K-THEORY AND G-THEORY OF DERIVED ALGEBRAIC STACKS

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Abstract. These are some notes on the basic properties of algebraic K-theory and G-theory of derived algebraic spaces and stacks, and the theory of fundamental classes in this setting.

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Introduction

In [Kh5] I studied the cohomology and Borel–Moore homology of derived schemes and algebraic spaces, as well as Borel-type extensions to derived algebraic stacks. In these notes I describe the K-theoretic counterpart to that formalism. In this analogy, the K-theory of perfect complexes behaves like cohomology while G-theory (K-theory of coherent sheaves) behaves like Borel–Moore homology. This relationship can actually be made quite precise using the formalism of motivic categories: the theory of *KGL-modules* provides a category of coefficients for K-theory[1], and Borel–Moore homology in that setting (i.e., cohomology with coefficients in the dualizing complex) is G-theory. See [CD, §13.3] and [Ji].

The notes are almost entirely expository and were originally prepared as background material for the paper [Kh5][2]. I begin in Section 1 with some preliminaries on quasi-coherent complexes on derived algebraic stacks and various finiteness conditions. Thanks primarily to work of Hall and Rydh over the past several years, these finiteness conditions are well-understood for very general algebraic stacks.

In the second and third sections I define K-theory and G-theory and record their basic properties. Most of these go back to [SGA6], [Q2], and [TT] in the case of classical schemes, and to [To1, To2, KrRa, HoKr] in the case of classical stacks.

The fourth section discusses the failure of homotopy invariance in K-theory of singular spaces and some of its ramifications. I explain how *forcing* homotopy invariance for arbitrary spaces results in a new cohomology theory, first introduced by Weibel [We2] for classical schemes, that agrees with K-theory on nonsingular spaces but behaves more "correctly" on singular ones.

In the fifth section I explain how K-theory and G-theory acquire some useful descent properties after passage to rational coefficients. This leads to étale-local variants of K-theory and G-theory of stacks, which do not agree however with K-theory and G-theory even rationally. A subtlety here is the existence and behaviour of direct images in étale G-theory, which are not compatible with those in G-theory. Note that, for stacks, these étale-localized theories

[1] or rather its homotopy invariant version KH (see Subsect. 4.2)

[2] To address the question of why they are longer than [Kh5], I would like to assure the reader that the next revision of [Kh5] will almost certainly be even longer.

are the ones that are actually compatible with the étale motivic theories studied in [Kh5].

The most important aspect of the formalism in [Kh5] is the theory of fundamental classes. The K-theoretic counterpart to that part of the story, at least some of which is known to the experts, is developed in Section 6. The K-theoretic virtual structure sheaf of [Lee] and its basic properties fall out of this formalism for free. I also give another description of Gysin maps in G-theory using deformation to the 1-shifted tangent bundle. Then I recall some variants of the Grothendieck–Riemann–Roch theorem. One formulation, proven in [Kh5], compares the virtual structure sheaf with the virtual fundamental class in Borel–Moore homology (or the Chow groups). Another is a direct generalization of the one proven in [SGA6] and involves the behaviour of K-theoretic fundamental classes with respect to the γ-filtration. This inspires some unsolicited speculations on a theory of derived algebraic cycles.

Conventions. I generally tried to follow the same conventions and notation as in [Kh5]. Since a substantial revision to that paper is in preparation, I was not always able to achieve this.

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1. Perfect and coherent complexes

1.1. **Complexes over derived commutative rings.** Let A be a derived commutative ring. We write $\mathbf{D}(A)$ for the derived ∞-category of complexes over A. We will say a complex $M \in \mathbf{D}(A)$ is *connective*, resp. *coconnective*, if we have $\pi_i(M) = 0$ for all $i < 0$, resp. for all $i > 0$. More generally we say M is n*-connective*, resp. n*-coconnective*, if we have $\pi_i(M) = 0$ for all $i < n$, resp. for all $i > n$. We write

$$\mathbf{D}(A)_{\geqslant n} \subseteq \mathbf{D}(A), \qquad \mathbf{D}(A)_{\leqslant n} \subseteq \mathbf{D}(A),$$

for the respective full subcategories. If M is both connective and coconnective, i.e., if $\pi_i(M) = 0$ for all $i \neq 0$, then we say M is *discrete*. The full subcategory of discrete complexes

$$\mathbf{D}(A)^{\heartsuit} \subseteq \mathbf{D}(A)$$

is equivalent to the abelian category $\mathrm{Mod}_{\pi_0(A)}$ of $\pi_0(A)$-modules. What we have just described is of course nothing else than the standard t-structure on $\mathbf{D}(A)$.

The cohomologically oriented reader will want to write

$$\mathrm{H}^{-i}(M) \coloneqq \pi_i(M)$$

and reset the notation as follows:

$$\mathbf{D}(A)^{\leqslant n} \coloneqq \mathbf{D}(A)_{\geqslant -n}, \qquad \mathbf{D}(A)^{\geqslant n} \coloneqq \mathbf{D}(A)_{\leqslant -n}.$$

We recall some finiteness conditions on complexes.

Definition 1.1. A complex M over A is *perfect* if it is in the thick subcategory of $\mathbf{D}(A)$ generated by A. That is, if it is built out of A under finite (co)limits and direct summands.

To define (pseudo)coherent complexes, it is convenient (but not necessary) to assume that A is noetherian. Here this means that the ordinary commutative ring $\pi_0(A)$ is noetherian and that the homotopy groups $\pi_i(A)$ are finitely generated as $\pi_0(A)$-modules for all $i \geqslant 0$. Whenever we discuss (pseudo)coherent complexes below, the reader should either assume the ring is noetherian[3] or should replace the definition of pseudocoherence below by [Lu4, Defn. 7.2.4.10].

Definition 1.2. (i) A complex M over A is *pseudocoherent*[4] if it is eventually connective, i.e., $\pi_i(M) = 0$ for $i \ll 0$, and $\pi_i(M)$ is finitely generated as a $\pi_0(A)$-module for all i. (ii) A complex M over A is *coherent* if it is pseudocoherent and also eventually coconnective, i.e., $\pi_i(M) = 0$ for $i \gg 0$.

Definition 1.3. (i) A complex M over A is *of Tor-amplitude* $\leqslant n$ if for every discrete complex $N \in \mathbf{D}(A)^{\heartsuit}$, the derived tensor product $M \otimes^{\mathbf{L}}_A N$ is n-coconnective. Equivalently, for every coconnective complex $N \in \mathbf{D}(A)_{\leqslant 0}$, the derived tensor product $M \otimes^{\mathbf{L}}_A N$ is n-coconnective (see [Lu4, Prop. 7.2.4.23 (5)]). (ii) A complex M over A is *of finite Tor-amplitude* if it is of Tor-amplitude $\leqslant n$ for some n.

The following lemma summarizes the relationships between these finiteness conditions.

Lemma 1.4.

(i) *The property of (pseudo)coherence is stable under finite (co)limits and direct summands. In other words, the (pseudo)coherent complexes form a thick subcategory of* $\mathbf{D}(A)$.

(ii) *Every perfect complex over A is pseudocoherent.*

(iii) *A pseudocoherent complex over A is perfect if and only if it is of finite Tor-amplitude.*

(iv) *If A is eventually coconnective, then every perfect complex over A is coherent.*

(v) *Assume A is eventually coconnective and that every coherent complex is perfect. Then A is discrete, and regular as an ordinary commutative ring.*

[3]Noetherianness can be replaced harmlessly by the slightly weaker property of coherence in the sense of [Lu4, Def. 7.2.4.13].

[4]In [Lu4, Lu5, HLP], the term "almost perfect" is used instead of pseudocoherent.

Proof. (i) See [Lu4, Lem. 7.2.4.11 (1-2)]. (ii) By definition of perfectness, it suffices to show that A itself is pseudocoherent over A. If A is noetherian, then this follows from the definition of pseudocoherence; see [Lu4, 7.2.4.11 (3)] for the non-noetherian case. (iii) See [Lu4, Prop. 7.2.4.23(4)]. (iv) Again it suffices to show that A is coherent over itself, which follows from the definitions when A is eventually coconnective. (v) See [Lu5, Lem. 11.3.3.3]. □

1.2. **Quasi-coherent, perfect, and coherent complexes.** Let $\mathcal{X}$ be a derived algebraic stack. We write $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ for the stable ∞-category of *quasi-coherent complexes* on $\mathcal{X}$, see e.g. [To6, Subsect. 3.1].

Over an affine derived scheme $\mathrm{Spec}(A)$, a quasi-coherent complex $\mathcal{F}$ is the same datum as that of the complex $\mathbf{R}\Gamma(\mathrm{Spec}(A), \mathcal{F})$ over A, i.e., there is an equivalence

$$\mathbf{D}_{\mathrm{qc}}(\mathrm{Spec}(A)) \simeq \mathbf{D}(A)$$

given by $\mathcal{F} \mapsto \mathbf{R}\Gamma(\mathrm{Spec}(A), \mathcal{F})$. We write $\mathcal{O}_X \in \mathbf{D}_{\mathrm{qc}}(\mathrm{Spec}(A))$ for the quasi-coherent complex whose derived global sections are $A \in \mathbf{D}(A)$.

Over a derived algebraic stack $\mathcal{X}$, a quasi-coherent complex $\mathcal{F}$ amounts to the data of quasi-coherent complexes $u^*(\mathcal{F}) \in \mathbf{D}_{\mathrm{qc}}(X)$ for every smooth morphism $u : X \to \mathcal{X}$ with X affine, together with a homotopy coherent system of compatibilities between these complexes as u varies. For example, the structure sheaf $\mathcal{O}_{\mathcal{X}} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ satisfies $u^*(\mathcal{O}_{\mathcal{X}}) \simeq \mathcal{O}_X$ for every u.

Definition 1.5. (i) A quasi-coherent complex $\mathcal{F}$ on an affine derived scheme $X = \mathrm{Spec}(A)$ is *perfect*, *pseudocoherent*, or *coherent* if $\mathbf{R}\Gamma(X, \mathcal{F})$ has the respective property as a complex over A. (ii) A quasi-coherent complex $\mathcal{F}$ on a derived algebraic stack $\mathcal{X}$ is *perfect*, *pseudocoherent*, or *coherent*, if $u^*(\mathcal{F})$ has the respective property for every smooth morphism $u : X \to \mathcal{X}$ from an affine X.

We write

$$\mathbf{D}_{\mathrm{perf}}(\mathcal{X}) \subseteq \mathbf{D}_{\mathrm{qc}}(\mathcal{X}) \text{ and } \mathbf{D}_{\mathrm{coh}}(\mathcal{X}) \subseteq \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$$

for the full subcategories of perfect and coherent complexes on $\mathcal{X}$.

Remark 1.6. If $\mathcal{X}$ is a classical algebraic stack, then $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ can be described as the derived ∞-category of complexes of $\mathcal{O}_{\mathcal{X}}$-modules (on the lisse-étale topos) with quasi-coherent cohomology. See [HR2, Prop. 1.3]. In that language, $\mathbf{D}_{\mathrm{coh}}(\mathcal{X})$ is the full subcategory of complexes with bounded and coherent cohomology.

Definition 1.7. Let $\mathcal{X}$ be a derived algebraic stack and $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ a quasi-coherent complex. For an integer $n \in \mathbf{Z}$, we say that $\mathcal{F}$ is *n-connective* if $\pi_i(u^*\mathcal{F}) = 0$ for all $i < n$ and smooth morphisms $u : X \to \mathcal{X}$ with X affine. We say $\mathcal{F}$ is *n-coconnective* if $\pi_i(u^*\mathcal{F}) = 0$ for all $i > n$ and smooth morphisms $u : X \to \mathcal{X}$ with X affine. The (0-)connective and (0-)coconnective complexes define full subcategories

$$\mathbf{D}_{\mathrm{qc}}(\mathcal{X})_{\geqslant 0} \subseteq \mathbf{D}_{\mathrm{qc}}(\mathcal{X}), \quad \mathbf{D}_{\mathrm{qc}}(\mathcal{X})_{\leqslant 0} \subseteq \mathbf{D}_{\mathrm{qc}}(\mathcal{X}),$$

respectively, which together form a canonical t-structure on $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$. This t-structure restricts to $\mathbf{D}_{\mathrm{coh}}(\mathcal{X})$, but typically not to $\mathbf{D}_{\mathrm{perf}}(\mathcal{X})$ unless $\mathcal{X}$ is a classical regular stack. We write

$$\mathrm{Qcoh}(\mathcal{X}) := \mathbf{D}_{\mathrm{qc}}(\mathcal{X})^{\heartsuit}, \quad \mathrm{Coh}(\mathcal{X}) := \mathbf{D}_{\mathrm{coh}}(\mathcal{X})^{\heartsuit}$$

for the hearts. Note that these are insensitive to the derived structure on $\mathcal{X}$: they are equivalent to the abelian categories of quasi-coherent and coherent sheaves, respectively, on the classical truncation $\mathcal{X}_{\mathrm{cl}}$.

Remark 1.8. Let $\mathcal{X}$ be a derived algebraic stack. Suppose that $\mathcal{X}$ has *bounded structure sheaf*, i.e., that the quasi-coherent complex $\mathcal{O}_{\mathcal{X}}$ is n-coconnective for some $n \gg 0$. In that case there is an inclusion $\mathbf{D}_{\mathrm{perf}}(\mathcal{X}) \subseteq \mathbf{D}_{\mathrm{coh}}(\mathcal{X})$. Indeed the properties of perfectness and coherence are both local by definition so this follows from Lemma 1.4.

1.3. **Tensor, Hom, and inverse/direct images.** We have the following basic operations on $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$.

For every derived algebraic stack $\mathcal{X}$, there is a (derived) tensor product $\otimes$ on $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$. It is left adjoint, as a bifunctor, to the internal Hom functor $\underline{\mathrm{Hom}}$. These are part of a closed symmetric monoidal structure on $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$. For every fixed perfect complex $\mathcal{F} \in \mathbf{D}_{\mathrm{perf}}(\mathcal{X})$, the operations

$$\mathcal{G} \mapsto \mathcal{F} \otimes \mathcal{G}$$

preserves perfectness and coherence. (One easily reduces to the case where $\mathcal{X} = X$ is affine and $\mathcal{F} = \mathcal{O}_X$.)

Given a morphism of derived algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$, there is an inverse image functor

$$\mathbf{L}f^* : \mathbf{D}_{\mathrm{qc}}(\mathcal{Y}) \to \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$$

which is symmetric monoidal. It is right t-exact, i.e., preserves connectivity. It also preserves perfect complexes and pseudocoherent complexes. If f is flat, then $\mathbf{L}f^* = f^*$ is also left t-exact.

The inverse image functor admits a right adjoint, the direct image functor

$$\mathbf{R}f_* : \mathbf{D}_{\mathrm{qc}}(\mathcal{X}) \to \mathbf{D}_{\mathrm{qc}}(\mathcal{Y}).$$

By adjunction, it is left t-exact, i.e., preserves coconnectivity. If it is affine, then $\mathbf{R}f_* = f_*$ is also right t-exact. If f is representable and proper (i.e., if it is representable, separated of finite type, and satisfies the valuative criterion), then $\mathbf{R}f_*$ preserves pseudocoherent and coherent complexes. See [Lu5, Thms. 5.6.0.2], and Subsect. 1.6 for a discussion of the non-representable case.

Remark 1.9. When f is representable (by qcqs derived algebraic spaces), $\mathbf{R}f_*$ commutes with colimits and satisfies base change and projection formulas. That is:

(i) The functor $\mathbf{L}f^*$ is compact, i.e., its right adjoint $\mathbf{R}f_*$ preserves colimits. See [Lu5, Cor. 3.4.2.2 (2)].

(ii) For every homotopy cartesian square of derived algebraic stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{Y}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array}$$

there is a canonical isomorphism

$$\mathbf{L}q^*\mathbf{R}f_* \to \mathbf{R}g_*\mathbf{L}p^*$$

of functors $\mathbf{D}_{\mathrm{qc}}(\mathcal{X}) \to \mathbf{D}_{\mathrm{qc}}(\mathcal{Y}')$. See [Lu5, Cor. 3.4.2.2 (3)].

(iii) For every $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ and $\mathcal{G} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{Y})$, there is a canonical isomorphism

$$\mathbf{R}f_*(\mathcal{F}) \otimes \mathcal{G} \to \mathbf{R}f_*(\mathcal{F} \otimes \mathbf{L}f^*(\mathcal{G}))$$

which is natural in $\mathcal{F}$ and $\mathcal{G}$. In other words, $\mathbf{R}f_*$ is $\mathbf{D}_{\mathrm{qc}}(\mathcal{Y})$-linear when $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ is regarded as a $\mathbf{D}_{\mathrm{qc}}(\mathcal{Y})$-module category via the symmetric monoidal functor $\mathbf{L}f^*$. See [Lu5, Rem. 3.4.2.6].

As we will see in the next subsection, representability can be considerably weakened.

