VOEVODSKY'S CRITERION FOR CONSTRUCTIBLE CATEGORIES OF COEFFICIENTS

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Abstract. We give a short account of Voevodsky's construction of six functor formalisms satisfying $\mathbf{A}^1$-homotopy invariance and localization, and apply it to the motivic stable homotopy category over derived algebraic spaces. Special care is taken to eliminate various finiteness and separatedness hypotheses.

Introduction

Cohomology theories of algebraic varieties take coefficients in various categories. These (∞-)categories are equipped with the formalism of six operations:

(a) *Basic functoriality.* For every morphism of schemes $f : X \to Y$, a pair of adjoint functors

$$f^* : \mathbf{D}(Y) \to \mathbf{D}(X), \quad f_* : \mathbf{D}(X) \to \mathbf{D}(Y).$$

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(b) *Exceptional functoriality.* For every locally of finite type morphism of schemes $f : X \to Y$, a pair of adjoint functors

$$f_! : \mathbf{D}(X) \to \mathbf{D}(Y), \quad f^! : \mathbf{D}(Y) \to \mathbf{D}(X).$$

(c) *Tensor and Hom.* For every scheme X, a pair of adjoint bifunctors

$$\otimes : \mathbf{D}(X) \times \mathbf{D}(X) \to \mathbf{D}(X), \quad \underline{\mathrm{Hom}} : \mathbf{D}(X) \times \mathbf{D}(X) \to \mathbf{D}(X).$$

(d) *Thom twist.*[1] For every scheme X and every perfect complex $\mathcal{E}$, an auto-equivalence $\langle \mathcal{E} \rangle : \mathbf{D}(X) \simeq \mathbf{D}(X)$.

This data is subject to the usual compatibilities and coherences. For example, the tensor and Hom are part of closed symmetric monoidal structures on $\mathbf{D}(X)$; the inverse image functor f^* is symmetric monoidal; the functors $f_!$ satisfy base change and projection formulas; there is an isomorphism $f_! \simeq f_*$ when f is proper and an isomorphism $f^! \simeq f^* \langle \Omega_f \rangle$ when f is smooth (where Ω_f is the relative cotangent sheaf); and the Thom twist commutes with the other six operations.

Up to this point the above description applies equally well to coherent cohomology and the "classical" theories (ℓ-adic, Betti, etc.). Characteristic of the latter settings are the properties of $\mathbf{A}^1$-*homotopy invariance* and *localization*. The first asserts that for every X, the projection $p : X \times \mathbf{A}^1 \to X$ of the affine line induces an isomorphism

$$\mathrm{unit} : \mathrm{id} \to p_* p^*.$$

The second says that for any closed immersion $i : Z \to X$ with open complement $j : U \to S$, there is an "exact sequence"

$$\mathbf{D}(Z) \xrightarrow{i_*} \mathbf{D}(X) \xrightarrow{j^*} \mathbf{D}(U)$$

where i_* is fully faithful with essential image equal to the kernel of j^*. Let's use the term *constructible* to describe categories of coefficients satisfying both these properties[2].

Typically the difficult part of constructing a six functor formalism is the exceptional functoriality. An interesting observation of Voevodsky [Vo2] is that, for constructible categories of coefficients, the exceptional operations come "for free" in the sense that they are uniquely determined by the rest of the data. More precisely, note the following consequence of the existence of the exceptional operations: for a smooth morphism f, the isomorphism $f^! \simeq f^* \langle \Omega_f \rangle$ implies that f^* also admits a *left* adjoint $f_\sharp$ (given by $f_! \langle \Omega_f \rangle$) which also satisfies the base change and projection formulas. Voevodsky's

[1] If this looks unfamiliar, here are two examples. In the étale setting, $\langle \mathcal{E} \rangle$ is canonically identified with $(r)[2r]$, where r is the rank of $\mathcal{E}$. In the coherent setting (i.e., ind-coherent or solid quasi-coherent sheaves), $\langle \mathcal{E} \rangle$ is tensoring with the (graded) determinant of $\mathcal{E}$. The Thom twist is not one of the "official" six operations, but it can be recovered from them.

[2] In [Kh1] I used the term *motivic* ∞-*category* (of coefficients). The reason for this rebranding attempt that this class of categories includes many examples that have nothing to do with motives, while any mention of motives seems to scare many people away. I'm also open to other suggestions though.

result says that it is enough to check this property to have the full formalism of six operations on a constructible category.

A complete account of Voevodsky's criterion was first given by Ayoub in his thesis [Ay], with some quasi-projectivity hypotheses that were later dropped by Cisinski–Déglise [CD]. In this note I propose to revisit the topic again with the benefit of recent technological advancements to give a proof that is considerably shorter (and essentially self-contained). It is also slightly more general in the following ways:

(a) I work in the setting of derived algebraic geometry. This generality is necessary for some applications (see [Kh4]), and comes for free in view of derived invariance (Lemma 2.13).

(b) I work with algebraic spaces instead of schemes. This extension also comes for free, since every qcqs algebraic space is Nisnevich-locally affine. However, there is one place where we will greatly benefit from working in this larger category: see the proof of Theorem 2.43, which uses Rydh's result that any unramified morphism admits a canonical, global, factorization through a closed immersion and an étale morphism, *in the category of algebraic spaces* (see [Ry]).

(c) Standard arguments are used to drop noetherianness hypotheses.

(d) The exceptional operations are usually only constructed for separated morphisms of finite type. This restriction is dropped using ∞-categorical descent techniques developed by Liu and Zheng [LZ2, LZ1].

(e) The purity isomorphism $f^! \simeq f^*\langle\Omega_f\rangle$ has also only been constructed for smooth morphisms that are *separated*. This restriction is dropped using Theorem 2.43 mentioned above. Note that in the étale setting, this can be achieved via gluing as in [LZ2], where the homotopy coherence problem is avoided using the perverse t-structure. The lack of a suitable t-structure on an arbitrary constructible categories prevents us from applying the same technique.

I also apply the construction to the universal example, which is the motivic stable homotopy category. The verification of Voevodsky's conditions in that case are done in Sect. 1. The proof of Voevodsky's theorem is undertaken in Sect. 2. I also discuss some complementary results such as descent properties and behaviour of constructible objects.

0.1. **Conventions.**

- A symmetric monoidal structure on a presentable ∞-category is *presentable* if the tensor product $\otimes$ preserves colimits in each argument. A *symmetric monoidal presentable ∞-category* is a presentable ∞-category equipped with a presentable symmetric monoidal structure.
- A colimit-preserving functor of presentable ∞-categories is *compact* if its right adjoint preserves filtered colimits.

- A morphism of derived algebraic spaces is of finite type/presentation if the induced morphism of classical truncations is of finite type/presentation.

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1. The motivic stable homotopy category

In this section we construct the ∞-category of motivic spectra over a qcqs derived algebraic space, and show that it is a constructible sheaf theory.

1.1. **Unstable category.** Let S be a qcqs derived algebraic space. We write $\mathrm{Asp}_{/S}$ for the ∞-category of derived algebraic spaces of finite presentation over S.

Definition 1.1 (Nisnevich topology)**.**

(i) A *Nisnevich square* over $X \in \mathrm{Asp}_{/S}$ is a cartesian square of derived algebraic spaces

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd} \tag{1.2}$$

where j is a quasi-compact open immersion, p is étale and of finite presentation, and there exists a closed immersion $Z \hookrightarrow X$ complementary to j such that the induced morphism $p^{-1}(Z) \to Z$ is invertible.

(ii) The *Nisnevich topology* on $\mathrm{Asp}_{/S}$ is the Grothendieck topology associated to the pretopology generated by the following covering families: (a) the empty family, covering the empty space $\varnothing$; (b) for any $X \in \mathrm{Asp}_{/S}$ and for any Nisnevich square over X of the form (1.2), the family $\{U \to X, V \to X\}$, covering X.

Remark 1.3. The Nisnevich topology admits many alternative characterizations, see [BH, App. A], [Lu2, §3.7.4], and [Kh2, §2.2].

Definition 1.4. We say that a full subcategory $\mathcal{A}_{/S} \subseteq \mathrm{Asp}_{/S}$ is *admissible* if it is essentially small[3] and satisfies the following conditions:

(i) S belongs to $\mathcal{A}_{/S}$.

[3] As in [Ho1, App. C], it can be useful to relax the condition that $\mathcal{A}_{/S}$ is essentially small to the following weaker condition: there exists an essentially small full subcategory $\mathcal{A}^0_{/S} \subseteq \mathcal{A}_{/S}$ such that every Nisnevich sheaf of anima on $\mathcal{A}_{/S}$ is the right Kan extension of its restriction to $\mathcal{A}^0_{/S}$. For example, in *loc. cit.* it is shown that for any scheme S (not necessarily qcqs), the category of all smooth schemes over S satisfies this weaker condition.

(ii) If X belongs to $\mathcal{A}_{/S}$ and Y is étale and of finite presentation over X, then Y belongs to $\mathcal{A}_{/S}$.

(iii) If X belongs to $\mathcal{A}_{/S}$, then $X \times \mathbf{A}^1$ belongs to $\mathcal{A}_{/S}$.

(iv) If X and Y belong to $\mathcal{A}_{/S}$, then $X \times_S Y$ belongs to $\mathcal{A}_{/S}$.

We say that a morphism $f : X \to S$ is *admissible* (with respect to the admissible subcategory $\mathcal{A}_{/S}$) if it exhibits X as an object of $\mathcal{A}_{/S}$.

Example 1.5. We denote by $\mathrm{Sm}_{/S} \subseteq \mathrm{Asp}_{/S}$ the admissible subcategory of smooth derived algebraic spaces of finite presentation over S.

Definition 1.6. Let $\mathcal{A}$ be an essentially small ∞-category admitting finite coproducts and $\mathcal{V}$ an ∞-category admitting finite products. Following Voevodsky [Vo3] we say that a $\mathcal{V}$-valued presheaf $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is *radditive* if it commutes with finite products, i.e., if $\mathbf{R}\Gamma(\varnothing, \mathcal{F})$ is contractible and the canonical maps

$$\mathbf{R}\Gamma(X \sqcup Y, \mathcal{F}) \to \mathbf{R}\Gamma(X, \mathcal{F}) \times \mathbf{R}\Gamma(Y, \mathcal{F})$$

are invertible for all $X, Y \in \mathcal{A}$. Here $\mathbf{R}\Gamma(X, \mathcal{F}) \in \mathcal{V}$ denotes the sections of $\mathcal{F}$ over any $X \in \mathcal{C}$.

Definition 1.7. Let S be a qcqs derived algebraic space and $\mathcal{A}_{/S} \subseteq \mathrm{Asp}_{/S}$ an admissible subcategory.

(i) An $\mathcal{A}$-*fibred anima* over S is a presheaf of anima on $\mathcal{A}_{/S}$.

(ii) An $\mathcal{A}$-*fibred anima* $\mathcal{F}$ satisfies *Nisnevich descent* if it satisfies Čech descent with respect to the Nisnevich topology (restricted to $\mathcal{A}_{/S}$). Equivalently, for every Nisnevich covering family $(U_\alpha \to X)_\alpha$, the augmented cosimplicial diagram

$$\mathbf{R}\Gamma(X, \mathcal{F}) \to \prod_{\alpha} \mathbf{R}\Gamma(U_\alpha, \mathcal{F}) \rightrightarrows \prod_{\alpha,\beta} \mathbf{R}\Gamma(U_\alpha \underset{X}{\times} U_\beta, \mathcal{F}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots.$$

exhibits $\mathbf{R}\Gamma(X, \mathcal{F})$ as the homotopy limit (totalization).

(iii) An $\mathcal{A}$-*fibred anima* $\mathcal{F}$ satisfies $\mathbf{A}^1$-*homotopy invariance* if, for every $X \in \mathcal{A}_{/S}$, the map

$$\mathbf{R}\Gamma(X, \mathcal{F}) \to \mathbf{R}\Gamma(X \times \mathbf{A}^1, \mathcal{F})$$

is invertible.

(iv) An $\mathcal{A}$-fibred anima $\mathcal{F}$ is *motivic* if it satisfies Nisnevich descent and $\mathbf{A}^1$-homotopy invariance. We write $\mathbf{H}(\mathcal{A}_{/S})$ for the ∞-category of $\mathcal{A}$-fibred motivic anima over S, and simply $\mathbf{H}(S)$ in case $\mathcal{A}_{/S} = \mathrm{Sm}_{/S}$.

(v) For any ∞-category $\mathcal{V}$ admitting limits, we can similarly define $\mathcal{A}$-*fibred motivic* $\mathcal{V}$-*objects* over S as $\mathcal{V}$-valued presheaves on $\mathcal{A}_{/S}$ satisfying Nisnevich descent and $\mathbf{A}^1$-homotopy invariance.

Remark 1.8. The ∞-category $\mathbf{H}(\mathcal{A}_{/S})$ is an accessible left Bousfield localization of the ∞-category of $\mathcal{A}$-fibred anima. In particular, it is presentable.

This follows from [Lu1, Props. 5.5.4.2, 5.5.4.15, 5.5.8.10]. The localization functor, which we denote $\mathcal{F} \mapsto \mathbf{L}(\mathcal{F})$, can be computed as the transfinite composite

$$\mathbf{L}(\mathcal{F}) \simeq \varinjlim_{n \geqslant 0} (\mathrm{L}_{\mathbf{A}^1} \circ \mathrm{L}_{\mathrm{Nis}})^{\circ n}(\mathcal{F}), \tag{1.9}$$

where $\mathrm{L}_{\mathrm{Nis}}$ and $\mathrm{L}_{\mathbf{A}^1}$ are the Nisnevich and $\mathbf{A}^1$-localization functors, respectively. See [Kh2, Rem. 2.4.3] and [CK, Rem. 2.1.16(iii)].

Example 1.10. For any derived algebraic space X over S which belongs to $\mathcal{A}_{/S}$, the presheaf represented by X defines an $\mathcal{A}$-fibred anima $\mathrm{h}_S(X)$. Its motivic localization

$$\mathbf{L}\mathrm{h}_S(X) \in \mathbf{H}(\mathcal{A}_{/S})$$

is called the *$\mathcal{A}$-fibred motivic anima represented by* X. When there is no risk of confusion, we will sometimes write simply X instead of $\mathbf{L}\mathrm{h}_S(X)$.

As X ranges over $\mathcal{A}_{/S}$, the objects $\mathbf{L}\mathrm{h}_S(X)$ generate $\mathbf{H}(\mathcal{A}_{/S})$ under sifted colimits. See [Kh2, Prop. 2.4.4] and [CK, Rem. 2.1.16(iv)]. Moreover, these objects are *compact*, as follows from the fact that the conditions of Nisnevich descent and $\mathbf{A}^1$-invariance are stable under filtered colimits (by [Kh2, Thm. 2.2.7] and by definition, respectively).

Example 1.11. We write

$$\mathrm{pt}_S \coloneqq \mathbf{L}\mathrm{h}_S(S) \simeq \mathrm{h}_S(S)$$

for the motivic $\mathcal{A}$-fibred space represented by the base S. This object is terminal in $\mathbf{H}(\mathcal{A}_{/S})$.

