a topological Artin stack over  $S$ . There is a canonical isomorphism in  $D(\Lambda)$*

$$C_\bullet(X/S) \simeq \varprojlim_{K \subseteq S} C_\bullet(X, X \setminus X_K)\langle -T_S \rangle,$$

where the limit is over compact subsets  $K \subseteq S$  and  $X_K = X \times_S K \subseteq X$ .<sup>25</sup>

*Proof.* We have  $C_\bullet(X/S) \simeq a_{S,*}f_!f^!a_S^!\Lambda\langle -T_S \rangle$ . Using the equivalence  $\mathrm{Shv}(S) \rightarrow \varprojlim_{K \subseteq S} \mathrm{Shv}(K)$  (via  $*$ -inverse image) and base change for the squares

$$\begin{array}{ccc} X_K & \xrightarrow{i_X} & X \\ \downarrow f_K & & \downarrow f \\ K & \xrightarrow{i} & S, \end{array}$$

we can write

$$f_!f^!a_S^!\Lambda \simeq \varprojlim_K i_*i^*f_!f^!a_S^!\Lambda \simeq \varprojlim_K i_*f_{K,!}i_X^*f^!a_S^!\Lambda.$$

Thus  $a_{S,*}f_!f^!a_S^!\Lambda$  is the limit over  $K$  of

$$a_{S,*}i_*f_{K,!}i_X^*f^!a_S^!\Lambda \simeq a_{S,!}i_!f_{K,!}i_X^*f^!a_S^!\Lambda \simeq a_{X,!}i_{X,!}i_X^*a_X^!\Lambda \simeq C_\bullet(X, X \setminus X_K)$$

using the fact that  $a_K = a_S \circ i$  is proper and that  $a_S \circ i \circ f_K = a_X \circ i_X$ .  $\square$

**Remark 2.10.** Assume for simplicity that  $S$  is an oriented  $C^\infty$ -manifold of dimension  $d$  (so that  $\langle -T_S \rangle \simeq [-d]$ ). At the level of homology groups, Lemma 2.9 means that there are canonical surjections

$$H_n(X/S) \twoheadrightarrow \varprojlim_{K \subseteq S} H_{n+d}(X, X \setminus X_K)$$

with generally nontrivial kernels

$$\varprojlim_K^1 H_{n+d+1}(X, X \setminus X_K).$$

Compare [Spa, Thm. 7.3].

<sup>25</sup>Since  $S$  is Hausdorff (by convention),  $K \hookrightarrow S$  is a closed immersion. Hence  $X_K \hookrightarrow X$  is a closed immersion by base change, and  $X \setminus X_K$  is the complementary open immersion as in Footnote 12.

## 3. EQUIVARIANT HOMOLOGY

**3.1. Definition.** Let  $G$  be a group acting on an Artin stack  $X$ .<sup>26</sup> Consider the projection

$$f : [X/G] \rightarrow BG := [\mathrm{pt}/G]$$

from the quotient stack to the classifying stack.<sup>27</sup> For every commutative ring  $\Lambda$ , the complexes of  $G$ -equivariant (Borel–Moore) chains on  $X$ , denoted by

$$C_\bullet^G(X) := C_\bullet([X/G]_{/BG}), \quad C_\bullet^{\mathrm{BM},G}(X) := C_\bullet([X/G]_{/BG}),$$

are the (derived) global sections of the objects

$$\begin{aligned} C_\bullet^G(X) &:= C_\bullet([X/G]_{/BG}) := f_! f^!(\Lambda_{BG}), \\ C_\bullet^{\mathrm{BM},G}(X) &:= C_\bullet^{\mathrm{BM}}([X/G]_{/BG}) := f_* f^!(\Lambda_{BG}), \end{aligned}$$

of  $\mathrm{Shv}(BG)$ .

