GENERALIZED COHOMOLOGY THEORIES FOR ALGEBRAIC STACKS

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Abstract. We extend the stable motivic homotopy category of Voevodsky to the class of scalloped algebraic stacks, and show that it admits the formalism of Grothendieck's six operations. Objects in this category represent generalized cohomology theories for stacks like algebraic K-theory, as well as new examples like genuine motivic cohomology and algebraic cobordism. These cohomology theories admit Gysin maps and satisfy homotopy invariance, localization, and Mayer–Vietoris. We also prove a fixed point localization formula for torus actions. Finally, the construction is contrasted with a "limit-extended" stable motivic homotopy category: we show for example that limit-extended motivic cohomology of quotient stacks is computed by the equivariant higher Chow groups of Edidin–Graham, and we also get a good new theory of Borel-equivariant algebraic cobordism.

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1. Introduction

1.1. **Motivic homotopy theory of stacks.** Motivic homotopy theory provides a framework for the study of generalized or extraordinary cohomology theories in algebraic geometry, such as motivic cohomology, algebraic K-theory, and algebraic cobordism. Objects of the motivic stable homotopy category $\mathbf{SH}(X)$, over a scheme or algebraic space X, are "generalized sheaves" that can be taken as coefficients for cohomology.

In this paper we are interested in generalized cohomology theories on algebraic stacks. To that end, we introduce an extension of the motivic stable homotopy category to a large class of algebraic stacks, called *scalloped* stacks (see Subsect. 1.7 below), which includes for instance tame Deligne–Mumford or Artin stacks with separated diagonal as well as quotients of qcqs algebraic spaces by nice linear algebraic groups. Our first main result is as follows (see Theorem 7.1 and Example 5.12):

Theorem A. *The assignment $\mathcal{X} \mapsto \mathbf{SH}(\mathcal{X})$, together with the formalism of six operations, extends from qcqs*[1] *algebraic spaces to scalloped algebraic stacks. More precisely, we have the following operations:*

(i) *For every scalloped stack $\mathcal{X}$, a pair of adjoint bifunctors $(\otimes, \underline{\mathrm{Hom}})$.*
(ii) *For every morphism of scalloped stacks $f : \mathcal{X} \to \mathcal{Y}$, an adjoint pair*
$$f^* : \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X}), \quad f_* : \mathbf{SH}(\mathcal{X}) \to \mathbf{SH}(\mathcal{Y}).$$
(iii) *For every representable morphism of finite type $f : \mathcal{X} \to \mathcal{Y}$ between scalloped stacks $\mathcal{X}$ and $\mathcal{Y}$, an adjoint pair*
$$f_! : \mathbf{SH}(\mathcal{X}) \to \mathbf{SH}(\mathcal{Y}), \quad f^! : \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X}).$$

Moreover, these satisfy various identities including the base change and projection formulas, homotopy invariance, purity isomorphism, and localization triangle.

In the case of noetherian schemes, the six functor formalism on $\mathbf{SH}$ was constructed in the work of Voevodsky, Ayoub, and Cisinski–Déglise (see [Vo2, Ay, CD]). For a self-contained account in the generality of qcqs

[1] quasi-compact and quasi-separated

algebraic spaces, see [Kh5]. Our proof of Theorem A is logically independent of these earlier works.

1.2. **Genuine cohomology theories.** Given a motivic spectrum $\mathcal{F} \in \mathbf{SH}(\mathcal{X})$ over a scalloped stack $\mathcal{X}$, we define the cohomology spectra of $\mathcal{X}$ with coefficients in $\mathcal{F}$ as the mapping spectra

$$\mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}) = \mathrm{Maps}_{\mathbf{SH}(\mathcal{X})}\left(\mathbf{1}_{\mathcal{X}}, \mathcal{F}\langle\alpha\rangle\right)$$

where $\alpha \in \mathrm{K}(\mathcal{X})$ is a K-theory class and $\mathcal{F}\langle\alpha\rangle$ denotes the Thom twist[2] by α. Theorem A yields (see Subsect. 9.2):

Corollary B. *Cohomology with coefficients in $\mathcal{F} \in \mathbf{SH}(\mathcal{X})$ has the following properties:*

(i) Functoriality. *For every representable morphism $f : \mathcal{X}' \to \mathcal{X}$, there are inverse image maps*

$$f^* : \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}) \to \mathbf{R}\Gamma^{\alpha}(\mathcal{X}', \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{Y})$.[3] *If f is smooth and proper, then there is also a Gysin map*

$$f_! : \mathbf{R}\Gamma^{\alpha+\Omega_{\mathcal{X}/\mathcal{Y}}}(\mathcal{X}, \mathcal{F}) \to \mathbf{R}\Gamma^{\alpha}(\mathcal{Y}, \mathcal{F})$$

where $\Omega_{\mathcal{X}/\mathcal{Y}}$ denotes the relative cotangent sheaf. These are functorial and satisfy base change and projection formulas.