1.2. **Pointed category.**

Definition 1.12. Let S be a qcqs derived algebraic space and $\mathcal{A}_{/S} \subseteq \mathrm{Asp}_{/S}$ an admissible subcategory. A *pointed motivic $\mathcal{A}$-fibred anima* over S is a pair $(\mathcal{F}, s)$, where $\mathcal{F} \in \mathbf{H}(\mathcal{A}_{/S})$ and $s : \mathrm{pt}_S \to \mathcal{F}$ is a morphism in $\mathbf{H}(\mathcal{A}_{/S})$. We write $\mathbf{H}(\mathcal{A}_{/S})_\bullet$ for the ∞-category of pointed motivic $\mathcal{A}$-fibred anima, and simply $\mathbf{H}(S)_\bullet$ in case $\mathcal{A}_{/S} = \mathrm{Sm}_{/S}$.

Remark 1.13. The ∞-category $\mathbf{H}(\mathcal{A}_{/S})_\bullet$ is presentable and admits a canonical presentable symmetric monoidal structure. We denote the monoidal product by $\wedge$. Moreover, the forgetful functor $\mathbf{H}(\mathcal{A}_{/S})_\bullet \to \mathbf{H}(\mathcal{A}_{/S})$ admits a symmetric monoidal left adjoint $\mathcal{F} \mapsto \mathcal{F}_+$ which freely adjoins a base point. From Example 1.10 it follows that the objects $\mathbf{L}\mathrm{h}_S(X)_+ \in \mathbf{H}(\mathcal{A}_{/S})_\bullet$ are compact and generate under sifted colimits as $X \in \mathcal{A}_{/S}$ varies. See [Ro, Cor. 2.32].

Example 1.14. Let $\mathcal{E}$ be a finite locally free sheaf on S. Write $E = \mathbf{V}_S(\mathcal{E})$ for its total space, $p : E \to S$ for the projection, and $E \smallsetminus S$ for the complement of the zero section. The *Thom anima* of $\mathcal{E}$ is the pointed motivic $\mathcal{A}$-fibred anima

$$\mathrm{Th}_S(\mathcal{E}) \coloneqq \mathbf{L}\mathrm{h}_S(E)/\mathbf{L}\mathrm{h}_S(E \smallsetminus S),$$

i.e., the cofibre of the inclusion $E \smallsetminus S \hookrightarrow E$ taken in the ∞-category $\mathbf{H}(\mathcal{A}_{/S})$. This is well-defined as an $\mathcal{A}$-fibred motivic space as long as p is admissible (i.e., $E \in \mathcal{A}_{/S}$). As a finite colimit of compact objects, $\mathrm{Th}_S(\mathcal{E})$ is a compact object of $\mathbf{H}(\mathcal{A}_{/S})_\bullet$.

Example 1.15. The Thom anima of the free sheaf of rank one is denoted simply

$$\mathbf{T}_S \coloneqq \mathrm{Th}_S(\mathcal{O}_S) = \mathbf{A}^1_S/(\mathbf{A}^1_S \smallsetminus S).$$

Note that we have a canonical isomorphism

$$\mathbf{T}_S \simeq \Sigma_{S^1}(\mathbf{A}^1_S \smallsetminus S)$$

in $\mathbf{H}(\mathcal{A}_{/S})_\bullet$, where Σ_{S^1} denotes suspension with respect to the topological circle.

1.3. **Stable category.**

Definition 1.16. Let S be a qcqs derived algebraic space and $\mathcal{A}_{/S} \subseteq \mathrm{Asp}_{/S}$ an admissible subcategory. An *$\mathcal{A}$-fibred motivic spectrum* over S is a $\mathbf{T}_S$-spectrum object in the ∞-category $\mathbf{H}(\mathcal{A}_{/S})_\bullet$. That is, it is a sequence $(\mathcal{F}_n)_{n\geqslant 0}$ of pointed motivic $\mathcal{A}$-fibred anima $\mathcal{F}_n$, together with isomorphisms

$$\Omega_{\mathbf{T}}(\mathcal{F}_{n+1}) \simeq \mathcal{F}_n$$

for every $n \geqslant 0$, where $\Omega_{\mathbf{T}}$ denotes the loop space functor formed with respect to $\mathbf{T}_S \in \mathbf{H}(\mathcal{A}_{/S})_\bullet$ (Example 1.15). We write $\mathbf{SH}(\mathcal{A}_{/S})$ for the ∞-category of $\mathcal{A}$-fibred motivic spectra, and simply $\mathbf{SH}(S)$ in case $\mathcal{A}_{/S} = \mathrm{Sm}_{/S}$.

Remark 1.17. The ∞-category $\mathbf{SH}(\mathcal{A}_{/S})$ can be defined more precisely as the cofiltered limit of the tower

$$\cdots \xrightarrow{\Omega_{\mathbf{T}}} \mathbf{H}(\mathcal{A}_{/S})_\bullet \xrightarrow{\Omega_{\mathbf{T}}} \mathbf{H}(\mathcal{A}_{/S})_\bullet$$

which can be taken either in the (very large) ∞-category of large ∞-categories, or in the subcategory of presentable ∞-categories and right adjoint functors (see [Lu1, Thm. 5.5.3.18]). Dually, $\mathbf{SH}(\mathcal{A}_{/S})$ can be described as the *colimit* of the tower of left adjoints

$$\mathbf{H}(\mathcal{A}_{/S})_\bullet \xrightarrow{\Sigma_{\mathbf{T}}} \mathbf{H}(\mathcal{A}_{/S})_\bullet \xrightarrow{\Sigma_{\mathbf{T}}} \cdots,$$

in the ∞-category of presentable ∞-categories and left adjoint functors (see [Lu1, Cor. 5.5.3.4]). In particular, $\mathbf{SH}(\mathcal{A}_{/S})$ is presentable.

Remark 1.18. For every $n \geqslant 0$ we have a pair of adjoint functors

$$\Sigma^{\infty-n}_{\mathbf{T}} : \mathbf{H}(\mathcal{A}_{/S})_\bullet \to \mathbf{SH}(\mathcal{A}_{/S}), \quad \Omega^{\infty-n}_{\mathbf{T}} : \mathbf{SH}(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{A}_{/S})_\bullet,$$

which are the inclusions/projections of the nth component of the colimit/limit. (For example, $\Omega^{\infty-n}((\mathcal{F}_m)_{m\geqslant 0}) = \mathcal{F}_n$ for every n.) Since the object $\mathbf{T}_S \in \mathbf{H}(\mathcal{A}_{/S})_\bullet$ is compact, $\Omega_{\mathbf{T}}$ commutes with filtered colimits, so by [Lu1, Prop. 5.5.7.6] each functor $\Omega^{\infty-n}$ preserves filtered colimits. In other words, each functor $\Sigma^{\infty-n}$ is compact.

By construction, the family of functors

$$\Omega^{\infty-n}_{\mathbf{T}} : \mathbf{SH}(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{A}_{/S})_\bullet$$

is conservative as $n \geqslant 0$ varies. Dually, the functors $\Sigma^{\infty-n}_{\mathbf{T}}$ generate under filtered colimits.

Remark 1.19. Combining the above with Remark 1.13, we find that the compact objects $\Sigma^{\infty-n}_{\mathbf{T}}\mathbf{L}\mathrm{h}_S(X)_+$ generate $\mathbf{SH}(\mathcal{A}_{/S})$ under colimits as $n \geqslant 0$ and $X \in \mathcal{A}_{/S}$ vary.

Remark 1.20. The presentable ∞-category $\mathbf{SH}(\mathcal{A}_{/S})$ admits a canonical symmetric monoidal structure. We denote the monoidal product by $\otimes$ and the unit by $\mathbf{1}_S$. This symmetric monoidal structure is presentable and we have moreover:

(i) The functor $\Sigma^{\infty}_{\mathbf{T}} : \mathbf{H}(\mathcal{A}_{/S})_\bullet \to \mathbf{SH}(\mathcal{A}_{/S})$ is symmetric monoidal.

(ii) The object $\Sigma^{\infty}_{\mathbf{T}}(\mathbf{T}_S) \in \mathbf{SH}(\mathcal{A}_{/S})$ is $\otimes$-invertible.

In fact, $\Sigma^{\infty}_{\mathbf{T}}$ is universal among symmetric monoidal functors that invert $\mathbf{T}_S$. This follows from [Ro, Cor. 2.22] in view of the fact that the cyclic permutation of $\mathbf{T}_S^{\wedge 3}$ is homotopic to the identity. This follows by functoriality from the case $S = \mathrm{Spec}(\mathbf{Z})$, see [Vo1, Lem. 4.4] or [Ay, Lem. 4.5.65].

Remark 1.21. From Lemma 1.28 below it will follow that in fact the Thom anima $\mathrm{Th}_S(\mathcal{E})$ becomes $\otimes$-invertible in $\mathbf{SH}(\mathcal{A}_{/S})$ for *every* finite locally free sheaf $\mathcal{E}$.

1.4. **Functoriality.**

Proposition 1.22. *Let $f : T \to S$ be a morphism of qcqs derived algebraic spaces. Let $\mathcal{A}_{/S} \subseteq \mathrm{Asp}_{/S}$ and $\mathcal{A}_{/T} \subseteq \mathrm{Asp}_{/T}$ be admissible subcategories such that, for every $X \in \mathcal{A}_{/S}$, the base change $X \times_S T \in \mathrm{Asp}_{/T}$ belongs to $\mathcal{A}_{/T}$. Then there exists a functor*

$$\mathbf{L}f^* : \mathbf{SH}(\mathcal{A}_{/S}) \to \mathbf{SH}(\mathcal{A}_{/T})$$

satisfying the following properties.

(i) *$\mathbf{L}f^*$ commutes with colimits, hence in particular admits a right adjoint*
$$f_* : \mathbf{SH}(\mathcal{A}_{/T}) \to \mathbf{SH}(\mathcal{A}_{/S}).$$

(ii) *For any $X \in \mathcal{A}_{/S}$ and $n \geqslant 0$, there is a canonical isomorphism*
$$\mathbf{L}f^*\big(\Sigma^{\infty-n}_{\mathbf{T}}\mathbf{L}\mathrm{h}_S(X)_+\big) \simeq \Sigma^{\infty-n}_{\mathbf{T}}\mathbf{L}\mathrm{h}_T(X \underset{S}{\times} T)_+.$$

(iii) *$\mathbf{L}f^*$ is compact, i.e., its right adjoint f_* preserves colimits.*

(iv) *$\mathbf{L}f^*$ is symmetric monoidal.*

Proof. By assumption, the base change functor $X \mapsto X \times_S T$ restricts to a functor $f^{-1} : \mathcal{A}_{/S} \to \mathcal{A}_{/T}$. Restriction of presheaves along f^{-1} induces a functor f_* from $\mathcal{A}$-fibred anima over T to $\mathcal{A}$-fibred anima over S. It has a left adjoint f^* given by left Kan extension, hence characterized uniquely by commutativity with colimits and the formula $f^*(\mathrm{h}_S(X)) \simeq \mathrm{h}_T(X \times_S T)$ for any $X \in \mathcal{A}_{/S}$.

Since base change preserves Nisnevich covers and $\mathbf{A}^1$-projections, we see that f_* preserves motivic anima and thus induces a functor

$$f_* : \mathbf{H}(\mathcal{A}_{/T}) \to \mathbf{H}(\mathcal{A}_{/S}).$$

Thus it has a left adjoint given by localizing f^*:

$$\mathbf{L}f^* : \mathbf{H}(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{A}_{/T}).$$

Since f^* and $\mathbf{L}$ commute with finite products, $\mathbf{L}f^*$ is cartesian monoidal.

Since $\mathbf{L}f^*$ and f_* both preserve terminal objects, they have obvious extensions to pointed motivic anima:

$$\mathbf{L}f^* : \mathbf{H}(\mathcal{A}_{/S})_\bullet \to \mathbf{H}(\mathcal{A}_{/T})_\bullet, \quad f_* : \mathbf{H}(\mathcal{A}_{/T})_\bullet \to \mathbf{H}(\mathcal{A}_{/S})_\bullet.$$

The left adjoint is the unique extension of the unpointed $\mathbf{L}f^*$ that commutes with the functor $\mathcal{F} \mapsto \mathcal{F}_+$, $\mathbf{H}(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{A}_{/S})_\bullet$. It also inherits a symmetric monoidal structure by the universal property of [Ro, Cor. 2.32].

Since $\mathbf{L}f^*(\mathbf{T}_S) = \mathbf{T}_T$, it follows that $\mathbf{L}f^*$ commutes with the $\mathbf{T}$-suspension functor $\Sigma_{\mathbf{T}}$. By adjunction, f_* commutes with $\Omega_{\mathbf{T}}$. Therefore we get unique extensions of $\mathbf{L}f^*$ and f_* from pointed motivic anima to motivic spectra:

$$\mathbf{L}f^* : \mathbf{SH}(\mathcal{A}_{/S}) \to \mathbf{SH}(\mathcal{A}_{/T}), \quad f_* : \mathbf{SH}(\mathcal{A}_{/T}) \to \mathbf{SH}(\mathcal{A}_{/S})$$

such that $\mathbf{L}f^*$ commutes with $\Sigma^\infty_{\mathbf{T}}$ and f_* commutes with $\Omega^\infty_{\mathbf{T}}$. By the universal property of Remark 1.20, $\mathbf{L}f^*$ inherits a symmetric monoidal structure. □

Proposition 1.23. *Let $f : T \to S$ be a morphism of qcqs derived algebraic spaces. Let $\mathcal{A}_{/S} \subseteq \mathrm{Asp}_{/S}$ and $\mathcal{A}_{/T} \subseteq \mathrm{Asp}_{/T}$ be admissible subcategories as in Proposition 1.22. If f is admissible, then the inverse image functor $\mathbf{L}f^*$ admits a left adjoint*

$$\mathbf{L}f_\sharp : \mathbf{SH}(\mathcal{A}_{/T}) \to \mathbf{SH}(\mathcal{A}_{/S}).$$

This functor is characterized uniquely by commutativity with colimits and the formula

$$\mathbf{L}f_\sharp\big(\Sigma^\infty_{\mathbf{T}}\mathbf{L}\mathrm{h}_T(Y)_+\big) \simeq \Sigma^\infty_{\mathbf{T}}\mathbf{L}\mathrm{h}_S(Y)_+$$

for any $Y \in \mathcal{A}_{/T}$. Moreover, it is $\mathbf{SH}(\mathcal{A}_{/S})$-linear; in particular, we have the projection formula

$$\mathbf{L}f_\sharp(\mathcal{G}) \otimes \mathcal{F} \simeq \mathbf{L}f_\sharp(\mathcal{G} \otimes \mathbf{L}f^*(\mathcal{F}))$$

for every $\mathcal{F} \in \mathbf{SH}(\mathcal{A}_{/S})$ and $\mathcal{G} \in \mathbf{SH}(\mathcal{A}_{/T})$.