**3.1.1.** Since  $a_{BG} : BG \rightarrow \mathrm{pt}$  is smooth with  $\langle -T_{BG} \rangle \simeq \langle T_G \rangle$ , we have  $\Lambda_{BG} \simeq \omega_{BG} \langle T_G \rangle$  by Poincaré duality. Thus

$$a_{BG,*} f_* f^!(\Lambda_{BG}) \simeq a_{BG,*} f_* f^! a_{BG}^!(\Lambda) \langle T_G \rangle,$$

or in other words,  $C_\bullet^{\mathrm{BM},G}(X) \simeq C_\bullet^{\mathrm{BM}}([X/G]) \langle T_G \rangle$ .

**3.1.2.** For homology, we get the less clean description

$$C_\bullet^G(X) = a_{BG,*} f_! f^!(\Lambda_{BG}) \simeq a_{BG,*} f_! f^! a_{BG}^!(\Lambda) \langle T_G \rangle.$$

When  $a_{BG} : BG \rightarrow \mathrm{pt}$  is not proper<sup>28</sup>, this cannot be identified with  $C_\bullet([X/G]) \langle T_G \rangle$ .

**3.2. The Borel construction.** We will now relate our definitions of equivariant (Borel–Moore) homology to a model for the Borel construction. To avoid orientation issues, we will work in the complex analytic or algebraic categories for simplicity.

Say a space  $A$  is  $k$ -acyclic if for every eventually coconnective complex  $K \in \mathrm{Shv}(\mathrm{pt})$ , the unit  $K \rightarrow a_* a^*(K)$  is  $k$ -coconnective, where  $a : A \rightarrow \mathrm{pt}$  is the projection. Let  $(E_n G)_n$  be a sequential diagram of oriented  $C^\infty$ -manifolds with free  $G$ -action such that for every integer  $k > 0$ ,  $E_n G$  is  $k$ -acyclic for sufficiently large  $n$ .

For example, if  $G$  is a linear algebraic group, then the construction of [Tot, Rem. 1.4] provides an example of such an  $(E_n G)_n$  by [KR, Prop. 4.15].<sup>29</sup>

<sup>26</sup>As before, “stack” can be read in the topological, complex analytic, or algebraic category as preferred, and the same goes for “group” (it does not mean “discrete group”).

<sup>27</sup>See e.g. [Kha3, §4] for the definition of group actions on higher stacks and their quotient stacks.

<sup>28</sup>This amounts to compactness of  $G$ . For example, the complex analytification of a linear algebraic group will be compact if and only if it is finite.

<sup>29</sup>See also [BL, §3.1] for a construction when  $G$  is a Lie group with finitely many components.

Note that the freeness assumption means that the quotient stacks<sup>30</sup>  $B_n G := [E_n G/G]$  have trivial stabilizers<sup>31</sup>. We denote by  $q_n : B_n G \rightarrow BG$  the projections.

**Proposition 3.1.** *For every  $\mathcal{F} \in \mathrm{Shv}(BG)$ , the canonical morphism  $\mathcal{F} \rightarrow \varprojlim_n q_{n,*} q_n^*(\mathcal{F})$  is invertible.*

**Corollary 3.2.** *Let  $X$  be an Artin stack with  $G$ -action. Then the canonical maps in  $\mathrm{Shv}(BG)$*

$$\begin{aligned} \mathbb{C}_{\bullet}^{\mathrm{BM},G}(X) &\rightarrow \varprojlim_n \mathbb{C}_{\bullet}^{\mathrm{BM}}(X \times^G E_n G)[-d_n + g] \\ \mathbb{C}_{\bullet}^G(X) &\rightarrow \varprojlim_n q_{n,*} \mathbb{C}_{\bullet}(X \times^G E_n G_{/B_n G}) \end{aligned}$$

are invertible, where  $d_n$  is the dimension of  $B_n G$  and  $g = \dim(G)$ .