1.4. **Finiteness of Tor-amplitude.** Recall that a morphism $A \to B$ of derived commutative rings is of Tor-amplitude $\leqslant n$ if B is of Tor-amplitude $\leqslant n$ as a complex over A (Definition 1.3). This condition is fpqc-local on the source and target, so we may extend to derived algebraic stacks in the usual manner. For example, a morphism is of Tor-amplitude $\leqslant 0$ if and only if it is flat. We also say a morphism is of *finite Tor-amplitude* if it is of Tor-amplitude $\leqslant n$ for some $n \geqslant 0$. We clearly have:

Proposition 1.10. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of derived algebraic stacks. If f is of Tor-amplitude $\leqslant n$, then $\mathbf{L}f^*$ restricts to a functor*

$$\mathbf{L}f^* : \mathbf{D}_{\mathrm{qc}}(\mathcal{Y})_{\leqslant 0} \to \mathbf{D}_{\mathrm{qc}}(\mathcal{X})_{\leqslant n},$$

i.e., it sends coconnective complexes to n-coconnective complexes.

Remark 1.11. Recall that $\mathbf{L}f^*$ always preserves pseudocoherent complexes. Thus when f is of finite Tor-amplitude, it also preserves coherent complexes.

Remark 1.12. If f is proper representable, locally almost of finite presentation, and of finite Tor-amplitude, then $\mathbf{R}f_*$ preserves perfect complexes. See [Lu5, Thm. 6.1.3.2].

Recall that a morphism of derived algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is *quasi-smooth* if it is of finite presentation and the relative cotangent complex $\mathcal{L}_{\mathcal{X}/\mathcal{Y}}$ is of Tor-amplitude $\leqslant 1$. Equivalently, it factors fpqc-locally on the source as the inclusion $\mathcal{X} \hookrightarrow \mathcal{Y}'$ of the derived zero locus of a section of a vector bundle over a stack $\mathcal{Y}'$ which is smooth over $\mathcal{Y}$; see [KhRy, Prop. 2.3.14]. This is a derived version of the notion of *local complete intersection* morphism.

Lemma 1.13. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived algebraic stacks. Then f is of finite Tor-amplitude.*

Proof. If f is smooth, then it is of Tor-amplitude $\leqslant 0$. Therefore we may assume that f is the inclusion of the derived zero locus $\mathcal{X} = X$ of n functions $g_1, \ldots, g_n$ on an affine derived scheme $\mathcal{Y} = Y$. We claim that f is of Tor-amplitude $\leqslant n$. By induction, we may assume that $n = 1$. Then the exact triangle

$$\mathcal{O}_Y \xrightarrow{g_1} \mathcal{O}_Y \to f_*(\mathcal{O}_X)$$

shows that f is of Tor-amplitude $\leqslant 1$. □

1.5. **Finiteness of cohomological dimension.** The functor f_* is well-behaved for many non-representable morphisms as well. The relevant condition is as follows.

Definition 1.14. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of derived algebraic stacks. We say that f is *of cohomological dimension* $\leqslant n$ if $\mathbf{R}f_*$ restricts to a functor

$$\mathbf{R}f_* : \mathbf{D}_{\mathrm{qc}}(\mathcal{X})_{\geqslant 0} \to \mathbf{D}_{\mathrm{qc}}(\mathcal{Y})_{\geqslant -n},$$

i.e., it sends connective complexes to $(-n)$-connective complexes. Equivalently, $\mathbf{R}f_*(\mathcal{F})$ is $(-n)$-connective for every *discrete* $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{X})$ (see [HLP, Lem. A.1.6]).

Example 1.15. We say that a derived algebraic stack $\mathcal{X}$ is of cohomological dimension $\leqslant n$ if the structural morphism $f : \mathcal{X} \to \mathrm{Spec}(\mathbf{Z})$ is. Note that this is the case if and only if the complex

$$\mathbf{R}\Gamma\big(\mathrm{Spec}(\mathbf{Z}), \mathbf{R}f_*(\mathcal{F})\big) \simeq \mathbf{R}\Gamma(\mathcal{X}, \mathcal{F})$$

is $(-n)$-connective for all quasi-coherent sheaves $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{X})$. In other words, if

$$\mathrm{H}^i(\mathcal{X}, \mathcal{F}) = \pi_{-i}\left(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{F})\right) = 0$$

for all $i > n$ and all $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{X})$.

Definition 1.16. We say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ is *universally of finite cohomological dimension* if for every morphism $\mathcal{Y}' \to \mathcal{Y}$ with $\mathcal{Y}'$ qcqs, the base change $f' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ is of finite cohomological dimension.

In [HR2], qcqs morphisms that are universally of finite cohomological dimension are called “concentrated”.

Remark 1.17. If $\mathcal{Y}$ is quasi-compact with quasi-affine diagonal, then any qcqs morphism $f : \mathcal{X} \to \mathcal{Y}$ of finite cohomological dimension is also *universally* of finite cohomological dimension. See [HR2, Lem. 2.5(v)].

Proposition 1.18. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact quasi-separated morphism of derived algebraic stacks. If f is universally of finite cohomological dimension, then $\mathbf{R}f_*$ commutes with colimits and satisfies the base change and projection formulas (as in Remark 1.9).*

Proof. See [HR2, Thm. 2.6] and [HLP, Prop. A.1.5]. □

Remark 1.19. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is of finite cohomological dimension if and only if the underlying morphism $f_{\mathrm{cl}} : \mathcal{X}_{\mathrm{cl}} \to \mathcal{Y}_{\mathrm{cl}}$ of classical truncations is of finite cohomological dimension. Indeed, this follows by consideration of the following commutative square

$$\begin{array}{ccc} \mathrm{Qcoh}(\mathcal{X}_{\mathrm{cl}}) & \xrightarrow{\mathbf{R}f_{\mathrm{cl},*}} & \mathbf{D}(\mathcal{Y}_{\mathrm{cl}})_{\leqslant 0} \\ \downarrow{\scriptstyle i_{\mathcal{X},*}} & & \downarrow{\scriptstyle i_{\mathcal{Y},*}} \\ \mathrm{Qcoh}(\mathcal{X}) & \xrightarrow{\mathbf{R}f_*} & \mathbf{D}(\mathcal{Y})_{\leqslant 0}, \end{array}$$

where the vertical arrows are direct image along the inclusions of the classical truncations. Recall that $i_{\mathcal{X},*}$ and $i_{\mathcal{Y},*}$ are t-exact (since $i_{\mathcal{X}}$ and $i_{\mathcal{Y}}$ are affine morphisms) and that the left-hand arrow is an equivalence (Definition 1.7).

Example 1.20. If $f : \mathcal{X} \to \mathcal{Y}$ is a representable morphism of qcqs derived algebraic stacks, then it is universally of finite cohomological dimension. See [Lu5, Prop. 2.5.4.4, Thm. 3.4.2.1].

Example 1.21. Let G be an fppf group scheme over an affine scheme S and consider the morphism $f : BG \to S$, the structural morphism of the classifying stack. If G is linearly reductive, then f is universally of cohomological dimension zero. In characteristic zero there are more examples, see [HR1, Sect. 1].

Example 1.22. Let $\mathcal{X}$ be a qcqs derived algebraic stack. Assume that any one of the following conditions hold:

(i) $\mathcal{X}$ is of characteristic zero and its stabilizers at all points are affine.

(ii) $\mathcal{X}$ is of positive characteristic and its stabilizers at all points are linearly reductive.

(iii) $\mathcal{X}$ is of mixed characteristic and its stabilizers at all points are "nice" (extensions of tame finite étale groups by groups of multiplicative type).

(iv) $\mathcal{X}$ is of mixed characteristic, its (classical) inertia stack is of finite presentation over $\mathcal{X}_{\mathrm{cl}}$, its stabilizers at all points are affine, and its stabilizers at points of positive characteristic are linearly reductive.

Then $\mathcal{X}$ is of finite cohomological dimension (see [DG, Thm. 1.4.2] and [HR1, Thm. 2.1]). Similarly, a morphism of qcqs derived algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is of finite cohomological dimension if for any morphism $v : Y \to \mathcal{Y}$ with Y affine, the fibre $\mathcal{X} \times_{\mathcal{Y}} Y$ satisfies one of the above conditions.

1.6. **Cohomological properness.** We would now like to study the question of when the direct image functor preserves coherent complexes. So, we introduce the following definition.

Definition 1.23. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of derived algebraic stacks. Assume that f is of finite cohomological dimension. We say that f is *cohomologically proper* if for every coherent sheaf $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, the direct image $\mathbf{R}f_*(\mathcal{F}) \in \mathbf{D}_{\mathrm{coh}}(\mathcal{Y})$ is a coherent complex.

Remark 1.24. Since f is of finite cohomological dimension, $\mathbf{R}f_*(\mathcal{F}) \in \mathbf{D}_{\mathrm{qc}}(\mathcal{Y})$ is bounded (cohomologically) for any coherent sheaf $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$. Thus the condition of Definition 1.23 is that it has coherent cohomologies, i.e., that for every smooth morphism $u : Y \to \mathcal{Y}$ with Y affine, the groups

$$\mathrm{H}^i(Y, u^*\mathbf{R}f_*(\mathcal{F})) = \pi_{-i}(\mathbf{R}\Gamma(Y, u^*\mathbf{R}f_*(\mathcal{F})))$$

are finitely generated over $\mathrm{H}^0(Y, \mathbf{R}f_*(\mathcal{F}))$ for all i.

Remark 1.25. If $f : \mathcal{X} \to \mathcal{Y}$ is cohomologically proper, then any coherent *complex* $\mathcal{F} \in \mathbf{D}_{\mathrm{coh}}(\mathcal{X})$ also has coherent direct image $\mathbf{R}f_*(\mathcal{F}) \in \mathbf{D}_{\mathrm{coh}}(\mathcal{Y})$. This follows by induction on the Postnikov tower, i.e., using the exact triangles

$$\pi_n(\mathcal{F})[n] \to \tau_{\leqslant n}(\mathcal{F}) \to \tau_{\leqslant n-1}(\mathcal{F})$$

coming from the (bounded) t-structure on $\mathbf{D}_{\mathrm{coh}}(\mathcal{X})$.

Remark 1.26. As in Remark 1.19, the condition of cohomological properness may be checked on classical truncations.

Proposition 1.27. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of finite cohomological dimension between noetherian derived algebraic stacks. If f is proper (of finite type, separated and satisfying the valuative criterion), then it is cohomologically proper.*

Proof. See [Fa, Thm. 1], [Ol, Thm. 1.2]. □

Proposition 1.28. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a cohomologically proper morphism of derived algebraic stacks that is universally of finite cohomological dimension. If f is moreover almost of finite presentation and of finite Tor-amplitude, then $\mathbf{R}f_*$ also preserves perfect complexes.*

Proof. Since f is universally of finite cohomological dimension, formation of $\mathbf{R}f_*$ is stable under base change (Proposition 1.18). Since perfectness is smooth-local by definition, we may assume that $\mathcal{Y} = Y$ is affine. By Lemma 1.4(iii) it will suffice to show that the coherent complex $\mathbf{R}f_*(\mathcal{F}) \in \mathbf{D}_{\mathrm{coh}}(Y)$ is of finite Tor-amplitude for every perfect complex $\mathcal{F} \in \mathbf{D}_{\mathrm{perf}}(\mathcal{X})$. Now the claim follows from [Lu5, Prop. 6.1.3.1]. □

1.7. The Thomason condition.

Following [HR2] we define:

Definition 1.29. Let $\mathcal{X}$ be a qcqs derived algebraic stack. We say that $\mathcal{X}$ *satisfies the Thomason condition* if the following properties hold:

(i) The stable ∞-category $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ is compactly generated.

(ii) For every cocompact closed subset $Z \subseteq |\mathcal{X}|$ there exists a compact object $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ which is set-theoretically supported exactly on Z.

Here a closed subset $Z \subseteq |\mathcal{X}|$ is *cocompact* if its open complement $\mathcal{X} \smallsetminus Z$ is quasi-compact. We write $\mathbf{D}_{\mathrm{qc}}(\mathcal{X} \text{ on } Z)$ for the full subcategory of $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ spanned by quasi-coherent complexes $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ with support contained in Z (i.e., whose restriction to $\mathcal{X} \smallsetminus Z$ is isomorphic to zero), and similarly $\mathbf{D}_{\mathrm{perf}}(\mathcal{X} \text{ on } Z)$ for the intersection of $\mathbf{D}_{\mathrm{perf}}(\mathcal{X})$ and $\mathbf{D}_{\mathrm{qc}}(\mathcal{X} \text{ on } Z)$.

Proposition 1.30. *If $\mathcal{X}$ satisfies the Thomason condition and is of finite cohomological dimension, then the canonical functor* $\mathrm{Ind}(\mathbf{D}_{\mathrm{perf}}(\mathcal{X})) \to \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ *is an equivalence. Moreover, for every cocompact closed subset $Z \subseteq |\mathcal{X}|$, the canonical functor* $\mathrm{Ind}(\mathbf{D}_{\mathrm{perf}}(\mathcal{X} \text{ on } Z)) \to \mathbf{D}_{\mathrm{qc}}(\mathcal{X} \text{ on } Z)$ *is an equivalence. In other words,* $\mathbf{D}_{\mathrm{qc}}(\mathcal{X} \text{ on } Z)$ *is compactly generated and the compact objects are precisely the perfect complexes supported on Z.*

Proof. See [HR2, Lems. 4.4 and 4.10(2)] or [HoKr, Prop. 3.5]. □

In view of Proposition 1.30 we might make the following definition.

Definition 1.31. Let $\mathcal{X}$ be a qcqs derived algebraic stack. If $\mathcal{X}$ satisfies the Thomason condition and is of finite cohomological dimension, then we say $\mathcal{X}$ is *perfect*.

Remark 1.32. There are many slightly different definitions of "perfect stack" in the literature. For example, $\mathcal{X}$ is perfect in the sense of [HoKr, Defn. 3.3] if it satisfies part (i) of the Thomason condition and is of finite cohomological dimension. However, I don't know any examples of stacks which are perfect in this sense and do not also satisfy part (ii) of the Thomason condition.

Example 1.33. Any qcqs derived scheme or algebraic space X is perfect. See [BVdB, Thm. 3.1.1], [To5, Thm. 4.7], [Lu5, Prop. 9.6.1.1].

In Example 1.22 we have seen that finite cohomological dimension can be guaranteed if $\mathcal{X}$ has nice enough stabilizers. Theorem C of [HR2] gives us a way to verify the Thomason condition locally for étale or quasi-finite flat covers. This gives the following class of examples of perfect stacks.

Example 1.34. Let $\mathcal{X}$ be a quasi-compact derived algebraic stack. Assume that $\mathcal{X}$ has affine diagonal and any one of the following conditions holds:

(i) $\mathcal{X}$ is of characteristic zero and its stabilizers at all closed points are reductive.

(ii) $\mathcal{X}$ is of positive characteristic and its stabilizers at all points are linearly reductive.

(iii) $\mathcal{X}$ is of mixed characteristic and its stabilizers at all points are nice (see Example 1.22).

Then $\mathcal{X}$ is perfect. This follows from [AHHR] (cf. [AHR1, Thm. 5.1], [AHR2, Prop. 14.1]); see [HoKr, Cor. 3.17] and [BKRS, Ex. A.1.4, Thm. A.3.2]. Note that this includes tame Deligne–Mumford stacks with affine diagonal and more generally algebraic stacks with affine diagonal that are tame in the sense of [AOV].

Example 1.35. For classical Deligne–Mumford stacks (or more generally algebraic stacks with quasi-finite diagonal), we can replace the local structure theorem of [AHHR] by [Ry2, Thm. 7.1]. Thus if $\mathcal{X}$ is a qcqs algebraic stack with quasi-finite and *separated* diagonal, it is perfect in the following cases:

(i) $\mathcal{X}$ is of characteristic zero.

(ii) $\mathcal{X}$ is of positive or mixed characteristic, and its stabilizers at all points are linearly reductive.

For example, this includes tame qcqs Deligne–Mumford stacks with separated diagonal. See [HoKr, Cor. 3.16].

2. Algebraic K-theory

For a commutative ring R, Quillen's algebraic K-theory spectrum $\mathrm{K}(R)$ (see [Q2]) can be defined succinctly as the homotopy-theoretic group completion [Q1, Ma, GGN] of the groupoid of finitely generated projective R-modules, regarded as an $\mathcal{E}_\infty$-monoid under tensor product. This can be viewed either as an $\mathcal{E}_\infty$-group or a connective spectrum. This description is also valid for a derived commutative ring R (see [Lu3, Lect. 19, Thm. 5]).

For a scheme X, doing the same construction with the groupoid of finite locally free $\mathcal{O}_X$-modules (or vector bundles over X of finite rank) gives rise to Quillen's original construction of $\mathrm{K}(X)$. The Thomason–Trobaugh definition [TT] is more sophisticated: it replaces vector bundles by perfect complexes and requires Waldhausen's $S_\bullet$-construction instead group completion. The advantage of Thomason–Trobaugh K-theory is its excellent behaviour over arbitrary (qcqs) schemes: for instance, there are localization and Mayer–Vietoris long exact sequences. Moreover, Thomason–Trobaugh K-theory and Quillen K-theory agree for all schemes that arise in practice: to be precise, the relevant condition is the *resolution property*, which requires that every coherent sheaf of finite type admits a surjection from a finite locally free sheaf. For example, any quasi-projective scheme admits the resolution property. Again, this discussion remains valid for X a derived scheme.