Proof. We adopt again the notation of the proof of Proposition 1.22. The assumption that $T \in \mathcal{A}_{/S}$ implies that the functor $f^{-1} : \mathcal{A}_{/S} \to \mathcal{A}_{/T}$ admits a left adjoint, the forgetful functor

$$(X \to T) \mapsto (X \to T \xrightarrow{f} S). \tag{1.24}$$

In this case f^* coincides, at the level of fibred anima, with the functor of restriction of presheaves along (1.24), and admits a left adjoint $f_\sharp$ characterized uniquely by commutativity with colimits and the formula $f_\sharp(\mathrm{h}_T(Y) \simeq \mathrm{h}_S(Y)$ for $Y \in \mathcal{A}_{/T}$. It is easy to check that $f_\sharp$ satisfies the projection formula. Moreover, since (1.24) preserves Nisnevich covering families and $\mathbf{A}^1$-projections, it follows that $f^* \simeq \mathbf{L}f^*$ sends motivic anima over S to motivic anima over T, and that it admits as left adjoint the functor

$$\mathbf{L}f_\sharp : \mathbf{H}(\mathcal{A}_{/T}) \to \mathbf{H}(\mathcal{A}_{/S})$$

which still satisfies the projection formula and is therefore a morphism of $\mathbf{H}(\mathcal{A}_{/S})$-modules. Since $\mathbf{L}f_\sharp$ preserves terminal objects, it extends to a functor

$$\mathbf{L}f_\sharp : \mathbf{H}(\mathcal{A}_{/T})_\bullet \to \mathbf{H}(\mathcal{A}_{/S})_\bullet$$

which is a morphism of $\mathbf{H}(\mathcal{A}_{/S})_\bullet$-modules and hence induces, by extension of scalars along the symmetric monoidal functor $\Sigma^\infty_{\mathbf{T}} : \mathbf{H}(\mathcal{A}_{/S})_\bullet \to \mathbf{SH}(\mathcal{A}_{/S})$, an $\mathbf{SH}(\mathcal{A}_{/S})$-linear functor

$$\mathbf{L}f_\sharp : \mathbf{SH}(\mathcal{A}_{/T}) \to \mathbf{SH}(\mathcal{A}_{/S}).$$

□

Remark 1.25. In the situation of Proposition 1.23, we will write $f^* := \mathbf{L}f^*$, as the proof shows that the underived functor already preserves motivic objects.

Proposition 1.26 (Admissible base change)**.** *Suppose given a cartesian square of qcqs derived algebraic spaces*

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "q"] & S' \arrow[d, "p"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

where p and q are admissible. Then there are canonical isomorphisms

$$\mathbf{L}q_\sharp \mathbf{L}g^* \to \mathbf{L}f^* \mathbf{L}p_\sharp,$$
$$f_* p^* \to q^* g_*.$$

Proof. The first morphism is the composite

$$\mathbf{L}q_\sharp \mathbf{L}g^* \xrightarrow{\text{unit}} \mathbf{L}q_\sharp \mathbf{L}g^* p^* \mathbf{L}p_\sharp \simeq \mathbf{L}q_\sharp q^* \mathbf{L}f^* \mathbf{L}p_\sharp \xrightarrow{\text{counit}} \mathbf{L}f^* \mathbf{L}p_\sharp$$

and the second is its right transpose. To show that it is invertible, it suffices to evaluate it on objects of the form $\Sigma^{\infty-n}_{\mathbf{T}} \mathbf{L}\mathrm{h}_S(X)_+$, where $X \in \mathcal{A}_{/S}$ and $n \geqslant 0$, since each functor involved commutes with colimits. Then the claim is clear. □

Corollary 1.27. *Let $j : U \to S$ be an open immersion of qcqs derived algebraic spaces. Then the functor j_* is fully faithful.*

Proof. Apply Proposition 1.26 to the self-intersection square

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d, equal] & U \arrow[d, "j"] \\ U \arrow[r, "j"] & S. \end{tikzcd}$$

□

Lemma 1.28. *Let $(f_\alpha : U_\alpha \to X)_\alpha$ be a Nisnevich covering family. Then the family of functors*

$$f_\alpha^* : \mathbf{SH}(\mathcal{A}_{/X}) \to \mathbf{SH}(\mathcal{A}_{/U_\alpha})$$

is jointly conservative as α varies.

Proof. Since the functors $\Omega^{\infty-n}_{\mathbf{T}} : \mathbf{SH}(\mathcal{A}_{/S}) \to \mathbf{H}(\mathcal{A}_{/S})$ are jointly conservative as $n \geqslant 0$ varies, it suffices to show the claim at the level of motivic anima. This follows from the definitions, see [Kh2, Prop. 2.5.7]. □

1.5. **Thom and Tate twists.**

Construction 1.29. Let $\mathcal{E}$ be a finite locally free sheaf on a qcqs derived algebraic space S. The *Thom twist* is the endofunctor

$$\mathcal{F} \mapsto \mathcal{F}\langle\mathcal{E}\rangle \coloneqq \mathcal{F} \otimes \Sigma^\infty_{\mathbf{T}} \mathrm{Th}_S(\mathcal{E})$$

of $\mathbf{SH}(\mathcal{A}_{/S})$. By Remark 1.21 this is an auto-equivalence, whose inverse we denote by $\mathcal{F} \mapsto \mathcal{F}\langle -\mathcal{E}\rangle$. For example, we have

$$\mathcal{F}\langle\mathcal{O}\rangle = \Sigma_{\mathbf{T}}(\mathcal{F}), \quad \mathcal{F}\langle -\mathcal{O}\rangle = \Omega_{\mathbf{T}}(\mathcal{F})$$

for all $\mathcal{F} \in \mathbf{SH}(\mathcal{A}_{/S})$.

Proposition 1.30. *Let $f : T \to S$ be a morphism of qcqs derived algebraic spaces and let $\mathcal{E}$ be a finite locally free sheaf on S. Then we have:*

(i) *There is a canonical isomorphism*

$$f^*(\mathcal{F}\langle\mathcal{E}\rangle) \simeq f^*(\mathcal{F})\langle f^*\mathcal{E}\rangle,$$

natural in $\mathcal{F} \in \mathbf{SH}(\mathcal{A}_{/S})$.

(ii) *There is a canonical isomorphism*

$$f_*(\mathcal{G}\langle f^*\mathcal{E}\rangle) \simeq f_*(\mathcal{G})\langle\mathcal{E}\rangle,$$

natural in $\mathcal{G} \in \mathbf{SH}(\mathcal{A}_{/T})$.

(iii) *If f is admissible, then there is a canonical isomorphism*

$$f_\sharp(\mathcal{G}\langle f^*\mathcal{E}\rangle) \simeq f_\sharp(\mathcal{G})\langle\mathcal{E}\rangle,$$

natural in $\mathcal{G} \in \mathbf{SH}(\mathcal{A}_{/T})$.

Proof. The first claim follows from the fact that f^* sends the Thom anima $\mathrm{Th}_S(\mathcal{E})$ to $\mathrm{Th}_T(f^*\mathcal{E})$. The second and third follow from the first by adjunction and by the projection formula, respectively. □

Proposition 1.31. *Let S be a qcqs derived algebraic space and let*

$$\mathcal{E}' \to \mathcal{E} \to \mathcal{E}''$$

be an exact triangle of finite locally free sheaves on S. Then there are canonical isomorphisms

$$\mathcal{F}\langle\mathcal{E}\rangle \simeq (\mathcal{F}\langle\mathcal{E}'\rangle)\langle\mathcal{E}''\rangle \simeq (\mathcal{F}\langle\mathcal{E}''\rangle)\langle\mathcal{E}'\rangle,$$

natural in $\mathcal{F} \in \mathbf{SH}(\mathcal{A}_{/S})$.

Proof. The claim is clear when the triangle is split. To reduce to this case, consider the derived algebraic space parametrizing splittings; since it is an affine bundle (being a torsor under the Hom-bundle $\underline{\mathrm{Hom}}_S(\mathcal{E}', \mathcal{E}'')$), inverse image along it is fully faithful by homotopy invariance. □

Remark 1.32. The Thom twist construction $\langle \mathcal{E} \rangle$ extends from locally free sheaves to perfect complexes. In fact, it extends to a canonical map of spectra

$$\mathrm{K}(S) \to \mathrm{Aut}_{\mathbf{SH}(\mathcal{A}_{/S})}(\mathbf{SH}(\mathcal{A}_{/S})) \simeq \mathrm{Pic}(\mathbf{SH}(\mathcal{A}_{/S})),$$

where the target is the ∞-groupoid of automorphisms of $\mathbf{SH}(\mathcal{A}_{/S})$ as a $\mathbf{SH}(\mathcal{A}_{/S})$-module. Moreover, the map is natural in S (with respect to inverse image). This essentially follows from Propositions 1.30(i) and 1.31. More precisely, by Nisnevich descent (which we will prove in a more general setting in Theorem 2.52) it is enough to construct the map on the site of affine derived schemes. In that case, the source is the group completion of the presheaf of finite finite locally free sheaves and the target is group-complete. Hence the map in question is induced by (the motivic localization of) the canonical map sending a finite locally free sheaf to the associated Thom anima (viewed as a fibred anima).

Notation 1.33. Writing

$$1 := [\mathcal{O}_S] \in \mathrm{K}(S), \quad n := n \cdot 1 \in \mathrm{K}(S)$$

as usual leads to the notation

$$\mathcal{F}\langle 1 \rangle := \mathcal{F}\langle \mathcal{O}_S \rangle = \Sigma_{\mathbf{T}}(\mathcal{F})$$

as well as more generally

$$\mathcal{F}\langle n \rangle := \mathcal{F}\langle \mathcal{O}_S^{\oplus n} \rangle, \quad \mathcal{F}\langle -n \rangle := \mathcal{F}\langle -\mathcal{O}_S^{\oplus n} \rangle,$$

for any integer $n \geqslant 0$.

Notation 1.34 (Tate twist)**.** For any $\mathcal{F} \in \mathbf{SH}(\mathcal{A}_{/S})$ and any integer $n \in \mathbf{Z}$ we write

$$\mathcal{F} \mapsto \mathcal{F}(n) := \mathcal{F}\langle n \rangle[-2n],$$

so that $\mathcal{F}(n)[2n] = \mathcal{F}\langle n \rangle$.

Remark 1.35. Tate twists can be defined without reference to Thom twists: we have

$$\mathcal{F}(1) \simeq \Sigma^{\infty}_{\mathbf{T}}(\mathbf{A}^1_S \smallsetminus S)[-1],$$

where $\mathbf{A}^1_S \smallsetminus S$ is pointed by any nonzero section (e.g. by the unit section). This follows from Example 1.15.

1.6. **Localization.** Let S be a qcqs derived algebraic space and suppose given a diagram of qcqs derived algebraic spaces

$$Z \xhookrightarrow{i} S \xhookleftarrow{j} U$$

where i is a closed immersion and j is the complementary open immersion. We refer to this data as a (qcqs) *closed/open pair* in S.

Theorem 1.36. *Let S be a qcqs derived algebraic space and (i,j) a closed/open pair in S as above. Assume that every $X \in \mathcal{A}_{/S}$ is* smooth. *Then we have:*

(i) *The direct image functor $i_* : \mathbf{SH}(\mathcal{A}_{/Z}) \to \mathbf{SH}(\mathcal{A}_{/S})$ is fully faithful.*
(ii) *An $\mathcal{A}$-fibred motivic spectrum $\mathcal{F} \in \mathbf{SH}(\mathcal{A}_{/S})$ belongs to the essential image of i_* if and only if $j^*(\mathcal{F}) \simeq 0$.*

Proof. These claims are proven at the level of pointed motivic anima in [Kh2, Thm. 3.2.4]. They imply that the functor

$$i_* : \mathbf{H}(\mathcal{A}_{/Z})_\bullet \to \mathbf{H}(\mathcal{A}_{/S})_\bullet$$

lifts to a morphism of $\mathbf{H}(\mathcal{A}_{/S})_\bullet$-modules (see [Kh2, Prop. 3.4.2]). Extending scalars along $\Sigma^\infty_{\mathbf{T}}$ then yields the claim for motivic spectra. □

Remark 1.37. An equivalent reformulation of Theorem 1.36 is that, for any closed/open pair (i,j) as above, there is a canonical exact triangle

$$j_\sharp j^*(\mathcal{F}) \xrightarrow{\text{counit}} \mathcal{F} \xrightarrow{\text{unit}} i_*i^*(\mathcal{F}),$$

natural in $\mathcal{F} \in \mathbf{SH}(\mathcal{A}_{/S})$.

Example 1.38. Let $p : E \to S$ be the total space of a finite locally free sheaf $\mathcal{E}$ on a qcqs derived algebraic space S. Denote by $s : S \to E$ the zero section and consider the following diagram.

$$\begin{tikzcd} S \arrow[r, hook, "s"] \arrow[dr, equal] & E \arrow[d, "p"] & E \smallsetminus S \arrow[l, hook', "j"'] \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

Assume that p is admissible. Applying Remark 1.37 to the upper row and then taking the $\sharp$-direct image along p, we have an exact triangle

$$q_\sharp(\mathbf{1}_{E\smallsetminus S}) \to p_\sharp(\mathbf{1}_E) \to p_\sharp s_*(\mathbf{1}_S).$$

This is identified with the image along $\Sigma^\infty_{\mathbf{T}}$ of the cofibre sequence of pointed motivic anima

$$\mathbf{L}\mathrm{h}_S(E \smallsetminus S)_+ \to \mathbf{L}\mathrm{h}_S(E)_+ \to \mathrm{Th}_S(\mathcal{E})$$

of Example 1.14. In particular, there is a canonical isomorphism

$$p_\sharp s_*(\mathbf{1}_S) \simeq \Sigma^\infty_{\mathbf{T}} \mathrm{Th}_S(\mathcal{E}).$$

In fact, by the projection formula, it follows that there is a canonical isomorphism

$$p_\sharp s_* \simeq \langle \mathcal{E} \rangle$$

of auto-equivalences of $\mathbf{SH}(\mathcal{A}_{/S})$.

1.7. Examples of motivic spectra.

Example 1.39 (K-theory)**.** Over every qcqs derived algebraic space S, there is a motivic spectrum $\mathrm{KGL}_S \in \mathbf{SH}(\mathcal{A}_{/S})$. It is stable under arbitrary inverse images: there are canonical isomorphisms

$$\mathbf{L}f^*(\mathrm{KGL}_S) \simeq \mathrm{KGL}_T$$

for every morphism $f : T \to S$ of qcqs derived algebraic spaces.

We recall the construction, which essentially follows Voevodsky [Vo1, 6.2] and Cisinski [Ci]. For a qcqs derived algebraic space S, let $\mathrm{KH}(S)$ denote its homotopy invariant K-theory spectrum (see [Kh3, §5.4]). Let KH_S denote the presheaf $X \mapsto \mathrm{KH}(X)$ restricted to the site $\mathcal{A}_{/S}$. This satisfies Nisnevich descent and $\mathbf{A}^1$-invariance, hence defines an $\mathcal{A}$-fibred motivic S^1-spectrum (in the sense of Definition 1.7(v) with $\mathcal{V}$ the ∞-category of spectra). Moreover, it can be described as the Bott periodization of the connective K-theory spectrum [Kh3, Proof of Thm. 5.13]. It follows therefore from [Ho3, Prop. 3.2] that KH_S deloops uniquely to a Bott-periodic motivic $\mathcal{E}_\infty$-ring spectrum $\mathrm{KGL}_S \in \mathbf{SH}(\mathcal{A}_{/S})$. Since KH_S is stable under inverse image in S [Kh3, Proof of Thm. 5.13], so is KGL_S.

