DESCENDABILITY AND DESCENT IN TOPOLOGICAL WEAVES

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Abstract. We prove a criterion for a finitely presented surjection of algebraic spaces to be descendable in a topological weave. We apply this to show that étale motivic spectra satisfy h-descent on noetherian finite-dimensional schemes with residue fields of uniformly bounded étale cohomological dimension. We also show that rational motivic cohomology satisfies arc-descent in weights $\leqslant 1$, and we construct the "forgetting supports" isomorphism $f_! \simeq f_*$ for a proper DM-type morphism of Artin stacks, in rational motivic sheaves.

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Introduction

A fundamental result in the theory of étale cohomology is the cohomological descent theorem of [SGA4, Exp. $\mathrm{V}^{\mathrm{bis}}$, Prop. 4.3.2], which says that the presheaf $S \mapsto R\Gamma_{\text{ét}}(S;\Lambda)$, where Λ is a torsion commutative ring, satisfies descent for the h-topology of Voevodsky [Voe]; this amounts to descent for both étale surjections and finitely presented proper surjections. Moreover, this can be upgraded to h-descent for the presheaf of ∞-categories $S \mapsto \mathrm{D}_{\text{ét}}(S;\Lambda)$, where the transition functors are $*$-pullbacks.

In this paper we study the question of h-descent for abstract topologically flavoured six-functor formalisms, i.e., for the topological weaves of [Kha]. We provide a general criterion for h-descent, which we apply in particular to various flavours of motivic sheaves. Consider first Morel's model for rational motivic sheaves: over an arbitrary base S, $\mathbf{SH}_{\mathbf{Q},+}(S)$ denotes the plus part of the ∞-category of rational motivic spectra over S (see [CD, §16]).

Theorem A. $\mathbf{SH}_{\mathbf{Q},+}(-)$ *satisfies h-descent on all algebraic spaces.*

Next consider $\mathbf{SH}_{\text{ét}}(S)$, the ∞-category of motivic spectra over S satisfying étale hyperdescent, and (for S noetherian of finite dimension) the ∞-category $\mathbf{DM}_{\text{ét}}(S;\Lambda)$ of étale motivic sheaves as defined in [CD2, §2].

Theorem B. $\mathbf{SH}_{\text{ét}}(-)$ *satisfies h-descent on noetherian finite-dimensional algebraic spaces whose residue fields have uniformly bounded étale cohomological dimension. The same holds for* $\mathbf{DM}_{\text{ét}}(-;\Lambda)$ *for any commutative ring* Λ.

The cases of $\mathbf{SH}_{\mathbf{Q},+}$ and $\mathbf{DM}_{\text{ét}}(-;\Lambda)$ are folklore.[1] On global sections, Theorem A recovers h-descent for motivic cohomology with rational coefficients, recorded in the literature on quasi-excellent noetherian finite-dimensional schemes by Cisinski–Déglise (see [CD, Thms. 14.3.4, 16.2.13]). We will also extend this to hyperdescent for the *arc*-topology of [BM], for rational motivic cohomology in weights $\leqslant 1$ (and conditionally in arbitrary weights); see Theorem 3.6. Theorem B recovers in particular h-descent for étale motivic cohomology, which was recorded for noetherian finite-dimensional schemes in [CD2, Cor. 5.5.7].

In fact, for both $\mathbf{SH}_{\mathbf{Q},+}$ and $\mathbf{SH}_{\text{ét}}$ we show the stronger assertion that any finitely presented surjection $f : Y \twoheadrightarrow X$ of qcqs algebraic spaces is *descendable* in the sense of Mathew [Mat]: that is, the unit object $\mathbf{1}_X \in \mathbf{D}(X)$ can be built out of $f_*(\mathbf{1}_Y)$ as follows: it lies in the smallest full subcategory containing $f_*(\mathbf{1}_Y)$ and closed under cofibres, fibres, direct summands, and $(-) \otimes \mathcal{F}$ for every $\mathcal{F} \in \mathbf{D}(X)$. When f is proper, this in particular implies Čech descent along f (see Proposition 2.6).

[1] For $\mathbf{SH}_{\mathbf{Q},+}$ one may use Nisnevich descent and continuity to reduce to the case of *noetherian finite-dimensional* schemes, where it can be easily derived from the work done in [CD] using the ∞-categorical techniques developed in [Lur, Lur2]. We will give a more direct argument. For $\mathbf{DM}_{\text{ét}}(-;\Lambda)$, it can similarly be derived from the work of Ayoub and Cisinski–Déglise [Ayo, CD2].

Switching our focus from descent to descendability has several advantages. First, it implies not only that every $\mathcal{F} \in \mathbf{D}(X)$ can be expressed as the totalization $\mathrm{Tot}(f_{\bullet,*}f_\bullet^*(\mathcal{F}))$, where $f_\bullet : Y_\bullet \to X$ is the Čech nerve, but also that this totalization commutes past any exact functor F (see Remark 2.11):

$$F(\mathcal{F}) \simeq \mathrm{Tot}\big(F\big(f_{\bullet,*}f_\bullet^*(\mathcal{F})\big)\big).$$

Second, descendability is preserved by any symmetric monoidal functor of stable ∞-categories. Theorem A therefore implies that any $\mathbf{Q}$-linear oriented topological weave satisfies h-descent on algebraic spaces; similarly Theorem B implies that a topological weave satisfies h-descent if and only if it satisfies étale descent, over bases with residue fields of uniformly bounded étale cohomological dimension.

The most important advantage of descendability in *proving* Theorems A and B is that it admits the following useful criterion.

Theorem C. *Let $\mathbf{D}$ be a topological weave. Then the following conditions are equivalent:*

(i) *Every finite étale surjection and every finite radicial surjection of qcqs algebraic spaces is descendable.*

(ii) *Every finitely presented surjection of qcqs algebraic spaces is descendable.*

We end the paper with the following application, which shows how the stronger form of Čech descent provided by descendability can be useful.

Theorem D. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper 1-Artin morphism with proper diagonal. Assume that f is representable by Deligne–Mumford stacks, or that f has tame relative inertia and $\mathcal{Y}$ is equicharacteristic. Then the canonical* forgetting supports *map $\alpha_f : f_!(\mathcal{F}) \to f_*(\mathcal{F})$ is invertible for every $\mathcal{F} \in \mathbf{SH}_{\mathbf{Q},+}(\mathcal{X})$.*

A similar result was claimed in [Kha3, Thm. A.7], with the following "proof" (we consider the DM variant). One can reduce to the case where $\mathcal{Y}$ is a scheme and $\mathcal{X}$ is a DM stack. It is known that one can find a proper representable surjection $g : Z \twoheadrightarrow \mathcal{X}$ where Z is a scheme, and then Čech descent gives $\mathcal{F} \simeq \mathrm{Tot}(g_{\bullet,*}g_\bullet^*(\mathcal{F}))$ where $g_\bullet : Z_\bullet \to \mathcal{X}$ is the Čech nerve. By functoriality of α this reduces us to the invertibility of α_{g_n} and $\alpha_{f\circ g_n}$. The gap here is that $f_!$ might not *a priori* commute past the totalization. This is precisely what is patched by the observation that g is *descendable* (Theorem 5.1).

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1. Descendability

1.1. **Definitions.** We briefly review the notion of *descendability* as studied by Mathew [Mat, §3.3]; proofs of the following statements can be found either there or in [BS, §11.2]. Let $\mathcal{D}$ be a symmetric monoidal presentable stable ∞-category, in which the tensor product is colimit-preserving in each argument.

Definition 1.1. A morphism $A \to B$ of E_∞-algebras in $\mathcal{D}$ is *descendable* if the morphism $\{A\} \to \{\mathrm{Tot}_{\leqslant n}(B^\bullet)\}_n$ is invertible in $\mathrm{Pro}(\mathcal{D})$. Here, $B^\bullet$ denotes the Čech nerve of $A \to B$, the cosimplicial object whose nth term is the $(n+1)$-fold tensor product of B over A, and $\{\mathrm{Tot}_{\leqslant n}(B^\bullet)\}_n$ is its tower of partial totalizations, regarded as a pro-object of $\mathcal{D}$.

Descendability is closed under composition and extension of scalars. Moreover, any symmetric monoidal functor preserves descendability. See [Mat, Cor. 3.21, Prop. 3.24]. We will also say that an E_∞-algebra A is *descendable* if the unit $\mathbf{1} \to A$ is descendable.

Remark 1.2. If $A \to B$ is descendable, then the pro-object $\{\mathrm{Tot}_{\leqslant n}(B^\bullet)\}_n$ is essentially constant; in particular, we have (see e.g. [Mat, Prop. 3.10]):

$$F\big(\mathrm{Tot}(B^\bullet)\big) \simeq \varprojlim_n F\big(\mathrm{Tot}_{\leqslant n}(B^\bullet)\big) \simeq \varprojlim_n \mathrm{Tot}_{\leqslant n}\big(F(B^\bullet)\big) \simeq \mathrm{Tot}\big(F(B^\bullet)\big)$$

for any left-exact functor $F : \mathcal{D} \to \mathcal{V}$ valued in an ∞-category with limits.

We have the following criterion for descendability:

Lemma 1.3. *Let $A \to B$ be a morphism of E_∞-algebras in $\mathcal{D}$. Let $\mathcal{C} \subseteq \mathrm{Mod}_A(\mathcal{D})$ be the smallest thick subcategory which contains B and is closed under $(-) \otimes M$ for every $M \in \mathrm{Mod}_A(\mathcal{D})$. Then $A \to B$ is descendable if and only if $\mathcal{C}$ contains A.*

We also have the following quantitative version:

Definition 1.4. Let $A \to B$ be a morphism of E_∞-algebras in $\mathcal{D}$. We say $A \to B$ is *descendable of index* $\leqslant N$, for a natural number $N > 0$, if the canonical map $A \to \mathrm{Tot}_{\leqslant N-1}(B^\bullet)$ admits a retraction in $\mathcal{D}$.

Since $\mathrm{Tot}_{\leqslant 0}(B^\bullet) \simeq B$, descendability of index $\leqslant 1$ is the condition that the unit $A \to B$ admits a retraction. The index also admits the following description (see [MNN, Prop. 4.9 (4)]).

Lemma 1.5. *Let $A \to B$ be a morphism of E_∞-algebras in $\mathcal{D}$, and let I denote its fibre in $\mathcal{D}$. Then $A \to B$ is descendable of index $\leqslant N$ if and only if the map $\varepsilon^N : I^{\otimes_A N} \to A^{\otimes_A N} \simeq A$ is null-homotopic.*

The connection between Definitions 1.1 and 1.4 is as follows (see [BS, Lem. 11.20]).

Lemma 1.6. *Let $A \to B$ be a morphism of E_∞-algebras in $\mathcal{D}$. Then $A \to B$ is descendable if and only if it is descendable of index $\leqslant N$ for some N.*

Lemma 1.7. *Let $A \to B$ be a morphism of E_∞-algebras in $\mathcal{D}$. If B is descendable of index $\leqslant N$, then so is A.*

Proof. There is an induced morphism $I_A \to I_B$ on fibres, and ε_A^N factors as

$$\varepsilon_A^N : I_A^{\otimes N} \to I_B^{\otimes N} \xrightarrow{\varepsilon_B^N} \mathbf{1}. \qquad \square$$

The following lemma shows that descendability is "descendable-local".

Lemma 1.8. *Let A and B be E_∞-algebras in $\mathcal{D}$. If B is descendable of index $\leqslant N$ and $B \to A \otimes B$ is descendable of index $\leqslant M$, then A is descendable of index $\leqslant NM$.*

Proof. Let I_A and I_B denote the fibres of the unit maps of A and B, respectively. We have the diagram with exact rows

$$\begin{tikzcd} I_A \arrow[r, "\varepsilon_A"] \arrow[d] & \mathbf{1} \arrow[r, "\eta_A"] \arrow[d, "\eta_B"] & A \arrow[d] \\ I_A \otimes B \arrow[r, "\varepsilon_A \otimes \mathrm{id}_B"] & B \arrow[r] & A \otimes B. \end{tikzcd}$$

By assumption, the map $\varepsilon_A^M \otimes \mathrm{id}_B : I_A^{\otimes M} \otimes B \to B$ is null-homotopic, hence so is $\eta_B \circ \varepsilon_A^M : I_A^{\otimes M} \to B$. Thus from the exact triangle $I_B \to \mathbf{1} \to B$, there exists a lift $\phi : I_A^{\otimes M} \to I_B$ with $\varepsilon_B \circ \phi \simeq \varepsilon_A^M : I_A^{\otimes M} \to \mathbf{1}$. We then have

$$\varepsilon_A^{NM} \simeq (\varepsilon_A^M)^{\otimes N} \simeq (\varepsilon_B \circ \phi)^{\otimes N} \simeq \varepsilon_B^N \circ \phi^{\otimes N} \simeq 0,$$

as claimed. $\square$

For a presentable weave $\mathbf{D}$, we define:

Definition 1.9. A morphism of algebraic spaces $f : Y \to X$ is *descendable* (of index $\leqslant N$) in the weave $\mathbf{D}$ if $\mathbf{1}_X \to f_*(\mathbf{1}_Y)$ is descendable (of index $\leqslant N$), as a morphism of E_∞-algebras in $\mathbf{D}(X)$.

This property is closed under composition and base change.