For algebraic stacks, the resolution property is much less common. In fact, among stacks with affine stabilizers, the resolution property holds precisely for quotients of quasi-affine schemes by actions of a general linear group GL_n [Tot1, Gr]. So for stacks, it is crucial to work with the Thomason–Trobaugh construction.

2.1. **Definitions.** Let $\mathcal{A}$ be a stable ∞-category. The Waldhausen $S_\bullet$-construction makes sense in this setting and can be used to produce the K-theory space of $\mathcal{A}$. Iterating it gives rise to an $\mathcal{E}_\infty$-group structure on this space, and hence to a connective spectrum $\mathrm{K}(\mathcal{A})$. See [Lu4, Rem. 1.2.2.5], [BGT, Sect. 7], and [Ba2, Sect. 10]. Using a generalization of the Bass construction (see [CiKh, Sect. 4]), one can also produce a Bass–Thomason–Trobaugh K-theory spectrum $\mathrm{K}^{\mathrm{B}}(\mathcal{A})$ whose connective cover $\mathrm{K}^{\mathrm{B}}(\mathcal{A})_{\geqslant 0}$ is $\mathrm{K}(\mathcal{A})$. Its homotopy groups are denoted

$$\mathrm{K}_i(\mathcal{A}) = \pi_i(\mathrm{K}^{\mathrm{B}}(\mathcal{A})), \qquad i \in \mathbf{Z},$$

so that $\mathrm{K}_i(\mathcal{A}) \simeq \pi_i(\mathrm{K}(\mathcal{A}))$ for $i \geqslant 0$. Any exact functor of stable ∞-categories $\mathcal{A} \to \mathcal{B}$ induces maps of spectra $\mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$ and $\mathrm{K}^{\mathrm{B}}(\mathcal{A}) \to \mathrm{K}^{\mathrm{B}}(\mathcal{B})$.

Theorem 2.1. *Let $j^* : \mathcal{A} \to \mathcal{B}$ be a functor between compactly generated stable ∞-categories with fully faithful right adjoint j_*. Suppose that j^* is compact (i.e., preserves compact objects) and its kernel $\mathcal{A}_0$ is compactly generated. Then we have:*

(i) Proto-localization. *There is a fibre sequence of connective spectra*
$$\mathrm{K}(\mathcal{A}_0^\omega) \to \mathrm{K}(\mathcal{A}^\omega) \to \mathrm{K}(\mathcal{B}^\omega).$$

(ii) Localization. *There is an exact triangle of spectra*
$$\mathrm{K}^{\mathrm{B}}(\mathcal{A}_0^\omega) \to \mathrm{K}^{\mathrm{B}}(\mathcal{A}^\omega) \to \mathrm{K}^{\mathrm{B}}(\mathcal{B}^\omega).$$

In particular, there is a long exact sequence
$$\cdots \xrightarrow{\partial} \mathrm{K}_i(\mathcal{A}_0^\omega) \to \mathrm{K}_i(\mathcal{A}^\omega) \to \mathrm{K}_i(\mathcal{B}^\omega) \xrightarrow{\partial} \mathrm{K}_{i-1}(\mathcal{A}_0^\omega) \to \cdots.$$

Proof. See [Ba2, Prop. 10.20] for the first statement. The second follows from the first and [CiKh, Thm. 4.5.7]. □

Corollary 2.2 (Additivity)**.** *For any semi-orthogonal decomposition of $\mathcal{A}$ into stable subcategories, the spectra $\mathrm{K}(\mathcal{A})$ and $\mathrm{K}^{\mathrm{B}}(\mathcal{A})$ both split into direct sums.*

Proof. As in [Kh4, Lem. 2.8], we may reduce to the case of a semi-orthogonal decomposition of length 2. Applying Theorem 2.1 to the ind-completions, we get a fibre sequence (resp. exact triangle), which is split by the map $j_* : \mathrm{K}(\mathcal{B}) \to \mathrm{K}(\mathcal{A})$ (resp. $j_* : \mathrm{K}^{\mathrm{B}}(\mathcal{B}) \to \mathrm{K}^{\mathrm{B}}(\mathcal{A})$). □

Remark 2.3. If the kernel $\mathcal{A}_0$ is not compactly generated, one can still identify the fibre in Theorem 2.1 as the Efimov K-theory of $\mathcal{A}_0$ (see [Ho2]).

Theorem 2.4. *Suppose given a commutative square*
$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f^*} & \mathcal{B} \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle q^*} \\ \mathcal{A}' & \xrightarrow{g^*} & \mathcal{B}' \end{array}$$
of compactly generated stable ∞-categories and compact colimit-preserving functors. If the square is cartesian (in the ∞-category of large ∞-categories) and g^* has fully faithful right adjoint g_*, then the induced square of spectra
$$\begin{array}{ccc} \mathrm{K}^{\mathrm{B}}(\mathcal{A}^\omega) & \xrightarrow{f^*} & \mathrm{K}^{\mathrm{B}}(\mathcal{B}^\omega) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle q^*} \\ \mathrm{K}^{\mathrm{B}}(\mathcal{A}'^\omega) & \xrightarrow{g^*} & \mathrm{K}^{\mathrm{B}}(\mathcal{B}'^\omega) \end{array}$$
is cartesian.

Proof. The cartesianness of the square implies that f_* is also fully faithful. Since the ∞-category of spectra is stable, it will suffice to show that the horizontal fibres are isomorphic. If the kernels of f^* and g^* are compactly generated, this follows immediately from Theorem 2.1. By Remark 2.3, this holds even otherwise. (See [Ho2, Cor. 13] and, for another proof, [Ta, Thm. 18].) □

Definition 2.5. For a derived algebraic stack $\mathcal{X}$, write

$$\mathrm{K}(\mathcal{X}) = \mathrm{K}(\mathbf{D}_{\mathrm{perf}}(\mathcal{X})), \quad \mathrm{K}^{\mathrm{B}}(\mathcal{X}) = \mathrm{K}^{\mathrm{B}}(\mathbf{D}_{\mathrm{perf}}(\mathcal{X}))$$

for the Thomason–Trobaugh and Bass–Thomason–Trobaugh K-theory spectra.

From now on we'll mostly restrict our attention to K^{B}, since statements about K can be recovered simply by passing to connective covers.

2.2. **Operations.** The various operations on perfect complexes give rise to operations in K-theory.

The tensor product on $\mathbf{D}_{\mathrm{perf}}(\mathcal{X})$ induces a cup product

$$\cup : \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \otimes \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \to \mathrm{K}^{\mathrm{B}}(\mathcal{X})$$

which is part of an $\mathcal{E}_\infty$-ring structure on $\mathrm{K}^{\mathrm{B}}(\mathcal{X})$.

For any morphism $f : \mathcal{X} \to \mathcal{Y}$, the inverse image functor $f^* : \mathbf{D}_{\mathrm{perf}}(\mathcal{Y}) \to \mathbf{D}_{\mathrm{perf}}(\mathcal{X})$ gives rise to a map

$$f^* : \mathrm{K}^{\mathrm{B}}(\mathcal{Y}) \to \mathrm{K}^{\mathrm{B}}(\mathcal{X}).$$

Since f^* is symmetric monoidal, this map is multiplicative (a homomorphism of $\mathcal{E}_\infty$-ring spectra).

If f is proper, of finite cohomological dimension, almost of finite presentation, and of finite Tor-amplitude, then there is a Gysin map

$$f_* : \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \to \mathrm{K}^{\mathrm{B}}(\mathcal{Y}).$$

By Proposition 1.18 this map is $\mathrm{K}^{\mathrm{B}}(\mathcal{Y})$-linear (a homomorphism of $\mathrm{K}^{\mathrm{B}}(\mathcal{Y})$-module spectra). In other words:

Proposition 2.6 (Projection formula)**.** *If $f : \mathcal{X} \to \mathcal{Y}$ is proper, of finite cohomological dimension, almost of finite presentation, and of finite Tor-amplitude, then we have canonical identifications*

$$f_*(x) \cup y \simeq f_*(x \cup f^*(y))$$

in $\mathrm{K}^{\mathrm{B}}(\mathcal{Y})$, for all $x \in \mathrm{K}^{\mathrm{B}}(\mathcal{X})$, $y \in \mathrm{K}^{\mathrm{B}}(\mathcal{Y})$.

We also have (by Proposition 1.18):

Proposition 2.7 (Base change formula)**.** *Suppose given a homotopy cartesian square of derived algebraic stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{Y}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

If f is proper, of finite cohomological dimension, almost of finite presentation, and of finite Tor-amplitude, then we have a canonical homotopy

$$q^* f_* \simeq g_* p^*$$

of maps $\mathrm{K}^{\mathrm{B}}(\mathcal{X}) \to \mathrm{K}^{\mathrm{B}}(\mathcal{Y}')$.

If $Z \subseteq |\mathcal{X}|$ is a cocompact closed subset, then we write

$$\mathrm{K}^{\mathrm{B}}(\mathcal{X} \text{ on } \mathcal{Z}) \coloneqq \mathrm{K}^{\mathrm{B}}(\mathbf{D}_{\mathrm{perf}}(\mathcal{X} \text{ on } \mathcal{Z}))$$

for the Bass–Thomason–Trobaugh K-theory spectrum of $\mathbf{D}_{\mathrm{perf}}(\mathcal{X} \text{ on } \mathcal{Z})$, see Subsect. 1.7.

Theorem 2.8 (Localization)**.** *Let $Z \subseteq |\mathcal{X}|$ be a cocompact closed subset. If $\mathcal{X}$ is perfect, then there is a canonical exact triangle of spectra*

$$\mathrm{K}^{\mathrm{B}}(\mathcal{X} \text{ on } Z) \to \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \xrightarrow{j^*} \mathrm{K}^{\mathrm{B}}(\mathcal{X} \smallsetminus Z).$$

Proof. Note that if $\mathcal{X}$ is perfect, then so is $\mathcal{X} \smallsetminus Z$. By Proposition 1.30 we have $\mathbf{D}_{\mathrm{perf}}(\mathcal{X}) \simeq \mathbf{D}_{\mathrm{qc}}(\mathcal{X})^{\omega}$ (and similarly for $\mathcal{X} \smallsetminus Z$) and $\mathbf{D}_{\mathrm{perf}}(\mathcal{X} \text{ on } Z) \simeq \mathbf{D}_{\mathrm{qc}}(\mathcal{X} \text{ on } Z)^{\omega}$. Thus we may apply Theorem 2.1. □

2.3. **Projective bundles and blow-ups.** Let $\mathcal{X}$ be a derived algebraic stack. For any locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ of rank $n+1$, $n \geqslant 0$, we may consider the associated projective bundle

$$q : \mathbf{P}_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}.$$

This is the universal derived stack over $\mathcal{X}$ equipped with a surjection $q^*(\mathcal{E}) \twoheadrightarrow \mathcal{O}(1)$ onto an invertible sheaf. The morphism q is smooth of relative dimension n, proper and schematic (see [Kh4, Prop. 3.1]).

The following is a generalization of [TT, Thm. 7.3], [T2].

Theorem 2.9. *The maps*

$$\mathrm{K}^{\mathrm{B}}(X) \to \mathrm{K}^{\mathrm{B}}(\mathbf{P}_{\mathcal{X}}(\mathcal{E})), \quad x \mapsto q^*(x) \cup [\mathcal{O}(-k)],$$

induce an isomorphism

$$\mathrm{K}^{\mathrm{B}}(\mathbf{P}_{\mathcal{X}}(\mathcal{E})) \simeq \bigoplus_{k=0}^{n} \mathrm{K}^{\mathrm{B}}(\mathcal{X}).$$

Proof. The standard semi-orthogonal decomposition of Orlov–Thomason on $\mathbf{D}_{\mathrm{perf}}(\mathbf{P}_{\mathcal{X}}(\mathcal{E}))$ extends to this setting by [Kh4, Thm. B]. Thus the formula follows from Corollary 2.2. See [Kh4, Cor. 3.6]. □

For any quasi-smooth closed immersion of derived algebraic stacks $i : \mathcal{Z} \to \mathcal{X}$, say of virtual codimension n, we may form the derived blow-up $\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}$ as in [KhRy]. This fits in a commutative square

$$\begin{array}{ccc}
\mathbf{P}_{\mathcal{Z}}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) & \xrightarrow{i_D} & \mathrm{Bl}_{\mathcal{Z}} \mathcal{X} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
\mathcal{Z} & \xrightarrow{i} & \mathcal{X}
\end{array}$$

which is cartesian on classical truncations (but not homotopy cartesian unless $n = 1$). The upper closed immersion i_D is a virtual Cartier divisor, i.e., a quasi-smooth closed immersion of virtual codimension 1. The morphism $q : \mathbf{P}_{\mathcal{Z}}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) \to \mathcal{Z}$ is the projection of the projective bundle associated to the conormal sheaf $\mathcal{N}_{\mathcal{Z}/\mathcal{X}}$. The projection $p : \mathrm{Bl}_{\mathcal{Z}} \mathcal{X} \to \mathcal{X}$ is quasi-smooth of relative virtual dimension 0, proper, and schematic. The following is a generalization of [T3].

Theorem 2.10. *The maps* $p^* : \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \to \mathrm{K}^{\mathrm{B}}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X})$ *and*

$$\mathrm{K}^{\mathrm{B}}(\mathcal{Z}) \to \mathrm{K}^{\mathrm{B}}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}), \quad x \mapsto i_{D,*}\left(q^*(x) \cup [\mathcal{O}(-k)]\right),$$

induce an isomorphism

$$\mathrm{K}^{\mathrm{B}}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}) \simeq \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \oplus \bigoplus_{k=1}^{n-1} \mathrm{K}^{\mathrm{B}}(\mathcal{Z}).$$

Proof. The standard semi-orthogonal decomposition of Orlov–Thomason on $\mathbf{D}_{\mathrm{perf}}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X})$ extends to this setting by [Kh4, Thm. C]. Thus the formula follows from Corollary 2.2. See [Kh4, Cor. 4.4]. □

Corollary 2.11. *The induced square*

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{B}}(\mathcal{X}) & \xrightarrow{i^*} & \mathrm{K}^{\mathrm{B}}(\mathcal{Z}) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle q^*} \\ \mathrm{K}^{\mathrm{B}}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}) & \xrightarrow{i_D^*} & \mathrm{K}^{\mathrm{B}}(\mathbf{P}_{\mathcal{Z}}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})) \end{array}$$

is cartesian.

Proof. Combine Theorems 2.9 and 2.10. □

2.4. **Excising closed subsets.**

Theorem 2.12. *Let $\mathcal{X}$ and $\mathcal{X}'$ be perfect derived algebraic stacks. Let $f : \mathcal{X}' \to \mathcal{X}$ be a morphism and $Z \subseteq |\mathcal{X}|$ a cocompact closed subset. Suppose one of the following conditions holds:*

(i) Formal neighbourhood. *The morphism f is representable and is an isomorphism infinitely near Z. That is, the induced morphism $\mathcal{X}'^{\wedge}_{f^{-1}(Z)} \to \mathcal{X}^{\wedge}_{Z}$ on formal completions is invertible.*

(ii) Étale neighbourhood. *The morphism f is étale and restricts to an isomorphism $f^{-1}(Z)_{\mathrm{red}} \to Z_{\mathrm{red}}$.*

Then the induced square

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{B}}(\mathcal{X}) & \longrightarrow & \mathrm{K}^{\mathrm{B}}(\mathcal{X} \smallsetminus Z) \\ \downarrow{\scriptstyle f^*} & & \downarrow \\ \mathrm{K}^{\mathrm{B}}(\mathcal{X}') & \longrightarrow & \mathrm{K}^{\mathrm{B}}(\mathcal{X}' \smallsetminus f^{-1}(Z)) \end{array}$$

is cartesian.

Proof. Can be deduced using Theorem 2.4, see [BKRS, Thm. 4.1.1, Rem. 4.1.4]. □

Corollary 2.13. *Let $\mathcal{X}$ be a perfect derived algebraic stack. If $\mathcal{U} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{X}$ are open substacks such that $\mathcal{X} = \mathcal{U} \cup \mathcal{V}$, then the square*

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{B}}(\mathcal{X}) & \longrightarrow & \mathrm{K}^{\mathrm{B}}(\mathcal{U}) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{B}}(\mathcal{V}) & \longrightarrow & \mathrm{K}^{\mathrm{B}}(\mathcal{U} \cap \mathcal{V}) \end{array}$$

is cartesian. In particular, there is a long exact Mayer–Vietoris sequence

$$\cdots \xrightarrow{\partial} \mathrm{K}_i(\mathcal{X}) \to \mathrm{K}_i(\mathcal{U}) \oplus \mathrm{K}_i(\mathcal{V}) \to \mathrm{K}_i(\mathcal{U} \cap \mathcal{V}) \xrightarrow{\partial} \mathrm{K}_{i-1}(\mathcal{X}) \to \cdots.$$

Proof. Apply Theorem 2.12 with $\mathcal{Z} \to \mathcal{X}$ a closed immersion complementary to $\mathcal{U}$ and $\mathcal{X}' = \mathcal{V}$. □

Remark 2.14. By [Lu2, Thm. 7.3.5.2], Corollary 2.13 implies that K^{B} satisfies Zariski descent on the site of perfect derived algebraic stacks. Similarly Theorem 2.12(ii) implies that it satisfies descent for Nisnevich's completely decomposed étale topology. Indeed this topology is generated by families $\{f : \mathcal{X}' \to \mathcal{X}, j : \mathcal{X} \smallsetminus Z \hookrightarrow \mathcal{X}\}$, where f is an étale neighbourhood of a cocompact closed $Z \subseteq |\mathcal{X}|$ and j is the inclusion of the complement (see [HoKr, Prop. 2.9] for the case of stacks, [MV, §3, Prop. 1.4] for the case of noetherian finite-dimensional schemes). Hence the claim follows from [Kh1, Thm. 2.2.7].