Example 1.40 (Motivic cohomology)**.** We define the motivic cohomology spectrum $\mathbf{Z}^{\mathrm{mot}}_S \in \mathbf{SH}(\mathrm{Sm}_{/S})$, over a qcqs derived algebraic space S, as the inverse image of Spitzweck's motivic cohomology spectrum over $\mathrm{Spec}(\mathbf{Z})$ [Sp]. For S the spectrum of a field (resp. Dedekind domain), $\mathbf{Z}^{\mathrm{mot}}_S$ can be described in terms of the Bloch (resp. Bloch–Levine) cycle complexes (see [Sp, Sect. 5, Thm. 7.18]). For S a noetherian classical scheme of finite dimension, $\mathbf{Z}^{\mathrm{mot}}_S$ agrees with the cdh-local motivic cohomology spectrum of Cisinski–Déglise, at least up to inverting the exponential characteristic [CD1, Rem. 3.7]. It is stable under inverse image by definition.

Example 1.41 (Algebraic cobordism)**.** We define the algebraic cobordism spectrum $\mathrm{MGL}_S \in \mathbf{SH}(\mathcal{A}_{/S})$, over a qcqs derived algebraic space S, as the homotopy colimit

$$\mathrm{MGL}_S = \varinjlim_{(X,\mathcal{E})} f_\sharp(\mathbf{1}_X)\langle\mathcal{E}\rangle$$

over the ∞-category[4] of pairs $(X, \mathcal{E})$ with $f : X \to S$ an admissible morphism and $\mathcal{E} \in \mathrm{K}(X)$ a K-theory class of virtual rank 0. This is stable under inverse image and, in case $\mathcal{A}_{/S} = \mathrm{Sm}_{/S}$, one can show as in [BH, Thm. 16.13] that this it agrees with Voevodsky's description [Vo1, Subsect. 6.3].

2. The six operations

2.1. $(*, \sharp, \otimes)$-formalisms and Voevodsky's conditions.

Notation 2.1.

[4] i.e., the "total space" of the cartesian fibration associated to the presheaf sending $X \in \mathcal{A}_{/S}$ to the virtual rank 0 part of $\mathrm{K}(X)$

(i) Fix a qcqs derived algebraic space S_0 and a full subcategory $\mathcal{S}$ of the ∞-category of qcqs derived algebraic spaces over S_0 which is closed under coproducts and fibred products. Assume also that for any $S \in \mathcal{S}$, we have: (a) $U \in \mathcal{S}$ for every quasi-compact open $U \subseteq S$; (b) $Z \in \mathcal{S}$ for every closed subspace $Z \subseteq S$; (c) $\mathbf{P}(\mathcal{E}) \in \mathcal{S}$ for every finite locally free sheaf $\mathcal{E}$ on S.

(ii) Fix a class of *admissible* morphisms in $\mathcal{S}$, containing all open immersions and projections $X \times \mathbf{P}^n \to X$ $(n \geqslant 0)$, closed under composition and base change, and satisfying the 2-out-of-3 property. Let $\mathcal{A} \subseteq \mathcal{S}$ denote the (non-full) subcategory of $\mathcal{S}$ spanned by admissible morphisms.

(iii) Given a presheaf of ∞-categories $\mathbf{D}^*$ on $\mathcal{S}$, we will write

$$\mathbf{D}(S) := \mathbf{D}^*(S)$$

for every $S \in \mathcal{S}$. For every morphism $f : T \to S$ in $\mathcal{S}$, we denote by

$$f^* := \mathbf{D}^*(f) : \mathbf{D}(S) \to \mathbf{D}(T)$$

the functor of *inverse image* along f.

(iv) If $\mathbf{D}^*$ takes values in *presentable* ∞-categories and colimit-preserving functors, then we say simply that $\mathbf{D}^*$ is a *presheaf of presentable ∞-categories*. In this case, every inverse image functor f^* admits a right adjoint f_* called *direct image* along f.

(v) If $\mathbf{D}^*$ moreover factors through the ∞-category of *symmetric monoidal* presentable ∞-categories, then we say that $\mathbf{D}^*$ is a *presheaf of symmetric monoidal presentable ∞-categories*. We write $\otimes : \mathbf{D}(S) \otimes \mathbf{D}(S) \to \mathbf{D}(S)$ for the monoidal product and $\mathbf{1}_S \in \mathbf{D}(S)$ for the monoidal unit over any $S \in \mathcal{S}$. Since $\otimes$ commutes with colimits in each argument (see conventions), it admits as right adjoint an internal hom bifunctor $\underline{\mathrm{Hom}} : \mathbf{D}(S)^{\mathrm{op}} \times \mathbf{D}(S) \to \mathbf{D}(S)$.

Definition 2.2. A $(*, \sharp, \otimes)$-*formalism* on $(\mathcal{S}, \mathcal{A})$ is a presheaf $\mathbf{D}^*$ of symmetric monoidal presentable ∞-categories on $\mathcal{S}$ satisfying the following properties.

(i) For every admissible morphism $f : T \to S$ in $\mathcal{S}$, the inverse image functor f^* admits a left adjoint

$$f_\sharp : \mathbf{D}(T) \to \mathbf{D}(S)$$

called $\sharp$-*direct image*.

(ii) The $\sharp$-direct image functors satisfy the projection formula. That is, $f_\sharp : \mathbf{D}(T) \to \mathbf{D}(S)$ is a morphism of $\mathbf{D}(S)$-modules, where $\mathbf{D}(T)$ is regarded as a $\mathbf{D}(S)$-module via the symmetric monoidal functor $f^* : \mathbf{D}(S) \to \mathbf{D}(T)$.

(iii) The $\sharp$-direct image functors satisfy the base change formula, as in Proposition 1.26.

(iv) *Additivity.* For any finite family $(S_\alpha)_\alpha$ in $\mathcal{S}$, the canonical functor

$$\mathbf{D}\big(\coprod_\alpha S_\alpha\big) \to \prod_\alpha \mathbf{D}(S_\alpha)$$

is an equivalence.

By default, when we speak of $(*, \sharp, \otimes)$-formalisms on $\mathcal{S}$, the admissible morphisms will be taken to be the smooth morphisms in $\mathcal{S}$ (i.e., $\mathcal{A} = \mathrm{Sm}$).

Definition 2.3 (Thom twist)**.** Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $(\mathcal{S}, \mathcal{A})$. Let $S \in \mathcal{S}$ and let $\mathcal{E}$ be a finite locally free sheaf on S. Write $E = \mathbf{V}_S(\mathcal{E})$ for its total space, $p : E \to S$ for the projection, and $s : S \to E$ for the zero section. Define the *Thom twist* $\langle \mathcal{E} \rangle$ as the endofunctor on $\mathbf{D}(S)$ given by

$$\mathcal{F} \mapsto \mathcal{F}\langle \mathcal{E} \rangle \coloneqq p_\sharp s_*(\mathcal{F}).$$

This is well-defined as long as p is admissible.

Definition 2.4 (Voevodsky conditions)**.** Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$ and consider the following conditions.

(i) *Homotopy invariance.* For every $S \in \mathcal{S}$ and every vector bundle $p : E \to S$, the unit map

$$\mathrm{id} \to p_* p^*$$

is invertible.

(ii) *Localization.* For every closed/open pair

$$Z \xhookrightarrow{i} S \xhookleftarrow{j} U$$

in $\mathcal{S}$, the functor i_* is fully faithful with essential image spanned by objects in the kernel of j^*.

(iii) *Thom stability.* For every $S \in \mathcal{S}$ and every finite locally free sheaf $\mathcal{E}$ on S, the endofunctor $\langle \mathcal{E} \rangle$ on $\mathbf{D}(S)$ is an equivalence.

When these hold, we say that $\mathbf{D}^*$ *satisfies the Voevodsky conditions.* (Compare [Vo2, §2, 1.2.1].)

Example 2.5 (Motivic spectra)**.** Let $\mathcal{S}$ be the ∞-category of qcqs derived algebraic spaces. Write $\mathbf{SH}^*$ for the $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$ given by the presheaf $S \mapsto \mathbf{SH}(S)$, $f \mapsto \mathbf{L}f^*$. For simplicity we will omit the decoration $\mathbf{L}$ from $\mathbf{L}f^*$ and $\mathbf{L}f_\sharp$ below. $\mathbf{SH}^*$ satisfies Voevodsky's conditions:

(i) *Homotopy invariance.* To show that $\mathcal{F} \to p_* p^*(\mathcal{F})$ is invertible for all $\mathcal{F} \in \mathbf{SH}(S)$, it will suffice to show that the induced map

$$\mathrm{Maps}(\Sigma^{\infty-n}_{\mathbf{T}}(X_+), \mathcal{F}) \to \mathrm{Maps}(\Sigma^{\infty-n}_{\mathbf{T}}(X_+), p_* p^*(\mathcal{F}))$$

is invertible for all $X \in \mathrm{Sm}_{/S}$ and $n \geqslant 0$. By adjunction, this is identified with the map

$$\mathbf{R}\Gamma(X, \Omega^{\infty-n}_{\mathbf{T}}(\mathcal{F})) \to \mathbf{R}\Gamma(X \underset{S}{\times} E, \Omega^{\infty-n}_{\mathbf{T}}(\mathcal{F})).$$

Since the motivic anima $\Omega^{\infty-n}_{\mathbf{T}}(\mathcal{F}) \in \mathbf{H}(S)$ satisfies Nisnevich descent, we may assume that E is a trivial vector bundle $\mathbf{A}^r_S$. For $r = 1$ the claim is just the $\mathbf{A}^1$-homotopy invariance property, and for $r > 1$ it follows by induction.

(ii) *Localization.* This is Theorem 1.36.

(iii) *Thom stability.* Follows from Remark 1.21 and Example 1.38.

Example 2.6 (Étale sheaves)**.** Let S_0 be a qcqs derived algebraic space, ℓ an integer which is invertible on S_0, and $\mathcal{S}$ the ∞-category of qcqs derived algebraic spaces over S_0. Let $\mathbf{D}^*_{\text{ét}}(-, \mathbf{Z}/\ell\mathbf{Z})$ denote the presheaf

$$S \mapsto \mathbf{D}_{\text{ét}}(S, \mathbf{Z}/\ell\mathbf{Z})$$

sending $S \in \mathcal{S}$ to the derived ∞-category of étale sheaves of $\mathbf{Z}/\ell\mathbf{Z}$-modules on the small étale site of S. Then the results of [SGA4] (cf. [CD2, §1]) imply that $\mathbf{D}^*_{\text{ét}}(-, \mathbf{Z}/\ell\mathbf{Z})$ defines a $(*, \sharp, \otimes)$-formalism which satisfies Voevodsky's conditions. Note that in this setting, the Thom twist $\langle \mathcal{E} \rangle$ by any finite locally free sheaf $\mathcal{E}$ of rank r is canonically identified with $(r)[2r]$. The functor $p_\sharp$, for a smooth morphism p of finite presentation, is given by

$$p_\sharp = p_!\langle \mathcal{L}_p \rangle \simeq p_!(d)[2d],$$

where d is the relative dimension of p. Similarly, the ℓ-adic derived ∞-category $\mathbf{D}^*_{\text{ét}}(-, \mathbf{Z}_\ell)$ also defines a $(*, \sharp, \otimes)$-formalism ∞-category satisfying Voevodsky conditions.

Remark 2.7. Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$. If $\mathbf{D}^*$ satisfies the Voevodsky conditions then the ∞-categories $\mathbf{D}(S)$ are stable for all $S \in \mathcal{S}$. Indeed, one has $\mathbf{1}\langle \mathcal{O}_S \rangle \simeq S^1 \otimes M_S(\mathbf{A}^1_S \smallsetminus S)$ by homotopy invariance (cf. Example 1.15), where $M_S(X)$ is shorthand for $p_\sharp(\mathbf{1}_X)$, for any smooth morphism $p : X \to S$. Hence if $\mathbf{D}^*$ is Thom stable, then the topological circle S^1 acts invertibly on $\mathbf{D}(S)$.

Remark 2.8. Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$. The base change formula for $\sharp$-direct image has the following consequences for a closed/open pair

$$Z \xhookrightarrow{i} S \xhookleftarrow{j} U$$

in $\mathcal{S}$. First, the functors $j_\sharp$ and j_* are fully faithful (consider the self-intersection, as in Corollary 1.27). Second, $j^* i_* \simeq 0$ and $i^* j_\sharp \simeq 0$ (consider the intersection of i and j).

Remark 2.9. Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$. If $\mathbf{D}^*$ satisfies the localization property, then for any closed/open pair (i, j) in $\mathcal{S}$ as above, there is a canonical exact triangle

$$j_\sharp j^* \xrightarrow{\text{counit}} \text{id} \xrightarrow{\text{unit}} i_* i^*$$

of endofunctors of $\mathbf{D}(S)$ (cf. Remark 1.37). Moreover, i_* admits as right adjoint the functor $i^!$ defined as the homotopy fibre

$$\text{Fib}(i^* \xrightarrow{\text{unit}} i^* j_* j^*).$$

By right transposition from the triangle above, we therefore get another triangle

$$i_*i^! \simeq i_!i^! \xrightarrow{\text{counit}} \mathrm{id} \xrightarrow{\text{unit}} j_*j^*.$$

Remark 2.10. For any closed/open pair (i,j) in $\mathcal{S}$ as above, the localization property implies that the stable ∞-category $\mathbf{D}(S)$ admits a semi-orthogonal decomposition

$$\mathbf{D}(S) = \big\langle \mathbf{D}(S)_+, \mathbf{D}(S)_- \big\rangle,$$

where $\mathbf{D}(S)_+$ is the essential image of i_* and $\mathbf{D}(S)_-$ is the essential image of j_*.

Lemma 2.11 (Constructible separation)**.** *Let $\mathbf{D}^*$ be a $(*,\sharp,\otimes)$-formalism on $\mathcal{S}$. If $\mathbf{D}^*$ satisfies the localization property, then for any $S \in \mathcal{S}$ and any constructible covering family*[5] *$(j_\alpha : S_\alpha \to S)_\alpha$, the family of inverse image functors*

$$j_\alpha^* : \mathbf{D}(S) \to \mathbf{D}(S_\alpha)$$

is jointly conservative as α varies.

Proof. It suffices to consider families of the form (i,j), where i and j form a closed pair, so the claim is immediate from localization. □

Lemma 2.12 (Nisnevich separation)**.** *Let $\mathbf{D}^*$ be a $(*,\sharp,\otimes)$-formalism on $\mathcal{S}$. If $\mathbf{D}^*$ satisfies the localization property, then for any $S \in \mathcal{S}$ and any Nisnevich covering family $(f_\alpha : S_\alpha \to S)_\alpha$, the family of inverse image functors*

$$f_\alpha^* : \mathbf{D}(S) \to \mathbf{D}(S_\alpha)$$

is jointly conservative as α varies.