(ii) Homotopy invariance. *For every scalloped stack $\mathcal{X}$ and every vector bundle $p : \mathcal{E} \to \mathcal{X}$, the inverse image map*

$$p^* : \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}) \to \mathbf{R}\Gamma^{\alpha}(\mathcal{E}, \mathcal{F})$$

is invertible for every $\alpha \in \mathrm{K}(\mathcal{X})$.

(iii) Localization. *For every scalloped stack $\mathcal{X}$ and every closed immersion $i : \mathcal{Z} \to \mathcal{X}$ with quasi-compact open complement $j : \mathcal{U} \to \mathcal{X}$, there is an exact triangle*

$$\mathbf{R}\Gamma^{\alpha}_{\mathcal{Z}}(\mathcal{X}, \mathcal{F}) \to \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}) \xrightarrow{j^*} \mathbf{R}\Gamma^{\alpha}(\mathcal{U}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{X})$, where

$$\mathbf{R}\Gamma^{\alpha}_{\mathcal{Z}}(\mathcal{X}, \mathcal{F}) = \mathrm{Maps}_{\mathbf{SH}(\mathcal{X})}\left(i_*(\mathbf{1}_{\mathcal{Z}}), \mathcal{F}\langle\alpha\rangle\right)$$

is cohomology with support in $\mathcal{Z}$.

(iv) Mayer–Vietoris. *Let $f : \mathcal{X}' \to \mathcal{X}$ be a representable étale morphism (resp. a proper representable morphism) of scalloped stacks which induces an isomorphism away from a quasi-compact open substack*

[2] For oriented examples (such as motivic cohomology, algebraic K-theory, and algebraic cobordism), a choice of orientation determines isomorphisms $\mathcal{F}\langle\alpha\rangle \simeq \mathcal{F}\langle r\rangle \simeq \mathcal{F}(r)[2r]$, where r is the virtual rank of α (when viewed as a K-theory class, $r = [\mathcal{O}_{\mathcal{X}}] + \cdots + [\mathcal{O}_{\mathcal{X}}] = [\mathcal{O}^r_{\mathcal{X}}]$). We also adopt the convention that for $\alpha = 0$, we write simply $\mathbf{R}\Gamma(\mathcal{X}, \mathcal{F})$.

[3] By abuse of notation, in the target α means $f^*(\alpha) \in \mathrm{K}(\mathcal{X}')$ and $\mathcal{F}$ means $f^*(\mathcal{F}) \in \mathbf{SH}(\mathcal{X}')$.

$\mathcal{Y} \subseteq \mathcal{X}$ (resp. a closed substack $\mathcal{Y} \subseteq \mathcal{X}$). Then there is a homotopy cartesian square

$$\begin{array}{ccc} \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}) & \longrightarrow & \mathbf{R}\Gamma^{\alpha}(\mathcal{Y}, \mathcal{F}) \\ \downarrow{\scriptstyle f^*} & & \downarrow \\ \mathbf{R}\Gamma^{\alpha}(\mathcal{X}', \mathcal{F}) & \longrightarrow & \mathbf{R}\Gamma^{\alpha}(f^{-1}(\mathcal{Y}), \mathcal{F}) \end{array}$$

for every $\alpha \in \mathrm{K}(\mathcal{X})$.

We are primarily interested in the cohomology theories arising from three main examples of motivic spectra over $\mathcal{X}$: the algebraic K-theory spectrum $\mathrm{KGL}_{\mathcal{X}}$, the (integral) motivic cohomology spectrum $\mathbf{Z}_{\mathcal{X}}$, and the algebraic cobordism spectrum $\mathrm{MGL}_{\mathcal{X}}$. Cohomology with coefficients in $\mathrm{KGL}_{\mathcal{X}}$ recovers (the $\mathbf{A}^1$-invariant version of) the well-known algebraic K-theory of stacks:

Example 1.1. For every scalloped stack $\mathcal{X}$, there is a canonical isomorphism

$$\mathbf{R}\Gamma(\mathcal{X}, \mathrm{KGL}) \simeq \mathrm{KH}(\mathcal{X}),$$

where the right-hand side is the homotopy invariant K-theory spectrum $\mathrm{KH}(\mathcal{X})$ as introduced in [KrRa, HK]. In particular, if $\mathcal{X}$ is nonsingular, then we have

$$\mathbf{R}\Gamma(\mathcal{X}, \mathrm{KGL}) \simeq \mathrm{K}(\mathcal{X}) \simeq \mathrm{G}(\mathcal{X})$$

where $\mathrm{K}(\mathcal{X})$ is the Thomason–Trobaugh K-theory spectrum of perfect complexes on $\mathcal{X}$ and $\mathrm{G}(\mathcal{X})$ is the Quillen K-theory spectrum of coherent sheaves on $\mathcal{X}$. See Subsect. 10.1.