2.5. **Excising open subsets.** The problem of excising open subsets is more subtle. We are given a morphism of derived algebraic stacks $f : \mathcal{X}' \to \mathcal{X}$ and quasi-compact open subsets $\mathcal{U} \subseteq \mathcal{X}$, $\mathcal{U}' \subseteq \mathcal{X}'$. Let $\mathcal{Z} \to \mathcal{X}$ and $\mathcal{Z}' \to \mathcal{X}'$ be closed immersions complementary to $\mathcal{U}$ and $\mathcal{U}'$, respectively, fitting in a commutative square

$$\begin{array}{ccc} \mathcal{Z}' & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow f \\ \mathcal{Z} & \longrightarrow & \mathcal{X} \end{array}$$

which is cartesian on underlying classical truncations.

For example, if f is affine and the square is also cocartesian, then it is called a *Milnor square*. Algebraic K-theory does not generally satisfy excision for Milnor squares (see [Sw]), but it does for Milnor squares that are *homotopy* cartesian:

Theorem 2.15. *Suppose given a Milnor square as above consisting of* perfect *derived algebraic stacks. If the square is moreover homotopy cartesian, then the induced square of spectra*

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{B}}(\mathcal{X}) & \longrightarrow & \mathrm{K}^{\mathrm{B}}(\mathcal{Z}) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{B}}(\mathcal{X}') & \longrightarrow & \mathrm{K}^{\mathrm{B}}(\mathcal{Z}') \end{array}$$

is cartesian.

Proof. See [LT, Thm. A] in the affine case and [BKRS] for the case of stacks. ☐

For a general Milnor square, we can still get an excision statement if we replace the closed substacks by their formal completions. In fact, we have the following refinement where we consider the natural pro-spectra $\widehat{\mathrm{K}^{\mathrm{B}}}(\mathcal{X}^{\wedge}_{\mathcal{Z}})$, $\widehat{\mathrm{K}^{\mathrm{B}}}(\mathcal{X}'^{\wedge}_{\mathcal{Z}'})$ associated to the formal completions (viewed as ind-stacks).

Theorem 2.16. *For any Milnor square as above consisting of noetherian derived algebraic stacks with bounded structure sheaves and satisfying the conditions of Example 1.34, the induced square of pro-spectra*

$$\begin{array}{ccc} \{\mathrm{K}^{\mathrm{B}}(\mathcal{X})\} & \longrightarrow & \widehat{\mathrm{K}^{\mathrm{B}}}(\mathcal{X}^{\wedge}_{\mathcal{Z}}) \\ \downarrow & & \downarrow \\ \{\mathrm{K}^{\mathrm{B}}(\mathcal{X}')\} & \longrightarrow & \widehat{\mathrm{K}^{\mathrm{B}}}(\mathcal{X}'^{\wedge}_{\mathcal{Z}'}) \end{array}$$

is cartesian.

Proof. See [BKRS]. In the affine case see also [LT, Thm. 2.32] where it is proven that the square is "weakly cartesian". ☐

Another interesting case is that of a proper representable morphism $f : \mathcal{X}' \to \mathcal{X}$ which restricts to an isomorphism $\mathcal{U}' \to \mathcal{U}$. Then the square is often called a *proper cdh square* or *abstract blow-up square*. K-theory typically doesn't satisfy excision with respect to such squares either, but again it holds if we pass to formal completions.

Theorem 2.17 (Proper excision)**.** *Let $f : \mathcal{X}' \to \mathcal{X}$ be a proper representable morphism of noetherian derived algebraic stacks with bounded structure sheaves. Assume that $\mathcal{X}$ and $\mathcal{X}'$ have affine diagonal and nice stabilizers, or that f is finite and $\mathcal{X}$ and $\mathcal{X}'$ satisfy the conditions of Example 1.34. Let $\mathcal{U} \subseteq \mathcal{X}$ and $\mathcal{U}' \subseteq \mathcal{X}'$ be quasi-compact open subsets such that f restricts to an isomorphism $\mathcal{U}' \to \mathcal{U}$. If Z and Z' are the respective set-theoretic complements of $\mathcal{U}$ and $\mathcal{U}'$, then the induced square of pro-spectra*

$$\begin{array}{ccc} \{\mathrm{K}^{\mathrm{B}}(\mathcal{X})\} & \longrightarrow & \widehat{\mathrm{K}^{\mathrm{B}}}(\mathcal{X}^{\wedge}_{Z}) \\ \downarrow & & \downarrow \\ \{\mathrm{K}^{\mathrm{B}}(\mathcal{X}')\} & \longrightarrow & \widehat{\mathrm{K}^{\mathrm{B}}}(\mathcal{X}'^{\wedge}_{Z'}) \end{array}$$

is cartesian.

Remark 2.18. For *regular* (nonsingular) schemes and stacks, we will see in Subsect. 4.2 that all these properties do hold without passing to formal completions.

Remark 2.19. If we restrict our attention to affine derived schemes and only look at "very low" K-groups, then Milnor excision and proper excision do hold; see [Mi, Thm 3.3], [B, Chap. XII, Thm. 8.3]. We will come back to this point in Subsect. 4.1.

2.6. **Bass fundamental sequence.** The Bass fundamental sequence is one of the few ways to understand negative K-groups. In fact, the construction of Bass–Thomason–Trobaugh K-theory K^{B} is rigged to make it hold for the negative K-groups.

Theorem 2.20. *Let $\mathcal{X}$ be a perfect stack. Then for every integer n there is an exact sequence of abelian groups*

$$0 \to \mathrm{K}_n(\mathcal{X}) \to \mathrm{K}_n(\mathcal{X}\times\mathbf{A}^1)\oplus\mathrm{K}_n(\mathcal{X}\times\mathbf{A}^1) \to \mathrm{K}_n(\mathcal{X}\times\mathbf{G}_m) \xrightarrow{\partial} \mathrm{K}_{n-1}(\mathcal{X}) \to 0,$$

functorial in $\mathcal{X}$ with respect to inverse images. Moreover, the map ∂ admits a natural $\mathrm{K}_*(\mathcal{X})$-module splitting.

Proof. By Corollary 2.13 there is a Mayer–Vietoris sequence for the standard affine cover of $\mathcal{X}\times\mathbf{P}^1$:

$$\cdots \to \mathrm{K}_{n+1}(\mathcal{X}\times\mathbf{G}_m) \xrightarrow{\partial} \mathrm{K}_n(\mathcal{X}\times\mathbf{P}^1) \to \mathrm{K}_n(\mathcal{X}\times\mathbf{A}^1)\oplus\mathrm{K}_n(\mathcal{X}\times\mathbf{A}^1) \to \mathrm{K}_n(\mathcal{X}\times\mathbf{G}_m) \xrightarrow{\partial} \cdots$$

Now apply the projective bundle formula (Theorem 2.9). The splitting comes from the Bott class $b\in\mathrm{K}(\mathcal{X}\times\mathbf{G}_m)[-1]$. See [TT, Thm. 7.5] or [CiKh, Thm. 4.3.1, Rem. 4.3.2] for details. □

2.7. **Nil-invariance.** The following result shows that, up to K_{n+1}, the K-groups of a derived commutative ring R are only sensitive to the first n homotopy groups of R.

Proposition 2.21. *Let R be a derived commutative ring. For every integer n, consider the canonical homomorphism $R\to\tau_{\leqslant n}(R)$ to the n-truncation (set $\tau_{\leqslant n}(R) \coloneqq \tau_{\leqslant 0}(R) = \pi_0(R)$ for $n<0$). Then the inverse image map*

$$\mathrm{K}^{\mathrm{B}}(\mathrm{Spec}(R)) \to \mathrm{K}^{\mathrm{B}}(\mathrm{Spec}(\tau_{\leqslant k}R))$$

induces an isomorphism $\mathrm{K}_i(\mathrm{Spec}(R)) \simeq \mathrm{K}_i(\mathrm{Spec}(\tau_{\leqslant k}R))$ for all $i\leqslant k+1$.

Proof. See [Kh2, Prop. 4.2] or [BKRS, Prop. 5.1.3]. □

In particular, an affine derived scheme X and its classical truncation X_{cl} have the same K-groups up to K_1.

Proposition 2.22 (Nil-invariance)**.** *Let $i : Z\to X$ be a surjective closed immersion of affine derived schemes. Then the map $i^* : \mathrm{K}^{\mathrm{B}}(X)\to\mathrm{K}^{\mathrm{B}}(Z)$ is 1-connective; i.e., it is surjective on K_1 and bijective on K_n for all $n\leqslant 0$.*

Proof. By Proposition 2.21 we may replace X and Z by their classical truncations. By the Bass fundamental sequence (Theorem 2.20) it suffices to show surjectivity on K_1 and bijectivity on K_0. This is [B, Chap. IX, Prop. 1.3]. □

Remark 2.23. Proposition 2.22 does not generalize to higher K-groups. Contrast with Theorem 4.8.

Remark 2.24. For non-affine derived schemes or algebraic spaces, Proposition 2.22 still holds below the Krull dimension d (i.e., any surjective closed immersion will induce a bijection on K_n for $n < -d$). For derived algebraic stacks (satisfying the conditions of Example 1.34), the same holds if we replace d by the Nisnevich cohomological dimension (see [BKRS, Cor. 5.1.4]).

3. G-theory

3.1. **Definition and basic properties.** For a noetherian derived algebraic stack $\mathcal{X}$, the *G-theory spectrum* $G(\mathcal{X})$ is defined as the algebraic K-theory of the stable ∞-category of coherent complexes on $\mathcal{X}$:

$$G(\mathcal{X}) = K(\mathbf{D}_{coh}(\mathcal{X})).$$

Proposition 3.1. *Let $\mathcal{X}$ be a noetherian derived algebraic stack. Then the canonical map*

$$G(\mathcal{X}) = K(\mathbf{D}_{coh}(\mathcal{X})) \to K^B(\mathbf{D}_{coh}(\mathcal{X}))$$

is invertible. In other words, the spectrum $K^B(\mathbf{D}_{coh}(\mathcal{X}))$ is connective.

Proof. Since $\mathbf{D}_{coh}(\mathcal{X})$ admits a bounded t-structure with noetherian heart, this follows from [AGH, Thm. 1.2]. □

Remark 3.2. Most of our discussion on G-theory can be extended from noetherian to qcqs stacks without modification. Proposition 3.1 seems to be an exception: I don't know whether $K^B(\mathbf{D}_{coh}(\mathcal{X}))$ will be connective for $\mathcal{X}$ non-noetherian (cf. [AGH, Conj. B]).

Proposition 3.3. *Let $\mathcal{X}$ be a noetherian derived algebraic stack. Denote by* $\mathrm{Coh}(\mathcal{X})$ *the abelian category of coherent sheaves on $\mathcal{X}$ and by* $K(\mathrm{Coh}(\mathcal{X}))$ *its K-theory in the sense of Quillen. Then there is a canonical isomorphism*

$$K(\mathrm{Coh}(\mathcal{X})) \to G(\mathcal{X}),$$

functorial in $\mathcal{X}$.

Proof. Since $\mathbf{D}_{coh}(\mathcal{X})$ admits a bounded t-structure with heart $\mathrm{Coh}(\mathcal{X})$, this follows from [Ba1, Thm. 6.1]. □

Corollary 3.4 (Derived invariance)**.** *Let $\mathcal{X}$ be a noetherian derived algebraic stack and write $i : \mathcal{X}_{cl} \to \mathcal{X}$ for the inclusion of the classical truncation. Then the direct image map*

$$i_* : G(\mathcal{X}_{cl}) \to G(\mathcal{X})$$

is invertible.

Proof. Follows immediately from Proposition 3.3, since $i_* : \mathbf{D}_{\mathrm{coh}}(\mathcal{X}_{\mathrm{cl}}) \to \mathbf{D}_{\mathrm{coh}}(\mathcal{X})$ is t-exact and induces an equivalence $\mathrm{Coh}(\mathcal{X}_{\mathrm{cl}}) \simeq \mathrm{Coh}(\mathcal{X})$ on hearts. □

Theorem 3.5 (Poincaré duality)**.** *Let $\mathcal{X}$ be a noetherian derived algebraic stack. If $\mathcal{X}$ has bounded structure sheaf, then the inclusion $\mathbf{D}_{\mathrm{perf}}(\mathcal{X}) \subseteq \mathbf{D}_{\mathrm{coh}}(\mathcal{X})$ induces canonical maps*

$$\mathrm{K}(\mathcal{X}) \to \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \to \mathrm{K}^{\mathrm{B}}(\mathbf{D}_{\mathrm{coh}}(\mathcal{X})) \simeq \mathrm{G}(\mathcal{X})$$

which are invertible if $\mathcal{X}$ is regular.

Proof. If $\mathcal{X}$ is regular, then the inclusion $\mathbf{D}_{\mathrm{perf}}(\mathcal{X}) \subseteq \mathbf{D}_{\mathrm{coh}}(\mathcal{X})$ is an equality, hence the second map is invertible. From this it follows that $\mathrm{K}^{\mathrm{B}}(\mathcal{X})$ is connective, so the first map is also invertible (since it is a connective cover). □

Remark 3.6. The map $\mathrm{K}(\mathcal{X}) \to \mathrm{G}(\mathcal{X})$, or its factorization through $\mathrm{K}^{\mathrm{B}}(\mathcal{X})$, is sometimes called the *Cartan map*.

3.2. **Operations.** Since tensoring with a perfect complex preserves coherence, we get a cap product

$$\cap : \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \otimes \mathrm{G}(\mathcal{X}) \to \mathrm{G}(\mathcal{X})$$

which is part of a $\mathrm{K}^{\mathrm{B}}(\mathcal{X})$-module structure on $\mathrm{G}(\mathcal{X})$.

For any morphism $f : \mathcal{X} \to \mathcal{Y}$ of finite Tor-amplitude, the inverse image functor $f^* : \mathbf{D}_{\mathrm{coh}}(\mathcal{Y}) \to \mathbf{D}_{\mathrm{coh}}(\mathcal{X})$ gives rise to a Gysin map

$$f^* : \mathrm{G}(\mathcal{Y}) \to \mathrm{G}(\mathcal{X}).$$

This is compatible with the K-theoretic inverse image under the maps in Theorem 3.5.

Let f be a proper morphism. If f is representable, or more generally of finite cohomological dimension, then there is a direct image map

$$f_* : \mathrm{G}(\mathcal{X}) \to \mathrm{G}(\mathcal{Y}).$$

This is compatible with the K-theoretic direct image (when f is of finite Tor-amplitude and almost of finite presentation) under the maps in Theorem 3.5. By Proposition 1.18 it is also $\mathrm{K}^{\mathrm{B}}(\mathcal{Y})$-linear (a homomorphism of $\mathrm{K}^{\mathrm{B}}(\mathcal{Y})$-module spectra). In other words:

Proposition 3.7 (Projection formula)**.** *If $f : \mathcal{X} \to \mathcal{Y}$ is proper and of finite cohomological dimension, then we have canonical identifications*

$$y \cap f_*(x) \simeq f_*(f^*(y) \cap x)$$

in $\mathrm{G}(\mathcal{Y})$, for all $x \in \mathrm{G}(\mathcal{X})$, $y \in \mathrm{K}^{\mathrm{B}}(\mathcal{Y})$.

Proposition 1.18 similarly implies:

Proposition 3.8 (Base change formula)**.** *Suppose given a homotopy cartesian square of derived algebraic stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{Y}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

If f is proper and of finite cohomological dimension and q is of finite Tor-amplitude, then we have a canonical homotopy

$$q^* f_* \simeq g_* p^*$$

of maps $\mathrm{G}(\mathcal{X}) \to \mathrm{G}(\mathcal{Y}')$.

For G-theory we have the following analogue of Theorem 2.8:

Theorem 3.9 (Localization)**.** *Let $i : \mathcal{Z} \to \mathcal{X}$ be a closed immersion of derived algebraic stacks with quasi-compact complementary open immersion $j : \mathcal{U} \to \mathcal{X}$. Then there is a canonical exact triangle of spectra*

$$\mathrm{G}(\mathcal{Z}) \xrightarrow{i_*} \mathrm{G}(\mathcal{X}) \xrightarrow{j^*} \mathrm{G}(\mathcal{U}).$$

Proof. By Proposition 3.3, this follows from Quillen's dévissage and localization theorems [Q2, Sect. 5, Thms. 4 and 5]. □

3.3. **Excision.**

Theorem 3.10 (Étale excision)**.** *Let $\mathcal{X}$ be a derived algebraic stack. Let $j : \mathcal{U} \to \mathcal{X}$ be a quasi-compact open immersion, and $Z \subseteq |\mathcal{X}|$ the set-theoretic complement. Then for any étale neighbourhood $f : \mathcal{X}' \to \mathcal{X}$ of Z (i.e., f is an étale morphism which restricts to an isomorphism $f^{-1}(Z)_{\mathrm{red}} \to Z_{\mathrm{red}}$), the induced square*

$$\begin{array}{ccc} \mathrm{G}(\mathcal{X}) & \longrightarrow & \mathrm{G}(\mathcal{U}) \\ \downarrow & & \downarrow \\ \mathrm{G}(\mathcal{X}') & \longrightarrow & \mathrm{G}(f^{-1}(\mathcal{U})) \end{array}$$

is cartesian.