Proof. It suffices to consider families of the form $(j : U \to S, p : V \to S)$, where

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "j"] & S \end{tikzcd}$$

is a Nisnevich square. If $i : Z \to S$ is a closed immersion complementary to j, then it follows from localization that the pair (j^*, i^*) is jointly conservative. But since p is an isomorphism over Z, it follows that (j^*, p^*) is also conservative. □

Lemma 2.13 (Nil invariance)**.** *Let $\mathbf{D}^*$ be a $(*,\sharp,\otimes)$-formalism on $\mathcal{S}$. Let $i : S' \to S$ be a surjective closed immersion of derived algebraic spaces in $\mathcal{S}$. Then the pair of adjoint functors*

$$i^* : \mathbf{D}(S) \to \mathbf{D}(S'), \quad i_* : \mathbf{D}(S') \to \mathbf{D}(S)$$

is an equivalence of ∞-categories. In particular, for every $S \in \mathcal{S}$, there are canonical equivalences

$$\mathbf{D}(S) \simeq \mathbf{D}(S_{\mathrm{cl}}) \simeq \mathbf{D}(S_{\mathrm{cl,red}})$$

where S_{cl} is the classical truncation and $S_{\mathrm{cl,red}}$ is its reduction.

[5] i.e., any family that generates a covering for the constructible topology

Proof. By localization, the pair of functors (i^*, j^*) is conservative, where j is the complementary open immersion. But the open complement is empty by assumption. □

Remark 2.14 (Universal property of **SH**)**.** For any $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$ satisfying Voevodsky's conditions, there exists a unique system of colimit-preserving functors

$$R_S : \mathbf{SH}(S) \to \mathbf{D}(S)$$

for every $S \in \mathcal{S}$, which commute with $\sharp$-direct images (along smooth morphisms), inverse images, tensor products, and arbitrary Thom twists.

Indeed, consider the functor $M_S : \mathrm{Sm}_S \to \mathbf{D}(S)$ sending $(p : X \to S) \mapsto p_\sharp(\mathbf{1}_X)$, which is symmetric monoidal by the smooth projection and base change formulas. Its left Kan extension to the ∞-category of Sm-fibred anima sends $\mathbf{A}^1$-projections to isomorphisms in $\mathbf{D}(S)$ (by the homotopy invariance property of $\mathbf{D}^*$) and Nisnevich squares to cartesian squares in $\mathbf{D}(S)$ (by localization and smooth base change, cf. proof of Lemma 2.12). Thus M_S extends uniquely to a symmetric monoidal colimit-preserving functor $\mathbf{H}(S) \to \mathbf{D}(S)$. Since the target is pointed (has a zero object), this in particular factors through $\mathbf{H}(S)_\bullet$. By Thom stability, it sends $\mathbf{T}_S$ (Example 1.15) to a $\otimes$-invertible object and hence factors by Remark 1.20 through $\mathbf{SH}(S)$ as desired. The resulting functor R_S is, by construction, the unique symmetric monoidal colimit-preserving functor which sends $\Sigma^{\infty-n}_{\mathbf{T}}(X_+)$ to $p_\sharp(\mathbf{1}_X)\langle -n\rangle$ for every $(p : X \to S) \in \mathrm{Sm}_{/S}$ and every $n \geqslant 0$. It is immediate from this description that R_S commutes with $\sharp$-direct image, with inverse image (by smooth base change), and with Thom twists.

Definition 2.15 (Compact generation)**.** Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $(\mathcal{S}, \mathcal{A})$. We say that $\mathbf{D}^*$ is *compactly generated* if the following conditions hold:

(i) For every $S \in \mathcal{S}$, the ∞-category $\mathbf{D}(S)$ is compactly generated.
(ii) For every morphism $f : T \to S$ in $\mathcal{S}$, the functor $f^* : \mathbf{D}(S) \to \mathbf{D}(T)$ is compact, i.e., preserves compact objects.

See [DFJK, Def. A.5]

Definition 2.16 (Continuity)**.** Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $(\mathcal{S}, \mathcal{A})$. We say that $\mathbf{D}^*$ is *continuous* if, for every cofiltered system $(S_\alpha)_\alpha$ of affine derived schemes with limit S, the canonical functor

$$\varinjlim_{\alpha} \mathbf{D}(S_\alpha) \to \mathbf{D}(S)$$

is an equivalence, where the colimit is taken in the ∞-category of presentable ∞-categories and left adjoint functors. In other words, $\mathbf{D}^*$ restricts to a filtered colimit-preserving functor $R \mapsto \mathbf{D}(\mathrm{Spec}(R))$ from derived commutative rings to presentable ∞-categories. See [DFJK, App. A] for details.

Remark 2.17. Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$ satisfying Voevodsky's conditions. We will see (Theorem 2.52) that $\mathbf{D}^*$ satisfies Nisnevich descent.

Therefore, if $\mathbf{D}^*$ satisfies continuity as in Definition 2.16, then it will in fact satisfy continuity for any cofiltered system in $\mathcal{S}$ with affine transition maps.

Example 2.18. The $(*, \sharp, \otimes)$-formalism $\mathbf{SH}^*$ satisfies continuity. This essentially follows from the fact that $\mathrm{Sm}_{/S}$ is the colimit of $\mathrm{Sm}_{/S_\alpha}$ (see [To, Prop. 1.6]). See [Ho1, Prop. C.12(4)] for details.

2.2. Closed base change.

Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$ satisfying Voevodsky's conditions.

Lemma 2.19 (Closed base change)**.** *Suppose given a commutative square in $\mathcal{S}$*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "g"] & S' \arrow[d, "f"] \\ Z \arrow[r, "i"] & S \end{tikzcd}$$

which is cartesian on classical truncations. Then there is a canonical isomorphism

$$f^* i_* \to i'_* g^*$$

of functors $\mathbf{D}(Z) \to \mathbf{D}(S')$.

Proof. Consider the natural transformation

$$f^* i_* \xrightarrow{\text{unit}} i'_* i'^* f^* i_* \simeq i'_* g^* i^* i_* \xrightarrow{\text{counit}} i'_* g^*.$$

By smooth base change, we can work locally on S and assume in particular that i has quasi-compact open complement. Since $i^* i_* \simeq \mathrm{id}$ by the localization property, it will suffice to show that it is invertible after applying i^* on the right. It also suffices (by Lemma 2.11) to show it is invertible after applying either i'^* or j'^* on the left, where j' is the open immersion complementary to i'. The first claim is obvious and the second follows from Remark 2.8. ☐

Lemma 2.20 (Closed projection formula)**.** *Suppose given a closed immersion $i : Z \to S$ in $\mathcal{S}$. Then the functor $i_* : \mathbf{D}(Z) \to \mathbf{D}(S)$ is a morphism of $\mathbf{D}(S)$-module ∞-categories. In particular, there are canonical isomorphisms*

$$i_*(\mathcal{F}') \otimes \mathcal{F} \to i_*(\mathcal{F}' \otimes i^*(\mathcal{F}))$$

natural in $\mathcal{F}, \mathcal{G} \in \mathbf{D}(S)$ and $\mathcal{F}' \in \mathbf{D}(Z)$.

Proof. Recall that i^* is symmetric monoidal, so by abstract nonsense its right adjoint i_* admits a canonical structure of *lax* morphism of $\mathbf{D}(S)$-modules. More concretely, there is a canonical natural transformation

$$i_*(-) \otimes (-) \to i_*(- \otimes i^*(-)),$$

which is the right transpose of the natural transformation

$$i^*(i_*(-) \otimes (-)) \simeq i^* i_*(-) \otimes i^*(-) \xrightarrow{\text{counit}} (-) \otimes i^*(-).$$

By definition, this lax structure is strict if this morphism is invertible. By smooth base change, we can work locally on S and assume in particular that i has quasi-compact open complement j. It will suffice (by Lemma 2.11) to show it is invertible after applying either i^* or j^* on the left. The first claim is clear from $i^* i_* \simeq \mathrm{id}$ and the second from $j^* i_* \simeq 0$ (Remark 2.8). ☐

Lemma 2.21 (Smooth-closed base change)**.** *Suppose given a cartesian square in $\mathcal{S}$*

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "q"] & S' \arrow[d, "p"] \\ Z \arrow[r, "i"] & S \end{tikzcd}$$

where i is a closed immersion and p and q are smooth. Then there is a canonical isomorphism

$$p_\sharp i'_* \to i_* q_\sharp$$

of functors $\mathbf{D}(Z') \to \mathbf{D}(S)$.

Proof. Consider the natural transformation

$$p_\sharp i'_* \xrightarrow{\text{unit}} i_* i^* p_\sharp i'_* \simeq i_* q_\sharp i'^* i'_* \xrightarrow{\text{counit}} i_* q_\sharp.$$

By smooth base change, we can work locally on S and assume in particular that i has quasi-compact open complement j. It will suffice (by Lemma 2.11) to show it is invertible after applying either i^* or j^* on the left. Again, the claims follow from $i^* i_* \simeq \mathrm{id}$ and $j^* i_* \simeq 0$ (Remark 2.8), respectively. ☐

Remark 2.22. Recall the *Thom twist* functor $\langle \mathcal{E} \rangle : \mathbf{D}(S) \to \mathbf{D}(S)$ associated to any finite locally free sheaf $\mathcal{E}$ on a derived algebraic space $S \in \mathcal{S}$. It follows from the smooth and closed projection formulas that we have canonical isomorphisms

$$\mathcal{F}\langle \mathcal{E} \rangle \simeq \mathcal{F} \otimes \mathbf{1}_S \langle \mathcal{E} \rangle,$$

natural in $\mathcal{F} \in \mathbf{D}(S)$. In other words, the operation $\langle \mathcal{E} \rangle$ is $\mathbf{D}(S)$-linear.

Remark 2.23. Let $\mathrm{Aut}_{\mathbf{D}}(\mathbf{D})$ denote the presheaf that sends $S \in \mathcal{S}$ to the ∞-groupoid $\mathrm{Aut}_{\mathbf{D}(S)}(\mathbf{D}(S))$ of $\mathbf{D}(S)$-linear auto-equivalences of $\mathbf{D}(S)$. Of course, this is canonically isomorphic to the Picard ∞-groupoid $\mathrm{Pic}(\mathbf{D}(S))$ of $\otimes$-invertible objects. Note that the assignment $\mathcal{E} \mapsto \langle \mathcal{E} \rangle$ extends to a canonical map of presheaves

$$\mathrm{K} \to \mathrm{Aut}_{\mathbf{D}}(\mathbf{D}) \simeq \mathrm{Pic}(\mathbf{D}).$$

In the universal case $\mathbf{D} = \mathbf{SH}^*$ this is Remark 1.32. In general, the desired map is the composite

$$\mathrm{K} \to \mathrm{Aut}_{\mathbf{SH}}(\mathbf{SH}) \xrightarrow{R} \mathrm{Aut}_{\mathbf{D}}(\mathbf{D})$$

with the map induced by the morphism of $(*, \sharp, \otimes)$-formalisms ∞-categories $R : \mathbf{SH}^* \to \mathbf{D}^*$ (Remark 2.14).

2.3. **Proper base change.** Fix again a $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$ satisfying Voevodsky's conditions. In this subsection we extend the lemmas from the previous subsection from closed immersions to proper morphisms.

Theorem 2.24. *Let $f : X \to Y$ be a proper morphism in $\mathcal{S}$. Consider the following assertions:*

(i) Proper base change. *For any commutative square in* $\mathcal{S}$

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

which is cartesian on classical truncations, there is a canonical isomorphism

$$\mathrm{Ex}^*_* : v^* f_* \to g_* u^*$$

of functors $\mathbf{D}(X) \to \mathbf{D}(Y')$.

(ii) Smooth-proper base change. *For any cartesian square in* $\mathcal{S}$

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "p"] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where p *and* q *smooth, there is a canonical isomorphism*

$$\mathrm{Ex}_{\sharp,*} : q_\sharp g_* \to f_* p_\sharp$$

of functors $\mathbf{D}(X') \to \mathbf{D}(Y)$.

(iii) Atiyah duality. *If the proper morphism* f *is* smooth, *then the canonical morphism of functors* $\mathbf{D}(X) \to \mathbf{D}(Y)$ *(see Construction 2.26)*

$$\varepsilon_f : f_\sharp \langle \mathcal{L}_f \rangle \to f_*$$

is invertible. In particular, $f_* \langle -\mathcal{L}_f \rangle$ *is left adjoint to* f^*.

These assertions hold under any one of the following technical assumptions:

(a) *The morphism* f *is projective.*

(b) *The derived algebraic space* X *is noetherian.*

(c) *The* $(*, \sharp, \otimes)$*-formalism* $\mathbf{D}^*$ *is continuous and compactly generated.*

We begin with a construction of the morphism which is asserted to be invertible in Theorem 2.24(iii). This will require a preliminary result:

Theorem 2.25 (Relative purity)**.** *Let* $S \in \mathcal{S}$*, let* $X, Y \in \mathrm{Sm}_{/S}$ *with structural morphisms* $p : X \to S$ *and* $q : Y \to S$*, and let* $i : X \to Y$ *be a closed immersion over* S*. Then there is a canonical isomorphism*

$$q_\sharp i_* \simeq p_\sharp \langle \mathcal{N}_{X/Y} \rangle,$$

where $\mathcal{N}_{X/Y}$ *is the conormal sheaf of* i.

Proof. Note that if we set $P_S(X, Y) := q_\sharp i_*$, then we have

$$P_S(X, N_{X/Y}) = p_\sharp \pi_\sharp s_* \simeq p_\sharp \langle \mathcal{N}_{X/Y} \rangle,$$

where $\pi : N_{X/Y} \to X$ is the projection of the normal bundle (i.e., total space of $\mathcal{N}_{X/Y}$) and $s : X \to N_{X/Y}$ is the zero section. Let $D_{X/Y}$ denote the deformation to the normal bundle [KhRy, Thm. 4.1.13], so that there are morphisms of pairs (i.e., homotopy cartesian squares)

$$(X,Y) \to (X \times \mathbf{A}^1, D_{X/Y}) \leftarrow (X, N_{X/Y})$$

given by the inclusions of the fibres over 0 and 1 of $D_{X/Y} \to \mathbf{A}^1$. It will suffice to show that the induced morphism (cf. [CD, 2.4.32])

$$P_S(X,Y) \to P_S(X \times \mathbf{A}^1, D_{X/Y}) \leftarrow P_S(X, N_{X/Y})$$

is invertible. For this, one may either follow the proof of [CD, Thm. 2.4.35] or first use Nisnevich separation (Lemma 2.12) and derived invariance (Lemma 2.13) to literally reduce to the case of *loc. cit.* □

Construction 2.26. Let $f : X \to Y$ be a smooth proper morphism in $\mathcal{S}$. Its diagonal is a closed immersion $\Delta_f : X \to X \times_Y X$ (since f is separated) whose normal bundle is canonically identified with the cotangent complex $\mathcal{L}_f$. Consider the homotopy cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

The exchange transformation $\mathrm{Ex}_{\sharp,*}$ associated to this square (defined just as in Lemma 2.21) gives rise to the desired morphism

$$\varepsilon_f : f_\sharp = f_\sharp \mathrm{pr}_{2,*} \Delta_{f,*} \xrightarrow{\mathrm{Ex}_{\sharp,*}} f_* \mathrm{pr}_{1,\sharp} \Delta_{f,*} \simeq f_* \langle \mathcal{L}_f \rangle$$

where the isomorphism comes from relative purity (Theorem 2.25). Note that formation of ε_f commutes with smooth inverse images, by the smooth base change formula.