Corollary 1.10. *Let $f : Y \to X$ and $p : X' \to X$ be morphisms, and let $f' : Y' \to X'$ denote the base change of f along p. Suppose that p_* satisfies the projection formula (Pr2) (see* [Kha, Def. 2.18]*), and the exchange morphism* $\mathrm{Ex}^*_* : p^* f_* \mathbf{1}_Y \to f'_* \mathbf{1}_{Y'}$ *is invertible. If p is descendable of index $\leqslant N$ and f' is descendable of index $\leqslant M$, then f is descendable of index $\leqslant NM$.*

Proof. Let $A = f_* \mathbf{1}_Y$ and $B = p_* \mathbf{1}_{X'}$; by the assumptions, $A \otimes B \simeq p_*(f'_* \mathbf{1}_{Y'})$. Hence Lemma 1.8 shows that $B \to A \otimes B$ is descendable of index $\leqslant M$ since f' is. $\square$

1.2. **Criteria for finite étale covers.** Let $f : Y \to X$ be a finite étale morphism of algebraic spaces. In order for f to be descendable of index $\leqslant 1$, it suffices that the composite

$$\mathbf{1}_X \xrightarrow{\text{unit}} f_*f^*(\mathbf{1}_X) \simeq f_!f^!(\mathbf{1}_X) \xrightarrow{\text{counit}} \mathbf{1}_X, \tag{1.11}$$

be invertible. (This is the Euler characteristic of the dualizable object $f_*(\mathbf{1}_Y)$.) This condition can be made more concrete using the following fact from [EK, Prop. 2.2.5] (see also [EK, Rem. 3.2.2]):

Lemma 1.12. *Let $\mathbf{D}$ be a topological weave. Let $X = \mathrm{Spec}(\kappa)$ be the spectrum of a field and $f : Y \to X$ a finite étale morphism of degree d. Then the Euler characteristic (1.11) is the class $d_\varepsilon \in \mathrm{End}_{\mathbf{D}(k(x))}(\mathbf{1}_{k(x)})$.*

We say that $\mathbf{D}$ satisfies *conservativity of stalks* for an algebraic space X if the functors $x^* : \mathbf{D}(X) \to \mathbf{D}(k(x))$, $x \in X$, are jointly conservative. For example, this holds for X locally noetherian when $\mathbf{D}$ is a topological weave satisfying continuity and geometric generation (see [Kha, Prop. 3.13]).

Corollary 1.13. *Let $\mathbf{D}$ be a topological weave satisfying conservativity of stalks for an algebraic space X. Let $f : Y \to X$ be a finite étale morphism of degree d. Then the following conditions are equivalent:*

(i) *The Euler characteristic (1.11) is invertible.*

(ii) *The class $d_\varepsilon \in \mathrm{End}_{\mathbf{D}(X)}(\mathbf{1}_X)$ is invertible.*

(iii) *For every point $x \in X$, the class $d_\varepsilon \in \mathrm{End}_{\mathbf{D}(k(x))}(\mathbf{1}_{k(x)})$ is invertible.*

Proof. Since (1.11) and d_ε are both functorial in X, the first two claims are both equivalent to the third by conservativity of stalks. □

Recall that an *orientation* of $\mathbf{D}$ is a factorization of the Thom twist map $\mathrm{K} \to \underline{\mathrm{Pic}}$ through the rank map $\mathrm{K} \to \mathbf{Z}$. By construction of d_ε, such an orientation identifies $d_\varepsilon \simeq d$ in $\mathrm{End}_{\mathbf{D}(X)}(\mathbf{1}_X)$ for every X. In particular:

Corollary 1.14. *Let $\mathbf{D}$ be an oriented topological weave satisfying conservativity of stalks for an algebraic space X. Let $f : Y \to X$ be a finite étale morphism of degree d. Then the Euler characteristic (1.11) is invertible if and only if the class $d \in \mathrm{End}_{\mathbf{D}(X)}(\mathbf{1}_X)$ is invertible.*

Corollary 1.15. *Let $\mathbf{D}$ be an oriented $\mathbf{Q}$-linear topological weave satisfying conservativity of stalks for an algebraic space X. Then for every finite étale surjection $f : Y \twoheadrightarrow X$, the Euler characteristic (1.11) is invertible.*

Corollary 1.16. *Let $\mathbf{D}$ be a topological weave satisfying continuity and geometric generation. Let $f : Y \to X$ be a finite étale morphism of constant degree d between qcqs algebraic spaces over a noetherian algebraic space S. If the class $d_\varepsilon \in \mathrm{End}_{\mathbf{D}(\kappa)}(\mathbf{1}_\kappa)$ is invertible for every field κ over S, then the Euler characteristic (1.11) of f is invertible; in particular, f is descendable of index $\leqslant 1$.*

Proof. When X is locally noetherian, $\mathbf{D}$ satisfies conservativity of stalks for X (see [Kha, Prop. 3.13]), so the claim follows from Corollary 1.13. In general, write X as the limit of a cofiltered system $(X_\alpha)_\alpha$ of algebraic spaces of finite presentation over S, with affine transition maps (see [Ry2, Thm. D]). By [Ry2, Props. B.2, B.3] f is the base change of some finite étale $f_\alpha : Y_\alpha \to X_\alpha$ along $u_\alpha : X \to X_\alpha$ (for some α). After replacing X_α by the closed and open subspace over which f_α is of degree d (which contains the image of u_α), we may moreover assume that f_α is of constant degree d. Since X_α is noetherian, the Euler characteristic of f_α is invertible by the previous case, hence so is that of its base change f. □

Corollary 1.17. *Let $\mathbf{D}$ be an oriented $\mathbf{Q}$-linear topological weave satisfying continuity and geometric generation. Then for every finite étale surjection $f : Y \twoheadrightarrow X$ of qcqs algebraic spaces, the Euler characteristic (1.11) is invertible; in particular, f is descendable of index $\leqslant 1$.*

Proof. By radditivity, we may assume that f is of constant degree $d > 0$. As in Corollary 1.14, the orientation identifies $d_\varepsilon \simeq d$ in $\mathrm{End}_{\mathbf{D}(\kappa)}(\mathbf{1}_\kappa)$ for every field κ, and d is invertible by $\mathbf{Q}$-linearity. The claim now follows from Corollary 1.16, applied with $B = \mathrm{Spec}(\mathbf{Z})$. □

Lemma 1.18. *Let $\mathbf{D}$ be a radditive weave. Then every finite étale surjection is descendable (of index $\leqslant N$) if and only if for every finite group G, every G-torsor is descendable (of index $\leqslant N$).*

Proof. Let $f : Y \twoheadrightarrow X$ be a finite étale surjection of degree d; by radditivity we may assume that d is constant. The permutation action of the symmetric group S_d on $Y^d := Y^{\times_X d}$ is free and transitive away from the big diagonal

$$\Delta^d := \bigcup_{1 \leqslant i < j \leqslant d} \mathrm{pr}_{i,j}^{-1}(Y) \subseteq Y^d,$$

where $\mathrm{pr}_{i,j} : Y^d \to Y \times_X Y$ are the projections, and Y is regarded as a subscheme of $Y \times_X Y$ via the (open and closed) diagonal $Y \to Y \times_X Y$. Thus the map

$$g : Y^{(d)} := Y^d \smallsetminus \Delta^d \subseteq Y^d \longrightarrow X$$

is an S_d-torsor. It can be expressed equivalently as the composite

$$g : Y^{(d)} \xrightarrow{p} Y \xrightarrow{f} X,$$

where p is the restriction of the first projection of Y^d. The unit of p gives rise to a morphism of E_∞-algebras

$$f_*\mathbf{1}_Y \to f_*p_*\mathbf{1}_{Y^{(d)}} \simeq g_*\mathbf{1}_{Y^{(d)}}.$$

Now Lemma 1.7 shows that if g is descendable of index $\leqslant N$, then so is f. □

1.3. Criteria for finite radicial covers.

Lemma 1.19. *Let $\mathbf{D}$ be a weave satisfying nil-invariance. Then for every finite radicial surjection $f : Y \twoheadrightarrow X$, the following conditions are equivalent:*

(i) *The functor f^* is conservative.*

(ii) *The functor f^* is an equivalence.*

Moreover, in this case there is a canonical isomorphism $f^* \simeq f^!$.

Proof. The diagonal $\Delta : Y \to Y \times_X Y$ is a nil-immersion. By nil-invariance, we have that $\Delta^* \simeq \Delta^!$ is an equivalence. From this it follows that the counit $f^* f_* \to \mathrm{id}$ is invertible (see proof of [EK, Prop. 2.2.1] for details). Then by the triangle identities, the unit $\mathrm{id} \to f_* f^*$ is invertible if and only if f^* is conservative. When this holds, i.e. f^* is an equivalence, it follows formally that its right adjoint $f_* \simeq f_!$ is also an equivalence, and that its left and right adjoints f^* and $f^!$ are tautologically identified. □

We say that a weave $\mathbf{D}$ satisfies *topological invariance* if it satisfies nil-invariance and, for every finite radicial surjection $f : Y \twoheadrightarrow X$, the functor f^* is an equivalence. By [EK, Thm. 2.1.1, Rem. 2.2.9], we have:

Theorem 1.20. *Let $\mathbf{D}$ be a topological weave and $f : Y \twoheadrightarrow X$ a finite radicial surjection. If every prime number is invertible either in $\mathcal{O}_X$ or in $\mathrm{End}_{\mathbf{D}(X)}(\mathbf{1}_X)$, then f^* is an equivalence. In particular, $\mathbf{D}[\mathcal{P}^{-1}]$ satisfies topological invariance on the category of algebraic spaces over X, where $\mathcal{P}$ is the set of primes not invertible in $\mathcal{O}_X$.*

Proof. Let $\mathcal{Q}$ be the set of primes invertible in $\mathrm{End}_{\mathbf{D}(X)}(\mathbf{1}_X)$. Every $q \in \mathcal{Q}$ acts invertibly on $\mathbf{1}_X$, hence on $\mathbf{D}(X)$. Since $\mathbf{SH}$ is the universal topological weave, there exists a morphism of weaves $\mathbf{SH} \to \mathbf{D}$ which factors through $\mathbf{SH}[\mathcal{Q}^{-1}]$ on algebraic spaces over X. Thus the invertibility of the unit and counit of the adjunction $(f^*, f_* \simeq f_!)$ follows from the case of $\mathbf{SH}[\mathcal{Q}^{-1}]$, which is [EK, Thm. 2.1.1]. □

1.4. **Bootstrapping.** Our main result in this section is as follows:

Theorem 1.21. *Let $\mathbf{D}$ be a weave satisfying the localization property. Then the following conditions are equivalent:*

(i) *Every finite étale surjection and every finite radicial surjection of qcqs algebraic spaces is descendable.*

(ii) *Every finitely presented surjection $f : Y \twoheadrightarrow X$ of qcqs algebraic spaces is descendable.*

1.5. **Proof of Theorem 1.21.**

Lemma 1.22. *Let $f : Y \to X$ be a morphism admitting a section s. Then f is descendable of index $\leqslant 1$.*

Proof. The unit map of $s_*(\mathbf{1}_X)$ induces a retraction $f_*\mathbf{1}_Y \to f_*s_*\mathbf{1}_X \simeq \mathbf{1}_X$ of the unit of $f_*(\mathbf{1}_Y)$. □

Lemma 1.23. *Let $\mathbf{D}$ be a weave satisfying the localization property and $f : Y \to X$ a morphism. Given a closed immersion $i : Z \hookrightarrow X$ with open complement $j : U \hookrightarrow X$, let f_Z and f_U denote the base changes of f to Z and U. If f_Z and f_U are descendable of index $\leqslant N_Z$ and $\leqslant N_U$, respectively, then f is descendable of index $\leqslant N_Z + N_U$.*

Proof. By smooth base change, we have $j^*\mathcal{I}_f \simeq \mathcal{I}_{f_U}$. Consider the localization triangle $i_*i^!\mathbf{1}_X \to \mathbf{1}_X \to j_*\mathbf{1}_U$. The morphism $\varepsilon_f^{N_U}$ factors as

$$\varepsilon_f^{N_U} : \mathcal{I}_f^{\otimes N_U} \xrightarrow{\phi_f} i_*i^!\mathbf{1}_X \to \mathbf{1}_X.$$

Indeed, its composite with the unit $\mathbf{1}_X \to j_*\mathbf{1}_U$ factors through j_* of $j^*\varepsilon_f^{N_U} \simeq \varepsilon_{f_U}^{N_U} \simeq 0$, hence is null-homotopic. Thus the map $\varepsilon_f^{N_U+N_Z}$ factors through

$$\mathcal{I}_f^{\otimes N_Z+N_U} \simeq \mathcal{I}_f^{\otimes N_Z} \otimes \mathcal{I}_f^{\otimes N_U} \xrightarrow{\mathrm{id}\otimes\phi_f} \mathcal{I}_f^{\otimes N_Z} \otimes i_*i^!(\mathbf{1}_X) \simeq i_*(i^*\mathcal{I}_f^{\otimes N_Z} \otimes i^!\mathbf{1}_X)$$

and

$$i_*(i^*\mathcal{I}_f^{\otimes N_Z} \otimes i^!\mathbf{1}_X) \xrightarrow{i_*(i^*\varepsilon_f^{N_Z}\otimes\mathrm{id})} i_*(i^*\mathbf{1}_X \otimes i^!\mathbf{1}_X) \simeq i_*i^!(\mathbf{1}_X) \xrightarrow{\text{counit}} \mathbf{1}_X.$$

Since f_Z is descendable of index $\leqslant N_Z$, applying Lemma 1.7 to the morphism of algebras $\mathrm{Ex}^*_* : i^*f_*\mathbf{1}_Y \to f_{Z,*}\mathbf{1}_{Y_Z}$ yields that $i^*\varepsilon_f^{N_Z}$ is null-homotopic. The claim follows. □

Proof of Theorem 1.21. Since finite étale and finite radicial surjections are in particular finitely presented surjections, (ii) implies (i). Conversely, assume (i) and let $f : Y \twoheadrightarrow X$ be a finitely presented surjection. Suppose first that X is a qcqs scheme. Then there exists by [EGA, IV$_4$, Prop. 17.16.4][2] a stratification by locally closed subschemes X_α for which there are finite radicial and finite étale surjections, $g_\alpha : X'_\alpha \to X_\alpha$ and $h_\alpha : X''_\alpha \to X'_\alpha$, respectively, such that f admits a section after base change along each $X''_\alpha \to X$.