Proof. By Theorem 3.9, the horizontal fibres are isomorphic. □

Theorem 3.11 (Proper co-excision)**.** *Let $\mathcal{X}$ be a derived algebraic stack. Let $i : \mathcal{Z} \to \mathcal{X}$ be a closed immersion with quasi-compact open complement $j : \mathcal{U} \to \mathcal{X}$. Then for any proper morphism $f : \mathcal{X}' \to \mathcal{X}$ which restricts to an isomorphism $f^{-1}(\mathcal{U})_{\mathrm{red}} \to \mathcal{U}_{\mathrm{red}}$, the induced square*

$$\begin{array}{ccc} \mathrm{G}(f^{-1}(\mathcal{Z})) & \longrightarrow & \mathrm{G}(\mathcal{X}') \\ \downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{G}(\mathcal{Z}) & \xrightarrow{i_*} & \mathrm{G}(\mathcal{X}) \end{array}$$

is cocartesian (and hence cartesian).

Proof. By Theorem 3.9, the horizontal cofibres are isomorphic. ☐

3.4. **Projective bundles and blow-ups.** We have the following G-theory analogues of Theorems 2.9 and 2.10.

Theorem 3.12. *Let $\mathcal{X}$ be a derived algebraic stack, $\mathcal{E}$ a locally free sheaf on $\mathcal{X}$ of rank $n+1$, $n \geqslant 0$, and $q : \mathbf{P}_{\mathcal{X}}(\mathcal{E}) \to X$ the associated projective bundle. Then the maps*

$$\mathrm{G}(X) \to \mathrm{G}(\mathbf{P}_{\mathcal{X}}(\mathcal{E})), \quad x \mapsto q^*(x) \cup [\mathcal{O}(-k)]$$

induce an isomorphism

$$\mathrm{G}(\mathbf{P}_{\mathcal{X}}(\mathcal{E})) \simeq \bigoplus_{k=0}^{n} \mathrm{G}(\mathcal{X}).$$

Proof. The standard semi-orthogonal decomposition on $\mathbf{D}_{\mathrm{qc}}(\mathbf{P}_{\mathcal{X}}(\mathcal{E}))$ (see [Kh4, Thm. 3.3]) restricts to coherent complexes, so the formula follows from Corollary 2.2. ☐

Theorem 3.13. *Let $\mathcal{X}$ be a derived algebraic stack. For any quasi-smooth closed immersion $i : \mathcal{Z} \to \mathcal{X}$ of virtual codimension n, consider the blow-up square*

$$\begin{array}{ccc} \mathbf{P}_{\mathcal{Z}}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) & \xrightarrow{i_D} & \mathrm{Bl}_{\mathcal{Z}} \mathcal{X} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \mathcal{Z} & \xrightarrow{i} & \mathcal{X} \end{array} \tag{3.14}$$

Then the maps $p^* : \mathrm{G}(\mathcal{X}) \to \mathrm{G}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X})$ and

$$\mathrm{G}(\mathcal{Z}) \to \mathrm{G}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}), \quad x \mapsto i_{D,*}\left(q^*(x) \cup [\mathcal{O}(-k)]\right),$$

induce an isomorphism

$$\mathrm{G}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}) \simeq \mathrm{G}(\mathcal{X}) \oplus \bigoplus_{k=1}^{n-1} \mathrm{G}(\mathcal{Z}).$$

Proof. The standard semi-orthogonal decomposition on $\mathbf{D}_{\mathrm{qc}}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X})$ (see [Kh4, Thm. 4.3]) restricts to coherent complexes. Thus the formula follows from Corollary 2.2. ☐

Corollary 3.15. *Let $\mathcal{X}$ be a derived algebraic stack. For any quasi-smooth closed immersion $i : \mathcal{Z} \to \mathcal{X}$, the blow-up square (3.14) induces a cartesian square*

$$\begin{array}{ccc} \mathrm{G}(\mathcal{X}) & \xrightarrow{i^*} & \mathrm{G}(\mathcal{Z}) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle q^*} \\ \mathrm{G}(\mathrm{Bl}_{\mathcal{Z}} \mathcal{X}) & \xrightarrow{i_D^*} & \mathrm{G}(\mathbf{P}_{\mathcal{Z}}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})). \end{array}$$

Proof. Combine Theorems 3.12 and 3.13. ☐

3.5. **Homotopy invariance.** Let $\mathcal{X}$ be a derived algebraic stack. Let $\mathcal{E}$ be a locally free sheaf of finite rank, $\phi : \mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathcal{X}}$ a surjective[5] $\mathcal{O}_{\mathcal{X}}$-module homomorphism, and $\mathcal{F}$ its fibre (which is locally free of finite rank). The associated *affine bundle*

$$\pi : \mathbf{V}_{\mathcal{X}}(\mathcal{E}, \phi) \to \mathcal{X}$$

is the moduli of splittings of ϕ. It fits in a closed/open pair

$$\mathbf{P}_{\mathcal{X}}(\mathcal{F}^\vee) \hookrightarrow \mathbf{P}_{\mathcal{X}}(\mathcal{E}^\vee) \hookleftarrow \mathbf{V}_{\mathcal{X}}(\mathcal{E}, \phi)$$

and it is a torsor under the vector bundle $\mathbf{V}_{\mathcal{X}}(\mathcal{F}^\vee) \to \mathcal{X}$. Every vector bundle torsor arises via this construction, see e.g. the proof of [T1, Thm. 4.1].

Theorem 3.16. *Let $\mathcal{X}$ be a derived algebraic stack. For any $(\mathcal{E}, \phi)$ as above, let $\pi : \mathbf{V}_{\mathcal{X}}(\mathcal{E}, \phi) \to \mathcal{X}$ denote the associated affine bundle. Then the inverse image map*

$$\pi^* : \mathrm{G}(\mathcal{X}) \to \mathrm{G}(\mathbf{V}_{\mathcal{X}}(\mathcal{E}, \phi))$$

is invertible.

Proof. Note that π is smooth of finite presentation, so the map exists. Combine the localization triangle (Theorem 3.9)

$$\mathrm{G}(\mathbf{P}_{\mathcal{X}}(\mathcal{F}^\vee)) \to \mathrm{G}(\mathbf{P}_{\mathcal{X}}(\mathcal{E}^\vee)) \to \mathrm{G}(\mathbf{V}_{\mathcal{X}}(\mathcal{E}, \phi))$$

with the projective bundle formula (Theorem 3.12). □

For any finite locally free sheaf $\mathcal{E}$, the affine bundle associated to the projection $\mathcal{E} \oplus \mathcal{O}_{\mathcal{X}} \twoheadrightarrow \mathcal{O}_{\mathcal{X}}$ is the vector bundle $\mathbf{V}_{\mathcal{X}}(\mathcal{E}^\vee)$ classifying sections $\mathcal{O}_{\mathcal{X}} \to \mathcal{E}$. Therefore we in particular get homotopy invariance for vector bundles.

Corollary 3.17. *Let $\mathcal{X}$ be a derived algebraic stack. For any finite locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, let $\pi : \mathbf{V}_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}$ denote the vector bundle parametrizing cosections of $\mathcal{E}$. Then the inverse image map*

$$\pi^* : \mathrm{G}(\mathcal{X}) \to \mathrm{G}(\mathbf{V}_{\mathcal{X}}(\mathcal{E}))$$

is invertible.

We can also generalize this further to vector bundle stacks.

Corollary 3.18. *Let $\mathcal{X}$ be a derived algebraic stack. Assume that $\mathcal{X}$ has affine stabilizers. For any perfect complex $\mathcal{E}$ on $\mathcal{X}$ of Tor-amplitude $\leqslant 0$, let $\pi : \mathbf{V}_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}$ denote the vector bundle stack parametrizing cosections of $\mathcal{E}$. Then the inverse image map*

$$\pi^* : \mathrm{G}(\mathcal{X}) \to \mathrm{G}(\mathbf{V}_{\mathcal{X}}(\mathcal{E}))$$

is invertible.

Proof. If $\mathcal{E}$ is of Tor-amplitude $[0,0]$, then it is finite locally free and we are in the case of Corollary 3.17.

If $\mathcal{E}$ is of Tor-amplitude $[-1,-1]$, then $E := \mathbf{V}_{\mathcal{X}}(\mathcal{E})$ is the classifying stack of the vector bundle $E[-1] := \mathbf{V}_{\mathcal{X}}(\mathcal{E}[1])$. In this case, the canonical morphism

[5] on π_0

$\sigma : X \twoheadrightarrow [X/E[-1]] \simeq E$ is the projection of a vector bundle (it is the quotient by $E[-1]$ of the projection $E[-1] \to X$). Thus σ^* is invertible by Corollary 3.17. Since $\sigma^* \pi^* = \mathrm{id}$, we deduce that π^* is invertible. Repeating the same argument inductively also gives the case of Tor-amplitude $[-k, -k]$, for any $k \geqslant 0$.

Finally let $\mathcal{E}$ be of Tor-amplitude $[-k, 0]$ for some $k > 0$. Since $\mathcal{X}$ has affine stabilizers, its classical truncation admits a stratification by stacks with the resolution property (see [HR1, Prop. 2.6(i)]). By Corollary 3.4 and Theorem 3.9, we may therefore assume that $\mathcal{X}$ admits the resolution property. In this case we can find a morphism $\phi : \mathcal{E}_0[-k] \to \mathcal{E}$ with $\mathcal{E}_0$ finite locally free, whose cofibre $\mathcal{E}'$ is of Tor-amplitude $[-k+1, 0]$. Write $E := \mathbf{V}_{\mathcal{X}}(\mathcal{E})$, $E' := \mathbf{V}_{\mathcal{X}}(\mathcal{E}')$, and $E_0[k] := \mathbf{V}_{\mathcal{X}}(\mathcal{E}_0[-k])$. The projection $\pi : E \to \mathcal{X}$ factors through the morphism $\phi : E \to E_0[k]$ and the projection $\pi_0 : E_0[k] \to \mathcal{X}$. Since ϕ is the projection of a vector bundle (it is the quotient by $E_0[k-1]$ of the projection $\pi' : E' \to \mathcal{X}$), ϕ^* is invertible by the Tor-amplitude $[0, 0]$ case. By the $[-k, -k]$ case above, π_0^* is also invertible. Hence $\pi^* \simeq \phi^* \pi_0^*$ is invertible. □

4. Homotopy invariance and singularities

4.1. **Weibel's conjecture.** Let X be a noetherian algebraic space. If X is regular, then combining Poincaré duality (Theorem 3.5) with Proposition 3.1 and Corollary 3.17 yields:

(i) The negative K-groups $\mathrm{K}_{-n}(X)$ vanish for all $n > 0$.

(ii) For any vector bundle $\pi : E \to X$, the map $\pi^* : \mathrm{K}^{\mathrm{B}}(X) \to \mathrm{K}^{\mathrm{B}}(E)$ is invertible.

When X is singular, both these properties fail. In fact, as we will see in the next subsection, all the "pathological" behaviour of algebraic K-theory for singular schemes, such as failure of nil-invariance and excision of open subsets (see Subsects. 2.7 and 2.5) can be traced back to the failure of homotopy invariance. In particular, these excision statements will turn out to be true for nonsingular spaces.

On the other hand, many of these pathologies disappear on "low enough" K-groups (see Proposition 2.22 and Remark 2.19). This is in some sense "explained" by the following statement, which says that even for singular spaces, homotopy invariance does hold on low enough K-groups.

Theorem 4.1. *Let X be a noetherian algebraic space. Suppose that X is pro-smooth over a noetherian algebraic space of Krull dimension d. Then we have:*

(i) *The negative K-groups $\mathrm{K}_{-n}(X)$ vanish for all $n > d$.*

(ii) *For any vector bundle $\pi : E \to X$, the maps $\pi^* : \mathrm{K}_{-n}(X) \to \mathrm{K}_{-n}(E)$ are invertible for all $n \geqslant d$.*

Example 4.2. If X is smooth (or pro-smooth) over a field, then it is regular and Theorem 4.1 just recovers the nonsingular situation discussed above. (Conversely, if we restrict our attention to affine schemes over a field, then by Popescu's desingularization theorem [Sp, Thm. 1.1], regularity implies pro-smoothness.)

Example 4.3. For X singular of dimension d, Theorem 4.1 was conjectured by Weibel [We1, Questions 2.9] in the affine case. Weibel's conjecture was proven by Kerz–Strunk–Tamme [KST, Thm. B] for schemes, and generalized to a large class of stacks in [BKRS, Thm. D] (for a suitable notion of dimension).

The slightly more general statement of Theorem 4.1 can be proven by using continuity to reduce to the smooth case and then arguing as in [BKRS, Rem. 5.3.3].

4.2. **Homotopy invariant K-theory.** We can introduce a variant of K-theory by *forcing* homotopy invariance for all (possibly singular) spaces. This construction goes back to Weibel [We2]. See [KrRa, HoKr, Ho3] for the case of stacks, [Kh4] for the case of derived algebraic spaces, and [KhRa] for derived stacks. We restrict to algebraic spaces for simplicity of exposition.

Let $\Delta^\bullet$ denote the standard cosimplicial affine scheme whose nth term is affine n-space $\mathbf{A}^n$ (see e.g. [MV, p. 45]). For any qcqs derived algebraic space X, let $\mathrm{KH}(X)$ denote the geometric realization of the simplicial diagram $\mathrm{K}^{\mathrm{B}}(\Delta^\bullet)$:

$$\mathrm{KH}(X) = \varinjlim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{K}^{\mathrm{B}}(X \times \Delta^n).$$

The canonical map of presheaves $\mathrm{K}^{\mathrm{B}} \to \mathrm{KH}$ exhibits KH as the $\mathbf{A}^1$-localization of K-theory in the sense of $\mathbf{A}^1$-homotopy theory (see [Ci]).

Remark 4.4. If X has bounded structure sheaf then by the universal property of $\mathbf{A}^1$-localization and homotopy invariance for G-theory (Corollary 3.17), we find that the canonical map $\mathrm{K}^{\mathrm{B}}(X) \to \mathrm{G}(X)$ (Theorem 3.5) factors through a map

$$\mathrm{KH}(X) \to \mathrm{G}(X),$$

functorial with respect to finite Tor-amplitude inverse images. If X is regular, then so are $X \times \Delta^n$ for all n, so this map is invertible. Hence all the maps

$$\mathrm{K}(X) \to \mathrm{K}^{\mathrm{B}}(X) \to \mathrm{KH}(X) \to \mathrm{G}(X)$$

are invertible when X is regular.

Remark 4.5. Since colimits of spectra are exact, KH inherits the following properties from K^{B}: localization sequence (Theorem 2.8), excision of closed subsets (Theorem 2.12), projective bundle formula (Theorem 2.9), and blow-up formula (Theorem 2.10).

Remark 4.6. For any $\mathbf{Z}$-linear stable ∞-category $\mathcal{A}$ we can define $\mathrm{KH}(\mathcal{A})$ as the spectrum

$$\mathrm{KH}(\mathcal{A}) = \varinjlim_{n\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{K}^{\mathrm{B}}(\mathcal{A} \otimes \mathbf{D}_{\mathrm{perf}}(\Delta^n))$$

so that $\mathrm{KH}(X) = \mathrm{KH}(\mathbf{D}_{\mathrm{perf}}(X))$. Therefore KH also has cup products and direct images along proper morphisms that are almost of finite presentation and of finite Tor-amplitude. These satisfy the projection and base change formulas (Propositions 2.6 and 2.7).

4.3. **Properties of** KH**.** As hinted at earlier, passing from K^{B} to KH has the remarkable side effect of forcing several other properties that were only true in K-theory either for regular (nonsingular) stacks, or only for very low K-groups for affines.

Proposition 4.7 (Homotopy invariance)**.** *For any qcqs derived algebraic space X and any vector bundle $\pi : E \to X$, the map $\pi^* : \mathrm{KH}(X) \to \mathrm{KH}(E)$ is invertible.*

Proof. Using Theorem 2.12(ii), we may assume the bundle is trivial. By induction we may assume it is of rank one. In that case this holds by construction. □

Theorem 4.8 (Nil-invariance)**.** *Let $i : Z \to X$ be a surjective closed immersion of qcqs derived algebraic spaces. Then the map $i^* : \mathrm{KH}(X) \to \mathrm{KH}(Z)$ is invertible.*

Proof. See [Kh4, Thm. 5.13]. □

In particular, taking Z to be the classical truncation of X shows that KH is insensitive to derived structures.

We also have the following versions of "excising open subsets".