Lemma 2.27. *Let $f : X \to Y$ be a smooth and proper morphism. If $\mathbf{D}^*$ satisfies Atiyah duality (Theorem 2.24(iii)) for any base change of f, then it also satisfies proper base change and smooth-proper base change (Theorem 2.24(i)-(ii)) for f.*

Proof. The natural transformations are defined just as in Lemmas 2.19 and 2.21. The proof of (i) is similar to (ii) (cf. [CD, Lem. 2.4.23(1)]), so we only prove the latter. Consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\Delta_g"] \arrow[d, "p"] & X' \times_{Y'} X' \arrow[r, "\mathrm{pr}_1"] \arrow[d, "p'"] & X' \arrow[d, "p"] \\ X \arrow[r, "\Delta_f"] & X \times_Y X \arrow[r, "\mathrm{pr}_1"] & X. \end{tikzcd} \tag{2.28}$$

It will suffice to show that

$$\mathrm{Ex}_{\sharp,*} : q_\sharp g_* \mathrm{pr}_{1,\sharp} \Delta_{g,*} \to f_* p_\sharp \mathrm{pr}_{1,\sharp} \Delta_{g,*}$$

is invertible, since $\mathrm{pr}_{1,\sharp}\Delta_{g,*} \simeq \langle \mathcal{L}_g \rangle$ is invertible (Theorem 2.25). This morphism fits in the diagram

$$\begin{tikzcd}
q_\sharp g_* \mathrm{pr}_{1,\sharp}\Delta_{g,*} \arrow[r, "\mathrm{Ex}_{\sharp,*}"] & f_* p_\sharp \mathrm{pr}_{1,\sharp}\Delta_{g,*} \arrow[r, equal] & f_* \mathrm{pr}_{1,\sharp} p'_\sharp \Delta_{g,*} \arrow[r, "\mathrm{Ex}_{\sharp,*}"] & f_* \mathrm{pr}_{1,\sharp}\Delta_{f,*} p_\sharp \\
q_\sharp g_\sharp \arrow[u, "\varepsilon_g"] \arrow[rrr, equal] & & & f_\sharp p_\sharp \arrow[u, "\varepsilon_f"']
\end{tikzcd}$$

where the vertical arrows are as in Construction 2.26, hence are invertible by Atiyah duality for f and g. The right-hand upper horizontal arrow is also invertible by smooth-closed base change (Lemma 2.21) for the left-hand square in (2.28). In view of the definition of ε, the diagram commutes by abstract nonsense, so the claim follows. □

Lemma 2.29. *Let $f : X \to Y$ be a smooth proper morphism in $\mathcal{S}$. If $\mathbf{D}^*$ satisfies smooth-proper base change (Theorem 2.24(ii)) for f, then it also satisfies Atiyah duality for f (Theorem 2.24(iii)).*

Proof. This is immediate from the construction of ε_f (Construction 2.26). □

We now prove Atiyah duality for projective bundles, using Hoyois's Pontryagin–Thom construction.

Lemma 2.30. *Let $S \in \mathcal{S}$ be a derived algebraic space. For any finite locally free sheaf $\mathcal{E}$ on S, consider the projection $f : \mathbf{P}(\mathcal{E}) \to S$ of the associated projective bundle. Then $\mathbf{D}^*$ satisfies Atiyah duality for f.*

Proof. Set $X := \mathbf{P}(\mathcal{E})$ to simplify the notation. We first reduce to the case where S is affine and $\mathcal{E}$ is free, since the claim is local on S. Indeed, formation of ε_f commutes with smooth base change so this follows from Lemma 2.12. Now recall from [Ho2, §5.3] that there is a Pontryagin–Thom collapse map $\eta_f : \mathbf{1}_S \to f_\sharp(\mathbf{1}_X\langle -\mathcal{L}_f\rangle)$ in $\mathbf{SH}(S)$ (by derived invariance, we may assume S is classical if desired). By Remark 2.14 we get an induced morphism of the same form in $\mathbf{D}(S)$. By the smooth projection formula this induces a natural transformation

$$\eta_f : \mathrm{id}_{\mathbf{D}(S)} \to f_\sharp f^* \langle -\mathcal{L}_f\rangle.$$

Let $\varepsilon'_f : f^* f_\sharp \langle -\mathcal{L}_f\rangle \to \mathrm{id}$ be the transposition of $\varepsilon_f : f_* \to f_\sharp\langle \mathcal{L}_f\rangle$. We claim that ε'_f and η_f are the counit and unit of an adjunction $(f^*, f_\sharp\langle -\mathcal{L}_f\rangle)$; this will in particular imply that ε_f is invertible as desired. To verify the triangle identities for the adjunction, we may easily reduce to showing that the composite

$$f^* \xrightarrow{\eta_f} f^* f_\sharp \langle -\mathcal{L}_f\rangle f^* \xrightarrow{\varepsilon'_f} f^*$$

induces the identity when evaluated on the unit $\mathbf{1}_S$ (see the beginning of the proof of [Ho2, Thm. 5.22]). Again by Remark 2.14 it will suffice to show that the morphism

$$\mathbf{1}_X \xrightarrow{f^*(\eta_f)} f^* f_\sharp(\mathbf{1}_X)\langle -\mathcal{L}_f\rangle \xrightarrow{\varepsilon'_f} \mathbf{1}_X$$

is the identity of $\mathbf{1}_X \in \mathbf{SH}(X)$. This is proven in [Ho2, Thm. 6.9]. □

We can now bootstrap from Lemma 2.30 to get Theorem 2.24 for all *projective* morphisms.

Theorem 2.31. *Theorem 2.24 holds for any projective morphism $f : X \to Y$ in $\mathcal{S}$.*

Proof. Let us prove the proper base change formula (i) for f. By derived invariance we may reduce assume X and Y are classical algebraic spaces. By Nisnevich separation (Lemma 2.12) and smooth base change, we may assume that Y (and hence X) is an affine scheme. By definition, f factors through a closed immersion into a projective bundle over Y. By closed base change (Lemma 2.19) we reduce to the case where f is the projection of a projective bundle. This case follows from Lemmas 2.30 and 2.27.

Exactly the same argument reduces smooth-proper base change (ii) to projective bundles. This case follows again from Lemmas 2.30 and 2.27.

Finally, Atiyah duality (iii) follows from (ii) and Lemma 2.29. □

Finally, we prove Theorem 2.24 in general.

Proof of Theorem 2.24. Under assumption (a), the theorem holds by Theorem 2.31. Let us prove the theorem under assumptions (b) and (c).

Let $f : X \to Y$ be a proper morphism in $\mathcal{S}$. We will prove proper base change for f; smooth-proper base change can be proven by the same type of argument, and Atiyah duality will then follow by Lemma 2.29. Thus suppose given a square

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.32}$$

as in the statement of (i). By derived invariance (Lemma 2.13), we may assume that the diagram consists of classical algebraic spaces. Say that a (classical) closed subspace $Z \subseteq X$ is *good* if the induced morphism

$$\mathrm{Ex}^*_* : v^* f_* i_* \to g_* u^* i_*$$

is invertible, where $i : Z \to X$ is the inclusion. By closed base change, this is equivalent to proper base change for $f \circ i : Z \to Y$ instead of f.

By noetherian induction, it will suffice to demonstrate the following claim:

(∗) For any nonempty closed subspace $Z \subseteq X$, there exists a proper closed subspace $Z_0 \subsetneq Z$ such that goodness of $Z_0 \subseteq X$ implies goodness of $Z \subseteq X$.

Without loss of generality, we may recalibrate notation and show that there exists a proper closed subspace $Z \subsetneq X$ such that goodness of $i : Z \to X$ implies goodness of $\mathrm{id}_X : X \to X$. By the version of Chow's lemma given in [Lu2, Prop. 5.5.2.1], there exists a projective morphism $\pi : \widetilde{X} \to X$ which is an isomorphism over a nonempty quasi-compact open $j : U \to X$, such that

$f \circ \pi : \widetilde{X} \to X \to Y$ is projective. Let $\widetilde{j} : U \to \widetilde{X}$ be the induced morphism as in the diagram below (where the square is cartesian).

$$\begin{tikzcd} U \arrow[r, hook, "\widetilde{j}"] \arrow[d, equal] & \widetilde{X} \arrow[d, "\pi"] \arrow[dr, "\widetilde{f}"] & \\ U \arrow[r, hook, "j"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $i : Z \to X$ be a closed immersion complementary to $j : U \to X$. Consider the commutative diagram

$$\begin{tikzcd} v^* f_* i_* i^! \arrow[r] \arrow[d] & v^* f_* \arrow[r] \arrow[d] & v^* f_* j_* j^* \arrow[d] \\ g_* u^* i_* i^! \arrow[r] & g_* u^* \arrow[r] & g_* u^* j_* j^* \end{tikzcd}$$

induced by the naturality of the localization triangle (Remark 2.9). To show that goodness of $i : Z \to X$ implies goodness of id_X, it will suffice to show that the right-hand vertical arrow is invertible. Factoring $j = \pi \circ \widetilde{j}$, it will suffice to show that

$$\mathrm{Ex}^*_* : v^* f_* \pi_* \to g_* u^* \pi_*$$

is invertible. By Theorem 2.31 applied to the projective morphism π, this is identified with Ex^*_* for the square (2.32) where f is replaced by $f \circ \pi : \widetilde{X} \to Y$. Since the latter is also projective, another application of Theorem 2.31 shows that this is invertible. Hence $(*)$ is proven.

Finally, suppose that $\mathbf{D}^*$ is continuous and compactly generated as in assumption (c). By compact generation it is sufficient to show that for every object $\mathcal{F} \in \mathbf{D}(X)$ and every compact object $\mathcal{G} \in \mathbf{D}(X)$, we have

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{G}, K_{\mathcal{F}}) = 0,$$

where $K_{\mathcal{F}}$ is the cofibre of $\mathrm{Ex}^*_* : v^* f_*(\mathcal{F}) \to g_* u^*(\mathcal{F})$. Fix an element $\phi \in \mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{G}, K_{\mathcal{F}})$ and consider the poset P_ϕ of (classical) closed subspaces $Z \subseteq X$ such that the image of ϕ by

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{G}, K_{\mathcal{F}}) \to \mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{G}, K_{i_* i^* \mathcal{F}})$$

is nonzero. It follows from $(*)$ that P_ϕ has no minimal element. Thus to show that P_ϕ is empty it will suffice to show that for any (hypothetical) descending chain of closed subspaces $(Z_\alpha)_\alpha$ in P_ϕ, with inclusions $i_\alpha : Z_\alpha \to X$, the intersection Z also belongs to P_ϕ. In this case the homomorphism above is identified, by continuity of $\mathbf{D}^*$ and compactness of $\mathcal{G}$, with the colimit of the homomorphisms

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{G}, K_{\mathcal{F}}) \to \mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{G}, K_{i_{\alpha,*} i_\alpha^* \mathcal{F}}).$$

Thus if $Z \notin P_\phi$, i.e. the image of ϕ in $\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{G}, K_{i_* i^* \mathcal{F}})$ is zero, then its image in $\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{G}, K_{i_{\alpha,*} i_\alpha^* \mathcal{F}})$ is already zero, i.e., $Z_\alpha \notin P_\phi$, for some large enough α. This concludes the argument that P_ϕ is empty, hence $\phi = 0$ as desired. $\square$

2.4. **The exceptional operations.** Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$ satisfying Voevodsky's conditions. In this subsection we construction the exceptional operations $f_!$ and $f^!$ on $\mathbf{D}^*$. The idea can be summarized as follows. By a well-known result of Nagata, as extended to algebraic spaces in [CLO], any separated morphism of finite type admits a compactification. A deformation theory argument yields the same for *derived* algebraic spaces, cf. [GR, Pt. II, Chap. 5, 2.1.6]. Then following Deligne [SGA4, Exp. XVII, §5], there is a unique way to construct the functor $f_!$, at the level of triangulated categories for all separated morphisms of finite type, such that $f_! = f_*$ for f proper and $f_! = f_\sharp$ for f an open immersion. Using the machinery of [LZ1], this can be done ∞-categorically. Moreover, we can then use descsent as in [LZ2] to drop the separatedness hypothesis.

Throughout this subsection, we fix the following notation:

Notation 2.33. Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$ satisfying Voevodsky's conditions. Assume either that every morphism in $\mathcal{S}$ is quasi-projective, that every $S \in \mathcal{S}$ is noetherian, or that $\mathbf{D}^*$ is continuous and compactly generated.

Theorem 2.34. *For any finite type morphism $f : X \to Y$ in $\mathcal{S}$, there exists a pair of adjoint functors*

$$f_! : \mathbf{D}(X) \to \mathbf{D}(Y), \quad f^! : \mathbf{D}(Y) \to \mathbf{D}(X),$$

and a natural transformation $\alpha_f : f_! \to f_*$, satisfying the following conditions:

(i) *There are canonical isomorphisms $f_! \simeq f_\sharp$ and $f^! \simeq f^*$ if f is an open immersion.*

(ii) *The natural transformation $\alpha_f : f_! \to f_*$ is invertible if f is proper.*

(iii) *The functor $f_!$ satisfies base change. That is, for any commutative square in $\mathcal{S}$*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

which is cartesian on classical truncations, the canonical morphisms of functors $\mathbf{D}(X) \to \mathbf{D}(Y')$

$$\mathrm{Ex}^*_! : v^* f_! \to g_! u^*,$$
$$\mathrm{Ex}^!_* : u_* g^! \to f^! v_*$$

are invertible.

(iv) *The functor $f_!$ satisfies the projection formula. That is, $f_! : \mathbf{D}(X) \to \mathbf{D}(Y)$ is a morphism of $\mathbf{D}(Y)$-module ∞-categories, where $\mathbf{D}(X)$ is regarded as a $\mathbf{D}(Y)$-module via the symmetric monoidal functor $f^* : \mathbf{D}(Y) \to \mathbf{D}(X)$. In particular, the canonical morphisms*

$$\mathcal{F} \otimes f_!(\mathcal{G}) \to f_!(f^*(\mathcal{F}) \otimes \mathcal{G}),$$
$$\underline{\mathrm{Hom}}(f^*(\mathcal{F}), f^!(\mathcal{F}')) \to f^!(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}')),$$
$$f_*(\underline{\mathrm{Hom}}(\mathcal{F}, f^!(\mathcal{G}))) \to \underline{\mathrm{Hom}}(f_!(\mathcal{F}), \mathcal{G})$$

are invertible for all $\mathcal{F}, \mathcal{F}' \in \mathbf{D}(X)$ *and* $\mathcal{G} \in \mathbf{D}(Y)$.

