THE LATTICE PROPERTY FOR PERFECT COMPLEXES ON SINGULAR STACKS

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Abstract. Let $\mathcal{C}$ be the stable ∞-category of perfect complexes on a derived Deligne–Mumford stack $\mathcal{X}$ of finite type over the complex numbers. We prove that the complexified noncommutative topological Chern character $\mathrm{K}^{\mathrm{top}}(\mathcal{C}) \otimes \mathbf{C} \to \mathrm{HP}(\mathcal{C})$ is invertible. In the appendix we show the same property for $\mathcal{C}$ the stable ∞-category of coherent complexes on a derived algebraic space.

Let $\mathcal{C}$ be a $\mathbf{C}$-linear stable ∞-category (or pretriangulated dg-category). The topological K-theory of $\mathcal{C}$ in the sense of Blanc [Bla] is a spectrum $\mathrm{K}^{\mathrm{top}}(\mathcal{C})$ which admits a canonical map to the periodic cyclic homology spectrum

$$\mathrm{K}^{\mathrm{top}}(\mathcal{C}) \to \mathrm{HP}(\mathcal{C})$$

that may be regarded as a "noncommutative" analogue of the Chern character. Let us say that $\mathcal{C}$ satisfies the *lattice property* when the induced map $\mathrm{K}^{\mathrm{top}}(\mathcal{C}) \otimes \mathbf{C} \to \mathrm{HP}(\mathcal{C})$ is invertible.

The *lattice conjecture*, motivated by considerations in noncommutative Hodge theory, is the assertion that any *smooth* and *proper* $\mathcal{C}$ satisfies the lattice property (see [KKP, 2.2.6(b)], [Bla, Conj. 1.7]). For $\mathcal{C}$ the stable ∞-category of perfect complexes on a scheme or stack, smoothness and properness do not appear to be relevant. For example, the lattice property is known for the stable ∞-category $\mathrm{Perf}(X)$ of perfect complexes on any quasi-separated derived algebraic space X of finite type over $\mathbf{C}$ (see [Bla, Prop. 4.32], [Kon, Cor. 6.8]). Halpern–Leistner and Pomerleano extended this to *smooth* Deligne–Mumford stacks as well as certain *smooth* global quotient stacks (see [HLP, Thm. 2.17, Cor. 2.19]). In this paper we consider the singular case:

Theorem A. *For a derived stack $\mathcal{X}$ of finite type over $\mathbf{C}$, the $\mathbf{C}$-linear stable ∞-category $\mathrm{Perf}(\mathcal{X})$ satisfies the lattice property in the following cases:*

(i) *$\mathcal{X}$ is Deligne–Mumford with separated diagonal.*

(ii) *$\mathcal{X} = [X/G]$ where X is a quasi-separated derived algebraic space of finite type and G is an affine algebraic group with diagonalizable identity component.*

In fact we prove the result more generally for derived Artin stacks $\mathcal{X}$ with separated diagonal whose stabilizers are *nice* algebraic groups in the sense of [HR, Def. 1.1]. The main new tool is an equivariant cdh descent result for truncating invariants of stable ∞-categories. Case (ii) of Theorem A was conjectured by Halpern–Leistner and Pomerleano (without the diagonalizability hypothesis).

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In the appendix we record a proof of the following result. We write $\mathrm{DCoh}(X)$ for the stable ∞-category of coherent (= pseudocoherent with bounded cohomology) complexes on a derived algebraic space X.

Theorem B. *Let X be a quasi-separated derived algebraic space of finite type over $\mathbf{C}$. The $\mathbf{C}$-linear stable ∞-category $\mathrm{DCoh}(X)$ satisfies the lattice property.*

Recently, Brown and Walker [BW] proved this for X a local complete intersection scheme using a dévissage result for HP. Our main ingredient is a stronger dévissage result that follows from work of Preygel [Pre].

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1. Cdh descent

Let G be an fppf group scheme over an affine scheme S, which we assume noetherian and of finite Krull dimension for simplicity. We will assume that G is *embeddable*, i.e., can be embedded as a closed subgroup of $\mathrm{GL}_{n,S}$ for some n, and that G is *nice*, i.e., an extension of a tame finite étale group scheme by a group scheme of multiplicative type (see [AHR, Def. 2.1]). In particular, G is linearly reductive.

Let E be a localizing invariant of $\mathcal{O}_S$-linear stable ∞-categories[1]. On the ∞-category $\mathcal{S}^G$ of quasi-separated derived algebraic spaces of finite type over S with G-action, the presheaf

$$E^G(-) := E(\mathrm{Perf}([-/G])$$

satisfies Nisnevich descent by [Kha2, Thm. 1.40, Rem. 2.15]. By the generalized Sumihiro theorem (see [KR, Thm. 2.14(ii)], [BKRS, Prop. A.1.9]), every $X \in \mathcal{S}^G$ admits a G-equivariant scallop decomposition by quasi-affines. If $X \in \mathcal{S}^G$ is quasi-affine it moreover admits a G-equivariant scallop decomposition by affines (see [BKRS, Prop. A.1.9]). We will use these observations repeatedly in combination with Nisnevich descent to reduce statements about $X \in \mathcal{S}^G$ to the affine case.