Theorem 4.9 (Milnor excision)**.** *For any Milnor square of qcqs derived algebraic spaces as in Subsect. 2.5, the induced square*

$$\begin{array}{ccc} \mathrm{KH}(X) & \xrightarrow{i^*} & \mathrm{KH}(Z) \\ \downarrow{\scriptstyle f^*} & & \downarrow \\ \mathrm{KH}(X') & \longrightarrow & \mathrm{KH}(Z') \end{array}$$

is cartesian.

Proof. By Theorem 4.8 we may replace the square by its classical truncation. By Theorem 2.12(ii) and [Lu5, Thm. 3.4.2.1] we may assume it consists of affine schemes. In this case the claim is a classical result of Weibel (it is equivalent to [We2, Thm. 2.1]). □

Theorem 4.10 (Proper excision)**.** *Let $f : X' \to X$ be a proper morphism of qcqs derived algebraic spaces. Let $U \subseteq X$ and $U' \subseteq X'$ be quasi-compact open subsets such that f restricts to an isomorphism $U'_{\mathrm{red}} \to U_{\mathrm{red}}$, and let $Z \subseteq X$*

and $Z' \subseteq X'$ be their reduced complements. Then the square

$$\begin{array}{ccc} \mathrm{KH}(X) & \longrightarrow & \mathrm{KH}(Z) \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow \\ \mathrm{KH}(X') & \longrightarrow & \mathrm{KH}(Z') \end{array}$$

is cartesian.

Proof. For schemes this is the main result of [Ci] (where the noetherian assumption can be dropped by [Ho1, App. C]). For the case of algebraic spaces see [Kh4, Thm. D]. □

Finally we also get the following Mayer–Vietoris property for closed subspaces.

Corollary 4.11. *Let X be a qcqs derived algebraic space. For any closed subspaces Y and Z such that $X = Y \cup Z$, the square*

$$\begin{array}{ccc} \mathrm{KH}(X) & \longrightarrow & \mathrm{KH}(Y) \\ \big\downarrow & & \big\downarrow \\ \mathrm{KH}(Z) & \longrightarrow & \mathrm{KH}(Y \cap Z) \end{array}$$

*is cartesian. Here $Y \cap Z$ is the classical or derived scheme-theoretic intersection (*KH *is insensitive to the difference by Theorem 4.8).*

5. Rational étale K-theory and G-theory

Given a derived algebraic stack $\mathcal{X}$, let $\mathrm{K}^{\mathrm{B}}(\mathcal{X})_{\mathbf{Q}}$ and $\mathrm{G}(\mathcal{X})_{\mathbf{Q}}$ denote the rationalized K- and G-theory spectra, respectively. In this section we introduce étale-localized versions $\mathrm{K}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}}$ and $\mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}}$.

5.1. Descent on algebraic spaces.

Theorem 5.1 (Étale and flat descent)**.**

(i) *On the ∞-category of qcqs derived algebraic spaces, the presheaf $X \mapsto \mathrm{K}^{\mathrm{B}}(X)_{\mathbf{Q}}$ satisfies descent for the étale and finite flat topologies.*

(ii) *On the ∞-category of qcqs derived algebraic spaces, the presheaf $X \mapsto \mathrm{KH}(X)_{\mathbf{Q}}$ satisfies descent for the fppf topology.*

(iii) *On the ∞-category of noetherian derived algebraic spaces and finite Tor-amplitude morphisms, the presheaf $X \mapsto \mathrm{G}(X)_{\mathbf{Q}}$ satisfies descent for the fppf topology.*

Remark 5.2. Recall that any smooth surjection generates a covering for the étale topology. Thus in particular, Theorem 5.1 implies that for any smooth surjection $p : U \to X$, the inverse image map p^* induces an isomorphism between $\mathrm{K}^{\mathrm{B}}(X)_{\mathbf{Q}}$ and the limit of the cosimplicial diagram

$$\mathrm{K}^{\mathrm{B}}(U)_{\mathbf{Q}} \rightrightarrows \mathrm{K}^{\mathrm{B}}(U \underset{X}{\times} U)_{\mathbf{Q}} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathrm{K}^{\mathrm{B}}(U \underset{X}{\times} U \underset{X}{\times} U)_{\mathbf{Q}} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

and similarly for KH and G-theory.

Proof of Theorem 5.1. We first show that each presheaf satisfies finite flat descent. For this we may restrict to the small étale site of a fixed derived algebraic space S. Combining Theorem 2.12(ii) or its KH/G-theory analogue (Remark 4.5 or Corollary 3.10) with [Lu5, Thm. 3.4.2.1], we may assume that S is affine. Then the claim follows by applying [CMNN, Prop. 5.4] (taking $n = 0$, cf. Appendix A in *op. cit.*), which is a statement about localizing invariants of $\mathbf{D}_{\mathrm{perf}}(S)$-linear ∞-categories (see Def. A.1 of *op. cit.*) that take values in rational spectra. For the K-theory statement, take the localizing invariant that sends a $\mathbf{D}_{\mathrm{perf}}(S)$-linear ∞-category $\mathcal{A}$ to the spectrum $\mathrm{K}^{\mathrm{B}}(\mathcal{A})_{\mathbf{Q}}$. For KH, take the localizing invariant

$$\mathcal{A} \mapsto \varinjlim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{K}^{\mathrm{B}}(\mathcal{A} \otimes_{\mathbf{D}_{\mathrm{perf}}(S)} \mathbf{D}_{\mathrm{perf}}(S \times \Delta^n))_{\mathbf{Q}}.$$

For G-theory, take the localizing invariant K' that sends a $\mathbf{D}_{\mathrm{perf}}(S)$-linear ∞-category $\mathcal{A}$ to the spectrum $\mathrm{K}(\mathcal{A} \otimes_{\mathbf{D}_{\mathrm{perf}}(S)} \mathrm{Coh}(S))_{\mathbf{Q}}$. Indeed, for every X étale over S, there is a canonical isomorphism

$$\begin{aligned} \mathrm{K}'(\mathbf{D}_{\mathrm{perf}}(X)) &= \mathrm{K}(\mathbf{D}_{\mathrm{perf}}(X) \otimes_{\mathbf{D}_{\mathrm{perf}}(S)} \mathrm{Coh}(S))_{\mathbf{Q}} \\ &\simeq \mathrm{K}(\mathrm{Coh}(X))_{\mathbf{Q}} = \mathrm{G}(X)_{\mathbf{Q}} \end{aligned}$$

by [GR, Chap. 4, Rem. 3.3.3].

By [Lu5, Thm. B.6.4.1], combining finite flat descent with Nisnevich descent (Remark 2.14) yields étale descent for each presheaf.

For $\mathrm{KH}_{\mathbf{Q}}$ and $\mathrm{G}_{\mathbf{Q}}$, we may use derived invariance (Theorem 4.8 and Corollary 3.4) to restrict to the site of classical algebraic spaces. Since the fppf topology on this site is generated by Nisnevich coverings and finite flat coverings (see e.g. [SP, Tag 05WN]), the claim follows. ☐

Theorem 5.3 (Proper descent)**.** *On the ∞-category of qcqs derived algebraic spaces, the presheaf $X \mapsto \mathrm{KH}(X)_{\mathbf{Q}}$ satisfies descent along proper schematic surjections of finite presentation. That is, for any proper schematic surjection of finite presentation $p : Y \to X$, p^* induces an isomorphism between $\mathrm{KH}(X)_{\mathbf{Q}}$ and the limit of the cosimplicial diagram*

$$\mathrm{KH}(Y)_{\mathbf{Q}} \rightrightarrows \mathrm{KH}(Y \overset{\mathbf{R}}{\underset{X}{\times}} Y)_{\mathbf{Q}} \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \mathrm{KH}(Y \overset{\mathbf{R}}{\underset{X}{\times}} Y \overset{\mathbf{R}}{\underset{X}{\times}} Y)_{\mathbf{Q}} \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} \cdots.$$

Proof. By derived invariance (Theorem 4.8) and descent (Corollary 5.1), we may restrict to the site of classical schemes. Furthermore we may fix a scheme S and restrict to the site of S-schemes of finite presentation. Then we may apply the criterion of [BS, Thm. 2.9], which reduces the claim to fppf descent (Theorem 5.1) and proper cdh descent (Theorem 4.10). ☐

Remark 5.4. Presumably, Theorem 5.3 extends to proper *representable* surjections of finite presentation, but I did not check whether [BS, Thm. 2.9] extends to algebraic spaces.

Remark 5.5. In fact, the previous proof shows that $\mathrm{KH}_{\mathbf{Q}}$ satisfies descent for Voevodsky's h-topology (or rather, Rydh's non-noetherian generalization of it [Ry1, Sect. 8]) on the site of (derived) schemes.

5.2. **Étale K-theory and G-theory.** Theorem 5.1 does not extend to stacks. In fact, on derived algebraic stacks, rational G-theory only satisfies descent with respect to étale maps that are *isovariant* or "stabilizer-preserving" (see [Jo, Sect. 3]).

Nevertheless, Theorem 5.1 implies that $\mathrm{K}^{\mathrm{B}}(-)_{\mathbf{Q}}$, $\mathrm{KH}(-)_{\mathbf{Q}}$, and $\mathrm{G}(-)_{\mathbf{Q}}$ extend uniquely to étale sheaves on derived algebraic stacks, which we denote by

$$\mathrm{K}^{\text{ét}}(-)_{\mathbf{Q}},\ \mathrm{KH}^{\text{ét}}(-)_{\mathbf{Q}},\ \text{and } \mathrm{G}^{\text{ét}}(-)_{\mathbf{Q}}.$$

These could equivalently described as the étale localizations of the presheaves $\mathrm{K}^{\mathrm{B}}(-)_{\mathbf{Q}}$, $\mathrm{KH}(-)_{\mathbf{Q}}$, and $\mathrm{G}(-)_{\mathbf{Q}}$, respectively. Note that, as before, we only ever discuss descent with respect to Čech covers and not hypercovers.

Thus by construction, for any derived algebraic stack $\mathcal{X}$ and any smooth atlas $p : X \to \mathcal{X}$, $\mathrm{K}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}}$ is isomorphic to the homotopy limit of the cosimplicial diagram

$$\mathrm{K}^{\text{ét}}(X)_{\mathbf{Q}} \rightrightarrows \mathrm{K}^{\mathrm{B}}(X \underset{\mathcal{X}}{\times} X)_{\mathbf{Q}} \Rrightarrow \mathrm{K}^{\mathrm{B}}(X \underset{\mathcal{X}}{\times} X \underset{\mathcal{X}}{\times} X)_{\mathbf{Q}} \cdots$$

and similarly for KH and G-theory. For example, if $\mathcal{X} = [X/G]$ is the quotient of a derived algebraic space X by the action of an fppf group space G, then $\mathrm{K}^{\text{ét}}([X/G])_{\mathbf{Q}}$ is the Borel-type G-equivariant K-theory of X:

$$\mathrm{K}^{\text{ét}}(X)_{\mathbf{Q}} \rightrightarrows \mathrm{K}^{\mathrm{B}}(G \times X)_{\mathbf{Q}} \Rrightarrow \mathrm{K}^{\mathrm{B}}(G \times G \times X)_{\mathbf{Q}} \cdots$$

Similarly for $\mathrm{KH}^{\text{ét}}(-)_{\mathbf{Q}}$ and $\mathrm{G}^{\text{ét}}(-)_{\mathbf{Q}}$.

All properties of K-theory, G-theory, and KH-theory involving *inverse* image functoriality easily extend to the étale-local variants just by descent from the case of algebraic spaces. This applies for example to the projective bundle formula, excision for quasi-smooth blow-ups, and homotopy invariance. In particular, we can drop the extra hypothesis in the latter statement (compare with Corollary 3.18).

Corollary 5.6 (Homotopy invariance)**.** *Let $\mathcal{X}$ be a derived algebraic stack. Let $\mathcal{E}$ be a perfect complex on $\mathcal{X}$ of Tor-amplitude $\leqslant 0$, and $\pi : \mathbf{V}_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}$ the associated vector bundle stack. Then inverse image along π induces isomorphisms of spectra*

$$\pi^* : \mathrm{KH}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} \to \mathrm{KH}^{\text{ét}}(\mathbf{V}_{\mathcal{X}}(\mathcal{E}))_{\mathbf{Q}},$$
$$\pi^* : \mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} \to \mathrm{G}^{\text{ét}}(\mathbf{V}_{\mathcal{X}}(\mathcal{E}))_{\mathbf{Q}}.$$

Proof. One argues as in Corollary 3.18, except that the resolution property can be guaranteed simply by localizing on $\mathcal{X}$ (in the étale topology). □

5.3. **Direct images.** Since $\mathrm{G}^{\text{ét}}(-)_{\mathbf{Q}}$ is defined using descent along inverse images, the direct image functoriality is somewhat subtle.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism between noetherian derived algebraic stacks. Let's see if we can try to construct a direct image map

$$f_* : \mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} \to \mathrm{G}^{\text{ét}}(\mathcal{Y})_{\mathbf{Q}}.$$

Since properness is local on the target, descent allows us to reduce to the case where $\mathcal{Y} = Y$ is representable by a derived algebraic space. If f happens to be representable, then it follows that $\mathcal{X} = X$ is also an algebraic space. In that case, we simply take the map

$$\mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} \simeq \mathrm{G}(X)_{\mathbf{Q}} \xrightarrow{f_*} \mathrm{G}(Y)_{\mathbf{Q}} \simeq \mathrm{G}^{\text{ét}}(\mathcal{Y})_{\mathbf{Q}}.$$

In the non-representable case, we can still construct a proper direct image assuming that $\mathcal{X}$ is quasi-compact and has quasi-finite and separated diagonal (e.g. if $\mathcal{X}$ is a quasi-compact derived Deligne–Mumford stack with separated diagonal). In that case, there exists by [Ry3, Thm. B] a scheme Z and a finite surjection of finite presentation $g : Z \to \mathcal{X}$ which is generically flat. This morphism is representable, hence induces a map

$$g_* : \mathrm{G}^{\text{ét}}(Z)_{\mathbf{Q}} \to \mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}}$$

which we claim is invertible. By noetherian induction and the localization triangle (Theorem 3.9), we may assume that g is flat. Since g is affine, we may assume by étale descent that $\mathcal{X}$ is affine, in which it follows from the base change and projection formulas (Propositions 3.7 and 3.8) that g^*g_* and g_*g^* are both multiplication by the degree of g.

Now since g and $f \circ g$ are representable, it follows that there is an essentially unique map $f_* : \mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} \to \mathrm{G}^{\text{ét}}(\mathcal{Y})_{\mathbf{Q}}$ making the following triangle commute:

$$\begin{array}{ccc} \mathrm{G}^{\text{ét}}(Z)_{\mathbf{Q}} & \xrightarrow{g_*} & \mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} \\ & \searrow^{(f\circ g)_*} & \downarrow f_* \\ & & \mathrm{G}^{\text{ét}}(\mathcal{Y})_{\mathbf{Q}} \end{array}$$

Given another choice of Z and g, it is an easy exercise to show that the two morphisms f_* are homotopic. Similarly, if $f : \mathcal{X} \to \mathcal{Y}$ and $f' : \mathcal{Y} \to \mathcal{Z}$ are two proper morphisms of finite cohomological dimension (where $\mathcal{X}$ and $\mathcal{Y}$ both have quasi-finite separated diagonal), then the two maps $f'_* \circ f_*$ and $(f' \circ f)_*$ are homotopic. (We do not address coherence of these homotopies here.) These direct images also satisfy the base change and projection formulas (Propositions 3.7 and 3.8).

Warning 5.7. The direct images thus constructed are *not* compatible with those in "genuine" G-theory induced by proper push-forward of coherent complexes. That is, let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of finite cohomological dimension and assume $\mathcal{X}$ has quasi-finite separated diagonal. Then the square

$$\begin{array}{ccc} \mathrm{G}(\mathcal{X})_{\mathbf{Q}} & \xrightarrow{f_*} & \mathrm{G}(\mathcal{Y})_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ \mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} & \xrightarrow{f_*} & \mathrm{G}^{\text{ét}}(\mathcal{Y})_{\mathbf{Q}} \end{array}$$

typically does not commute (unless f is representable).

Consider the following standard example. Let k be a field, G a finite group, and $f : BG \to \mathrm{Spec}(k)$ the structural map of the classifying stack. If the order of G is prime to the characteristic of k, then f is of finite cohomological dimension. Let $g : \mathrm{Spec}(k) \twoheadrightarrow BG$ denote the quotient morphism, which is finite étale and hence induces an isomorphism

$$\mathbf{Q} \simeq \mathrm{G}^{\text{ét}}(k)_{\mathbf{Q}} \xrightarrow{g_*} \mathrm{G}^{\text{ét}}(BG)_{\mathbf{Q}}.$$

Thus by construction, $f_* : \mathrm{G}^{\text{ét}}(BG)_{\mathbf{Q}} \to \mathrm{G}^{\text{ét}}(k)_{\mathbf{Q}}$ is homotopic to a scalar multiple of g^*. Since the étale localization map $\mathrm{G}(BG)_{\mathbf{Q}} \to \mathrm{G}^{\text{ét}}(BG)_{\mathbf{Q}}$ commutes with g^* by definition, we see that the square commutes if and only if f_* and g^* are also homotopic at the level of $\mathrm{G}(-)_{\mathbf{Q}}$ (up to multiplication by a scalar). But under the identifications of $\mathrm{G}(BG)_{\mathbf{Q}}$ with the K-theory spectrum of G-equivariant k-vector spaces and $\mathrm{G}(k)_{\mathbf{Q}}$ with the K-theory spectrum of k-vector spaces, the map g^* sends the class of a G-representation V to the underlying vector space V while f_* sends it to the class of the invariant subspace V^G. No scalar multiplication identifies these (e.g. take V to be a nontrivial representation of dimension one so that $V^G = 0$).