Moreover, the assignment $f \mapsto f_!$ *(resp.* $f \mapsto f^!$*) extends to a functor* $\mathbf{D}_!$ *(resp.* $\mathbf{D}^!$*), from* $\mathcal{S}$ *to the* ∞*-category of presentable* ∞*-categories and left-adjoint functors (resp. right-adjoint functors).*

Proof. Suppose first that f is separated and of finite type. Then one can show that the ∞-category of compactifications of f is contractible, following the proof of [GR, Pt. II, Chap. 5, 2.1.6] and using the extension of Nagata to algebraic spaces in [CLO]. Then the machinery of multisimplicial nerves developed in [LZ1] yields the claim, exactly as in [LZ1, Thm. 9.4.8] or [LZ2, Eqn. (3.8)], in view of Theorem 2.24.

The extension to finite type morphisms follows then from [LZ2, Thm. 4.1.8], since every morphism is locally (in the Nisnevich topology) separated. ☐

Corollary 2.35. *Suppose given a commutative square in* $\mathcal{S}$

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

which is cartesian on classical truncations, where f *is finite type. Then there is a canonical natural transformation*

$$\mathrm{Ex}^{*!} : u^* f^! \to g^! v^*.$$

If u *and* v *are smooth and* f *is separated of finite type, then* $\mathrm{Ex}^{*!}$ *is invertible.*

Proof. Define $\mathrm{Ex}^{*!}$ as the composite

$$u^* f^! \xrightarrow{\text{unit}} g^! g_! u^* f^! \simeq g^! v^* f_! f^! \xrightarrow{\text{counit}} g^! v^*$$

where the isomorphism is the base change formula (Theorem 2.34(iii)). Suppose v is smooth and f is separated. Choosing a compactification of f we may assume that it is either an open immersion or proper. In the first case, this is clear from $f^! \simeq f^*$ (Theorem 2.34(i)). In the second it follows by transposition from smooth-proper base change (Theorem 2.24(ii)). ☐

Remark 2.36. Let $f : X \to Y$ be an étale morphism of finite presentation in $\mathcal{S}$. Let $\Delta : X \to X \times_Y X$ denote the diagonal. Since Δ is an open immersion, we have $\Delta^! \simeq \Delta^*$ (Theorem 2.34(i)). Consider the homotopy cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and take the composite

$$f^! = \Delta^* \mathrm{pr}_1^* f^! \xrightarrow{\mathrm{Ex}^{*!}} \Delta^* \mathrm{pr}_2^! f^* \simeq \Delta^! \mathrm{pr}_2^! f^* = f^*.$$

In particular, when f is *separated* étale, this gives a canonical isomorphism $f^! \simeq f^*$.

Corollary 2.37. *Suppose given a commutative square in $\mathcal{S}$*

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

which is cartesian on classical truncations, where f is finite type and u and v are smooth. Then the natural transformation $\mathrm{Ex}^{*!} : u^* f^! \to g^! v^*$ is invertible.

Proof. We may choose a finite type morphism $f_0 : U \to V$ where U and V are affine schemes equipped with Nisnevich coverings $p : U \twoheadrightarrow X$ and $q : V \twoheadrightarrow Y$ fitting in the commutative square forming the bottom face of the following commutative cube:

$$\begin{tikzcd}
& X' \arrow[rr, "g"] \arrow[dd, "u" near end] & & Y' \arrow[dd, "v"] \\
U' \arrow[ur, "p'"] \arrow[rr, "g_0" near end] \arrow[dd, "u'"'] & & V' \arrow[ur, "q'"'] \arrow[dd, "v'" near start] & \\
& X \arrow[rr, "f" near start] & & Y \\
U \arrow[ur, "p"] \arrow[rr, "f_0"] & & V \arrow[ur, "q"'] &
\end{tikzcd}$$

The rest of the cube is formed by taking the derived base along $v : Y' \to Y$. By Lemma 2.12 it will suffice to show that the morphism

$$\mathrm{Ex}^{*!} : p'^* u^* f^! \to p'^* g^! v^*$$

is invertible. Note that p and q are separated (since U and V are affine), so there are canonical isomorphisms $p^* \simeq p^!$ and $q^* \simeq q^!$ by Remark 2.36. Under these identifications the above morphism is identified with

$$\mathrm{Ex}^{*!} : u'^* f_0^! q^* \to g_0^! v'^* q^*,$$

which is invertible by Corollary 2.35 applied to the front face (since f_0 is separated). $\square$

Corollary 2.37 allows us to extend the isomorphism $f^! \simeq f^*$ to non-separated étale morphisms:

Corollary 2.38. *Let $f : X \to Y$ be an étale morphism of finite presentation in $\mathcal{S}$. Then the natural transformation $f^! \to f^*$ of Remark 2.36 is invertible.*

Corollary 2.39. *Suppose given a commutative square in $\mathcal{S}$*

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

which is cartesian on classical truncations, where f is finite type. Then there is a natural transformation

$$\mathrm{Ex}_{!*} : f_! u_* \to v_* g_!$$

which is invertible if v is proper.

Proof. Define $\mathrm{Ex}_{!*}$ as the composite

$$f_! u_* \xrightarrow{\text{unit}} v_* v^* f_! u_* \simeq v_* g_! u^* u^* \xrightarrow{\text{counit}} v_* g_!,$$

where the isomorphiism is the base change formula (Theorem 2.34(iii)). Assume v is proper. If f is separated of finite type, then we may choose a compactification to reduce to the case of f proper, which follows from $f_* \simeq f_!$ (Theorem 2.34(ii)), and the case of f an open immersion, which follows by transposition from smooth-proper base change (Theorem 2.24(ii)).

For the case of f general, it will now suffice to show that the claim is local on X. Let $p : U \twoheadrightarrow X$ be a Nisnevich covering such that $f_0 = f \circ p$ is separated of finite type. Let $p' : V \twoheadrightarrow X'$ be its base change, so that $g_0 = g \circ p'$ is separated of finite type. Let $u_0 : V \to U$ be the induced map so that we have the following diagram:

$$\begin{tikzcd} V \arrow[r, two heads, "p'"] \arrow[d, "u_0"] & X' \arrow[r, "g"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\ U \arrow[r, two heads, "p"] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

By Lemma 2.12 and Corollary 2.38, $p^* \simeq p^!$ is conservative. By adjunction, its left adjoint $p_! : \mathbf{D}(U) \to \mathbf{D}(X)$ generates $\mathbf{D}(X)$ under colimits. Hence it will suffice to show that $\mathrm{Ex}_{!*} : f_! u_* p_! \to v_* g_! p_!$ is invertible. Using the isomorphism $\mathrm{Ex}_{!*} : p_! u_{0,*} \simeq u_* p'_!$ (for the left-hand square above), we find that this identified with $\mathrm{Ex}_{!*} : f_{0,!} u_{0,*} \to v_* g_{0,!}$ (for the composite square), which is invertible again by the separated finite type case. ☐

Remark 2.40. Let $f : X \to Y$ be a morphism in $\mathcal{S}$. Locally on X, the natural transformation $\alpha_f : f_! \to f_*$ (ii) can be described as follows. Let $\Delta : X \to X \times_Y X$ be the diagonal and consider the homotopy cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

If f is *separated*, so that Δ is a closed immersion, then Corollary 2.39 gives rise to a canonical natural transformation

$$f_! = f_! \mathrm{pr}_{1,*} \Delta_* \xrightarrow{\mathrm{Ex}_{!*}} f_* \mathrm{pr}_{2,!} \Delta_* \simeq f_* \mathrm{pr}_{2,!} \Delta_! = f_!.$$

We can formulate an analogue of Lemma 2.11 using exceptional inverse image:

Lemma 2.41 (Constructible separation)**.** *For any $S \in \mathcal{S}$ and any constructible covering family $(j_\alpha : S_\alpha \to S)_\alpha$, the family of inverse image functors*

$$j_\alpha^! : \mathbf{D}(S) \to \mathbf{D}(S_\alpha)$$

is jointly conservative as α varies.

Proof. We reduce to the case of a closed/open pair (i, j) and use the exact triangle

$$i_! i^! \xrightarrow{\text{counit}} \mathrm{id} \xrightarrow{\text{unit}} j_* j^*$$

from Remark 2.9 (recall that $j^* \simeq j^!$). $\square$

Remark 2.42. Let $S \in \mathcal{S}$. For every perfect complex $\mathcal{E}$ on S, the Thom twist $\langle \mathcal{E} \rangle$ commutes with each of the six operations. Therefore, we will often abuse notation by writing e.g. $f^*\langle \mathcal{E} \rangle$ instead of $\langle f^* \mathcal{E} \rangle \circ f^*$ when $\mathcal{E}$ lives on the target. (Using Lemma 2.12 one can reduce to the case of finite locally free sheaves, and then the claim is a straightforward exercise using various base change and projection formulas.)

2.5. **Purity.**

Theorem 2.43. *Let $\mathbf{D}^*$ be as in Notation 2.33. Let $S \in \mathcal{S}$, let $X, Y \in \mathrm{Sm}_{/S}$ with structural morphisms $p : X \to S$ and $q : Y \to S$, and let $f : X \to Y$ be an unramified[6] morphism over S. Then there is a canonical isomorphism*

$$f^! q^* \simeq p^* \langle \mathcal{L}_{X/Y} \rangle,$$

where $\mathcal{L}_{X/Y}$ is the relative cotangent complex of f.

Proof. If f is a closed immersion, then follows from Theorem 2.25 by transposition. In general, there exists by the main result of [Ry] a canonical global factorization of f through a closed immersion i and an étale morphism of finite presentation g:

$$X \xrightarrow{i} X' \xrightarrow{g} Y.$$

Combining the closed immersion case and Corollary 2.38, we get a canonical isomorphism

$$f^! q^* = i^! g^! q^* \simeq i^! g^* q^* = i^! (p')^* \simeq p^* \langle -\mathcal{N}_{X/X'} \rangle \simeq p^* \langle \mathcal{L}_{X/Y} \rangle,$$

where $p' : X \to S$ is the structural morphism and the identification $-\mathcal{N}_{X/X'} \simeq \mathcal{L}_{X/Y}$ in $\mathrm{K}(X)$ is induced by the isomorphism of perfect complexes $\mathcal{N}_{X/X'}[1] = \mathcal{L}_{X/X'} \simeq \mathcal{L}_{X/Y}$ induced by the étale morphism g. $\square$

We can extend Corollary 2.38 to smooth morphisms:

Theorem 2.44 (Purity)**.** *Let $\mathbf{D}^*$ be as in Notation 2.33. For any smooth morphism $f : X \to Y$ in $\mathcal{S}$, there is a canonical isomorphism*

$$\mathrm{pur}_f : f^! \to f^* \langle \mathcal{L}_f \rangle$$

of functors $\mathbf{D}(Y) \to \mathbf{D}(X)$.

[6] Equivalently, the shifted cotangent complex $\mathcal{L}_{X/Y}[-1]$ is locally free of finite rank; see [KhRy, Prop. 5.2.4].

Proof. Applying Corollary 2.37 to the homotopy cartesian square

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

yields a canonical isomorphism $\mathrm{Ex}^{*!} : \mathrm{pr}_1^* f^! \simeq \mathrm{pr}_2^! f^*$. Since f is smooth, the diagonal $\Delta : X \to X \times_Y X$ is unramified with cotangent complex $\mathcal{L}_\Delta \simeq \mathcal{L}_f[1]$. Applying $\Delta^!$ and using the relative purity isomorphism $\Delta^! \mathrm{pr}_1^* \simeq \langle -\mathcal{L}_f \rangle$ (Theorem 2.43), we get the canonical isomorphism

$$f^! \langle -\mathcal{L}_f \rangle \simeq \Delta^! \mathrm{pr}_1^* f^! \xrightarrow{\mathrm{Ex}^{*!}} \Delta^! \mathrm{pr}_2^! f^* \simeq f^*.$$

The purity isomorphism $\mathrm{pur}_f : f^! \simeq f^* \langle \mathcal{L}_f \rangle$ is obtained by Thom twisting by $\mathcal{L}_f$. □

Remark 2.45. The natural transformation

$$\mathrm{tr}_f : f_! f^* \langle \mathcal{L}_f \rangle \to \mathrm{id},$$

obtained by transposition from the purity isomorphism $\mathrm{pur}_f : f^! \simeq f^* \langle \mathcal{L}_f \rangle$, is called the *trace* of f.

2.6. **Étale and proper excision.**

Theorem 2.46 (Étale excision)**.** *Let $\mathbf{D}^*$ be as in Notation 2.33. Let $f : X' \to X$ be an étale morphism of finite presentation in $\mathcal{S}$ which induces an isomorphism away from a quasi-compact open immersion $j : U \to X$ in $\mathcal{S}$. Then the commutative squares*

$$\begin{tikzcd} \mathrm{id} \arrow[r] \arrow[d] & j_* j^* \arrow[d] \\ f_* f^* \arrow[r] & g_* g^*, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id} \arrow[r] \arrow[d] & j_! j^! \arrow[d] \\ f_! f^! \arrow[r] & g_! g^! \end{tikzcd} \tag{2.47}$$

are homotopy cartesian in $\mathbf{D}(X)$, where $g : f^{-1}(U) \to X$.

Proof. Let $i : Z \to X$ denote a closed immersion complementary to j and $f_U : U' = f^{-1}(U) \to U$ the base change, so that we have cartesian squares

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "f_U"] & X' \arrow[d, "f"] & Z \arrow[l, "i'"'] \arrow[d, equal] \\ U \arrow[r, "j"] & X & Z. \arrow[l, "i"'] \end{tikzcd}$$

Consider the left-hand square in (2.47). By Lemma 2.11 it will suffice to show it is homotopy cartesian after applying either i^* or j^*. Both claims follow easily from the smooth base change formula.

For the right-hand square in (2.47), note that there is a canonical isomorphism $f_! f^! \simeq f_\sharp f^*$ by purity (Theorem 2.44), and similarly for each of the terms. Thus the square is obtained by right transposition from the left-hand one. □

Theorem 2.48 (Proper excision)**.** *Let $\mathbf{D}^*$ be as in Notation 2.33. Let $f : X' \to X$ be a proper morphism in $\mathcal{S}$ which is an isomorphism away from a closed immersion $i : Z \to X$ in $\mathcal{S}$. Then the commutative squares*

$$\begin{tikzcd} \mathrm{id} \arrow[r] \arrow[d] & i_*i^* \arrow[d] \\ f_*f^* \arrow[r] & g_*g^*, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id} \arrow[r] \arrow[d] & i_!i^! \arrow[d] \\ f_!f^! \arrow[r] & g_!g^! \end{tikzcd} \tag{2.49}$$

are homotopy cartesian in $\mathbf{D}(X)$, where $g : f^{-1}(Z) \to X$.