By Lemma 1.22, each $f \times_X X''_\alpha$ is descendable. Since g_α and h_α are descendable by assumption, Corollary 1.10, applied first along h_α and then along g_α, implies that each $f \times_X X_\alpha$ is descendable. Refining the stratification $(X_\alpha)_\alpha$ if necessary, we may choose a filtration by closed subsets $\varnothing = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n = X$ such that each $Z_k \smallsetminus Z_{k-1}$ is a disjoint union of strata X_α. Since the weave is radditive, $f \times_X (Z_k \smallsetminus Z_{k-1})$ is descendable, and applying Lemma 1.23 recursively shows that f is descendable.

For a general algebraic space X, there exists by [HR, Ex. 3.2, Prop. 3.3] a Nisnevich covering $p : X' \twoheadrightarrow X$ with X' a qcqs scheme and a monomorphic splitting sequence $\varnothing = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = X$ where each V_j is a quasi-compact open and $Z_j := (V_j \smallsetminus V_{j-1})_{\mathrm{red}}$ are schemes. By the case above, each base change $f \times_X Z_j$ is descendable. Applying Lemma 1.23 recursively to the closed subset $Z_j \subseteq V_j$ with open complement V_{j-1}, we deduce that f is descendable. □

Remark 1.24. In the context of Theorem 1.21, suppose additionally that the finite étale and finite radicial surjections in (i) are descendable *of index* $\leqslant 1$. In this case the proof yields the following uniform bound:

($*$) For every qcqs algebraic space X of finite Krull dimension there exists a natural number N_X (depending only on X) such that every finitely presented surjection $f : Y \twoheadrightarrow X$ is descendable of index $\leqslant N_X$.

[2]stated for morphisms of schemes, but we may apply it to the composite $Y' \twoheadrightarrow Y \twoheadrightarrow X$ where $Y' \twoheadrightarrow Y$ is an étale surjection from a scheme

If moreover X is a qcqs scheme of Krull dimension d_X, one may choose the filtration $(Z_k)_k$ appearing in the proof to be the filtration by dimension (i.e., Z_k is the union of the strata of dimension $\leqslant k$), yielding the bound $N_X = d_X + 1$.

2. Descent

2.1. **Abstract descent criteria.** Throughout this subsection, $\mathbf{D}$ is a *radditive* preweave, i.e., for every finite family $(X_i)_i$ of algebraic spaces the canonical morphism $\mathbf{D}(\sqcup_i X_i) \to \prod_i \mathbf{D}(X_i)$ is invertible ([Kha2, §2.3]). Given a collection of morphisms $(f_\alpha : Y_\alpha \to X)_\alpha$ of algebraic spaces, we say that a presheaf F is *separated* with respect to $(f_\alpha)_\alpha$ if the functors f_α^* are jointly conservative, and we use the notion of Čech descent along $(f_\alpha)_\alpha$ from [Kha2, §2.3].

We record the following variants of [Kha2, Lem. 2.3.3] for morphisms admitting $*$- and !-direct image.

Lemma 2.1. *Let $\mathbf{D}$ be a radditive left preweave. Let $(f_\alpha : Y_\alpha \to X)_\alpha$ be a collection of morphisms such that $\mathbf{D}$ admits $*$-direct image for every morphism obtained by base change and composition from morphisms in $(f_\alpha)_\alpha$. Given a subset of indices $\alpha_1, \ldots, \alpha_n$, we write*

$$f_{\alpha_1,\ldots,\alpha_n} : Y_{\alpha_1,\ldots,\alpha_n} := Y_{\alpha_1} \underset{X}{\times} \cdots \underset{X}{\times} Y_{\alpha_n} \to X.$$

Consider the following conditions:

(i) *The presheaf $\mathbf{D}^*$ satisfies Čech descent along $(f_\alpha : Y_\alpha \to X)_\alpha$.*

(ii) *For every $\mathcal{F} \in \mathbf{D}(X)$ the following is a limit diagram in $\mathbf{D}(X)$:*

$$\mathcal{F} \to \prod_\alpha f_{\alpha,*} f_\alpha^*(\mathcal{F}) \rightrightarrows \prod_{\alpha,\beta} f_{\alpha,\beta,*} f_{\alpha,\beta}^*(\mathcal{F}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow \\[-0.6ex]\textstyle\rightarrow}} \cdots$$

(iii) *The presheaf $\mathbf{D}^*$ is separated with respect to $(f_\alpha : Y_\alpha \to X)_\alpha$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If f_α^* preserves totalizations of f_α^*-split cosimplicial diagrams for each α, then all three listed conditions are equivalent.

Proof. We consider the adjoint functors

$$F^* : \mathbf{D}(X) \rightleftarrows \mathrm{Tot}\Big(\prod_{\alpha_1,\ldots,\alpha_\bullet} \mathbf{D}(Y_{\alpha_1,\ldots,\alpha_\bullet})\Big) : F_* \tag{2.2}$$

as in the proof of [Kha2, Lem. 2.3.3]. Note that (i) holds if and only if F^* is an equivalence, (ii) holds if and only if the unit $\mathrm{id} \to F_*F^*$ is invertible, and (iii) holds if and only if F^* is conservative. It is thus clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

For the converse direction, we apply (the dual of) [Lur2, Cor. 4.7.5.3]. Condition (1) holds by the assumption on f_α^*, and condition (2) holds because each $f_{\alpha,*}$ commutes with $*$-inverse image (since $\mathbf{D}$ admits $*$-direct image for f_α), and conservativity of $\mathbf{D}(X) \to \prod_\alpha \mathbf{D}(Y_\alpha)$ is equivalent to condition (iii). The conclusion is that (iii) implies that F^* is an equivalence, i.e. (iii) $\Rightarrow$ (i). □

Lemma 2.3. *Let $\mathbf{D}$ be a radditive weave. Let $(f_\alpha : Y_\alpha \to X)_\alpha$ be a collection of morphisms. Given a subset of indices $\alpha_1, \ldots, \alpha_n$, we write*

$$f_{\alpha_1,\ldots,\alpha_n} : Y_{\alpha_1,\ldots,\alpha_n} := Y_{\alpha_1} \underset{X}{\times} \cdots \underset{X}{\times} Y_{\alpha_n} \to X.$$

Consider the following conditions:

(i) *The presheaf $\mathbf{D}^!$ satisfies Čech descent along $(f_\alpha : Y_\alpha \to X)_\alpha$.*

(ii) *For every $\mathcal{F} \in \mathbf{D}(X)$ the following is a colimit diagram in $\mathbf{D}(X)$:*

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \bigoplus_{\alpha,\beta} f_{\alpha,\beta,!} f^!_{\alpha,\beta}(\mathcal{F}) \rightrightarrows \bigoplus_{\alpha} f_{\alpha,!} f^!_\alpha(\mathcal{F}) \to \mathcal{F}.$$

(iii) *The presheaf $\mathbf{D}^!$ is separated with respect to $(f_\alpha : Y_\alpha \to X)_\alpha$.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If the functor $f^!_\alpha$ preserves geometric realizations of $f^!_\alpha$-split simplicial diagrams for each α, then all three listed conditions are equivalent.

Proof. The proof is similar, using the functor

$$F^! : \mathbf{D}(X) \to \mathrm{Tot}\Big(\prod_{\alpha_1,\ldots,\alpha_\bullet} \mathbf{D}(Y_{\alpha_1,\ldots,\alpha_\bullet}) \Big) \tag{2.4}$$

and its left adjoint $F_!$. It is clear that (i) holds if and only if $F^!$ is an equivalence, (ii) holds if and only if the counit $F_! F^! \to \mathrm{id}$ is invertible, and (iii) holds if and only if $F^!$ is conservative. It is thus clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

For the converse direction, we again apply [Lur2, Cor. 4.7.5.3]. Condition (1) holds by the assumption on $f^!_\alpha$, and condition (2) holds because each $f_{\alpha,!}$ commutes with $*$-inverse image (by the base change formula), and conservativity of $\mathbf{D}(X) \to \prod_\alpha \mathbf{D}(Y_\alpha)$ is equivalent to condition (iii). The conclusion is that (iii) implies that $F^!$ is an equivalence, i.e. (iii) $\Rightarrow$ (i). □

We also record the following simple observation:

Lemma 2.5. *Let $\mathbf{D}$ be a radditive weave. Suppose given a morphism $g : X' \to X$ such that $\mathbf{D}^*$, resp. $\mathbf{D}^!$, satisfies Čech descent along any base change of g. Then $\mathbf{D}^*$, resp. $\mathbf{D}^!$, satisfies Čech descent along any base change of a family $(f_\alpha : Y_\alpha \to X)_\alpha$ if and only if it satisfies Čech descent along any base change of the induced family*

$$(Y_\alpha \underset{X}{\times} X' \to X')_\alpha.$$

Proof. The condition is obviously necessary, and the converse direction follows from the fact that limits commute with limits. □

2.2. Descendability and descent.

Proposition 2.6. *Let $\mathbf{D}$ be a radditive weave. Let $f : Y \to X$ be a proper morphism of algebraic spaces. If f is descendable in $\mathbf{D}$, then the presheaves $\mathbf{D}^*$ and $\mathbf{D}^!$ satisfy Čech descent along f.*

Proof. Let $\mathcal{K} \in \mathbf{D}(X)$ be an object such that $f^*\mathcal{K} \simeq 0$ and consider the full subcategory $\mathcal{C} \subseteq \mathbf{D}(X)$ spanned by $\mathcal{F}$ such that $\mathcal{K} \otimes \mathcal{F} \simeq 0$. Then $\mathcal{C}$ is a thick subcategory closed under $(-) \otimes \mathcal{F}'$ for every $\mathcal{F}' \in \mathbf{D}(X)$. By the projection formula for $f_* \simeq f_!$, $\mathcal{K} \otimes f_* \mathbf{1}_Y \simeq f_* f^* \mathcal{K} \simeq 0$, i.e., $\mathcal{C}$ contains $f_* \mathbf{1}_Y$. Since f is descendable, it follows that $\mathbf{1}_X \in \mathcal{C}$, i.e. $\mathcal{K} \simeq 0$.

Similarly, if $\mathcal{K}$ is an object with $f^! \mathcal{K} \simeq 0$, then

$$\underline{\mathrm{Hom}}(f_* \mathbf{1}_Y, \mathcal{K}) \simeq f_* \underline{\mathrm{Hom}}(\mathbf{1}_Y, f^! \mathcal{K}) \simeq f_* f^! \mathcal{K} \simeq 0.$$

Thus $f_* \mathbf{1}_Y$ belongs to the full subcategory $\mathcal{C}' \subseteq \mathbf{D}(X)$ spanned by $\mathcal{F}$ such that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{K}) \simeq 0$. This is also a thick subcategory containing $f_* \mathbf{1}_Y$, since for all $\mathcal{F}'$,

$$\underline{\mathrm{Hom}}(\mathcal{F} \otimes \mathcal{F}', \mathcal{K}) \simeq \underline{\mathrm{Hom}}(\mathcal{F}', \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{K})) \simeq 0.$$

Applying descendability again, we conclude that $\mathbf{1}_X \in \mathcal{C}'$, i.e. $\mathcal{K} \simeq 0$.

We now show that $\mathbf{D}^*$ satisfies Čech descent along f. By Lemma 2.1 and the conservativity of f^*, it suffices to show that f^* preserves totalizations of f^*-split cosimplicial objects. Let $\mathcal{K}^\bullet$ be such an object and let $\mathcal{D} \subseteq \mathbf{D}(X)$ be the full subcategory of those $\mathcal{F}$ for which the canonical morphism $f^* \operatorname{Tot}(\mathcal{K}^\bullet \otimes \mathcal{F}) \to \operatorname{Tot}(f^*(\mathcal{K}^\bullet \otimes \mathcal{F}))$ is invertible. Then $\mathcal{D}$ is thick, and contains the essential image of f_*: for every $\mathcal{G} \in \mathbf{D}(Y)$, $\mathcal{K}^\bullet \otimes f_* \mathcal{G} \simeq f_*(\mathcal{G} \otimes f^* \mathcal{K}^\bullet)$ is split since $f^* \mathcal{K}^\bullet$ is (and any functor tautologically preserves totalizations of split cosimplicial objects). Since the essential image of f_* is closed under $(-) \otimes \mathcal{F}'$ for every $\mathcal{F}' \in \mathbf{D}(X)$ (by the projection formula, $f_*(\mathcal{G}) \otimes \mathcal{F}' \simeq f_*(\mathcal{G} \otimes f^* \mathcal{F}')$), the descendability of f thus implies that $\mathcal{D}$ contains $\mathbf{1}_X$. That is, f^* preserves the totalization of $\mathcal{K}^\bullet$.