An *abstract blow-up square* in $\mathcal{S}^G$ is a commutative square

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array} \tag{1.1}$$

which is cartesian on classical truncations, where i is a closed immersion and f is a proper morphism inducing an isomorphism $X' \smallsetminus f^{-1}(Z) \simeq X \smallsetminus Z$. The

[1] with values in spectra, say, or any stable ∞-category with an exact conservative functor to spectra

cdh topology on $\mathcal{S}^G$ is generated by Nisnevich covers and, for every abstract blow-up square as above, the cover $Z \coprod X' \to X$. The following cdh descent criterion is from [Kha1, Thm. 5.6]:

Theorem 1.2. *The presheaf $X \mapsto E^G(X)$ satisfies cdh hyperdescent on $\mathcal{S}^G$ if and only if it satisfies nil-invariance: for every $X \in \mathcal{S}^G$ and every surjective closed immersion $i : Z \to X$, the induced map*

$$i^* : E^G(X) \to E^G(Z)$$

is invertible.

Proof. For any $X \in \mathcal{S}^G$ the inclusion $\varnothing \to X$ is a proper morphism inducing an isomorphism over $X \smallsetminus Z = \varnothing$, so the condition is necessary. For the other direction, we first apply nil-invariance for the inclusion of the classical truncation to restrict our attention to abstract blow-up squares over $X \in \mathcal{S}^G$ classical. By [BKRS, Thm. C, Rem. 0.0.8], we have *pro*-excision for any such blow-up square. By nil-invariance applied to infinitesimal thickening, this reduces to ordinary excision. □

We deduce a G-equivariant version of [LT, Thm. E].

Corollary 1.3. *If E is a truncating invariant in the sense of* [LT, Def. 3.1], *then $E^G(-)$ satisfies cdh hyperdescent on $\mathcal{S}^G$.*

Proof. Since E is truncating, we have nil-invariance by [ES] (combining Thm. 1.0.4 and Prop. 5.1.10). □

2. Proof of Theorem A

Denote by $F(-)$ the fibre of the natural transformation $\mathrm{K}^{\mathrm{top}}(-) \otimes \mathbf{C} \to \mathrm{HP}(-)$. This is a localizing invariant of $\mathbf{C}$-linear stable ∞-categories, which is truncating by (the proof of) [Kon, Cor. 5.6]. We will prove the following, which generalizes both cases of Theorem A:

Theorem 2.1. *Let $\mathcal{X}$ be a derived algebraic stack of finite type over $\mathbf{C}$ with separated diagonal and nice stabilizers. Then* $\mathrm{Perf}(\mathcal{X})$ *satisfies the lattice property, i.e., $F(\mathcal{X}) \simeq 0$.*

Proof. By [KR, Thm. 2.12(ii)] (based on [AHHLR, Thm. 1.9]), $\mathcal{X}$ is nicely scalloped; that is, it admits a scallop decomposition by quotient stacks $[X/G]$ with G a nice embeddable group scheme over an affine $\mathbf{C}$-scheme S, acting on a finite type quasi-affine derived scheme X over S. By Nisnevich descent it will thus be enough to show that $F^G(X) \coloneqq F([X/G]) \simeq 0$ with X and G as above. Repeating the same reasoning with [BKRS, Prop. A.1.9], we may moreover assume that X is affine.

Since F is truncating, $F^G(-)$ satisfies cdh hyperdescent and nil-invariance by Corollary 1.3. In particular, we may assume that X is classical and reduced. Since X admits a G-equivariant resolution of singularities (e.g. by [ATW, Thm. 8.1.1] applied to $[X/G]$), there exists a G-equivariant cdh hypercover $\widetilde{X}_\bullet \to X$ where each $\widetilde{X}_n$ is smooth. By cdh hyperdescent again, we may thus assume that X is smooth. Note that X need no longer be affine, but since G

is nice we may apply generalized Sumihiro and Nisnevich descent to assume X affine again.

Thus suppose X is smooth and affine. In case G is defined over $\mathbf{C}$ (e.g. $G = \mathrm{GL}_{n,S}$), the claim is a special case of [HLP, Thm. 2.17], where $[X/G]$ admits a "semicomplete KN stratification" by [HLP, Thm. 1.3] because G is reductive and X is affine.