Proof. Let $j : U \to X$ denote the open immersion complementary to i and $f_Z : Z' = f^{-1}(Z) \to Z$ the derived change, so that we have cartesian squares

$$\begin{tikzcd} Z' \arrow[r, "j'"] \arrow[d, "f_Z"] & X' \arrow[d, "f"] & U \arrow[l, "j'"'] \arrow[d, equal] \\ Z' \arrow[r, "i"] & X & U. \arrow[l, "j"'] \end{tikzcd}$$

Consider the left-hand square in (2.49). By Lemma 2.11 it will suffice to show it is homotopy cartesian after applying either i^* or j^*. The i^* case follows easily from the localization property and proper base change (Theorem 2.24(i)). The j^* case follows immediately from the smooth base change formula.

For the right-hand square in (2.49) it will again suffice to apply either i^* or j^*. The i^* case again follows from localization and proper base change. The j^* case follows by smooth base change. □

2.7. **Descent.**

Definition 2.50. The *cdh topology* on $\mathcal{S}$ is the Grothendieck topology associated to the pretopology generated by the following covering families: (a) the empty family, covering the empty space $\varnothing$; (b) for every $S \in \mathcal{S}$ and every étale morphism $f : S' \to S$ inducing an isomorphism away from a quasi-compact open immersion $j : U \hookrightarrow S$, the family $\{j, f\}$ covering S; (c) for every $S \in \mathcal{S}$ and every proper morphism $f : S' \to S$ inducing an isomorphism away from a closed immersion $i : Z \hookrightarrow S$, the family $\{i, f\}$ covering S.

Corollary 2.51. *Let $\mathbf{D}^*$ be as in Notation 2.33. Let $S \in \mathcal{S}$ and $\mathcal{F} \in \mathbf{D}(S)$. Then we have:*

(i) *The assignment $(f : X \to S) \mapsto f_*f^*(\mathcal{F})$, regarded as a $\mathbf{D}(S)$-valued presheaf on the ∞-category of derived algebraic spaces over S, satisfies cdh descent.*
(ii) *The assignment $(f : X \to S) \mapsto f_!f^!(\mathcal{F})$, regarded as a $\mathbf{D}(S)$-valued presheaf on the ∞-category of derived algebraic spaces finite type over S, satisfies cdh co-descent.*

Proof. By a theorem of Voevodsky [Kh2, Thm. 2.2.7] (which immediately generalizes to presheaves with values in an arbitrary ∞-category), the claim is equivalent to Theorems 2.46 and 2.48. $\square$

Theorem 2.52. *Let $\mathbf{D}^*$ be as in Notation 2.33. Then the presheaves of ∞-categories $\mathbf{D}^*$ and $\mathbf{D}^!$ satisfy cdh descent (on the $\mathcal{S}$, resp. on the full subcategory of $\mathcal{S}$ spanned by finite type spaces). Moreover, $\mathbf{D}_!$ satisfies cdh co-descent, when regarded as a co-presheaf with values in the ∞-category of presentable ∞-categories and left-adjoint functors.*

Proof. It will again suffice to show that $\mathbf{D}^*$ and $\mathbf{D}^!$ satisfy étale and proper excision (as presheaves of ∞-categories).

(i) *Case of* $\mathbf{D}^*$. We prove étale excision; the proper version is proven by the same type of argument. Suppose given an étale morphism of finite presentation $f : X' \to X$ in $\mathcal{S}$ which induces an isomorphism away from a quasi-compact open immersion $j : U \to X$ in $\mathcal{S}$. Let $U' = U \times_X X'$ and $g : U' \to X$. The claim is that j^*, f^*, and g^* induce an equivalence
$$\mathbf{D}(X) \to \mathbf{D}(U) \underset{\mathbf{D}(U')}{\times} \mathbf{D}(X').$$
Note that this functor admits a right adjoint which sends an object of $\mathbf{D}(U) \times_{\mathbf{D}(U')} \mathbf{D}(X')$, given by $\mathcal{F}_U \in \mathbf{D}(U)$, $\mathcal{F}_{X'} \in \mathbf{D}(X')$, $\mathcal{F}_{U'} \in \mathbf{D}(U')$, and isomorphisms $\mathcal{F}_U|U' \simeq \mathcal{F}_{U'} \simeq \mathcal{F}_{X'}|U'$ in $\mathbf{D}(U')$, to the object
$$j_*(\mathcal{F}_U) \underset{g_*(\mathcal{F}_{U'})}{\times} f_*(\mathcal{F}_{X'}) \in \mathbf{D}(X).$$
Note that the unit of this adjunction is invertible by Theorem 2.46. To show that the counit is invertible it will suffice to show that the canonical morphisms
$$j^*(j_*(\mathcal{F}_U) \underset{g_*(\mathcal{F}_{U'})}{\times} f_*(\mathcal{F}_{X'})) \to \mathcal{F}_U$$
$$f^*(j_*(\mathcal{F}_U) \underset{g_*(\mathcal{F}_{U'})}{\times} f_*(\mathcal{F}_{X'})) \to \mathcal{F}_{X'}$$
are invertible. These are both easy exercises using Lemma 2.11 and the smooth base change formula.

(ii) *Case of* $\mathbf{D}^!$. We prove proper excision; the étale version is proven by the same type of argument. Suppose given a proper morphism $f : X' \to X$ in $\mathcal{S}$ which is an isomorphism away from a closed immersion $i : Z \to X$ in $\mathcal{S}$. Let $Z' = Z \times_X X'$ and $g : Z' \to X$. The claim is that $i^!$, $f^!$, and $g^!$ induce an equivalence
$$\mathbf{D}(X) \to \mathbf{D}(Z) \underset{\mathbf{D}(Z')}{\times} \mathbf{D}(X').$$
Note that this functor admits a left adjoint which sends an object of $\mathbf{D}(Z) \times_{\mathbf{D}(Z')} \mathbf{D}(X')$, given by $\mathcal{F}_Z \in \mathbf{D}(Z)$, $\mathcal{F}_{X'} \in \mathbf{D}(X')$, $\mathcal{F}_{Z'} \in \mathbf{D}(Z')$, and isomorphisms $\mathcal{F}_Z|Z' \simeq \mathcal{F}_{Z'} \simeq \mathcal{F}_{X'}|Z'$ in $\mathbf{D}(Z')$, to the object
$$i_!(\mathcal{F}_Z) \underset{g_!(\mathcal{F}_{Z'})}{\times} f_!(\mathcal{F}_{X'}) \in \mathbf{D}(X).$$

Note that the counit of this adjunction is invertible by Theorem 2.48. To show that the unit is invertible it will suffice to show that the canonical morphisms

$$\mathcal{F}_Z \to i^!(i_!(\mathcal{F}_Z) \underset{g_!(\mathcal{F}_{Z'})}{\times} f_!(\mathcal{F}_{X'}))$$
$$\mathcal{F}_{X'} \to f^!(i_!(\mathcal{F}_Z) \underset{g_!(\mathcal{F}_{Z'})}{\times} f_!(\mathcal{F}_{X'}))$$

are invertible. For the first, this follows from the localization property and base change formula (Theorem 2.34(iii)). For the second, one uses Lemma 2.41 along with localization and base change again.

(iii) *Case of* $\mathbf{D}_!$. By [Lu1, Thm. 5.5.3.18], this assertion is equivalent to cdh descent for $\mathbf{D}^!$, regarded as a presheaf with values in the ∞-category of presentable ∞-categories and *right*-adjoint functors. By [Lu1, Cor. 5.5.3.4], the latter assertion is equivalent to cdh descent of $\mathbf{D}^!$ as a presheaf of ∞-categories, which was just proven above.

□

2.8. **Constructible objects.** Throughout the subsection, fix $\mathbf{D}^*$ as in Notation 2.33.

Lemma 2.53. *Given any $S \in \mathcal{S}$ and any $\mathcal{F} \in \mathbf{D}(S)$, the following conditions are equivalent:*

(i) *The object $\mathcal{F}$ lies in the thick subcategory generated by objects of the form $p_!p^!(\mathbf{1}_S)\langle -n\rangle \simeq p_\sharp p^*(\mathbf{1}_S)\langle -n\rangle$, where $p : X \to S$ is a smooth morphism of finite presentation in $\mathcal{S}$ and $n \geqslant 0$ is an integer.*

(ii) *The object $\mathcal{F}$ lies in the thick subcategory generated by objects of the form $p_!p^!(\mathbf{1}_S)\langle -n\rangle \simeq p_\sharp p^*(\mathbf{1}_S)\langle -n\rangle$, where $p : X \to S$ is a smooth morphism of finite presentation in $\mathcal{S}$ with X* affine, *and $n \geqslant 0$ is an integer.*

Proof. Let $p : X \to S$ be a smooth morphism of finite presentation with $X \in \mathcal{S}$ not necessarily affine. Let us show that $p_!p^!(\mathbf{1}_S) \in \mathbf{D}(S)$ belongs to the thick subcategory described in (ii). By [Lu2, Thm. 3.4.2.1] there exists a stratification of U by open subspaces

$$\varnothing = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$$

and, for each $0 < i \leqslant n$, an affine derived scheme V_i and an étale morphism $V_i \to U_i$ inducing an isomorphism away from U_{i-1}. Now by étale excision (Theorem 2.46) the claim follows by a simple induction. □

Definition 2.54. Let $S \in \mathcal{S}$. An object $\mathcal{F} \in \mathbf{D}(S)$ is called *constructible* if it satisfies the equivalent conditions of Lemma 2.53. We write $\mathbf{D}_{\mathrm{cons}}(S) \subseteq \mathbf{D}(S)$ for the full subcategory spanned by constructible objects.

Definition 2.55 (Constructible generation)**.** We say that $\mathbf{D}^*$ is *constructibly generated* if it is compactly generated and every constructible object is compact. In this case it follows that compactness is equivalent to constructibility (see e.g. [Ne, Lem. 2.2]). See [DFJK, Def. A.7].

Example 2.56. The $(*, \sharp, \otimes)$-formalism $\mathbf{SH}^*$ is constructibly generated by Remark 1.19.

Proposition 2.57. *Let $S \in \mathcal{S}$. For an object $\mathcal{F} \in \mathbf{D}(S)$, the property of constructibility is stable under the following operations:*

(i) *Tensor product with any constructible object $\mathcal{G} \in \mathbf{D}_{\mathrm{cons}}(S)$.*

(ii) *Inverse image along any morphism $f : S' \to S$ in $\mathcal{S}$.*

(iii) *$\sharp$-direct image along any smooth morphism of finite presentation $g : S \to T$ in $\mathcal{S}$.*

Proof. Follows easily from the definitions, see e.g. [CD, Props. 4.2.3 and 4.2.4]. ☐

Corollary 2.58. *Let $S \in \mathcal{S}$. For an object $\mathcal{F} \in \mathbf{D}(S)$, the property of constructibility is stable under the operation i_*i^* for any closed immersion $i : Z \to S$ in $\mathcal{S}$.*

Proof. If j is the complementary open immersion, this follows from the localization triangle (Remark 2.9) since $j_\sharp j^*$ preserves constructibility by Proposition 2.57. ☐

From Proposition 2.57 it follows that $\mathbf{D}^*$ induces a presheaf of symmetric monoidal *small* ∞-categories $\mathbf{D}^*_{\mathrm{cons}}$ on $\mathcal{S}$.

Proposition 2.59. *The presheaf $\mathbf{D}^*_{\mathrm{cons}}$ satisfies Nisnevich descent.*

Proof. In view of Nisnevich descent for $\mathbf{D}^*$ (Corollary 2.52) it will suffice to show that constructibility can be detected by Nisnevich covering families. It is enough to restrict our attention to covering families $\{j, f\}$ arising from Nisnevich squares, i.e., $f : S' \to S$ is an étale morphism in $\mathcal{S}$ inducing an isomorphism outside a quasi-compact open immersion $j : U \hookrightarrow S$. In this situation we have by Theorem 2.46 (and purity) a cartesian square

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & j_\sharp j^*(\mathcal{F}) \arrow[d] \\ f_\sharp f^*(\mathcal{F}) \arrow[r] & g_\sharp g^*(\mathcal{F}). \end{tikzcd}$$

Thus for any $\mathcal{F} \in \mathbf{D}(S)$ such that $f^*(\mathcal{F})$ and $j^*(\mathcal{F})$ are constructible, it follows by Proposition 2.57 that $\mathcal{F}$ is a finite homotopy limit of constructible objects, hence is constructible. ☐

Proposition 2.60. *Let $S \in \mathcal{S}$. For an object $\mathcal{F} \in \mathbf{D}(S)$, the property of constructibility is stable under the operation of Thom twist by any perfect complex $\mathcal{E}$ on S.*

Proof. By Proposition 2.59 we may localize on S so that $\langle\mathcal{E}\rangle \simeq \langle n\rangle$ for some integer $n \in \mathbf{Z}$ (the virtual rank of $\mathcal{E}$). In that case, the claim follows by definition of constructibility. □

Theorem 2.61. *Let $S \in \mathcal{S}$. For an object $\mathcal{F} \in \mathbf{D}(S)$, the property of constructibility is stable under the operation of exceptional direct image $g_!$ along any morphism of finite type $g : S \to T$.*

Proof. If g is smooth, then by purity $g_! \simeq g_\sharp\langle -\mathcal{L}_g\rangle$ (Theorem 2.44) so this follows from Propositions 2.57 and 2.60.

For the general case, note that the claim is local on T by Proposition 2.59 (and the base change formula) so that we may assume T is affine. By Lemma 2.53 it will suffice to show that for any smooth morphism $p : X \to S$ with X affine, the object

$$g_!p_!p^!(\mathbf{1}_S) \simeq (g \circ p)_!(\mathbf{1}_S)\langle\mathcal{L}_p\rangle \in \mathbf{D}(T)$$

is constructible (since $g_!$ is exact and commutes with Thom twists by Remark 2.42). Since $\langle\mathcal{L}_p\rangle$ preserves constructibility (Proposition 2.60), we may replace g by $g \circ p$ and reduce to showing that $g_!(\mathbf{1}_S) \in \mathbf{D}(T)$ is constructible for any $g : S \to T$ in $\mathcal{S}$ of finite type and S and T both *affine*. By derived invariance (Lemma 2.13) we may moreover assume S and T are classical. Then since g is of finite type, it factors through a closed immersion $i : X \to \mathbf{A}^n_Y$ and the projection $\pi : \mathbf{A}^n_Y \to Y$. We know that $\pi_!$ preserves constructibility by the smooth case and we know that $i_!(\mathbf{1}_X) \simeq i_*i^*(\mathbf{1}_{\mathbf{A}^n_Y})$ is constructible by Corollary 2.58. □

Corollary 2.62. *If $\mathbf{D}^*$ is constructibly generated, then for any morphism of finite type $f : S \to T$ in $\mathcal{S}$, the functor $f_!$ is compact, i.e., its right adjoint $f^!$ commutes with colimits.*

Proof. Since $\mathbf{D}^*$ is compactly generated this is equivalent to the assertion that $f_!$ preserves compact objects. Since $\mathbf{D}^*$ is constructibly generated, these are the same as constructible objects. Thus the claim is Theorem 2.61. □

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