Dually, for $\mathbf{D}^!$, it suffices using Lemma 2.3 and the conservativity of $f^!$ to check that $f^!$ preserves geometric realizations of $f^!$-split simplicial objects. Let $\mathcal{K}_\bullet$ be such an object and let $\mathcal{D}' \subseteq \mathbf{D}(X)$ be the full subcategory of those $\mathcal{F}$ for which the canonical morphism $|f^! \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{K}_\bullet)| \to f^! |\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{K}_\bullet)|$ is invertible. Then $\mathcal{D}'$ is thick, and contains the essential image of f_*: by the proper projection formula

$$\underline{\mathrm{Hom}}(f_* \mathcal{G}, \mathcal{K}_\bullet) \simeq f_* \underline{\mathrm{Hom}}(\mathcal{G}, f^! \mathcal{K}_\bullet), \qquad \mathcal{G} \in \mathbf{D}(Y),$$

is split, since $f^! \mathcal{K}_\bullet$ is. By descendability we conclude again that $\mathcal{D}'$ contains $\mathbf{1}_X$, i.e., $f^!$ preserves the geometric realization of $\mathcal{K}_\bullet$. □

2.3. **Étale descent.** We say that the weave $\mathbf{D}$ satisfies *étale descent* if $\mathbf{D}^*$ (equivalently $\mathbf{D}^!$) satisfies Čech descent along any étale surjection.

Recall that a smooth morphism is surjective if and only if it is surjective on κ-valued points for all separably closed fields κ. We say it is *cd-surjective* (where *cd* stands for "completely decomposed") if it is surjective on κ-valued points for all fields κ, and $\mathbf{D}$ satisfies *Nisnevich descent* if $\mathbf{D}^*$ (equivalently $\mathbf{D}^!$) satisfies Čech descent along any cd-surjective étale morphism.

Proposition 2.7. *Let $\mathbf{D}$ be a weave. Consider the following conditions:*

(i) *The weave $\mathbf{D}$ satisfies étale descent.*

(ii) *The presheaf $\mathbf{D}^*$ (equivalently $\mathbf{D}^!$) satisfies Čech descent along finite étale surjections.*

(iii) *The presheaf $\mathbf{D}^*$ is separated with respect to finite étale surjections.*

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If $\mathbf{D}$ *satisfies Nisnevich descent, then also (iii) $\Rightarrow$ (i).*

Proof. The forward implications are clear. Suppose $\mathbf{D}$ satisfies Nisnevich descent. Since the étale topology is generated by Nisnevich covers and finite étale covers, we then have (i) $\Leftrightarrow$ (ii). By [Kha2, Lem. 2.3.3], we also have (ii) $\Leftrightarrow$ (iii). □

Corollary 2.8. *Let* $\mathbf{D}$ *be a radditive weave. If every finite étale surjection is descendable, then* $\mathbf{D}$ *satisfies étale descent.*

Proof. Combine Proposition 2.6 and Proposition 2.7. □

Corollary 2.9. *Let* $\mathbf{D}$ *be an oriented* $\mathbf{Q}$*-linear topological weave satisfying continuity and geometric generation. Then it satisfies étale descent on qcqs algebraic spaces.*

Proof. Combine Corollary 2.8 with Corollary 1.17. □

2.4. Proper descent.

Corollary 2.10. *Let* $\mathbf{D}$ *be a weave satisfying the localization property. If finite étale surjections and finite radicial surjections are descendable, then the presheaves* $\mathbf{D}^*$ *and* $\mathbf{D}^!$ *satisfy Čech descent for proper surjections of finite presentation.*

Proof. Let $f : Y \twoheadrightarrow X$ be a proper surjection of finite presentation. Since f is proper and descendable (Theorem 1.21), the claim follows from Proposition 2.6. □

Remark 2.11. In the situation of Corollary 2.10, Čech descent for a proper surjection of finite presentation $f : Y \twoheadrightarrow X$ gives in particular

$$\mathcal{F} \simeq \mathrm{Tot}(f_{\bullet,*} f_\bullet^* \mathcal{F}), \qquad \text{for all } \mathcal{F} \in \mathbf{D}(X),$$

where $Y_\bullet$ is the Čech nerve of f and $f_\bullet : Y_\bullet \to X$ the canonical morphisms. The fact that f is *descendable* (Theorem 1.21) means that this totalization is preserved by any exact functor of presentable stable ∞-categories $F : \mathbf{D}(X) \to \mathcal{V}$:

$$F\big(\mathrm{Tot}(f_{\bullet,*} f_\bullet^* \mathcal{F})\big) \simeq \mathrm{Tot}\big(F(f_{\bullet,*} f_\bullet^* \mathcal{F})\big).$$

This follows from Remark 1.2 together with the proper projection formula for $f_{\bullet,*}$.

2.5. h-descent.

From the previous results we can derive descent for Voevodsky's h-topology. Later, we will also consider the finer v-topology of [Ry3, BS], and the even finer arc-topology of [BM].

Definition 2.12.

(i) A morphism of algebraic spaces $f : Y \twoheadrightarrow X$ is an *arc-cover* if it is quasi-compact and, for every morphism $\mathrm{Spec}(V) \to X$ with V a valuation ring of rank $\leqslant 1$, there is an extension $V \hookrightarrow W$ of rank-$\leqslant 1$ valuation rings and a lift $\mathrm{Spec}(W) \to Y$.

(ii) A morphism of algebraic spaces $f : Y \twoheadrightarrow X$ is a *v-cover* if it is quasi-compact and, for every morphism $\mathrm{Spec}(V) \to X$ with V a valuation ring, there is an extension $V \hookrightarrow W$ of valuation rings and a lift $\mathrm{Spec}(W) \to Y$.

(iii) An *h-cover* is a v-cover of finite presentation.

By [Ry3, Thm. 2.8], v-covers are the same as *universally subtrusive* quasi-compact morphisms in the sense of [Ry3, Def. 2.2]. Quasi-compact faithfully flat morphisms and proper surjections are v-covers. On noetherian schemes, arc-covers are the same as v-covers (see [Ry3, Thm. 2.8]), and h-covers are v-covers of finite type.

Proposition 2.13. *Let $\mathbf{D}$ be a weave satisfying Nisnevich descent. Then $\mathbf{D}^*$ (resp. $\mathbf{D}^!$) satisfies Čech descent along h-covers if and only if it satisfies Čech descent along proper finitely presented surjections.*

Proof. Let $f : Y \twoheadrightarrow X$ be an h-cover of algebraic spaces. By [DH, Thm. 1.2], the algebraic space X admits an étale morphism $U \twoheadrightarrow X$ with Nisnevich-local sections, where U is a scheme. Choose an affine open cover $U = \bigcup_i U_i$. Since $\mathbf{D}^*$ satisfies Nisnevich descent, we may use Lemma 2.5 to replace f by the base changes $f \times_X U_i$ and thereby assume X is an affine scheme. By [Ry3, Thm. 3.12], f admits a refinement $f' : Y' \to X$ which factors as a quasi-compact open covering $p : Y' \twoheadrightarrow X'$ followed by a finitely presented proper surjection $q : X' \twoheadrightarrow X$. By assumption, p and q satisfy Čech descent after any base change, so [LZ, Lem. 3.1.2 (4)] now shows that f' satisfies Čech descent, hence by [LZ, Lem. 3.1.2 (3)] f itself satisfies Čech descent. □

3. Rational motivic sheaves

For an algebraic space S, we denote by $\mathbf{SH}(S)$ the stable ∞-category of motivic spectra over S. We denote by $\mathbf{SH}(S)_{\mathbf{Q}}$ its rationalization, which admits the functorial decomposition (see e.g. [CD, §16.2])

$$\mathbf{SH}(S)_{\mathbf{Q}} \simeq \mathbf{SH}(S)_{\mathbf{Q},+} \times \mathbf{SH}(S)_{\mathbf{Q},-}.$$

We will write $\mathbf{DM}(S)_{\mathbf{Q}} \coloneqq \mathbf{SH}(S)_{\mathbf{Q},+}$ and refer to it as the ∞-category of *rational motivic sheaves* on S. We will write $\mathbf{Q} \coloneqq \mathbf{1}_{\mathbf{Q},+}$ for its unit object.

This construction defines a topological weave over arbitrary algebraic spaces, and agrees with other models of rational motivic sheaves defined under more restrictive hypotheses. For example, it is equivalent to Voevodsky's category when S is geometrically unibranch and excellent (see [CD, Thm. 16.1.4, Thm. 16.2.13]).

3.1. Descendability and h-descent.

The weave $\mathbf{SH}$ itself satisfies neither étale descent nor topological invariance. We mention the following positive result.

Proposition 3.1. *Let X be a scheme over $\mathbf{F}_p$ and $f : Y \twoheadrightarrow X$ a finite étale surjection of degree $d = p^k$, $k \geqslant 1$. Then $\mathbf{SH}^*[1/p]$ satisfies Čech descent along f.*

Proof. We may assume that X is qcqs. For every field κ over $\mathbf{F}_p$, the class d_ε is invertible in $\mathrm{End}_{\mathbf{SH}(\kappa)[1/p]}(\mathbf{1}_\kappa)$ by [EK, Lem. 2.2.8], since d is a power of p by assumption. As $\mathbf{SH}[1/p]$ satisfies continuity and geometric generation (see e.g. [DFJK, Exs. A.4, A.9]), it follows by Corollary 1.16 applied with $B = \mathrm{Spec}(\mathbf{F}_p)$ that the Euler characteristic (1.11) is invertible, whence f is descendable of index $\leqslant 1$. We conclude by Proposition 2.6. ☐

After passing to the rationalization $\mathbf{SH}_{\mathbf{Q}}$, topological invariance holds by [EK] (cf. Theorem 1.20). However, $\mathbf{SH}_{\mathbf{Q}}$ still does not satisfy étale descent. The plus part $\mathbf{SH}_{\mathbf{Q},+} \simeq \mathbf{DM}_{\mathbf{Q}}$ is an oriented $\mathbf{Q}$-linear topological weave (see e.g. [CD, §14, §16.2]) satisfying continuity and geometric generation (see e.g. [DFJK, Exs. A.4, A.9]). Hence any finite étale surjection of qcqs algebraic spaces has Euler characteristic (1.11) invertible in $\mathbf{DM}_{\mathbf{Q}}$ (Corollary 1.17); in particular, it is descendable of index $\leqslant 1$. By topological invariance, the same holds for finite radicial surjections (Theorem 1.20). Thus Theorem 1.21, together with Remark 1.24, yields:

Theorem 3.2. *Let $f : Y \twoheadrightarrow X$ be a finitely presented surjection of qcqs algebraic spaces. Then f is descendable in $\mathbf{DM}_{\mathbf{Q}}$. Moreover, if X is of Krull dimension d then there exists a natural number N_X (depending only on X) such that every finitely presented surjection $f : Y \twoheadrightarrow X$ is descendable of index $\leqslant N_X$ in $\mathbf{DM}_{\mathbf{Q}}$.*

By Corollary 2.8, Corollary 2.10, and Proposition 2.13 we thus have:

Corollary 3.3. *The presheaves $\mathbf{DM}^*_{\mathbf{Q}}$ and $\mathbf{DM}^!_{\mathbf{Q}}$ satisfy h-descent on algebraic spaces.*

Corollary 3.3 is a folklore result which, using ∞-categorical machinery, is not difficult to derive from the work of Cisinski–Déglise in [CD] (compare [CD, Thms. 14.3.4 and 16.2.13], over quasi-excellent schemes). In Sect. 5 below, we will require the stronger form of descent provided by Theorem 3.2.

3.2. **v- and arc-descent.** We next turn to the question of descent for the finer v- and arc-topologies.

Question 3.4. *Does the presheaf $\mathbf{DM}^*_{\mathbf{Q}}$ satisfy v-descent (or arc-descent)?*

We do not see any particular reason to believe the answer is positive. The following conjecture seems more credible. We write

$$\mathrm{C}^\bullet_{\mathrm{mot}}(X;\mathcal{F}) \coloneqq \Gamma(X;\mathcal{F}), \qquad \mathrm{H}^s_{\mathrm{mot}}(X;\mathcal{F}) \coloneqq \pi_{-s}\,\mathrm{C}^\bullet_{\mathrm{mot}}(X;\mathcal{F})$$

for $\mathcal{F} \in \mathbf{DM}(X)_{\mathbf{Q}}$ and $s \in \mathbf{Z}$.

Conjecture 3.5. *For every integer $r \in \mathbf{Z}$, the presheaf $\mathrm{C}^\bullet_{\mathrm{mot}}(-;\mathbf{Q}(r))$ satisfies arc-descent.*

We will take a rather heavy-handed approach to this problem, assuming the following generalized Beilinson–Soulé type vanishing conditions for valuation rings, for each $r \in \mathbf{Z}$:

(BS$_r$) There exists an integer N_r such that for every finite rank henselian valuation ring V, $\mathrm{H}^m_{\mathrm{mot}}(V;\mathbf{Q}(r)) = 0$ for all $m < -N_r$.

Theorem 3.6. *Let $r \in \mathbf{Z}$. If (BS$_r$) holds, then the presheaf $\mathrm{C}^\bullet_{\mathrm{mot}}(-;\mathbf{Q}(r))$ satisfies v- and arc-hyperdescent on algebraic spaces.*

Unconditionally, we deduce:

Corollary 3.7. *For $r \leqslant 1$, the presheaf $\mathrm{C}^\bullet_{\mathrm{mot}}(-;\mathbf{Q}(r))$ satisfies v- and arc-hyperdescent.*

Proof. By Theorem 3.6 it suffices to verify (BS$_r$) for $r \leqslant 1$. In fact, for every qcqs scheme U we have:

(a) For $r < 0$, $\mathrm{H}^s_{\mathrm{mot}}(U;\mathbf{Q}(r)) \simeq 0$ for all $s \in \mathbf{Z}$. In the stated generality, this may be seen as follows: the Tate twist $\mathbf{Q}(r)$ is identified with $\mathrm{KGL}^{(r)}_{\mathbf{Q}}[-2r]$, the shifted r-th Adams eigenspace of $\mathrm{KGL}_{\mathbf{Q}}$ (see [Rio], [CD, Lem. 14.1.4]). By [BEM, Cor. 4.52(2), Rem. 4.22], this is the rationalization of the r-th slice $s^r\mathrm{KGL}$, and $\Gamma(U, s^r\mathrm{KGL}) := \mathrm{Maps}_{\mathbf{SH}(U)}(\mathbf{1}_U, s^r\mathrm{KGL}) \simeq 0$ holds by construction of the slice filtration.