In the case of $\mathbf{Z}_{\mathcal{X}}$ and $\mathrm{MGL}_{\mathcal{X}}$, we get new extensions of motivic cohomology and cobordism to stacks, which are "genuine" refinements of previously known cohomology theories even in the case of quotient stacks. The word "genuine" here is used in the sense of genuine equivariant homotopy theory ([Se2, HHR, NS]), as opposed to Borel-equivariant homotopy theory; compare Subsect. 1.4 below and see also the discussion in Subsect. 1.10.

There are also "quadratic" (SL-oriented) refinements of these three theories: hermitian K-theory $\mathrm{KQ}_{\mathcal{X}}$ (Remark 10.8), Milnor–Witt motivic cohomology $\widetilde{\mathbf{Z}}_{\mathcal{X}}$ (Remark 10.19), and special linear algebraic cobordism $\mathrm{MSL}_{\mathcal{X}}$ (Remark 10.12).

1.3. **Fixed point localization.** Let k be a field and let $T = \mathbf{G}_{m,k}$ be the multiplicative group over $\mathrm{Spec}(k)$. Given a motivic spectrum $\mathcal{F} \in \mathbf{SH}(BT)$, we can define T-equivariant Borel–Moore homology with coefficients in $\mathcal{F}$ by the formula:

$$\mathbf{R}\Gamma^T_{\alpha}(X, \mathcal{F}) \coloneqq \mathrm{Maps}_{\mathbf{SH}([X/T])}\big(\mathbf{1}\langle\alpha\rangle, f^!(\mathcal{F})\big),$$

where X is a qcqs algebraic space with T-action and $f : [X/T] \to BT$ is the projection. We prove an analogue of Thomason's concentration theorem (see [Th2, Thm. 2.1]) in this setting. It relates the equivariant Borel–Moore homology of an algebraic space with that of its fixed locus:

Theorem C (Concentration)**.** *Let X be a T-equivariant algebraic space of finite type over a field k. Denote by $i : X^T \to X$ the inclusion of the locus of fixed points. Then for every motivic spectrum $\mathcal{F} \in \mathbf{SH}(BT)$, the morphism of $\mathbf{R}\Gamma^*(BT,\mathcal{F})$-modules*

$$i_* : \mathbf{R}\Gamma^T_*(X^T,\mathcal{F}) \to \mathbf{R}\Gamma^T_*(X,\mathcal{F})$$

becomes invertible after inverting the Euler classes of powers of the tautological line bundle $[\mathbf{A}^1/\mathbf{G}_m]$.

See Corollary 11.3.

1.4. **Limit-extended cohomology theories.** A different way to extend **SH** and generalized cohomology theories to stacks is via the following formal procedure.

Given any qcqs algebraic stack $\mathcal{X}$ (not necessarily scalloped), denote by $\mathrm{Lis}_{\mathcal{X}}$ the ∞-category of pairs $(U, u : U \to \mathcal{X})$, where U is a qcqs algebraic space and $u : U \to \mathcal{X}$ is a smooth morphism. The *limit extension* $\mathbf{SH}_{\triangleleft}(\mathcal{X})$ is the homotopy limit of ∞-categories

$$\mathbf{SH}_{\triangleleft}(\mathcal{X}) = \varprojlim_{(U,u)\in\mathrm{Lis}_{\mathcal{X}}} \mathbf{SH}(U),$$

over the $*$-inverse image functors. The motivic cohomology, algebraic K-theory, and algebraic cobordism spectra immediately give rise to limit-extended motivic spectra

$$\mathbf{Z}^{\triangleleft}_{\mathcal{X}},\ \mathrm{KGL}^{\triangleleft}_{\mathcal{X}},\ \mathrm{MGL}^{\triangleleft}_{\mathcal{X}} \in \mathbf{SH}_{\triangleleft}(\mathcal{X})$$

over $\mathcal{X}$, simply by virtue of stability under $*$-inverse image. If $\mathcal{X}$ is scalloped, then these are moreover the images of their genuine counterparts by a canonical functor (which commutes with colimits and $*$-inverse image)

$$\mathbf{SH}(\mathcal{X}) \to \mathbf{SH}_{\triangleleft}(\mathcal{X}).$$

However, this functor is far from being an equivalence, so that the corresponding cohomology theories are very different (as the example below of K-theory shows).