6. Virtual fundamental classes

6.1. **Fundamental classes in G-theory.** Let $\mathcal{X}$ be a regular (nonsingular) algebraic stack. K-theoretic Poincaré duality (Theorem 3.5) can be reformulated as the assertion that there is a canonical class $[\mathcal{X}]_{\mathrm{G}} \in \mathrm{G}(\mathcal{X})$ such that cap product gives an isomorphism

$$[\mathcal{X}]_{\mathrm{G}} \cap (-) : \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \to \mathrm{G}(\mathcal{X}).$$

Of course, $[\mathcal{X}]_{\mathrm{G}}$ is just the class of the structure sheaf $[\mathcal{O}_{\mathcal{X}}]$.

This construction can be extended to (derived) stacks that are possibly *singular* but still quasi-smooth. First note that by Lemma 1.13 we have:

Corollary 6.1. *Let $\mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of derived algebraic stacks. If $\mathcal{Y}$ has bounded structure sheaf, then so does $\mathcal{X}$.*

Construction 6.2 (G-theoretic fundamental class)**.** Let $\mathcal{X}$ be a quasi-smooth derived algebraic stack over a regular noetherian algebraic stack $\mathcal{S}$. Then $\mathcal{X}$ is noetherian and its structure sheaf $\mathcal{O}_{\mathcal{X}}$ is coherent by Corollary 6.1 and Remark 1.8. In particular, there is a *G-theoretic fundamental class*

$$[\mathcal{X}]_{\mathrm{G}} \coloneqq [\mathcal{O}_{\mathcal{X}}] \in \mathrm{G}(\mathcal{X}).$$

Remark 6.3. Under the canonical isomorphism $\mathrm{G}(\mathcal{X}) \simeq \mathrm{G}(\mathcal{X}_{\mathrm{cl}})$ (Corollary 3.4), the fundamental class $[\mathcal{X}]_{\mathrm{G}}$ corresponds uniquely to a class

$$[\mathcal{X}]_{\mathrm{G}}^{\mathrm{vir}} \in \mathrm{G}(\mathcal{X}_{\mathrm{cl}}),$$

which we call the G-theoretic *virtual* fundamental class. Explicitly, it can be described as the alternating sum

$$[\mathcal{X}]_{\mathrm{G}}^{\mathrm{vir}} = \sum_{i \leqslant 0} (-1)^i [\mathrm{H}^i(\mathcal{O}_{\mathcal{X}})],$$

which is finite by assumption, where $\mathrm{H}^i(\mathcal{O}_{\mathcal{X}}) = \pi_{-i}(\mathcal{O}_{\mathcal{X}})$ are viewed as coherent sheaves on $\mathcal{X}_{\mathrm{cl}}$ via the equivalence $\mathrm{Coh}(\mathcal{X}_{\mathrm{cl}}) \simeq \mathbf{D}_{\mathrm{coh}}(\mathcal{X})^{\heartsuit}$.

This virtual G-theory class agrees with the *virtual structure sheaf* studied in detail by Y.-P. Lee [Lee]. Construction 6.2 goes back to [Ko, 1.4.2] and was written down more precisely by Toën in [To3, §4.4, para. 3]. Note that the various properties of virtual structure sheaves [Lee, Subsect. 2.4] are completely transparent from Construction 6.2.

Remark 6.4. Since the structural morphism $f : \mathcal{X} \to \mathcal{S}$ is quasi-smooth and hence of finite Tor-amplitude (Lemma 1.13), there is an inverse image map

$$f^* : \mathrm{G}(\mathcal{S}) \to \mathrm{G}(\mathcal{X}).$$

and the fundamental class $[\mathcal{X}]_{\mathrm{G}}$ is of course the same as the image of the fundamental class $[\mathcal{O}_{\mathcal{S}}] \in \mathrm{G}(\mathcal{S})$ defined above.

Remark 6.5. More generally, for any quasi-smooth morphism of noetherian derived algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ there is again by Lemma 1.13 a map

$$f^* : \mathrm{G}(\mathcal{Y}) \to \mathrm{G}(\mathcal{X}).$$

which corresponds under the isomorphisms of Corollary 3.4 to a map

$$\mathrm{G}(\mathcal{Y}_{\mathrm{cl}}) \to \mathrm{G}(\mathcal{X}_{\mathrm{cl}})$$

which can be thought of as a "virtual pullback" or virtual Gysin map along the morphism of classical stacks $f_{\mathrm{cl}} : \mathcal{X}_{\mathrm{cl}} \to \mathcal{Y}_{\mathrm{cl}}$. It agrees with the virtual pullbacks of [Qu] (as will follow from the next subsection).

6.2. **Fundamental classes via deformation to the 1-shifted tangent bundle.** In this subsection we give an alternative construction of the maps $f^* : \mathrm{G}(\mathcal{Y}) \to \mathrm{G}(\mathcal{X})$ in the case when $f : \mathcal{X} \to \mathcal{Y}$ is quasi-smooth. This description was implicitly used in the proof of the Grothendieck–Riemann–Roch formula proven in [Kh5] (see Subsect. 6.3 below).

Recall that for any quasi-smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ there is a canonical "deformation to the 1-shifted tangent bundle" which sits in a commutative diagram

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & \mathcal{X}\times\mathbf{A}^1 & \longleftarrow & \mathcal{X}\times\mathbf{G}_m \\ \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow{\scriptstyle f\times\mathrm{id}} \\ T_{\mathcal{X}/\mathcal{Y}}[1] & \xrightarrow{\hat{i}} & D_{\mathcal{X}/\mathcal{Y}} & \xleftarrow{\hat{j}} & \mathcal{Y}\times\mathbf{G}_m \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} & & \| \\ \mathcal{Y} & \xrightarrow{i} & \mathcal{Y}\times\mathbf{A}^1 & \xleftarrow{j} & \mathcal{Y}\times\mathbf{G}_m \end{array} \tag{6.6}$$

This is a family of quasi-smooth morphisms $\mathcal{X}\times\mathbf{A}^1\to D_{\mathcal{X}/\mathcal{Y}}$ parametrized by the affine line whose generic fibre is the morphism $f:\mathcal{X}\to\mathcal{Y}$ and whose special fibre is the zero section $0:\mathcal{X}\to T_{\mathcal{X}/\mathcal{Y}}[1]$ of the 1-shifted tangent bundle. The latter is by definition the vector bundle stack $\mathbf{V}(\mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1])$ associated to the (-1)-shifted cotangent complex. When f is a closed immersion, $T_{\mathcal{X}/\mathcal{Y}}[1]$ is just the normal bundle and $D_{\mathcal{X}/\mathcal{Y}}$ is the usual deformation to the normal bundle. Note that the left-hand arrow v is the composite of the projection $\pi:T_{\mathcal{X}/\mathcal{Y}}[1]\to\mathcal{X}$ with $f:\mathcal{X}\to\mathcal{Y}$. See [Kh5, Thm. 1.3][6].

Using this deformation we can construct a specialization map in G-theory in parallel to the case of Borel–Moore homology in [Kh5]. Consider the localization triangle (Theorem 3.9) associated to the lower row of (6.6):

$$\mathrm{G}(\mathcal{X}\times\mathbf{G}_m)[-1]\xrightarrow{\partial}\mathrm{G}(\mathcal{X})\xrightarrow{i_*}\mathrm{G}(\mathcal{X}\times\mathbf{A}^1)\xrightarrow{j^*}\mathrm{G}(\mathcal{X}\times\mathbf{G}_m).$$

The boundary map ∂ admits a section

$$\gamma_b:\mathrm{G}(\mathcal{X})\xrightarrow{q^*}\mathrm{G}(\mathcal{X}\times\mathbf{G}_m)\xrightarrow{\cap b}\mathrm{G}(\mathcal{X}\times\mathbf{G}_m)[-1]$$

where $q:\mathcal{X}\times\mathbf{G}_m\to\mathcal{X}$ is the projection and $b\in\mathrm{K}(\mathcal{X}\times\mathbf{G}_m)[-1]$ is the Bott class (inverse image of $b\in\mathrm{K}(\mathbf{Z}[T^{\pm1}])[-1]$).

Consider also the localization triangle associated to the middle row of (6.6). These fit into a commutative diagram

$$\begin{array}{ccccccc} \mathrm{G}(\mathcal{Y}\times\mathbf{G}_m)[-1] & \xrightarrow{\partial} & \mathrm{G}(\mathcal{Y}) & \xrightarrow{i_*} & \mathrm{G}(\mathcal{Y}\times\mathbf{A}^1) & \xrightarrow{j^*} & \mathrm{G}(\mathcal{Y}\times\mathbf{G}_m) \\ \| & & \downarrow{\scriptstyle v^*} & & \downarrow{\scriptstyle u^*} & & \| \\ \mathrm{G}(\mathcal{Y}\times\mathbf{G}_m)[-1] & \xrightarrow{\hat{\partial}} & \mathrm{G}(T_{\mathcal{X}/\mathcal{Y}}[1]) & \xrightarrow{\hat{i}_*} & \mathrm{G}(D_{\mathcal{X}/\mathcal{Y}}) & \xrightarrow{\hat{j}^*} & \mathrm{G}(\mathcal{Y}\times\mathbf{G}_m) \end{array}$$

where ∂ and $\hat{\partial}$ are the respective boundary maps. The right-hand square commutes by functoriality of inverse image, the middle square commutes by the base change formula, and the left-hand square commutes as a consequence.

Definition 6.7. Let $f:\mathcal{X}\to\mathcal{Y}$ be a quasi-smooth morphism of noetherian derived algebraic stacks. The *specialization map* is the composite

$$\mathrm{sp}_{\mathcal{X}/\mathcal{Y}}:\mathrm{G}(\mathcal{Y})\xrightarrow{\gamma_b}\mathrm{G}(\mathcal{Y}\times\mathbf{G}_m)[-1]\xrightarrow{\hat{\partial}}\mathrm{G}(T_{\mathcal{X}/\mathcal{Y}}[1]), \tag{6.8}$$

where the notation is as above.

[6] In the draft currently available on arXiv, $T_{\mathcal{X}/\mathcal{Y}}[1]$ is called the normal bundle stack and is denoted $N_{\mathcal{X}/\mathcal{Y}}$. That was before I learned that $T_{\mathcal{X}/\mathcal{Y}}[1]$, or rather its dual $T^*_{\mathcal{X}/\mathcal{Y}}[-1]$, the (-1)-shifted cotangent bundle, is well-known to shifted symplectic geometers.

Since γ_b is a section of ∂, commutativity of the diagram above immediately yields a canonical identification

$$\mathrm{sp}_{\mathcal{X}/\mathcal{Y}} = v^* = \pi^* \circ f^*,$$

where $\pi : T_{\mathcal{X}/\mathcal{Y}}[1] \to \mathcal{X}$ is the projection. As long as we assume $\mathcal{X}$ has affine stabilizers (e.g. is a derived Deligne–Mumford stack), we know that the map π^* is invertible by homotopy invariance (Corollary 5.6). Thus we find that the inverse image map f^* can be described in terms of the specialization map:

Proposition 6.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth morphism of noetherian derived algebraic stacks with affine stabilizers. Then $f^* : \mathrm{G}(\mathcal{Y}) \to \mathrm{G}(\mathcal{X})$ is canonically homotopic to the composite*

$$\mathrm{G}(\mathcal{Y}) \xrightarrow{\mathrm{sp}_{\mathcal{X}/\mathcal{Y}}} \mathrm{G}(T_{\mathcal{X}/\mathcal{Y}}[1]) \xrightarrow{(\pi^*)^{-1}} \mathrm{G}(\mathcal{X}).$$

Remark 6.10. Throughout the above discussion we can replace G-theory by rational étale G-theory. In that case we do not need to assume affineness of stabilizers, since homotopy invariance for vector bundle stacks always holds in $\mathrm{G}^{\text{ét}}$ (Corollary 5.6).

6.3. **Grothendieck–Riemann–Roch formulas.** Let k be a field and $\mathcal{X}$ a quasi-smooth derived algebraic stack over k. Via stable motivic homotopy theory (see [Ri], [Kh5]) one can construct canonical isomorphisms

$$\tau_{\mathcal{X}} : \mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} \to \bigoplus_{n \in \mathbf{Z}} \mathbf{R}\Gamma^{\mathrm{BM}}(\mathcal{X}, \mathbf{Q}(n))[-2n],$$

where on the right-hand side are spectra whose homotopy groups are the motivic Borel–Moore homology groups. Combining with the étale localization map $\mathrm{G}(\mathcal{X})_{\mathbf{Q}} \to \mathrm{G}^{\text{ét}}(\mathcal{X})_{\mathbf{Q}}$, we get a map

$$\tau_{\mathcal{X}} : \mathrm{G}(\mathcal{X})_{\mathbf{Q}} \to \bigoplus_{n \in \mathbf{Z}} \mathbf{R}\Gamma^{\mathrm{BM}}(\mathcal{X}, \mathbf{Q}(n))[-2n]$$

which on π_0 gives a map

$$\tau_{\mathcal{X}} : \mathrm{G}_0(\mathcal{X})_{\mathbf{Q}} \to \mathrm{H}^{\mathrm{BM}}_{2n}(\mathcal{X}, \mathbf{Q}(n)) \simeq \mathrm{A}_*(\mathcal{X}_{\mathrm{cl}})_{\mathbf{Q}}$$

where, if $\mathcal{X}$ has affine stabilizers, the target is now identified with the Chow groups of the classical truncation $\mathcal{X}_{\mathrm{cl}}$. This is a derived and stacky extension of the Baum–Fulton–MacPherson transformation [BFM] and we have the following analogue of the main result of *op. cit.*

Theorem 6.11.

(i) *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of finite cohomological dimension between quasi-smooth derived algebraic stacks over k, and assume that $\mathcal{X}$ has quasi-finite separated diagonal (e.g., $\mathcal{X}$ is Deligne–Mumford with separated diagonal). Then there is a commutative square*

$$\begin{array}{ccc} \mathrm{G}_0^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} & \xrightarrow{f_*} & \mathrm{G}_0^{\text{ét}}(\mathcal{Y})_{\mathbf{Q}} \\ \downarrow{\scriptstyle \tau_{\mathcal{X}}} & & \downarrow{\scriptstyle \tau_{\mathcal{Y}}} \\ \mathrm{H}^{\mathrm{BM}}_{2n}(\mathcal{X}, \mathbf{Q}(n)) & \xrightarrow{f_*} & \mathrm{H}^{\mathrm{BM}}_{2n}(\mathcal{Y}, \mathbf{Q}(n)). \end{array}$$

(ii) *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper* representable *morphism between quasi-smooth derived algebraic stacks over k. Then there is a commutative square*

$$\begin{array}{ccc} \mathrm{G}_0(\mathcal{X})_{\mathbf{Q}} & \xrightarrow{f_*} & \mathrm{G}_0(\mathcal{X})_{\mathbf{Q}} \\ \downarrow{\scriptstyle \tau_{\mathcal{X}}} & & \downarrow{\scriptstyle \tau_{\mathcal{Y}}} \\ \mathrm{H}^{\mathrm{BM}}_{2n}(\mathcal{X}, \mathbf{Q}(n)) & \xrightarrow{f_*} & \mathrm{H}^{\mathrm{BM}}_{2n}(\mathcal{Y}, \mathbf{Q}(n)). \end{array}$$

Proof. For the first claim, one may identify $\mathrm{G}_0^{\text{ét}}(\mathcal{X})$ with $\mathbf{R}\Gamma^{\mathrm{BM}}(\mathcal{X}, \mathrm{KGL})$ in a way that is natural with respect to proper direct images (see [Kh5, Ex. 2.13]). Then the claim follows immediately from the construction of proper direct images in Borel–Moore homology theories given in [Kh5]. The second claim follows by composing with the first with the square

$$\begin{array}{ccc} \mathrm{G}_0(\mathcal{X})_{\mathbf{Q}} & \xrightarrow{f_*} & \mathrm{G}_0(\mathcal{Y})_{\mathbf{Q}} \\ \downarrow{\scriptstyle \tau_{\mathcal{X}}} & & \downarrow{\scriptstyle \tau_{\mathcal{Y}}} \\ \mathrm{G}_0^{\text{ét}}(\mathcal{X})_{\mathbf{Q}} & \xrightarrow{f_*} & \mathrm{G}_0^{\text{ét}}(\mathcal{Y})_{\mathbf{Q}} \end{array}$$

which commutes by construction when f is representable (see Subsect. 5.3). □

Remark 6.12. The representability hypothesis in claim (ii) is necessary, see Warning 5.7. However, following [To1], it is possible to prove a variant where the lower horizontal map is replaced by the direct image of the induced morphism of inertia stacks. See [KPRY].