(b) For $r = 0$, $\mathrm{C}^\bullet_{\mathrm{mot}}(U;\mathbf{Q}) \simeq \Gamma_{\mathrm{cdh}}(U,\mathbf{Q})$ is the cdh hypercohomology of the constant sheaf $\mathbf{Q}$ (see e.g. [BEM, Thm. 1.1(6)]), hence $\mathrm{H}^s_{\mathrm{mot}}(U;\mathbf{Q}) = 0$ for $s < 0$.

(c) For $r = 1$, $\mathrm{C}^\bullet_{\mathrm{mot}}(-;\mathbf{Q}(1)) \simeq \Gamma_{\mathrm{cdh}}(-;\mathbf{G}_m \otimes \mathbf{Q})[-1]$ (see e.g. [BEM, Thm. 1.1(6)]), hence $\mathrm{H}^s_{\mathrm{mot}}(U;\mathbf{Q}(1)) \simeq \mathrm{H}^{s-1}_{\mathrm{cdh}}(U;\mathbf{G}_m)_{\mathbf{Q}} \simeq 0$ for $s \leqslant 0$ since $\mathbf{G}_m$ is discrete. □

The proof of Theorem 3.6 will make use of the following lemmas.

Lemma 3.8. *Let $r \in \mathbf{Z}$. The condition (BS$_r$) is equivalent to the existence of an integer N_r such that $\mathrm{C}^\bullet_{\mathrm{mot}}(X;\mathbf{Q}(r))$ is N_r-coconnective for every qcqs scheme X.*

Proof. Set $F := \mathrm{C}^\bullet_{\mathrm{mot}}(-;\mathbf{Q}(r))$ and let N_r be an integer such that $F(V)$ is N_r-coconnective for every henselian valuation ring V of finite rank. Since $\mathbf{DM}_{\mathbf{Q}}$ satisfies continuity and cdh descent, the same holds for F. Throughout the following, we regard F as a presheaf on the site of qcqs schemes, or equivalently by continuity, on the site of schemes finitely presented over $\mathrm{Spec}(\mathbf{Z})$ (see [EHIK2, Prop. 2.1.6]).

Let $K := \tau_{\geqslant N_r+1}(F)$ denote the objectwise (N_r+1)-connective cover of F, whose cdh sheafification $K_{\mathrm{cdh}} := \tau^{\mathrm{cdh}}_{\geqslant N_r+1}(F)$ is the connective cover in cdh sheaves of spectra. Since $\tau_{\geqslant N_r+1}$ commutes with filtered colimits of spectra, K still satisfies continuity, and hence so does K_{cdh} by [BEM, Prop. 2.15].

By noetherian approximation, any qcqs scheme X may be written as the limit of a cofiltered system of schemes of finite type over $\mathbf{Z}$ with affine transition maps. Fix an index α and a finite rank henselian valuation ring V over X_α. Since henselian valuation rings are the points of the cdh topos (see [EHIK2, §2.1]), we have $K_{\mathrm{cdh}}(V) \simeq K(V) \simeq \tau_{\geqslant N_r+1}F(V)$, which vanishes by (BS$_r$). Since X_α is of finite valuative dimension (see [EHIK2, Prop. 2.3.2 (9)]), we may now apply [EHIK2, Cor. 2.4.19] to deduce that $K_{\mathrm{cdh}}|_{X_\alpha}$ vanishes as a cdh sheaf on schemes over X_α. In particular, $K_{\mathrm{cdh}}(X_\alpha) \simeq 0$. Since this holds for all α, the continuity of K_{cdh} now implies that $K_{\mathrm{cdh}}(X) \simeq 0$ for every qcqs scheme X.

Consider now the exact triangle $\tau^{\mathrm{cdh}}_{\geqslant N_r+1}(F) \to F \to \tau^{\mathrm{cdh}}_{\leqslant N_r}(F)$ of cdh sheaves of spectra. By the previous paragraph, we have $F(X) \simeq \tau^{\mathrm{cdh}}_{\leqslant N_r} F(X)$, which is N_r-truncated. □

Lemma 3.9. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geqslant 0}, \mathcal{C}_{\leqslant 0})$ which is compatible with filtered colimits and right-separated. Let $(M^\bullet_\alpha)_\alpha$ be a filtered diagram of cosimplicial objects of $\mathcal{C}$ and N an integer such that $M^n_\alpha \in \mathcal{C}_{\leqslant N}$ for all α and $[n] \in \mathbf{\Delta}$. Then the canonical morphism*

$$\varinjlim_\alpha \mathrm{Tot}(M^\bullet_\alpha) \to \mathrm{Tot}\big(\varinjlim_\alpha M^\bullet_\alpha\big) \tag{3.10}$$

is invertible.

Proof. For any cosimplicial object $M^\bullet$, the fibre of $\mathrm{Tot}_{\leqslant m}(M^\bullet) \to \mathrm{Tot}_{\leqslant m-1}(M^\bullet)$ is $M'[-m]$ for M' a direct summand of M^m (by the dual of [Lur2, Rem. 1.2.4.3]). If $M^n \in \mathcal{C}_{\leqslant N}$ for all $n \in \mathbf{\Delta}$, it follows that the fibre of $\mathrm{Tot}_{\leqslant m}(M^\bullet) \to \mathrm{Tot}_{\leqslant m'}(M^\bullet)$ lies in $\mathcal{C}_{\leqslant N-m'-1}$ for all $m > m' \geqslant 0$. Passing to the limit over m, the same holds for the fibre of $\mathrm{Tot}(M^\bullet) \to \mathrm{Tot}_{\leqslant m'}(M^\bullet)$ (since coconnectivity is stable under limits). In particular $\mathrm{Tot}(M^\bullet) \in \mathcal{C}_{\leqslant N}$ and $\pi_k \mathrm{Tot}(M^\bullet) \to \pi_k \mathrm{Tot}_{\leqslant m'}(M^\bullet)$ is invertible whenever $m' \geqslant N - k + 1$.

The source and target of (3.10) both lie in $\mathcal{C}_{\leqslant N}$, since $\mathcal{C}_{\leqslant N}$ is closed under limits and filtered colimits. Fix k and set $m' = N - k + 1$. Since $\mathrm{Tot}_{\leqslant m'}$ commutes with filtered colimits (as a finite limit), and π_k commutes with filtered colimits by assumption, we have

$$\pi_k\big(\varinjlim_\alpha \mathrm{Tot}(M^\bullet_\alpha)\big) \simeq \varinjlim_\alpha \pi_k \mathrm{Tot}_{\leqslant m'}(M^\bullet_\alpha) \simeq \pi_k \mathrm{Tot}_{\leqslant m'}\big(\varinjlim_\alpha M^\bullet_\alpha\big) \simeq \pi_k \mathrm{Tot}\big(\varinjlim_\alpha M^\bullet_\alpha\big),$$

compatibly with the canonical morphism, where the last isomorphism uses $\varinjlim_\alpha M^p_\alpha \in \mathcal{C}_{\leqslant N}$. The fibre of (3.10) therefore lies in $\mathcal{C}_{\leqslant N}$ and has vanishing homotopy objects; in particular it lies in $\bigcap_n \mathcal{C}_{\leqslant -n}$, hence is zero by the right-separatedness assumption. □

Proof of Theorem 3.6. Condition (BS$_r$) implies that the presheaf $\mathrm{C}^\bullet_{\mathrm{mot}}(-;\mathbf{Q}(r))$ takes values in N_r-coconnective spectra (Lemma 3.8). Hence by [EHIK2, Lem. 3.1.7 (2), Ex. 3.1.6][3], it is a v- or arc-hypersheaf if and only if it is a v- or arc-sheaf.

Let $f : Y \twoheadrightarrow X$ be a v-cover of algebraic spaces. The claim is étale-local on X by Corollary 2.9 and smooth base change, so we may assume X affine. Since f is quasi-compact, Y is quasi-compact and hence admits an étale surjection $p : Y' \twoheadrightarrow Y$ with Y' affine. By Čech descent for p and its base

[3]Following [BM, Def. 4.16], we define the v- and arc-toposes of sheaves on the (non-essentially-small) site of qcqs schemes by fixing an uncountable strong limit cardinal κ and restricting to κ-bounded qcqs schemes as in *loc. cit.* Since $\mathrm{C}^\bullet_{\mathrm{mot}}(-;\mathbf{Q}(r))$ is continuous, the statement is independent of κ.

changes, [LZ, Lem. 3.1.2 (3),(4)] reduces Čech descent along f to Čech descent along $g := f \circ p$. Thus we may assume X and Y are affine. By [BS, Lem. 2.12], f is then the limit of a cofiltered diagram of h-covers $(f_\alpha : Y_\alpha \to X)_\alpha$ with affine transition maps. By continuity ([Kha, Rem. 2.76]; see [DFJK, Exs. A.4, A.9]) the canonical maps

$$\varinjlim_{\alpha} \mathrm{C}^\bullet_{\mathrm{mot}}(Y_{\alpha,n}; \mathbf{Q}) \to \mathrm{C}^\bullet_{\mathrm{mot}}(Y_n; \mathbf{Q})$$

are invertible for each $[n] \in \mathbf{\Delta}$. By Čech descent along each f_α (Corollary 3.3) it will suffice to show that the filtered colimit on the left-hand side commutes with the totalization over $[n] \in \mathbf{\Delta}$. This follows from Lemma 3.9 in view of the fact that $\mathrm{C}^\bullet_{\mathrm{mot}}(-; \mathbf{Q}(r))$ takes values in N_r-coconnective spectra.

For arc-descent, we apply the criterion of [BM, Thm. 4.1] to $\mathrm{C}^\bullet_{\mathrm{mot}}(-; \mathbf{Q}(r))$ regarded as a presheaf with values in N_r-coconnective spectra; this ∞-category is compactly generated by cotruncated objects (see [EHIK2, Ex. 3.1.6]). Since $\mathrm{C}^\bullet_{\mathrm{mot}}(-; \mathbf{Q}(r))$ satisfies continuity and v-descent, it remains to check that for every absolutely integrally closed valuation ring V and every prime ideal $\mathfrak{p} \subseteq V$, the square

$$\begin{tikzcd}
\mathrm{C}^\bullet_{\mathrm{mot}}(V; \mathbf{Q}(r)) \arrow[r] \arrow[d] & \mathrm{C}^\bullet_{\mathrm{mot}}(V/\mathfrak{p}; \mathbf{Q}(r)) \arrow[d] \\
\mathrm{C}^\bullet_{\mathrm{mot}}(V_\mathfrak{p}; \mathbf{Q}(r)) \arrow[r] & \mathrm{C}^\bullet_{\mathrm{mot}}(\kappa(\mathfrak{p}); \mathbf{Q}(r))
\end{tikzcd}$$

is cartesian. By [BM, Prop. 2.8], this is a special case of Milnor excision for $\mathrm{C}^\bullet_{\mathrm{mot}}(-; \mathbf{Q}(r))$ (see [EHIK, Cor. 1.2]). □

4. Étale motivic spectra

Let X be an algebraic space. We denote by $\mathbf{SH}_{\text{ét}}(X)$ the ∞-category of étale motivic spectra over X; that is, the full subcategory of $\mathbf{SH}(X)$ spanned by motivic spectra satisfying étale hyperdescent. The assignment $X \mapsto \mathbf{SH}_{\text{ét}}(X)$ forms a topological weave satisfying étale hyperdescent (e.g. by [Kha2, Prop. 2.3.2]). We write $\mathcal{S}$ for the category of noetherian algebraic spaces of finite Krull dimension, $\mathcal{S}_0$ for the full subcategory of those whose residue fields have uniformly bounded virtual étale cohomological dimension, and $\mathcal{S}_1 \subseteq \mathcal{S}_0$ for the further subcategory of those whose residue fields have uniformly bounded étale cohomological dimension.

4.1. **Descendability and h-descent.** Our main result is as follows:

Theorem 4.1. *For $X \in \mathcal{S}_1$, every finite type surjection $f : Y \twoheadrightarrow X$ is descendable in $\mathbf{SH}_{\text{ét}}$. For $X \in \mathcal{S}_0$, every finite type surjection $f : Y \twoheadrightarrow X$ is descendable in $\mathbf{SH}_{\text{ét}}[1/2]$.*

By Corollary 2.10 and Proposition 2.13 this implies:

Corollary 4.2. *On the category $\mathcal{S}_1$, the presheaves $\mathbf{SH}^*_{\text{ét}}$ and $\mathbf{SH}^!_{\text{ét}}$ satisfy h-descent. Similarly for the presheaves $\mathbf{SH}^*_{\text{ét}}[1/2]$ and $\mathbf{SH}^!_{\text{ét}}[1/2]$ on $\mathcal{S}_0$.*

4.2. **Proof of Theorem 4.1.** We first observe that any $X \in \mathcal{S}_0$ is of uniformly bounded p-étale cohomological dimension $\leqslant c$ for all odd p: by [Bac3, Cor. 2.13] it suffices to check that $\sup \operatorname{cd}_p k(x) < \infty$, over $x \in X$ and odd primes p; but $\operatorname{cd}_p k(x) \leqslant \operatorname{vcd} k(x)$, so this follows by the assumption. If moreover $X \in \mathcal{S}_1$, then the same holds at all primes p, since $\operatorname{cd}_p k(x) \leqslant \operatorname{cd} k(x)$ is then uniformly bounded by definition.