*Proof.* Applying Proposition 3.1 to  $\mathcal{F} = \mathbb{C}_{\bullet}^{\mathrm{BM},G}(X) = f_* f^!(\Lambda_{BG})$ , where  $f : [X/G] \rightarrow BG$ , yields

$$f_* f^!(\Lambda_{BG}) \simeq \varprojlim_n q_{n,*} q_n^* f_* f^!(\Lambda_{BG}).$$

Using the Poincaré duality isomorphism  $q_n^* \simeq q_n^![-d_n]$  and the base change formula, we may write

$$q_{n,*} q_n^* f_* f^!(\Lambda_{BG}) \simeq q_{n,*} f_{n,*} f_n^! q_n^!(\Lambda_{BG})[-d_n]$$

where  $f_n : X \times^G E_n \rightarrow B_n G$  is the base change of  $f$ . Translating back, this is

$$\mathbb{C}_{\bullet}^{\mathrm{BM},G}(X \times^G E_n G)[-d_n] \simeq \mathbb{C}_{\bullet}^{\mathrm{BM}}(X \times^G E_n G)[-d_n + g]$$

by (3.1.1).

Similarly, Proposition 3.1 expresses  $\mathcal{F} = \mathbb{C}_{\bullet}^G(X) = f_! f^!(\Lambda_{BG})$  as the limit over  $n$  of

$$q_{n,*} q_n^* f_! f^!(\Lambda_{BG}) \varprojlim_n q_{n,*} f_{n,!} f_n^! q_n^!(\Lambda_{BG})[-d_n] \varprojlim_n q_{n,*} f_{n,!} f_n^!(\Lambda_{B_n G}).$$

Translating back, this is  $\mathbb{C}_{\bullet}(X \times^G E_n G_{/B_n G})$ .  $\square$

The proof of Proposition 3.1 involves some analysis using the cohomological t-structure on  $\mathrm{Shv}(X)$  (see Footnote 16). Given a sequential diagram  $\{f_n : A_n \rightarrow S\}_n$  of Artin stacks, we say that  $\{f_n\}_n$  is *k-pro-acyclic* if for every eventually coconnective  $\mathcal{F} \in \mathrm{Shv}(S)$ , there exists an index  $N$  such that the unit map  $\mathcal{F} \rightarrow f_{n,*} f_n^*(\mathcal{F})$  has  $k$ -coconnective fibre for all  $n \geq N$ . We make the following three observations:

- (a) If  $\{f_n\}_n$  is  $k$ -pro-acyclic, then the induced map  $\mathcal{F} \rightarrow \varprojlim_n f_{n,*} f_n^*(\mathcal{F})$  has  $k$ -coconnective fibre. Indeed, coconnectivity is stable under limits.

<sup>30</sup>The quotients can be formed in the  $C^\infty$  category, as the “associated topological stack” functor is colimit-preserving. Similarly, if  $X$  and  $G$  are algebraic or complex analytic, then the quotients could be taken equivalently in the corresponding category.

<sup>31</sup>and are Hausdorff (even manifolds) when  $G$  acts properly on  $E_n G$  (e.g. if  $G$  is compact).



- (b) If  $\{f_n\}_n$  is  $k$ -pro-acyclic, then any smooth base change is  $k$ -pro-acyclic. Indeed, smooth  $*$ -inverse image is t-exact and commutes with  $*$ -direct image (using Poincaré duality and the fact that  $!$ -inverse image commutes with  $*$ -direct image).
- (c) If the base change of  $\{f_n\}_n$  along a smooth surjection  $S' \rightarrow S$  becomes  $k$ -pro-acyclic, then  $\{f_n : A_n \rightarrow S\}_n$  was already pro-acyclic. Indeed,  $*$ -inverse image along smooth surjections is t-exact and conservative.

*Proof of Proposition 3.1.* It will suffice to show that  $\{q_n\}_n$  is  $k$ -pro-acyclic for every  $k$ . By base change along the surjective submersion  $\text{pt} \rightarrow BG$ , this follows from the assumption that  $\{a_{E_n G} : E_n G \rightarrow \text{pt}\}_n$  is  $k$ -pro-acyclic.  $\square$

Passing to hypercohomology groups, the isomorphism of Proposition 3.1 gives rise for every  $i \in \mathbf{Z}$  to a surjective map

$$H^i(BG, \mathcal{F}) \xrightarrow{\sim} H^i(BG, \varprojlim_n q_{n,*} q_n^* \mathcal{F}) \twoheadrightarrow \varprojlim_n H^i(BG, q_{n,*} q_n^* \mathcal{F})$$

whose kernel is a  $\varprojlim^1$  term.