In fact, we show that for quotient stacks, cohomology with coefficients in any of the limit-extended cohomology theories above can be computed via Totaro's algebraic approximation of the Borel construction (see Theorem 12.9). For example, in the case of motivic cohomology we have:

Theorem D. *Let G be a linear algebraic group over a field k of characteristic zero. Let X be a smooth G-quasi-projective k-scheme. Then for every $n, s \in \mathbf{Z}$ there are canonical isomorphisms*

$$\pi_s \mathbf{R}\Gamma^n_{\triangleleft}([X/G],\mathbf{Z}) \simeq \mathrm{A}^n_G(X,s)$$

where on the right-hand side are the Edidin–Graham equivariant higher Chow groups [EG].

See Example 12.15. The result also holds for fields of positive characteristic, up to inverting the characteristic.

Example 1.2. In the case of $\mathrm{KGL}^{\triangleleft}_{[X/G]}$, the canonical map

$$\pi_0 \,\mathrm{K}([X/G]) \to \pi_0 \,\mathbf{R}\Gamma_{\triangleleft}([X/G], \mathrm{KGL}),$$

induced by the functor $\mathbf{SH}([X/G]) \to \mathbf{SH}_{\triangleleft}([X/G])$ (for G nice), is not bijective if G is nontrivial. In fact, it exhibits the target as a completion of the source. See Example 12.18.

Remark 1.3. For general coefficients $\mathcal{F}$, Theorem 12.9 gives isomorphisms

$$\mathbf{R}\Gamma^n_{\triangleleft}([X/G], \mathcal{F}) \simeq \varprojlim_i \mathbf{R}\Gamma^n(\mathcal{X} \underset{BG}{\times} U_i)$$

where $(U_i)_i$ is a sequence of algebraic approximations to the Borel construction. On π_0 we have surjections

$$\mathrm{H}^n_{\triangleleft}([X/G], \mathcal{F}) \twoheadrightarrow \varprojlim_i \mathrm{H}^n(\mathcal{X} \underset{BG}{\times} U_i)$$

which however we do not know to be injective for general $\mathcal{F}$. For example, the fact that this holds in the case of motivic cohomology relies on strong vanishing statements which do not hold e.g. in algebraic cobordism.

In particular, although the right-hand side has been considered in the case of algebraic cobordism in [HML, Kri] (and in an abstract setting in [KP, Cor. 3.8]), it is not known to satisfy the right-exact localization sequence (see Footnote 22). In contrast, limit-extended cobordism does have right-exact localization sequences which in fact even extend to long-exact sequences using the higher groups (see Proposition 12.17). In general, we regard the limit extension as a good way to define "Borel-type" extensions of arbitrary generalized cohomology theories.

Remark 1.4. Theorem 12.9 is deduced from a stronger comparison of motivic stable homotopy types. For example, the limit-extended motivic stable homotopy type of a classifying stack BG is compared with the Morel–Voevodsky construction. See Theorem 12.14.

1.5. **Derived stacks and virtual functoriality.** Throughout the paper, we work in the setting of derived algebraic geometry. In particular, Theorem A and Corollary B remain valid for scalloped *derived* stacks. Working in this generality allows us to construct an enhanced functoriality for our cohomology theories (see Theorem 8.4):

Theorem E. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth, representably smoothable*[4] *morphism of scalloped derived stacks with affine diagonal. Then there exists a natural transformation*

$$\mathrm{gys}_{\mathcal{X}/\mathcal{Y}} : f^*\langle \mathcal{L}_{\mathcal{X}/\mathcal{Y}} \rangle \to f^!$$

of functors $\mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X})$, where $\mathcal{L}_{\mathcal{X}/\mathcal{Y}}$ is the relative cotangent complex. If f is smooth, then $\mathrm{gys}_{\mathcal{X}/\mathcal{Y}}$ is the purity isomorphism (Theorem 7.11).

[4]Here representably smoothable means that f admits a global factorization through an unramified representable morphism followed by a smooth representable morphism (see Definition 8.3).

Moreover, for every motivic spectrum $\mathcal{F} \in \mathbf{SH}(\mathcal{Y})$, for $f : \mathcal{X} \to \mathcal{Y}$ proper quasi-smooth and representably smoothable, the Gysin transformation yields Gysin maps

$$f_! : \mathbf{R}\Gamma^{\alpha+\mathcal{L}_{\mathcal{X}/\mathcal{Y}}}(\mathcal{X}, \mathcal{F}) \to \mathbf{R}\Gamma^{\alpha}(\mathcal{Y}, \mathcal{F})$$

extending those of Corollary B.