Lemma 4.3. *On the category $\mathcal{S}$, there is a canonical equivalence $\mathbf{SH}_{\text{ét}}[1/2]_- \simeq 0$.*

Proof. There is a functorial splitting $\mathbf{SH}_{\text{ét}}[1/2] \simeq \mathbf{SH}_{\text{ét}}[1/2]_+ \times \mathbf{SH}_{\text{ét}}[1/2]_-$, where $\langle -1 \rangle \simeq 1$ on the plus part. The minus part is equivalently identified with the η-periodization $\mathbf{SH}_{\text{ét}}[1/2][\eta^{-1}]$ for η the algebraic Hopf map (see e.g. [ALP, Rem. 3.6], [DFJK, (1.0.a)]); this vanishes by [MT, Thm. A], i.e. $\mathbf{SH}_{\text{ét}}[1/2] \simeq \mathbf{SH}_{\text{ét}}[1/2]_+$. □

Lemma 4.4. *The weave $\mathbf{SH}_{\text{ét}}$ satisfies conservativity of stalks for every $X \in \mathcal{S}_0$.*

Proof. By [Bac3, Cor. 5.12], it will suffice to show that X admits an étale cover by schemes over which every finite type scheme is of uniformly bounded étale cohomological dimension. Consider the étale cover $\{X_i, X_\omega\}$ of X, where

$$X_i := X \otimes_{\mathbf{Z}} \mathbf{Z}[\tfrac{1}{2}, x]/(x^2+1), \qquad X_\omega := X \otimes_{\mathbf{Z}} \mathbf{Z}[\tfrac{1}{3}, x]/(x^2+x+1),$$

cf. [Bac3, Ex. 2.14]. Every residue field K of X_i (resp. X_ω) is generated over a residue field κ of X by a primitive fourth (resp. third) root of unity; in particular, -1 (resp. -3) is a square in K, so K is not formally real, and its absolute Galois group is therefore torsion-free by the Artin–Schreier theorem. By [Ser, I.§3.3, Props. 14 and 14′], it then follows that

$$\operatorname{cd}_p(K) = \operatorname{cd}_p(K(\sqrt{-1})) \leqslant \operatorname{vcd}(K) \leqslant \operatorname{vcd}(\kappa)$$

for every prime p. Thus X_i and X_ω have residue fields of uniformly bounded étale cohomological dimension, and the claim now follows by [Bac3, Cor. 2.13]. □

Proposition 4.5. *On the category $\mathcal{S}_0$, the weave $\mathbf{SH}^*_{\text{ét}}$ satisfies topological invariance.*

Proof. For every prime ℓ, ℓ is invertible in $\operatorname{End}_{\mathbf{SH}_{\text{ét}}(S)}(\mathbf{1}_S)$ where $S = \operatorname{Spec}(\mathbf{F}_\ell)$ by [BH2, Thm. A.1]. It follows that for every field κ of characteristic $p > 0$, p is invertible in $\operatorname{End}_{\mathbf{SH}_{\text{ét}}(S)}(\mathbf{1}_S)$ where $S = \operatorname{Spec}(\kappa)$. By Theorem 1.20 it follows that f^* is conservative for any finite radicial surjection $f : Y \twoheadrightarrow X$ with X the spectrum of a field. The general case then follows by Lemma 1.19 and conservativity of stalks (Lemma 4.4). □

In order to show descendability for finitely presented surjections in $\mathbf{SH}_{\text{ét}}$, the only obstruction after Theorem 1.21, Lemma 1.18, and Proposition 4.5, is the case of torsors under finite groups.

Proposition 4.6. *Let $X \in \mathcal{S}_0$ with uniformly bounded p-étale cohomological dimension $\leqslant c$ for every odd prime p. Let $f : Y \to X$ be a G-torsor where*

G is a finite group of order d. Then f is descendable of index $\leqslant 2(c+1)$ in $\mathbf{SH}_{\text{ét}}(X)[1/2]$. If X is moreover of uniformly bounded 2-étale cohomological dimension $\leqslant c$, then f is descendable of index $\leqslant 2(c+1)$ in $\mathbf{SH}_{\text{ét}}(X)$.

We treat the two cases in parallel; all objects below are formed in the weave $\mathbf{SH}_{\text{ét}}[1/2]$, resp. $\mathbf{SH}_{\text{ét}}$ in the second case. To simplify notation we write $\mathbf{1} := \mathbf{1}_X$ and $A := f_*(\mathbf{1}_Y)$. By Lemma 1.8, we may divide the proof of Proposition 4.6 into the following two statements:

(4.6.1) *Let $\mathbf{1}^\wedge_d$ be the d-completion of $\mathbf{1}$ and consider the E_∞-algebra $P_d := \mathbf{1}[1/d] \times \mathbf{1}^\wedge_d$. The unit map $\mathbf{1} \to P_d$ is descendable of index $\leqslant 2$.*

(4.6.2) *The unit $\mathbf{1} \to A$ becomes descendable of index $\leqslant c+1$ after extension of scalars along $\mathbf{1} \to P_d$.*

4.3. **Proof of (4.6.1).** Recall the cartesian square (see e.g. [BH1, Lem. 3.19])

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d] & \mathbf{1}^\wedge_d \arrow[d] \\ \mathbf{1}[1/d] \arrow[r] & \mathbf{1}^\wedge_d[1/d]. \end{tikzcd}$$

This yields the exact triangle $\mathbf{1} \to P_d \to Q_d$, where $Q_d := \mathbf{1}^\wedge_d[1/d]$. The claim is that for $I_d \simeq Q_d[-1]$ the fibre of the unit map $\mathbf{1} \to P_d$, the map $\varepsilon^2 : I_d^{\otimes 2} \to \mathbf{1}$ is null-homotopic.

Now Q_d is a module over $\mathbf{1}^\wedge_d$, hence a direct summand of $\mathbf{1}^\wedge_d \otimes Q_d$ via the identity

$$\mathrm{id} \simeq \big[Q_d \simeq \mathbf{1} \otimes Q_d \xrightarrow{\mathrm{unit}\otimes\mathrm{id}} \mathbf{1}^\wedge_d \otimes Q_d \xrightarrow{\mathrm{act}} Q_d\big].$$

Tensoring the splitting $P_d = \mathbf{1}[1/d] \oplus \mathbf{1}^\wedge_d$ with Q_d exhibits $\mathbf{1}^\wedge_d \otimes Q_d$, and hence Q_d, as a direct summand of the free P_d-module $P_d \otimes Q_d$. Shifting, $I_d \simeq Q_d[-1]$ is a direct summand of the P_d-module $P_d \otimes I_d$.

To prove that $\varepsilon^2 : I_d^{\otimes 2} \to \mathbf{1}$ is null-homotopic we make use of the following observation:

($*$) For every P_d-module N, the map $\varepsilon \otimes \mathrm{id}_N : I_d \otimes N \to \mathbf{1} \otimes N \simeq N$ is null-homotopic.

Indeed, consider the composite

$$I_d \otimes N \xrightarrow{\varepsilon\otimes\mathrm{id}_N} \mathbf{1} \otimes N \xrightarrow{\mathrm{unit}\otimes\mathrm{id}_N} P_d \otimes N \xrightarrow{\mathrm{act}} N.$$

The composite of the two rightmost arrows is the identity of N, while the composite of the two leftmost is null-homotopic (via the null-homotopy of $I_d \to \mathbf{1} \to P_d$). It follows that the entire composite is $\varepsilon \otimes \mathrm{id}_N$ and that it is null-homotopic.

We apply ($*$) to $N := P_d \otimes I_d$. Since I_d is a direct summand of N, the map $\varepsilon \otimes \mathrm{id}_I : I_d^{\otimes 2} \to I_d$ is a direct summand of $\varepsilon \otimes \mathrm{id}_N$, hence also null-homotopic. Hence the composite

$$\varepsilon^{\otimes 2} : I_d^{\otimes 2} \xrightarrow{\varepsilon\otimes\mathrm{id}_I} I_d \xrightarrow{\varepsilon} \mathbf{1}$$

is also null-homotopic, as claimed.

4.4. **Proof of (4.6.2).** The claim is that

$$A \otimes P_d \simeq f_*(\mathbf{1}_Y) \otimes (\mathbf{1}[1/d] \times \mathbf{1}^\wedge_d) \simeq f_*\big(\mathbf{1}_Y[1/d]\big) \times f_*\big((\mathbf{1}_Y)^\wedge_d\big)$$

is descendable of index $\leqslant c+1$ over P_d. This amounts to the claim that f is descendable of index $\leqslant c+1$ in the weaves $\mathbf{SH}_{\text{ét}}[1/2, 1/d]$ and $\mathbf{SH}_{\text{ét}}[1/2]^\wedge_d$ (resp. $\mathbf{SH}_{\text{ét}}[1/d]$ and $\mathbf{SH}_{\text{ét}}{}^\wedge_d$, in the second case). For $\mathbf{SH}_{\text{ét}}[1/2, 1/d]$ (resp. $\mathbf{SH}_{\text{ét}}[1/d]$) the claim follows from Corollary 1.13 (where conservativity of stalks for X holds by Lemma 4.4) and the following lemma, which was suggested to us by Bachmann.

Lemma 4.7. *For every field κ and every natural number d, the class d_ε is invertible in $\mathrm{End}_{\mathbf{SH}_{\text{ét}}(\kappa)[1/d]}(\mathbf{1}_\kappa)$ (where the notation is as in Lemma 1.12).*

Proof. Recall that d_ε is by definition the sum of $\langle(-1)^{i-1}\rangle$ over $1 \leqslant i \leqslant d$, and that the class $\langle a \rangle \in \mathrm{End}_{\mathbf{SH}(\kappa)}(\mathbf{1}_\kappa)$ of a unit $a \in \kappa^\times$ depends only on its square class: under the identification $\mathrm{End}_{\mathbf{SH}(\kappa)}(\mathbf{1}_\kappa) \simeq \mathrm{GW}(\kappa)$ of Morel (cf. [BH, Thm. 10.12]), it is the quadratic form $\langle a \rangle$. In particular, if -1 is a square in κ (e.g. if κ is of characteristic 2), then $\langle -1 \rangle \simeq \langle 1 \rangle \simeq 1$, so $d_\varepsilon \simeq d$ is invertible in $\mathrm{End}_{\mathbf{SH}_{\text{ét}}(\kappa)[1/d]}(\mathbf{1}_\kappa)$.

In general, we may assume κ is not of characteristic 2, so that $\kappa' := \kappa[x]/(x^2+1)$ is a finite étale κ-algebra of degree 2, in every residue field of which -1 is a square. Since $\mathbf{SH}_{\text{ét}}$ satisfies étale descent, the pullback functor p^* along $p : \mathrm{Spec}(\kappa') \twoheadrightarrow \mathrm{Spec}(\kappa)$ is conservative, and remains so after restricting to $[1/d]$-local objects. The class d_ε is stable under base change, so by the previous paragraph d_ε becomes a unit in $\mathbf{SH}_{\text{ét}}(\kappa')[1/d]$, hence by conservativity is invertible already in $\mathbf{SH}_{\text{ét}}(\kappa)[1/d]$. □

For $\mathbf{SH}_{\text{ét}}[1/2]^\wedge_d$, which is by definition the product of $\mathbf{SH}_{\text{ét}}[1/2]^\wedge_p$ over all odd primes p dividing d (resp. $\mathbf{SH}_{\text{ét}}{}^\wedge_d$, the product of $\mathbf{SH}_{\text{ét}}{}^\wedge_p$ over all primes p dividing d), we argue as follows, uniformly in p. By [BH2, Thm. A.1], $\mathbf{SH}_{\text{ét}}(Z_p)^\wedge_p \simeq 0$ for $Z_p := X \otimes_{\mathbf{Z}} \mathbf{F}_p$, the characteristic p part of X. Hence by Lemma 1.23 (since p-completion is exact, $\mathbf{SH}_{\text{ét}}(-)^\wedge_p$ still satisfies the localization property), we may replace X by $X[1/p]$ and thereby assume p is invertible on X. Then Bachmann's rigidity equivalence

$$\mathbf{SH}_{\text{ét}}(X)^\wedge_p \simeq \widehat{\mathrm{Shv}}(X_{\text{ét}})^\wedge_p$$

identifies $\mathbf{SH}_{\text{ét}}(X)^\wedge_p$ with the p-completion of the ∞-category of hypercomplete sheaves of spectra on the small étale site of X (see [Bac2, Thm. 3.1], which requires only that p be invertible on X; cf. [Bac3, Thm. 6.6]). This reduces us to the following statement (where $f : Y \to X$ is still as in Proposition 4.6):

(4.6.2.a) *The morphism f is descendable of index $\leqslant c+1$ in $\widehat{\mathrm{Shv}}(X_{\text{ét}})^\wedge_p$.*

We write $\mathcal{C} := \widehat{\mathrm{Shv}}(X_{\text{ét}})^\wedge_p$ and denote its unit object by $\mathbf{1}$. Assertion (4.6.2.a) is precisely that the map $\mathbf{1} \to \mathrm{Tot}_{\leqslant c}(A^\bullet)$ admits a retraction in $\mathcal{C}$, where $A^\bullet$ is the Čech nerve, with $A^n = A^{\otimes n+1}$. Writing $f_\bullet : Y_\bullet \to X$ for the Čech nerve of $f : Y \twoheadrightarrow X$, and $\Sigma^\infty_+ Y_n$ for the object of $\mathcal{C}$ represented by Y_n, we have

$$\underline{\mathrm{Hom}}(\Sigma^\infty_+ Y_\bullet, \mathbf{1}) \simeq f_{\bullet,*} f^*_\bullet(\mathbf{1}) \simeq A^\bullet$$

using $f_{n,*} f^*_n \simeq f_{n,!} f^!_n$ (since each f_n is finite étale) and the projection formula. Writing $|\mathrm{sk}_i(Y_\bullet)|$ for the geometric realization of the i-skeleton, it will thus suffice to show that for F_i the fibres

$$F_i \to |\mathrm{sk}_i \Sigma^\infty_+ Y_\bullet| \to \mathbf{1},$$

the boundary map $\partial : \mathbf{1}[-1] \to F_c$ is null-homotopic. We claim more generally that for every $i \geqslant -1$ the map $F_i \to F_{i+c+1}$ is null-homotopic (for $i = -1$, $F_{-1} \simeq \mathbf{1}[-1]$ and this is the boundary ∂). By a standard Postnikov filtration argument, it will suffice to show the following:

(i) F_{i+c+1} is Postnikov-complete.