**Lemma 3.3.** *If  $\mathcal{F} \in \text{Shv}(BG)$  is eventually coconnective, then for every integer  $i \in \mathbf{Z}$ , there exists a sufficiently large index  $N$  such that*

$$H^i(BG, \mathcal{F}) \rightarrow H^i(BG, q_{n,*} q_n^* \mathcal{F}) \simeq H^i(B_n G, q_n^* \mathcal{F})$$

*is invertible for all  $n \geq N$ .*

*Proof.* As explained in [KR, Rem. 4.10], this is a consequence of the fact that  $\{q_n\}_n$  is  $k$ -pro-acyclic for all  $k$ .  $\square$

**Corollary 3.4.** *Let  $X$  be an Artin stack with  $G$ -action.*

- (i) *Suppose that  $H_i(X) = 0$  for  $i \gg 0$ . Then for every integer  $i \in \mathbf{Z}$  there exists a sufficiently large index  $N$  such that the map*

$$H_i^G(X) \rightarrow H_i(X \times^G E_n G / B_n G)$$

*is invertible for all  $n \geq N$ .*

- (ii) *Suppose that  $H_i^{\text{BM}}(X) = 0$  for  $i \gg 0$ .<sup>32</sup> Then for every integer  $i \in \mathbf{Z}$  there exists a sufficiently large index  $N$  such that the map*

$$H_i^{\text{BM}, G}(X) \rightarrow H_{i+d_n-g}^{\text{BM}}(X \times^G E_n G)$$

*is invertible for all  $n \geq N$ .*

*Proof.* To apply Lemma 3.3, we need to check that the sheaves  $f_! f^!(\Lambda_{BG})$  and  $f_* f^!(\Lambda_{BG})$  are eventually coconnective. Note that  $f_!$ ,  $f_*$  and  $f^!$  all commute with smooth  $*$ -inverse image (see (1.2.5) for the latter two). Since  $*$ -inverse image along  $\text{pt} \rightarrow BG$  is t-exact and conservative, this follows from the eventual coconnectivity of  $a_! a^!(\Lambda) = C_\bullet(X)$  and  $a_* a^!(\Lambda) = C_\bullet^{\text{BM}}(X)$ , where  $a : X \rightarrow \text{pt}$  is the projection.  $\square$

<sup>32</sup>Since formation of global sections is always left t-exact, it is enough that the dualizing complex  $\omega_X$  is eventually coconnective. In the algebraic category, if  $X$  is an Artin stack of finite dimension in the sense of [SP, Tag 0AFL], then  $\omega_X$  is eventually coconnective by [KR, Thm. 4.4].