Quasi-smoothness is a derived version of the notion of local complete intersection morphism, which in particular gives rise on classical truncations to a relative perfect obstruction theory in the sense of [BF]. Since for every scalloped derived stack $\mathcal{X}$, the inclusion of the classical truncation $i : \mathcal{X}_{\mathrm{cl}} \to \mathcal{X}$ induces canonical isomorphisms

$$i^* : \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}) \simeq \mathbf{R}\Gamma^{\alpha}(\mathcal{X}_{\mathrm{cl}}, \mathcal{F}), \tag{1.5}$$

for all $\alpha \in \mathrm{K}(\mathcal{X})$, the Gysin functoriality for quasi-smooth morphisms can be interpreted as "virtual" functoriality (cf. [Ma]).

Note that the Gysin transformation $\mathrm{gys}_{\mathcal{X}/\mathcal{Y}}$ for any quasi-smooth, representably smoothable, morphism $f : \mathcal{X} \to \mathcal{Y}$, gives rise to a canonical "bivariant" *virtual fundamental class* which lives in the relative Borel–Moore homology theory

$$\mathbf{R}\Gamma^{-\mathcal{L}_{\mathcal{X}/\mathcal{Y}}}(\mathcal{X}, f^!(\mathcal{F}))$$

for any motivic ring spectrum $\mathcal{F} \in \mathbf{SH}(\mathcal{Y})$. This can be viewed as an "orientation" for f in the sense of the bivariant formalism of Fulton–MacPherson [FM].

1.6. **Homotopy invariant K-theory.** When applied to cohomology with coefficients in KGL (Example 1.1), the isomorphisms (1.5) yield the following corollary (see also Remark 10.5), which was obtained in 2018 in the early stages of this project. Since then, it has also been independently reproven[5] via categorical methods by Elmanto and Sosnilo [ES].

Corollary F. *For every scalloped derived stack $\mathcal{X}$ with classical truncation $\mathcal{X}_{\mathrm{cl}}$, there is a canonical isomorphism of spectra*

$$\mathrm{KH}(\mathcal{X}) \simeq \mathrm{KH}(\mathcal{X}_{\mathrm{cl}}).$$

The original modest motivation for this work was to give a proof of the following theorem, a slight generalization of a result of Hoyois–Krishna (see [Ho4, HK], and [Ci] for the case of schemes).

Corollary G. *The presheaf of spectra $\mathcal{X} \mapsto \mathrm{KH}(\mathcal{X})$ satisfies cdh descent on the site of scalloped stacks.*

In fact, although the result is an immediate consequence of Corollary B(iv), we also give a more direct argument, independent of the formalism of six operations, by using the cdh descent criterion in [Kh2] (see Remark 10.6).

[5]Although the result in *op. cit.* is not stated in this generality, the same proof applies to scalloped derived stacks. In fact, [ES, Thm. 5.2.2] generalizes to show that any "truncating" invariant of stable ∞-categories satisfies derived nilpotent invariance for scalloped derived stacks.

1.7. **Outline of Part I.** The first part of the paper begins in Sect. 2 by introducing the class of *scalloped* (derived) stacks. As a first approximation, scalloped stacks are those that are built, locally in some sense, out of quotient stacks of the form $[X/G]$ where G is a nice embeddable group scheme over an affine scheme S and X is a *quasi-affine* scheme with G-action. We recall (see Subsect. 2.1):

- Nice groups are a certain class of affine fppf group schemes which are linearly reductive (meaning that the functor of G-invariants is exact) which is stable under passage to closed subgroups and extensions. For example, algebraic tori are nice, as are finite étale groups of order invertible on S.
- Embeddability means that G is a closed subgroup of the general linear group of some vector bundle on S. For G nice, this always holds Nisnevich-locally on S (see [AHR, Cor. 13.2]), and even globally if S is the spectrum of a field.

Such quotients will be called *basic*. The class of scalloped stacks is then built as the closure of the class of basic stacks under the following property: given a quasi-compact algebraic stack $\mathcal{X}$ with separated diagonal, and a representable étale neighbourhood $p : \mathcal{V} \to \mathcal{X}$ of a closed substack $\mathcal{Z} \subseteq \mathcal{X}$ with $\mathcal{V}$ basic, we require that if $\mathcal{X} \smallsetminus \mathcal{Z}$ is scalloped, then so is $\mathcal{X}$. For example, this class includes quotients $[X/G]$ as above where now X is a qcqs *algebraic space*. It also includes tame Deligne–Mumford stacks, as well as tame Artin stacks in the sense of [AOV], with separated diagonal. In fact, the main result of [AHHLR] implies that the class of scalloped stacks is precisely the class of qcqs stacks with separated diagonal and nice stabilizers. See Sects. 2.3 and 2.5.