(ii) F_i is of cohomological dimension $\leqslant c + i$ for every $i \geqslant -1$.

(iii) F_{i+c+1} is $(i + c + 1)$-connective.

Here we say that an object $\mathcal{F} \in \mathcal{C}$ is of *cohomological dimension* $\leqslant N$ if

$$H^s(\mathcal{F}; \mathcal{G}) := \pi_0 \mathrm{Maps}_{\mathcal{C}}(F, \mathcal{G}[s]) \simeq 0$$

for all $\mathcal{G} \in \mathcal{C}^\heartsuit$ and all $s > N$. This condition is closed under extensions and direct summands.

Lemma 4.8. *For every finitely presented étale $U \to X$, the object $\Sigma^\infty_+ U \in \mathcal{C}$ is of cohomological dimension $\leqslant c$.*

Proof. Since X is of uniformly bounded p-étale cohomological dimension $\leqslant c$ (by the hypothesis of Proposition 4.6), we have

$$H^s(\Sigma^\infty_+ U; \mathcal{G}) \simeq H^s_{\text{ét}}(U; \mathcal{G}) \simeq 0$$

for all $s > c$ and all *p-torsion* $\mathcal{G} \in \mathcal{C}^\heartsuit$. For $\mathcal{G} \in \mathcal{C}^\heartsuit$ arbitrary, denote by $\mathcal{G}/\!\!/p^k$ the cofibre

$$\mathcal{G} \xrightarrow{p^k} \mathcal{G} \to \mathcal{G}/\!\!/p^k.$$

Note that $\pi_0(\mathcal{G}/\!\!/p^k) \simeq \mathcal{G}/p^k\mathcal{G}$ and $\pi_1(\mathcal{G}/\!\!/p^k)$ is the p^k-torsion subsheaf $\mathcal{G}_{\mathrm{tors}} \subseteq \mathcal{G}$, so $\mathcal{G}/\!\!/p^k$ is p^k-torsion. From the exact triangle $\mathcal{G}_{\mathrm{tors}}[1] \to \mathcal{G}/\!\!/p^k \to \mathcal{G}/p^k\mathcal{G}$ where each term is p^k-torsion, we read off: $H^s_{\text{ét}}(U; \mathcal{G}/\!\!/p^k) \simeq 0$ for $s > c$ and $H^c_{\text{ét}}(U; \mathcal{G}/\!\!/p^k) \simeq H^c_{\text{ét}}(U; \mathcal{G}/p^k\mathcal{G})$, the latter using the vanishing of $H^{>c}_{\text{ét}}(U; \mathcal{G}_{\mathrm{tors}})$. Note that this second isomorphism is compatible with transition maps, i.e. an isomorphism of pro-systems.

As $\mathcal{G}$ is derived p-complete, i.e. $\mathcal{G} \simeq \varprojlim_k \mathcal{G}/\!\!/p^k$, we have

$$\mathrm{Maps}_{\mathcal{C}}(\Sigma^\infty_+ U, \mathcal{G}) \simeq \varprojlim_k \mathrm{Maps}_{\mathcal{C}}(\Sigma^\infty_+ U, \mathcal{G}/\!\!/p^k) \simeq \varprojlim_k R\Gamma_{\text{ét}}(U; \mathcal{G}/\!\!/p^k)$$

and its homotopy groups sit in the Milnor sequence

$$0 \to \varprojlim_k{}^1 H^{s-1}_{\text{ét}}(U; \mathcal{G}/\!\!/p^k) \to H^s(\Sigma^\infty_+ U; \mathcal{G}) \to \varprojlim_k H^s_{\text{ét}}(U; \mathcal{G}/\!\!/p^k) \to 0.$$

We know that the right-hand term vanishes for $s > c$ and the left-hand term vanishes for $s \geqslant c+2$. In particular, it remains to prove that $H^{c+1}(\Sigma^\infty_+ U; \mathcal{G}) \simeq \varprojlim^1_k H^c_{\text{ét}}(U; \mathcal{G}/p^k\mathcal{G})$ vanishes.

Using the short exact sequence

$$0 \to p^k\mathcal{G}/p^{k+1}\mathcal{G} \hookrightarrow \mathcal{G}/p^{k+1}\mathcal{G} \twoheadrightarrow \mathcal{G}/p^k\mathcal{G} \to 0,$$

where the kernel is p-torsion, we obtain the long exact sequence

$$\cdots \to H^c_{\text{ét}}(U;\mathcal{G}/p^{k+1}\mathcal{G}) \xrightarrow{\phi_k} H^c_{\text{ét}}(U;\mathcal{G}/p^k\mathcal{G}) \xrightarrow{\partial} H^{c+1}_{\text{ét}}(U;p^k\mathcal{G}/p^{k+1}\mathcal{G}) \to \cdots,$$

where $H^{c+1}_{\text{ét}}(U;p^k\mathcal{G}/p^{k+1}\mathcal{G}) \simeq 0$. Thus the transition maps ϕ_k are surjective, the pro-system $\{H^c_{\text{ét}}(U;\mathcal{G}/p^k\mathcal{G})\}_k$ satisfies the Mittag–Leffler condition, and $\varprojlim^1_k H^c_{\text{ét}}(U;\mathcal{G}/p^k\mathcal{G}) \simeq 0$. In particular, $H^s(\Sigma^\infty_+ U;\mathcal{G}) \simeq 0$ for all $s > c$. □

We now use Lemma 4.8 to conclude the proof as follows. First, $\mathrm{sk}_0(\Sigma^\infty_+ Y_\bullet) \simeq \Sigma^\infty_+ Y$ is of cohomological dimension $\leqslant c$. By construction there are exact triangles (see [Lur2, Thm. 1.2.4.1, Rem. 1.2.4.3])

$$|\mathrm{sk}_{i-1}\,\Sigma^\infty_+ Y_\bullet| \to |\mathrm{sk}_i\,\Sigma^\infty_+ Y_\bullet| \to C_i[i],$$

where each C_i is a direct summand of $\Sigma^\infty_+ Y_i$. Thus by Lemma 4.8 again, $C_i[i]$ is of cohomological dimension $\leqslant c+i$. By induction, we conclude that $|\mathrm{sk}_i(\Sigma^\infty_+ Y_\bullet)|$ is of cohomological dimension $\leqslant c+i$, for every $i \geqslant 0$.

Proof of (ii). For $i = -1$, $F_{-1} \simeq \mathbf{1}[-1] \simeq \Sigma^\infty_+ X[-1]$ is of cohomological dimension $\leqslant c-1$. Thus in the exact triangle $\mathbf{1}[-1] \to F_i \to |\mathrm{sk}_i\,\Sigma^\infty_+ Y_\bullet|$, the outer terms are both of cohomological dimension $\leqslant c+i$, hence so is F_i.

Proof of (iii). The cofibre $F_i[1]$ of $|\mathrm{sk}_i\,\Sigma^\infty_+ Y_\bullet| \to |\Sigma^\infty_+ Y_\bullet| \simeq \mathbf{1}$ is built out of the objects $C_j[j]$, $j > i$, under colimits. Each $C_j[j]$ is j-connective, as a direct summand of $\Sigma^\infty_+ Y_i[j]$, and hence $(i+1)$-connective. Hence $F_i[1]$ is $(i+1)$-connective, i.e. F_i is i-connective.

Proof of (i). More generally, every $\mathcal{F} \in \mathcal{C}$ built out of finite colimits and limits from $\Sigma^\infty_+ U$, where $U \to X$ is finitely presented étale, is Postnikov-complete. This follows from [CM, Prop. 2.10], since $\mathcal{F}$ is hypercomplete by assumption and, by Lemma 4.8, the objects $\Sigma^\infty_+ U \in \mathcal{C}$ are of cohomological dimension $\leqslant c$ for every finitely presented étale $U \to X$.

5. Lisse extensions and forgetting supports

We fix a presentable weave $\mathbf{D}$ on algebraic spaces and consider its lisse extension to Artin stacks. For every 1-Artin morphism $f : \mathcal{X} \to \mathcal{Y}$ with proper diagonal, there exists a natural transformation $\alpha_f : f_! \to f_*$, called *forgetting supports* (see [Kha2, §2.2]). In this section we provide sufficient conditions for this map to be invertible when f is proper. In particular, this will imply that f admits $*$-direct image in $\mathbf{D}$, in the sense of [Kha2, §1.4].

5.1. **Descendability.** We begin with the following stacky generalization of Theorem 1.21. Note that this does not require $\mathbf{D}$ to be lisse-extended from algebraic spaces.

Theorem 5.1. *Let $\mathbf{D}$ be a presentable weave on Artin stacks satisfying localization, and suppose every finite surjection which is either étale or radicial is descendable. Let $\mathcal{Y}$ be a qcqs Deligne–Mumford stack and $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ a representable finitely presented surjection. Then f is descendable.*

Proof. Let $\mathcal{C} \subseteq \mathrm{Art}_{/\mathcal{Y}}$ be the full subcategory of Artin stacks which are étale and finitely presented over $\mathcal{Y}$, and denote by $\mathcal{C}_0 \subseteq \mathcal{C}$ the full subcategory spanned by $\mathcal{Y}' \to \mathcal{Y}$ such that $f_{\mathcal{Y}'} := f \times_{\mathcal{Y}} \mathcal{Y}'$ is descendable. We will show that $\mathcal{C}_0 = \mathcal{C}$, so in particular f is descendable, by applying the dévissage theorem of [Ry, Thm. D].

Since $\mathcal{Y}$ is a quasi-compact DM stack, there exists an affine scheme Y and a representable étale surjection $y : Y \twoheadrightarrow \mathcal{Y}$. Then $Y \in \mathcal{C}$ and f_Y is a finitely presented surjection of algebraic spaces, so $Y \in \mathcal{C}_0$ by Theorem 1.21.

It remains to verify the following closure conditions from [Ry, Thm. D]:

- For any morphism $\mathcal{Y}'' \to \mathcal{Y}'$ in $\mathcal{C}$ with $\mathcal{Y}' \in \mathcal{C}_0$, we have $\mathcal{Y}'' \in \mathcal{C}_0$. Indeed, $f_{\mathcal{Y}'}$ is descendable, hence so is its base change $f_{\mathcal{Y}'} \times_{\mathcal{Y}'} \mathcal{Y}'' = f_{\mathcal{Y}''}$.
- For any finite étale surjection $g : \mathcal{Y}'' \to \mathcal{Y}'$ in $\mathcal{C}$ with $\mathcal{Y}'' \in \mathcal{C}_0$, we have $\mathcal{Y}' \in \mathcal{C}_0$. Indeed, g is finite étale, so it satisfies the projection formula and base change, and it is descendable by assumption; since $f_{\mathcal{Y}''}$ is the base change of $f_{\mathcal{Y}'}$ along g, Corollary 1.10 shows that $f_{\mathcal{Y}'}$ is descendable.
- For $\mathcal{Y}' \in \mathcal{C}$, any open immersion $\mathcal{U} \hookrightarrow \mathcal{Y}'$ with $\mathcal{U} \in \mathcal{C}_0$, and any étale neighbourhood $\mathcal{V} \to \mathcal{Y}'$ of $\mathcal{Y}' \setminus \mathcal{U}$ with $\mathcal{V} \in \mathcal{C}_0$, we have $\mathcal{Y}' \in \mathcal{C}_0$. Indeed, the étale neighbourhood lifts $\mathcal{Z} = (\mathcal{Y}' \setminus \mathcal{U})_{\mathrm{red}}$ to a closed substack of $\mathcal{V}$, so $f_{\mathcal{V}} \times_{\mathcal{V}} \mathcal{Z} = f_{\mathcal{Z}}$ is descendable by base change. By Lemma 1.23, the descendability of $f_{\mathcal{U}}$ and $f_{\mathcal{Z}}$ then yields descendability of $f_{\mathcal{Y}'}$. ☐

5.2. Forgetting supports, DM case.

Theorem 5.2. *Let $\mathbf{D}$ be a lisse-extended presentable weave on Artin stacks satisfying localization, and suppose every finite surjection which is either étale or radicial is descendable. If $f : \mathcal{X} \to \mathcal{Y}$ is a proper Deligne–Mumford morphism of 1-Artin stacks, then $\alpha_f : f_! \to f_*$ is invertible.*

Proof. Let $(Y_i \to \mathcal{Y})_\alpha$ be a smooth covering family where Y_i are affine schemes, and consider the base changes $f_i : \mathcal{X} \times_{\mathcal{Y}} Y_i \to Y_i$. Since the $*$-pull-backs to Y_i are jointly conservative and send $(\alpha_f$ to α_{f_i} by smooth base change, we may assume $\mathcal{Y}$ is an affine scheme. Then $\mathcal{X}$ is a quasi-compact DM stack with quasi-compact and separated diagonal.