In [Kh5] a fundamental class $[\mathcal{X}]$ is constructed in motivic Borel–Moore homology and the following comparison with the G-theoretic fundamental class $[\mathcal{X}]_{\mathrm{G}} = [\mathcal{O}_{\mathcal{X}}] \in \mathrm{G}(\mathcal{X})$ (conjectured in [To3, Question 4.7]) is proven.

Theorem 6.13. *There is an equality*

$$[\mathcal{X}] = \mathrm{Td}_{\mathcal{X}}^{-1} \cap \tau_{\mathcal{X}}[\mathcal{X}]_{\mathrm{G}},$$

where $\mathrm{Td}_{\mathcal{X}}$ is the Todd class of the cotangent complex $\mathcal{L}_{\mathcal{X}}$.

Remark 6.14. Via the identification mentioned above we may view the fundamental class $[\mathcal{X}]$ as a class $[\mathcal{X}_{\mathrm{cl}}]^{\mathrm{vir}} \in \mathrm{A}_*(\mathcal{X}_{\mathrm{cl}})_{\mathbf{Q}}$. For $\mathcal{X}$ Deligne–Mumford, this coincides with the virtual fundamental class of Behrend–Fantechi [BF] (with respect to obstruction theory on $\mathcal{X}_{\mathrm{cl}}$ induced by the cotangent complex of $\mathcal{X}$) as explained in [Kh5, §3.3]. We can then read Theorem 6.13 as a comparison with the G-theoretic virtual fundamental class (Remark 6.3) via the formula

$$[\mathcal{X}_{\mathrm{cl}}]^{\mathrm{vir}} = (\mathrm{Td}^{\mathrm{vir}}_{\mathcal{X}_{\mathrm{cl}}})^{-1} \cap \tau_{\mathcal{X}_{\mathrm{cl}}}\left([\mathcal{X}_{\mathrm{cl}}]^{\mathrm{vir}}_{\mathrm{G}}\right)$$

in $\mathrm{A}_*(\mathcal{X}_{\mathrm{cl}})_{\mathbf{Q}}$, where $\mathrm{Td}^{\mathrm{vir}}_{\mathcal{X}_{\mathrm{cl}}}$ is the Todd class of $\mathcal{L}_{\mathcal{X}}|_{\mathcal{X}_{\mathrm{cl}}}$. This recovers the virtual Grothendieck–Riemann–Roch formulas of [FG, CFK, LS], all proven in the case where $\mathcal{X}$ is a derived scheme admitting an embedding into a smooth ambient scheme, as well as the extension to the case of quotient stacks in [RS].

6.4. **K-theoretic fundamental classes.** The following construction is a K-theory analogue of the "cohomological fundamental class" studied in [Kh5, §3.4].

Construction 6.15. Let $\mathcal{X}$ be a derived algebraic stack. For every quasi-smooth proper representable morphism $f : \mathcal{Z} \to \mathcal{X}$, there is a canonical class

$$[\mathcal{Z}/\mathcal{X}]^{\mathrm{K}} := f_*(1) = [f_*(\mathcal{O}_{\mathcal{Z}})] \in \mathrm{K}^{\mathrm{B}}(\mathcal{X}),$$

where $f_* : \mathrm{K}^{\mathrm{B}}(\mathcal{Z}) \to \mathrm{K}^{\mathrm{B}}(\mathcal{X})$ is the direct image map (which exists by Lemma 1.13) and $1 = [\mathcal{O}_{\mathcal{Z}}] \in \mathrm{K}^{\mathrm{B}}(\mathcal{Z})$ is the unit. If f is a closed immersion, then $[\mathcal{Z}/\mathcal{X}]^{\mathrm{K}}$ is supported on $\mathcal{Z}$, i.e., it lives in $\mathrm{K}^{\mathrm{B}}(\mathcal{X} \text{ on } \mathcal{Z})$.

When $\mathcal{X}$ is nonsingular, this class is just the direct image of the G-theoretic fundamental class of $\mathcal{Z}$:

Remark 6.16. Let $\mathcal{X}$ be a regular algebraic stack. For any quasi-smooth proper representable morphism $f : \mathcal{Z} \to \mathcal{X}$, consider the commutative square

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{B}}(\mathcal{Z}) & \xrightarrow{f_*} & \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \mathrm{G}(\mathcal{Z}) & \xrightarrow{f_*} & \mathrm{G}(\mathcal{X}) \end{array}$$

where the vertical arrows are the Cartan maps (Theorem 3.5). Since $\mathcal{X}$ is regular, the right-hand vertical arrow is invertible. Under this identification, it follows that the K-theoretic fundamental class $[\mathcal{Z}/\mathcal{X}]^{\mathrm{K}} \in \mathrm{K}^{\mathrm{B}}(\mathcal{X})$ can be identified with the direct image of the G-theoretic fundamental class $[\mathcal{Z}]_{\mathrm{G}} \in \mathrm{G}(\mathcal{Z})$ (Construction 6.2).

Proposition 6.17. *Suppose given a commutative square of derived algebraic stacks*

$$\begin{array}{ccc} \mathcal{Z}' & \xrightarrow{i'} & \mathcal{X}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{Z} & \xrightarrow[i]{} & \mathcal{X} \end{array}$$

where i and i' are quasi-smooth closed immersions. If the square is homotopy cartesian, then we have a canonical identification

$$q^*[\mathcal{Z}/\mathcal{X}] \simeq [\mathcal{Z}'/\mathcal{X}']$$

in $\mathrm{K}^{\mathrm{B}}(\mathcal{X}')$.

Proof. Evaluate the base change formula $q^* i_* \simeq i'_* p^*$ (Proposition 2.7) on the unit $1 = [\mathcal{O}_{\mathcal{Z}}] \in \mathrm{K}^{\mathrm{B}}(\mathcal{Z})$. ☐

Proposition 6.18 (Non-transverse Bézout formula)**.** *Let $\mathcal{X}$ be a derived algebraic stack. Let $f : \mathcal{Y} \to \mathcal{X}$ and $g : \mathcal{Z} \to \mathcal{X}$ be quasi-smooth proper representable morphisms. Then we have*

$$[\mathcal{Y}/\mathcal{X}]^{\mathrm{K}} \cup [\mathcal{Z}/\mathcal{X}]^{\mathrm{K}} \simeq [\mathcal{Y} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathcal{Z}/\mathcal{X}]^{\mathrm{K}}$$

in $\mathrm{K}^{\mathrm{B}}(\mathcal{X})$.

Proof. By the projection and base change formulas (Propositions 2.6 and 2.7), we have

$$f_*(1) \cup g_*(1) \simeq f_* f^* g_*(1) \simeq h_*(1)$$

where $h : \mathcal{Y} \times^{\mathbf{R}}_{\mathcal{X}} \mathcal{Z} \to \mathcal{X}$ is the projection. □

To deal with non-proper intersections there is an *excess intersection formula* which expresses the failure of the base change formula (Proposition 6.17) in terms of the K-theoretic Euler class of the excess bundle, see [T3] and [Kh6]. For example:

Proposition 6.19 (Self-intersection formula)**.** *Let $i : \mathcal{Z} \to \mathcal{X}$ be a quasi-smooth closed immersion. Then there is a canonical identification*

$$i^* i_*(-) \simeq e(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) \cup (-)$$

of maps $\mathrm{K}^{\mathrm{B}}(\mathcal{Z}) \to \mathrm{K}^{\mathrm{B}}(\mathcal{Z})$. In particular,

$$i^*[\mathcal{Z}/\mathcal{X}] \simeq e(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})$$

in $\mathrm{K}^{\mathrm{B}}(\mathcal{Z})$ and

$$[\mathcal{Z}/\mathcal{X}] \cup [\mathcal{Z}/\mathcal{X}] \simeq [\mathcal{Z} \overset{\mathbf{R}}{\underset{\mathcal{X}}{\times}} \mathcal{Z}/\mathcal{X}] \simeq i_*(e(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}))$$

in $\mathrm{K}^{\mathrm{B}}(\mathcal{X})$ (or $\mathrm{K}^{\mathrm{B}}(\mathcal{X}$ on $\mathcal{Z}))$.

6.5. **The γ-filtration and Chow cohomology of singular schemes.** This subsection is a digression that attempts to justify my interest in these "K-theoretic fundamental classes" (Subsect. 6.4), which do not seem to have received much attention in the literature so far.

6.5.1. *The γ-filtration and Chow cohomology of nonsingular schemes.* Let X be a scheme of finite type over a field k and consider the group $\mathrm{Z}_*(X)$ of algebraic cycles on X. Recall that there is a map

$$\mathrm{Z}_*(X) \to \mathrm{G}_0(X), \quad [Z] \mapsto [\mathcal{O}_Z].$$

This map does not respect rational equivalence but does so if we pass from $\mathrm{G}_0(X)$ to the graded pieces of the coniveau or γ-filtration (both agree with rational coefficients); there are canonical surjections

$$\mathrm{A}_i(X) \to \mathrm{Gr}^{d-i}(\mathrm{G}_0(X)), \quad [Z] \mapsto [\mathcal{O}_Z]$$

where $\mathrm{A}_*(X)$ denotes the quotient of $\mathrm{Z}_*(X)$ modulo rational equivalence.

When X is nonsingular, the Grothendieck–Riemann–Roch theorem implies that the kernel of this map is torsion. Moreover the γ-filtration is compatible with the cup product on $\mathrm{K}_0(X)$ and on the "Poincaré dual" theories there is an isomorphism of graded rings

$$\mathrm{A}^*(X)_{\mathbf{Q}} \to \mathrm{Gr}^*(\mathrm{K}_0(X))_{\mathbf{Q}}. \tag{6.20}$$

Here $\mathrm{A}^*(X) = \mathrm{A}_{d-*}(X)$ by definition and $\mathrm{K}_0(X) \simeq \mathrm{G}_0(X)$ by Theorem 3.5. See [SGA6, Exp. 0, Appendix and Exp. 14, §4] and [Fu2, Ex. 15.3.6].

6.5.2. *Chow cohomology of singular schemes.* For singular X the right-hand side of (6.20) still makes sense but it is less clear how to define "Chow cohomology" rings $\mathrm{A}^*(X)$ such that (6.20) holds.[7] There is a reasonable definition of top-degree Chow cohomology $\mathrm{A}^d(X)$, where X is a singular d-dimensional variety, due to Levine and Weibel, see e.g. [LW, Sr, GK]. Here I want to briefly explain how the theory of K-theoretic fundamental classes suggests a construction of a "derived" Chow cohomology theory.

Let $\mathrm{Z}^n_{\mathrm{der}}(X)$ denote the free abelian group on quasi-smooth projective morphisms $f : Z \to X$ of relative virtual dimension $-n$. Denote the class of $f : Z \to X$ by $[Z/X] \in \mathrm{Z}^n_{\mathrm{der}}(X)$. There is an obvious product

$$[Z/X] \cup [Z'/X] = [Z \overset{\mathbf{R}}{\underset{X}{\times}} Z'/X],$$

as well as a map

$$\mathrm{Z}^*_{\mathrm{der}}(X) \to \mathrm{K}_0(X), \quad [Z/X] \mapsto [Z/X]^{\mathrm{K}}$$

where $[Z/X]^K = [f_*(\mathcal{O}_Z)]$ is the K-theoretic fundamental class (Subsect. 6.4). There should be some natural quotient $\mathrm{Z}^*_{\mathrm{der}}(X) \twoheadrightarrow \mathrm{A}^*_{\mathrm{der}}(X)$ and an isomorphism

$$\mathrm{A}^*_{\mathrm{der}}(X)_{\mathbf{Q}} \to \mathrm{Gr}^*(\mathrm{K}_0(X))_{\mathbf{Q}}.$$

The following is a sort of sanity check for this hypothesis.

Theorem 6.21. *For every qcqs algebraic space X and every $[Z/X] \in \mathrm{Z}^n_{\mathrm{der}}(X)$, we have*

$$[Z/X]^{\mathrm{K}} \in \mathrm{Gr}^n(\mathrm{K}_0(X))_{\mathbf{Q}}.$$

More generally, the map $f_* : \mathrm{K}_0(Z) \to \mathrm{K}_0(X)$ sends $\mathrm{Fil}^k \mathrm{K}_0(Z)_{\mathbf{Q}}$ to $\mathrm{Fil}^{k+n} \mathrm{K}_0(X)_{\mathbf{Q}}$ for every k.

Proof. By embedding Z into a projective bundle over X and using the projective bundle formula one reduces to the case of a quasi-smooth closed immersion. That case is proven in [Kh3, Thm. 2] using derived blow-ups [KhRy] and the excess intersection formula [Kh6] to further reduce to the case of virtual Cartier divisors. □

[7] Fulton's operational Chow groups [Fu2, Chap. 17] are commonly used as a substitute for a "genuine" Chow cohomology theory, but they satisfy neither this property nor many others (for example they do not even map to singular cohomology, see [Tot2]). Voevodsky's motivic cohomology groups [Vo] are almost what one is looking for except that rationally they compare to the *homotopy invariant* version of K-theory. On singular schemes, motivic cohomology can be computed (as a presheaf of complexes) by the following procedure: restrict to smooth k-schemes, take the left Kan extension to k-schemes of finite type, and finally take the $\mathbf{A}^1$-homotopy localization of the result (see [Kh7, Prop. 5]). Stopping before the last step gives a non-homotopy invariant version of motivic cohomology. This is a variant of an old construction of Fulton from [Fu1, §3.1]. A similar procedure works for homotopy invariant algebraic K-theory (see e.g. [CiKh, Props. 5.2.2 and 5.3.7]) and in that case stopping before the last step does recover algebraic K-theory (see [Fu1, §3.2], [EHKSY, App. A]). The derived Chow ring constructed below should hopefully compute these non-homotopy invariant motivic cohomology groups for nice enough schemes.

In particular, there is an induced Gysin map $f_*^\gamma : \mathrm{Gr}^*\mathrm{K}_0(Z)_{\mathbf{Q}} \to \mathrm{Gr}^*\mathrm{K}_0(X)_{\mathbf{Q}}$. One has moreover the following version of the Grothendieck–Riemann–Roch theorem, a direct generalization of the original formulation in [SGA6, Exp. VIII, Thm. 3.6]:

Theorem 6.22. *Let X be a qcqs algebraic space. For every quasi-smooth projective morphism $f : Z \to X$ of relative virtual dimension n, there is a commutative square*

$$\begin{array}{ccc} \mathrm{K}_0(Z) & \xrightarrow{f_*} & \mathrm{K}_0(X) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ \mathrm{Gr}^*\mathrm{K}_0(Z)_{\mathbf{Q}} & \xrightarrow{f_*^\gamma(-\cdot \mathrm{Td}(\mathcal{L}_{Z/X}^\vee))} & \mathrm{Gr}^*\mathrm{K}_0(X)_{\mathbf{Q}} \end{array}$$

where $\mathrm{Td}(\mathcal{L}_{Z/X}^\vee)$ is the Todd class of the relative tangent complex.

Proof. Again, one reduces to the case of projective bundles, see [SGA6, Exp. VIII, §5], and the case of quasi-smooth closed immersions, see [Kh3, Thm. 3]. □

The theory of fundamental classes in motivic cohomology [Kh5] produces a *ring* homomorphism

$$\mathrm{Z}^*_{\mathrm{der}}(X) \to \mathrm{H}^{2*}(X, \mathbf{Z}(*))$$

which should descend to $\mathrm{A}^*_{\mathrm{der}}(X)$. When X is smooth the resulting map

$$\mathrm{A}^*_{\mathrm{der}}(X) \to \mathrm{H}^{2*}(X, \mathbf{Z}(*)) \simeq \mathrm{A}^*(X) \tag{6.23}$$

should be an isomorphism. Note that if the base field has resolution of singularities then every cycle in $\mathrm{A}^n(X)$ can be represented by the fundamental class of a projective lci map, so there is at least some quotient of $\mathrm{Z}^*_{\mathrm{der}}(X)$ for which (6.23) is an isomorphism. Whenever this is true, it means the intersection product on $\mathrm{A}^*(X)$ can be represented geometrically by derived fibred products.

Admittedly $\mathrm{A}^*_{\mathrm{der}}$ looks more like a cobordism theory than a cycle theory. In fact two completely different recent approaches to algebraic cobordism both yield constructions closely resembling $\mathrm{A}^*_{\mathrm{der}}(X)$. In [EHKSY] the authors considered a variant where the generators are required to be *finite* instead of projective. That theory gives a model for cobordism of *smooth* schemes and unfortunately requires a Nisnevich localization procedure which renders the end result fairly intractible. On the other hand Annala has recently constructed good theories of algebraic cobordism and Chow cohomology of singular schemes in characteristic zero [An1]. After further simplification and partial extension to general bases in [AnYo, An2], the generators are similar to those of $\mathrm{A}^*_{\mathrm{der}}$ but a simple description of the relations needed to pass from cobordism to Chow does not yet seem to be known. Any relation with filtrations on K-theory has not been studied beyond the remarks above as far as I know and comparisons with the Levine–Weibel Chow groups [LW] also have yet to be investigated.

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