We conclude Sect. 2 by extending compact generation results for the derived category of quasi-coherent complexes on a qcqs scheme [TT, BVdB] to the class of scalloped derived stacks (see Theorem 2.24). This was previously known in the cases of tame quasi-Deligne–Mumford classical stacks (see [HR2, Thm. A]) and quasi-compact derived stacks with affine diagonal and nice stabilizers (see [BKRS, Thm. A.3.2]). See also [DG]. The étale-local compact generation criterion of [HR2, Thm. C], used in the previously mentioned proofs, does not apply to general scalloped stacks. Our proof is in fact much simpler and follows the same lines as the case of schemes or algebraic spaces (as in e.g. [BVdB, Thm. 3.1.1] and [Lu3, Thm. 9.6.1.1]), using étale neighbourhoods inductively to reduce to the case of basic stacks.

In the rest of Part I, which consists of Sects. 3 and 4, we begin working towards the proof of Theorem A (which will be completed in Part II) by giving the construction of the unstable and stable motivic homotopy categories over a scalloped derived stack. Recall that if X is a qcqs algebraic space, the motivic homotopy category $\mathbf{H}(X)$ is the ∞-category of $\mathbf{A}^1$-homotopy invariant Nisnevich sheaves on the site $\mathrm{Sm}_{/X}$ of smooth algebraic spaces of finite presentation over X. The stabilization $\mathbf{SH}(X)$ is defined by adjoining a $\otimes$-inverse of the Thom space $\mathrm{Th}_X(\mathcal{O}_X) \simeq X \times (\mathbf{P}^1, \infty)$ of the trivial line bundle. See [Kh5].

Over a scalloped stack $\mathcal{X}$, the unstable category $\mathbf{H}(\mathcal{X})$ is defined similarly as the ∞-category of $\mathbf{A}^1$-invariant Nisnevich sheaves on $\mathrm{Sm}_{/\mathcal{X}}$, where $\mathrm{Sm}_{/\mathcal{X}}$ is the site of smooth representable morphisms of finite presentation over $\mathcal{X}$ (see Subsect. 3.1). However, the correct definition of $\mathbf{SH}(\mathcal{X})$ is considerably more involved due to the fact that vector bundles on a stack need not be Nisnevich-locally trivial. When $\mathcal{X}$ is the quotient of a qcqs algebraic space by a nice embeddable group, $\mathbf{SH}(\mathcal{X})$ is defined by adjoining $\otimes$-inverses of the Thom spaces of all vector bundles on $\mathcal{X}$ (Remark 4.7). For general $\mathcal{X}$, the construction is less explicit, but is determined by the requirement that the assignment $\mathcal{X} \mapsto \mathbf{SH}(\mathcal{X})$ satisfies Nisnevich descent. See Theorem 4.4.

For the quotient of a G-quasi-projective scheme X by a nice group G, $\mathbf{SH}([X/G])$ recovers Hoyois's equivariant stable motivic homotopy category $\mathbf{SH}^G(X)$ (Remark 4.8). Thus in this case, our construction in particular removes the quasi-projectivity hypotheses in [Ho3]. Due to the more involved construction of $\mathbf{SH}(\mathcal{X})$ for general $\mathcal{X}$, the standard functoriality results such as the smooth base change formula (see Theorem 4.9) require much more work than in the quotient stack case.

1.8. **Outline of Part II.** In Part II we complete the proof of Theorem A by constructing the formalism of Grothendieck's six operations on the stable motivic homotopy category.

We begin in Sect. 5 with an axiomatization of the system of categories $\mathbf{SH}(\mathcal{X})$ with its basic operations, via a structure called a $(*, \sharp, \otimes)$-formalism. Following Voevodsky's original insight in the case of schemes [Vo2], we identify sufficient conditions from which we will derive the proper base change theorem for a $(*, \sharp, \otimes)$-formalism (see Subsect. 5.2).

In Sect. 6 we prove the base change formula for proper representable morphisms (see Theorem 6.1). The general proof roughly follows [CD, §2.4] in the case of schemes. A key new input necessary to reduce the case of a proper representable morphism $f : \mathcal{X} \to \mathcal{Y}$ to the projective case is Theorem 6.11, which asserts that we can always find a covering $\widetilde{\mathcal{X}} \to \mathcal{X}$ in the proper cdh topology such that the composite $\widetilde{\mathcal{X}} \to \mathcal{Y}$ becomes projective. This relies on a variant of Chow's lemma for stacks proven by Rydh.

With proper base change in hand, we proceed in Sect. 7 to the construction of the !-operations. Following Deligne, the construction for *compactifiable* morphisms is relatively straightforward. Using the ∞-categorical machinery of [LZ], we further extend this to morphisms that are only Nisnevich-locally compactifiable, such as arbitrary representable morphisms of finite type between scalloped stacks. We also prove purity for smooth representable morphisms between scalloped stacks with affine diagonal (Theorem 7.11).