Now by [Ry2, Thm. B] there exists a finitely presented finite surjection $g : Z \twoheadrightarrow \mathcal{X}$ with Z a scheme. By Theorem 5.1, g is descendable. Denote by $Z_\bullet$ the Čech nerve of g; its terms are algebraic spaces since g is representable. By proper descent along g (Proposition 2.6), every $\mathcal{F} \in \mathbf{D}(\mathcal{X})$ is the totalization of $g_{\bullet,*}g_\bullet^*(\mathcal{F})$. This totalization is preserved by the limit-preserving functor f_* and, since g is descendable, also by the exact functor $f_!$ (Remark 2.11). Since each $g_\bullet : Z_\bullet \to \mathcal{X}$ is proper representable and each $f \circ g_\bullet : Z_\bullet \to Y$ is a proper morphism of algebraic spaces, $\alpha_{g_\bullet}$ and $\alpha_{f \circ g_\bullet}$ are invertible by [Kha, Lem. 2.21] and [Kha2, Thm. 3.4.5(ii), Lem. 2.2.2]. The claim follows by functoriality of α. ☐

Example 5.3. For example, for $\mathbf{SH}_{\mathbf{Q},+}$ all finite étale surjections and all finite radicial surjections are descendable of index $\leqslant 1$. Indeed, for finite étale surjections the Euler characteristic (1.11) is invertible, and for finite radicial surjections f, the unit $\mathrm{id} \to f_*f^*$ is invertible (both statements are checked by descent from the case of schemes).

5.3. **Forgetting supports, tame Artin case.** Suppose next that $\mathcal{Y}$ is not necessarily DM but only 1-*Artin*. We are then able to prove a result analogous to Theorem 5.2 in the "tame" case. For this we only require the following slightly weaker conditions on $\mathbf{D}$:

(D1) Let $f : \mathcal{Y} \twoheadrightarrow \mathcal{X}$ be a torsor under a finite étale group scheme H over $\mathcal{X}$ whose order $|H|$ is invertible in $\mathrm{End}_{\mathbf{D}(\mathcal{X})}(\mathbf{1}_{\mathcal{X}})$. Then f is descendable.

(D2) Let $f : \mathcal{Y} \twoheadrightarrow \mathcal{X}$ be a finite[4] radicial surjection such that every prime number is invertible either in $\mathcal{O}_{\mathcal{X}}$ or in $\mathrm{End}_{\mathbf{D}(\mathcal{X})}(\mathbf{1}_{\mathcal{X}})$. Then f is descendable.

Theorem 5.4. *Let $\mathbf{D}$ be a lisse-extended presentable weave on Artin stacks satisfying localization and (D1) and (D2). Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper 1-Artin morphism with proper diagonal. If f has tame relative inertia[5], stabilizer orders invertible in $\mathrm{End}_{\mathbf{D}(\mathcal{Y})}(\mathbf{1}_{\mathcal{Y}})$, and $\mathcal{Y}$ equicharacteristic, then $\alpha_f : f_! \to f_*$ is invertible.*

Proof. As in the proof of Theorem 5.2 we may assume $\mathcal{Y} = Y$ is an equicharacteristic algebraic space and $\mathcal{X}$ is 1-Artin with finite inertia. Then by Keel–Mori there exists a coarse moduli space $\pi : \mathcal{X} \to M$, and $f : \mathcal{X} \to Y$ factors as the proper surjection π followed by a proper morphism of algebraic spaces $g : M \to Y$. Since α_g is an isomorphism by [Kha, Lem. 2.21] and α is functorial in f, we may replace f by π and assume that $f : \mathcal{X} \to M$ is the (equicharacteristic) coarse moduli space of a tame 1-Artin stack $\mathcal{X}$.

Since $\mathcal{X}$ is tame, there exists by [AOV, Thm. 3.2] an étale surjection of algebraic spaces $p : M' \twoheadrightarrow M$ such that $\mathcal{X} \times_M M' \simeq [U/G]$ where U is a scheme of finite presentation over M' and G is a linearly reductive finite fppf group scheme over M' acting on U. Moreover, the proof of *loc. cit.* shows that over each connected component of M' there is a geometric point x of $\mathcal{X}$ such that the fibre $G_{f(x)}$ is the stabilizer $\underline{\mathrm{Aut}}_{\mathcal{X}}(x)$. Since $G \to M'$ is finite locally free, its order $|G|$ is locally constant and hence is equal, on the component containing x, to the order of this stabilizer; in particular it is invertible in $\mathrm{End}_{\mathbf{D}(M')}(\mathbf{1}_{M'})$ by hypothesis.

By smooth base change again we may replace M by M' and thereby assume that $\mathcal{X} = [U/G]$, with G finite linearly reductive of order invertible in $\mathrm{End}_{\mathbf{D}(M)}(\mathbf{1}_M)$. By [AHR, Thm. 19.9 (2,6)] (which apply since M is still equicharacteristic), there is a short exact sequence $1 \to G^0 \to G \twoheadrightarrow H \to 1$ of group schemes over M with H tame finite étale and G^0 finite radicial. Thus the quotient map $q : U \twoheadrightarrow [U/G]$ factors through the finite radicial surjection $U \twoheadrightarrow [U/G^0]$ followed by the finite étale surjection $[U/G^0] \to [U/G]$ (which is an H-torsor). Since $|H|$ and $|G^0|$ divide $|G|$, they are also invertible in

[4]We recall that a morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is *finite* if it is schematic, and the base change $f \times_{\mathcal{Y}} Y$ is a finite morphism of schemes for every scheme Y over $\mathcal{Y}$.

[5]That is: the relative inertia is finite, and for every geometric point y of $\mathcal{Y}$, the fibre $\mathcal{X}_y$ has linearly reductive stabilizers (and hence is a tame Artin stack in the sense of [AOV]).

$\mathrm{End}_{\mathbf{D}(M)}(\mathbf{1}_M)$. Thus assumptions (D1) and (D2) imply that q is descendable. For $U \twoheadrightarrow [U/G^0]$, note that (D2) applies because every prime is invertible in $\mathcal{O}_M$ or in $\mathrm{End}_{\mathbf{D}(M)}(\mathbf{1}_M)$. Indeed, since M is equicharacteristic, every prime other than its residue characteristic is invertible in $\mathcal{O}_M$; in characteristic $p > 0$ the finite radicial group G^0 has p-power order, so either it is trivial, or p divides $|G^0|$ and is invertible in $\mathrm{End}_{\mathbf{D}(M)}(\mathbf{1}_M)$.

Denote by $U_\bullet$ the Čech nerve of $q : U \twoheadrightarrow [U/G] = \mathcal{X}$. By proper descent along q (Proposition 2.6), every $\mathcal{F} \in \mathbf{D}(\mathcal{X})$ is the totalization of the cosimplicial diagram $q_{\bullet,*}q_\bullet^*(\mathcal{F})$. We now conclude just as in the proof of Theorem 5.2, using the descendability of q proven above. □

References

[AHR] J. Alper, J. Hall, D. Rydh, *The étale local structure of algebraic stacks*. Ann. Sci. Éc. Norm. Supér. (4) **59** (2026), no. 1, 125–198.

[ALP] A. Ananyevskiy, M. Levine, and I. Panin, *Witt sheaves and the η-inverted sphere spectrum*. J. Topol. **10** (2017), no. 2, 370–385.

[AOV] D. Abramovich, M. Olsson, A. Vistoli, *Tame stacks in positive characteristic*. Ann. Inst. Fourier **58** (2008), no. 4, 1057–1091.

[Ayo] J. Ayoub, *La réalisation étale et les opérations de Grothendieck*. Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 1, 1–145.

[BEM] T. Bachmann, E. Elmanto, M. Morrow, *$\mathbf{A}^1$-invariant motivic cohomology of schemes*. arXiv:2508.09915 (2025).

[BH] T. Bachmann, M. Hoyois, *Norms in motivic homotopy theory*. Astérisque **425** (2021).

[BH1] T. Bachmann, M. Hopkins, *η-periodic motivic stable homotopy theory over fields*. arXiv:2005.06778 (2020).

[BH2] T. Bachmann, M. Hoyois, *Vanishing of $\mathscr{SH}_{ét}^\wedge(\mathbb{F}_p)_p^\wedge$*, appendix A to *Remarks on étale motivic stable homotopy theory* by T. Bachmann, arXiv:2104.06002 (2021).

[BM] B. Bhatt, A. Mathew, *The arc-topology*. Duke Math. J. **170** (2021), no. 9, 1899–1988. arXiv:1807.04725.

[BS] B. Bhatt, P. Scholze, *Projectivity of the Witt vector affine Grassmannian*. Invent. Math. **209** (2017), no. 2, 329–423.

[Bac2] T. Bachmann, *Remarks on étale motivic stable homotopy theory*. arXiv:2104.06002 (2021).

[Bac3] T. Bachmann, *Rigidity in étale motivic stable homotopy theory*. Algebr. Geom. Topol. **21** (2021), no. 1, 173–209. arXiv:1810.08028.

[CD] D.-C. Cisinski, F. Déglise, *Triangulated categories of mixed motives*. Springer Monogr. Math. (2019). arXiv:0912.2110.

[CD2] D.-C. Cisinski, F. Déglise, *Étale motives*. Compos. Math. **152** (2016), no. 3, 556–666. arXiv:1305.5361.

[CM] D. Clausen, A. Mathew, *Hyperdescent and étale K-theory*. Invent. Math. **225** (2021), no. 3, 981–1076.

[DFJK] F. Déglise, J. Fasel, F. Jin, A. A. Khan, *On the rational motivic homotopy category*. J. Éc. Polytech., Math. **8** (2021), 533–583.

[DH] N. Deshmukh, J. Hall, *On the motivic homotopy type of algebraic stacks*. J. Algebra **693** (2026), 790–804.

[EGA] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique*, Publ. Math. IHÉS **4** (Chapter 0, 1–7, and I, 1–10), **8** (II, 1–8), **11** (Chapter 0, 8–13, and III, 1–5), **17** (III, 6–7), **20** (Chapter 0, 14–23, and IV, 1), **24** (IV, 2–7), **28** (IV, 8–15), and **32** (IV, 16–21), 1960–1967.

[EHIK] E. Elmanto, M. Hoyois, R. Iwasa, S. Kelly, *Milnor excision for motivic spectra*. J. reine angew. Math. **779** (2021), 223–235. arXiv:2004.12098.

[EHIK2] E. Elmanto, M. Hoyois, R. Iwasa, S. Kelly, *Cdh descent, cdarc descent, and Milnor excision*. Math. Ann. **379** (2021), no. 3–4, 1011–1045. arXiv:2002.11647.

[EK] E. Elmanto, A. A. Khan, *Perfection in motivic homotopy theory*. Proc. Lond. Math. Soc. **120** (2020), no. 1, 28–38.

[HR] J. Hall, D. Rydh, *Addendum to: "Étale dévissage, descent and pushouts of stacks"*. J. Algebra **498** (2018), 398–412.

[Kha] A. A. Khan, *Weaves*. Available at: `https://www.preschema.com/papers/weaves.pdf` (2023).

[Kha2] A. A. Khan, *Lisse extensions of weaves*. arXiv:2501.04114 (2025).

[Kha3] A. A. Khan, *Virtual fundamental classes of derived stacks I*. arXiv:1909.01332 (2019).

[LZ] Y. Liu, W. Zheng, *Enhanced six operations and base change theorem for higher Artin stacks*. arXiv:1211.5948 (2012).

[Lur] J. Lurie, *Higher topos theory*. Annals of Mathematics Studies **170**, Princeton University Press (2009).

[Lur2] J. Lurie, *Higher algebra*, version of 2017-09-18. Available at: `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[MNN] A. Mathew, N. Naumann, and J. Noel, *Nilpotence and descent in equivariant stable homotopy theory*. Adv. Math. **305** (2017), 994–1084. arXiv:1507.06869.

[MT] K. Mattis and S. Tubach, *Nilpotence of η in étale motivic spectra*. arXiv:2511.09476 (2025).

[Mat] A. Mathew, *The Galois group of a stable homotopy theory*. Adv. Math. **291** (2016), 403–541.

[Rio] J. Riou, *Algebraic K-theory, $\mathbf{A}^1$-homotopy and Riemann–Roch theorems*. J. Topol. **3** (2010), no. 2, 229–264.

[Ry] D. Rydh, *Étale dévissage, descent and pushouts of stacks*. J. Algebra **331** (2011), no. 1, 194–223. arXiv:1005.2171.

[Ry2] D. Rydh, *Noetherian approximation of algebraic spaces and stacks*. J. Algebra **422** (2015), 105–147.

[Ry3] D. Rydh, *Submersions and effective descent of étale morphisms*. Bull. Soc. Math. France **138** (2010), no. 2, 181–230.

[SGA4] M. Artin, A. Grothendieck, J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas (SGA 4)*. Lecture Notes in Math. **269, 270, 305**, Springer (1972–1973).

[Ser] J.-P. Serre, *Galois cohomology*. Springer Monogr. Math., Springer (1997).

[Voe] V. Voevodsky, *Homology of schemes*. Selecta Math. (N.S.) **2** (1996), no. 1, 111–153.

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