## REFERENCES

- [BL] J. Bernstein, V. Lunts, *Equivariant sheaves and functors*. Lect. Notes in Math. **1578** (1994).
- [Bla] A. Blanc, *Topological K-theory of complex noncommutative spaces*. Compos. Math. **152** (2016), no. 3, 489–555.
- [Boj] A. Bojko, *Wall-crossing for Calabi–Yau fourfolds: framework, tools, and applications*. Forthcoming.
- [Bre] G. E. Bredon, *Sheaf theory*. Grad. Texts Math. **170** (1997), Springer, 502 p.
- [Gode] R. Godement, *Topologie algébrique et théorie des faisceaux*. Hermann Paris (1958).
- [Hai] P. Haine, *Descent for sheaves on compact Hausdorff spaces*. [arXiv:2210.00186](https://arxiv.org/abs/2210.00186) (2022).
- [Joy] D. Joyce, *Enumerative invariants and wall-crossing formulae in abelian categories*. [arXiv:2111.04694](https://arxiv.org/abs/2111.04694) (2021).
- [KR] A. A. Khan, C. Ravi, *Equivariant generalized cohomology via stacks*. [arXiv:2209.07801](https://arxiv.org/abs/2209.07801) (2022).
- [KS] M. Kashiwara, P. Schapira, *Sheaves on manifolds*. Grundlehren Math. Wiss. **292** (1990), Springer-Verlag.
- [KV] M. Kapranov, E. Vasserot, *The cohomological Hall algebra of a surface and factorization cohomology*, [arXiv:1901.07641](https://arxiv.org/abs/1901.07641) (2019).
- [Kha1] A. A. Khan, *Virtual fundamental classes for derived stacks I*. [arXiv:1909.01332](https://arxiv.org/abs/1909.01332) (2019).
- [Kha2] A. A. Khan, *Cohomological intersection theory and derived algebraic geometry*. Lecture notes, available at: <https://www.preschema.com/lecture-notes/cohdag2022/cohdag2022.pdf> (2022).
- [Kha3] A. A. Khan, *Lectures on algebraic stacks*. [arXiv:2310.12456](https://arxiv.org/abs/2310.12456) (2023).
- [Kha4] A. A. Khan, *Weaves*. Available at: <https://www.preschema.com/papers/weaves.pdf> (2023).
- [Kha5] A. A. Khan, *Lisse extensions of weaves*. [arXiv:2501.04114](https://arxiv.org/abs/2501.04114) (2025).
- [LO] Y. Laszlo, M. Olsson, *The six operations for sheaves on Artin stacks. I. Finite coefficients*. Publ. Math. Inst. Hautes Études Sci., no. **107** (2008), 109–168.
- [LZ] Y. Liu, W. Zheng, *Enhanced six operations and base change theorem for higher Artin stacks*. [arXiv:1211.5948](https://arxiv.org/abs/1211.5948) (2012).
- [Lur] J. Lurie, *Spectral algebraic geometry*, version of 2018-02-03. Available at: <https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf>.
- [Noo1] B. Noohi, *Foundations of topological stacks*. [arXiv:math/0503247](https://arxiv.org/abs/math/0503247) (2005).
- [Noo2] B. Noohi, *Homotopy types of topological stacks*. Adv. Math. **230** (2012), no. 4-6, 2014–2047.
- [Ols] M. C. Olsson, *On proper coverings of Artin stacks*. Adv. Math. **198** (2005), no. 1, 93–106.
- [PY1] M. Porta, T. Y. Yu, *Representability theorem in derived analytic geometry*. J. Eur. Math. Soc. (JEMS) **22** (2020), no. 12, 3867–3951.
- [PY2] M. Porta, T. Y. Yu, *Non-archimedean Gromov–Witten invariants*. [arXiv:2209.13176](https://arxiv.org/abs/2209.13176) (2022).
- [SP] J. de Jong, *The Stacks Project*, <https://stacks.math.columbia.edu>.
- [Sim] C. Simpson, *The topological realization of a simplicial presheaf*. [q-alg/9609004](https://arxiv.org/abs/q-alg/9609004) (1996).
- [Spa] E. Spanier, *Singular homology and cohomology with local coefficients and duality for manifolds*. Pac. J. Math. **160** (1993), no. 1, 165–200.
- [Ste] P. Steffens, *Derived  $C^\infty$ -Geometry I: Foundations*. [arXiv:2304.08671](https://arxiv.org/abs/2304.08671) (2023).
- [Tot] B. Totaro, *The Chow ring of a classifying space*, in: Algebraic K-theory, Proc. Symp. Pure Math. **67** (1999), 249–281.
- [Vol] M. Volpe, *Six functor formalism for sheaves with non-presentable coefficients*. [arXiv:2110.10212](https://arxiv.org/abs/2110.10212) (2021).

Institute of Mathematics, Academia Sinica, Taipei 10617, Taiwan