Finally in Sect. 8 we adapt the constructions of [DJK] and [Kh3] to construct Euler and Gysin transformations in our setting. These will give rise in Part III to Euler classes and Gysin maps in cohomology.

1.9. **Outline of Part III.** In Part III we finally turn our attention towards generalized cohomology theories represented by motivic spectra.

In Sect. 9 we give the definitions and record the various operations and properties asserted in Corollary B. We also define relative Borel–Moore homology, for a representable morphism of finite type $f : \mathcal{X} \to \mathcal{Y}$, via cohomology with coefficients in $f^!(\mathcal{F})$. When f is smooth, this is related to cohomology via a Poincaré duality statement (Proposition 9.12).

Sect. 10 contains the constructions of our main examples of motivic spectra mentioned in Subsect. 1.2. For the construction of the algebraic K-theory spectrum KGL (Subsect. 10.1), we first use our compact generation result (Theorem 2.24) to show that algebraic K-theory satisfies Nisnevich descent on scalloped derived stacks (Theorem 10.2). This already yields unstable representability of KH (see Construction 10.3), which is enough to deduce Corollaries F and G (see Remarks 10.5 and 10.6). The stable representability (Theorem 10.7) is not much more difficult than in the case of schemes or algebraic spaces [Ci, Kh2] and quotient stacks [Ho4].

The construction of the algebraic cobordism spectrum MGL is more subtle because the original definition of Voevodsky [Vo, §6.3] in the case of schemes is reasonable for quotient stacks but not in general (see Remark 10.11). Instead, in Subsect. 10.2 we give a different definition following [BH, §16].

The definition of the motivic cohomology spectrum is even less obvious. Indeed, even for schemes the construction is highly nontrivial as Voevodsky's theory of finite correspondences, for example, hinges on the intersection theory of relative cycles, which is delicate over general bases [CD]. Recently, Hoyois [Ho5] has given a definition of the Spitzweck motivic cohomology spectrum [Sp] that relies on the much more robust theory of *framed* correspondences introduced in [EHKSY]. In Subsect. 10.3, we sketch an extension of this construction to stacks, although we do not undertake a full investigation of the theory of framed correspondences between stacks here.

In Sect. 11 we prove Theorem C. In fact, we prove the statement more generally for T any split torus over a connected noetherian affine base. Although our formulation (and proof) of this result is most directly inspired by [Th2], such localization theorems are ubiquitous in the setting of equivariant cohomology: see also [EG2, Bo, Se].

Sect. 12 deals with the construction of the limit-extended theory $\mathbf{SH}_{\triangleleft}$ and proves Theorem D.

1.10. **Related work.** Traditionally, equivariant cohomology theories in algebraic geometry are defined via algebraic versions of the Borel construction.

For example, equivariant Chow groups [EG] and equivariant algebraic cobordism [Kri, HML] are all defined this way. The main exception is algebraic K-theory, which is "genuine" by nature (cf. [Kri2]).

Candidates for genuine (or "Bredon-type") equivariant Chow or motivic cohomology theories (for finite discrete group actions) have been given by Levine–Serpé [LS] and Heller–Voineagu–Østvær [HVØ]. Although it is tempting to guess that the equivariant homotopy coniveau filtration of [LS] computes (the extension to stacks of) the slice filtration of Voevodsky on the equivariant motivic stable homotopy category, as in [Le], this cannot be true because the Levine–Serpé theory fails to be homotopy invariant for nontrivial vector bundles (see [LS, Cor. 5.6]). In particular, the Levine–Serpé theory cannot be representable by a genuine motivic spectrum.

For algebraic stacks with affine stabilizers, Kresch [Kre] has defined extensions of the Chow groups which agree with the Edidin–Graham Chow groups for quotient stacks. We expect Kresch's Chow groups to agree with limit-extended motivic Borel–Moore homology over a field.[6] The limit-extended category was considered in [Kh3] in the special case of categories satisfying étale descent (cf. Example 12.2), and used to defined limit-extended variants of *rational* motivic cohomology, K-theory, and cobordism. Some Bredon-type cohomology theories for stacks have been constructed by Joshua [Jo].

There are several constructions of *motives* of stacks [Toë2, Ch, HPL, CDH]. These are all Borel-type and should be recovered as the image of the limit-extended motivic stable homotopy type (Notation 12.13) by the "linearization" functor

$$\mathbf{SH}(k) \to \mathbf{DM}(k)$$

to Voevodsky motives.