Otherwise, choose an embedding $G \subseteq \mathrm{GL}_{n,S}$ and write

$$[X/G] \simeq [(X \overset{G}{\underset{S}{\times}} \mathrm{GL}_{n,S})/\mathrm{GL}_{n,S}],$$

where $X \times_S^G \mathrm{GL}_{n,S} = [(X \times_S \mathrm{GL}_{n,S})/G]$, with G acting on $X \times \mathrm{GL}_{n,S}$ by $h\cdot(x,g) = (h\cdot x, g\cdot h^{-1})$ and $\mathrm{GL}_{n,S}$ acting on $X\times\mathrm{GL}_{n,S}$ by $h\cdot(x,g) = (x, h\cdot g)$ (this passes to $X \times^G \mathrm{GL}_{n,S}$ since the actions commute). Since G is linearly reductive, $G/\mathrm{GL}_{n,S}$ is affine by Matsushima, so $X\times^G \mathrm{GL}_{n,S}$ is affine (and still smooth). Thus

$$F^G(X) \simeq F^{\mathrm{GL}_n}(X \overset{G}{\times} \mathrm{GL}_n) \simeq 0$$

as desired. □

Appendix A. The lattice property for DCoh(X)

A.1. **Dévissage for periodic cyclic cohomology.** Given a localizing invariant E, we write $E^{\mathrm{BM}}(-) \coloneqq E(\mathrm{DCoh}(-))$. For any closed immersion of (derived) algebraic spaces $i : Z \to X$ there is a canonical map

$$E^{\mathrm{BM}}(Z) = E(\mathrm{DCoh}(Z)) \to E(\mathrm{DCoh}(X \text{ on } Z)) \tag{A.1}$$

where $\mathrm{DCoh}(X \text{ on } Z)$ is the kernel of the restriction functor $\mathrm{DCoh}(X) \to \mathrm{DCoh}(X \smallsetminus Z)$. Since E is localizing, the target is identified with

$$E(\mathrm{DCoh}(X \text{ on } Z)) \simeq \mathrm{Fib}(E^{\mathrm{BM}}(X) \to E^{\mathrm{BM}}(X \smallsetminus Z)).$$

Thus (A.1) is invertible if and only if the sequence (which is canonically null-homotopic)

$$E^{\mathrm{BM}}(Z) \to E^{\mathrm{BM}}(X) \to E^{\mathrm{BM}}(X \smallsetminus Z)$$

is exact.

For algebraic K-theory, hence also for $\mathrm{K}^{\mathrm{top}}$, Quillen's dévissage theorem implies (A.1) is invertible. This is not the case for arbitrary localizing invariants (see [Kel, Ex. 1.11] for a counterexample in $E = \mathrm{HH}$).

Theorem A.2. *Let X be an algebraic space of finite type over $\mathbf{C}$. For any closed immersion $i : Z \hookrightarrow X$, the canonical map $\mathrm{HP}^{\mathrm{BM}}(Z) \to \mathrm{HP}(\mathrm{DCoh}(X \text{ on } Z))$ is invertible.*

Remark A.3. When X is smooth and $i : Z \hookrightarrow X$ is a a quasi-smooth closed immersion (with Z possibly derived), Brown and Walker recently gave a different proof of Theorem A.2. Indeed, using Proposition A.4 below and the local structure of quasi-smooth closed immersions (see [KRy, 2.3.6]), one reduces to the local calculation of [BW, Thm. 4.2(i)]. If we know that $\mathrm{HP}^{\mathrm{BM}}$

is insensitive to derived structures, this gives another proof of Theorem A.2 for X smooth (and Z any closed subspace), because every closed subspace of X admits some quasi-smooth derived structure locally on X. In fact, $\mathrm{HP}^{\mathrm{BM}}$ is indeed insensitive to derived structures if we admit Theorem A.5 below (in view of the localization triangle (A.6) for the closed immersion $X_{\mathrm{cl}} \to X$), but we do not know a direct proof that does not go through Preygel's comparison.

The following is a consequence of dévissage (as in [Kha2, Cor. 3.11]), but in fact holds more generally:

Proposition A.4. *Let E be a localizing invariant of k-linear stable ∞-categories (for a commutative ring k). Then $E^{\mathrm{BM}}(-)$ satisfies Nisnevich descent on qcqs derived algebraic spaces over k.*

Proof. Let X be a qcqs derived algebraic space. For every étale U over X, there is a canonical equivalence

$$\mathrm{Perf}(U) \otimes_{\mathrm{Perf}(X)} \mathrm{DCoh}(X) \simeq \mathrm{DCoh}(U)$$

by [GR, Chap. 4, Rem. 3.3.3]. Consider then the localizing invariant E' of $\mathrm{Perf}(X)$-linear stable ∞-categories[2] given by

$$E'(\mathcal{C}) \coloneqq E(\mathcal{C} \otimes_{\mathrm{Perf}(X)} \mathrm{DCoh}(X)),$$

so that $E'(\mathrm{Perf}(-)) \simeq E^{\mathrm{BM}}(-)$ on the small étale site of X. By [CMNN, Prop. A.15], $E'(\mathrm{Perf}(-))$ satisfies Nisnevich descent, hence so does $E^{\mathrm{BM}}(-)$. □

The following is [Pre, Thm. 1.1.2, Thm. 6.3.2]:

Theorem A.5 (Preygel)**.** *Let X be a quasi-separated derived algebraic space locally of finite type over $\mathbf{C}$. Then there is a canonical isomorphism*

$$\mathrm{HP}^{\mathrm{BM}}(X) \to \mathrm{C}_{\bullet}^{\mathrm{BM,dR}}(X) \otimes_k k(\!(u)\!)$$

where u is in homological degree -2. Moreover, it is covariantly functorial with respect to proper push-forwards and contravariantly functorial with respect to quasi-smooth pull-backs.

This immediately implies Theorem A.2. To see this we recall the definition of the complex of de Rham Borel–Moore chains on X, for X locally of finite type over a field k of characteristic zero:

$$\mathrm{C}_{\bullet}^{\mathrm{BM,dR}}(X) \coloneqq R\Gamma(X, \omega_X^{\mathrm{dR}})$$

where ω_X^{dR} denotes the dualizing complex of X in the ∞-category of D-modules. Following [GR], the latter is by definition the ∞-category $\mathrm{IndCoh}(X_{\mathrm{dR}})$ of ind-coherent sheaves on the *de Rham prestack* X_{dR} of X (see [Pre, 1.2.1], [GR, §1]). The de Rham dualizing complex ω_X^{dR} is just the dualizing complex of X_{dR} in $\mathrm{IndCoh}(X_{\mathrm{dR}})$. Thus more explicitly,

$$\mathrm{C}_{\bullet}^{\mathrm{BM,dR}}(X) = R\Gamma(X_{\mathrm{dR}}, \omega_{X_{\mathrm{dR}}}^{\mathrm{DCoh}}).$$

[2]See [CMNN, App. A] for this notion.

Let $i : Z \to X$ be a closed immersion and j the inclusion of the complement $X \smallsetminus Z$. Kashiwara's lemma [GR, Prop. 2.5.6] implies that we have an exact triangle of functors

$$i_* i^! \to \mathrm{id} \to j_* j^!$$

where the functoriality is at the level of D-modules. This gives rise to the localization exact triangle

$$\mathrm{C}^{\mathrm{BM,dR}}_\bullet(Z) \to \mathrm{C}^{\mathrm{BM,dR}}_\bullet(X) \to \mathrm{C}^{\mathrm{BM,dR}}_\bullet(X \smallsetminus Z). \tag{A.6}$$

Proof of Theorem A.2. By Theorem A.5, the sequence

$$\mathrm{HP}^{\mathrm{BM}}(Z) \to \mathrm{HP}^{\mathrm{BM}}(X) \to \mathrm{HP}^{\mathrm{BM}}(X \smallsetminus Z)$$

is identified with (A.6) $\otimes\, k(\!(u)\!)$. □

A.2. **Proof of Theorem B.** Let X be a quasi-separated derived algebraic space of finite type over $\mathbf{C}$. Let $F(-)$ denote the fibre of $\mathrm{K}^{\mathrm{top}}(-) \otimes \mathbf{C} \to \mathrm{HP}(-)$, regarded as a localizing invariant of $\mathbf{C}$-linear stable ∞-categories. We will show that $F^{\mathrm{BM}}(X) := F(\mathrm{DCoh}(X)) \simeq 0$.

By Proposition A.4 the claim is Nisnevich-local on X, so we may assume that X is an affine scheme. In particular, there exists a closed immersion $X \hookrightarrow Y$ where Y is a smooth affine $\mathbf{C}$-scheme. Since both $\mathrm{K}^{\mathrm{top,BM}}$ and $\mathrm{HP}^{\mathrm{BM}}$ satisfy dévissage (the former follows from the case of $\mathrm{K}^{\mathrm{BM}} = \mathrm{G}$ and the latter is Theorem A.2), so does F^{BM}. That is, we have an exact triangle

$$F^{\mathrm{BM}}(X) \to F^{\mathrm{BM}}(Y) \to F^{\mathrm{BM}}(Y \smallsetminus X).$$

Since Y and $Y \smallsetminus X$ are regular, we have $F^{\mathrm{BM}}(Y) \simeq F(\mathrm{Perf}(Y)) \simeq 0$ and $F^{\mathrm{BM}}(Y \smallsetminus X) \simeq 0$ by [Bla, Prop. 4.32]. The claim follows.

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