Our work is mainly inspired by the (genuine) equivariant theories of cohomology and stable homotopy theory in algebraic topology (see e.g. [HHR] or [HHR2, §§2-4]), as well as the genuine equivariant motivic homotopy theory of quasi-projective schemes constructed by Hoyois [Ho3]. In the topological setting, the distinction between Borel and genuine cohomology theories, and in particular the advantages of the latter, have long been well-understood: see for instance [LMM] and [CW] (as well as [NS, Thm. II.2.7]). Moreover, the Atiyah–Segal completion theorem [AS] and related results like the Segal conjecture (proven by Carlsson [Ca]) explain the precise manner in which the genuine theory is a refinement of the Borel one.

In the algebraic setting, an analogue of the Atiyah–Segal theorem, describing Borel-equivariant K-theory as a completion of (genuine) equivariant K-theory, has been proven by Krishna [Kri2]. Conversely, in the case of actions with finite stabilizers, (genuine) equivariant K-theory can be described as the

[6]One can define a cycle class map from the former to the latter, which induces an isomorphism on quotient stacks (at least assuming resolution of singularities). The claim should then follow by stratifying the stack by quotient stacks and using the localization sequence. However, we have not checked that the (explicitly defined) boundary map in Kresch's localization sequence agrees with the one in motivic Borel–Moore homology.

Borel-equivariant K-theory of the *inertia* (a.k.a. "twisted sectors"), up to rationalizing both sides and tensoring further with the maximal abelian extension $\mathbf{Q}(\mu_\infty)$ (see [BC, AS2, Vi, Toë]). It would be interesting to know if these results have analogues in other genuine theories like motivic cohomology and algebraic cobordism.

The six operations have been constructed in genuine equivariant motivic homotopy theory for quasi-projective G-schemes by Hoyois [Ho3]. When G is nice, our Theorem A extends his formalism in two orthogonal directions: we can take qcqs algebraic spaces with G-action (and arbitrary G-equivariant morphisms between them), and we also allow them to be derived. We note however that some of our results, namely localization and Atiyah duality (Theorems 3.19 and 6.1(iii)), are proven by reduction to the case considered by Hoyois. Further aspects of genuine equivariant motivic theory have been developed by Gepner–Heller [GH] and in work in progress of Bachmann [Bac].

The six operational viewpoint on Euler classes and Gysin maps that we follow was introduced in [DJK], and extended to derived algebraic spaces in [Kh3]. In Appendix A of *loc. cit.*, the author also extended this construction to stacks by working in the limit extension of the étale-local stable motivic homotopy category $\mathbf{SH}_{\text{ét}}$. Working in the limit extension of $\mathbf{SH}$ itself, one can use the same approach to define virtual fundamental classes in generalized cohomology theories not satisfying étale descent (details will appear elsewhere). In [Le3], Levine has used Hoyois's equivariant motivic homotopy category to define virtual fundamental classes for quasi-projective G-schemes equipped with an equivariant perfect obstruction theory. When the latter arises from the cotangent complex of an equivariant quasi-smooth structure on the scheme, one can adapt the argument of [Kh3, §3.3] to show that his construction agrees with Definition 9.11.

1.11. Conventions.

- We freely use the language of ∞-categories as in [Lu].
- We use the term *animum* (plural: anima) as an abbreviation for animated sets in the sense of [ÇS, §5.1.4].[7] The ∞-category of anima is equivalent to the ∞-category of ∞-groupoids, and can be modelled by the homotopy theory of Kan complexes.
- We adopt the convention that a symmetric monoidal presentable ∞-category is a presentable ∞-category with a symmetric monoidal structure for which the monoidal product $\otimes$ commutes with colimits in both arguments.
- A colimit-preserving functor between presentable ∞-categories is called *compact* if its right adjoint preserves filtered colimits (see e.g. [Lu3, Defn. C.3.4.2]).

[7] This is a minor modification of the terminology introduced in *loc. cit.*, where "anima" is both the singular and plural form. Our proposal of "animum/anima" is intended to match "spectrum/spectra".

- A derived algebraic stack is a derived 1-Artin stack as in [GR, Chap. 2, 4.1].
- We often use the abbreviation "qcqs" for a (derived) algebraic space or stack to mean quasi-compact and quasi-separated.
- Given a derived algebraic stack $\mathcal{X}$, we write $\mathrm{K}(\mathcal{X})$ for the K-theory animum of $\mathcal{X}$, defined by the Waldhausen $S_\bullet$-construction on the stable ∞-category of perfect complexes on $\mathcal{X}$. See e.g. [Bar] or [Kh4, §2.1]. By abuse of language, we will refer to points in $\mathrm{K}(\mathcal{X})$, rather than elements of $\pi_0\mathrm{K}(\mathcal{X})$, as "K-theory classes".

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2. Scalloped algebraic stacks

2.1. **Nice group schemes.** We recall the definition of nice and embeddable group schemes. We refer to [AHR] for more details. Fix an affine scheme S.

Definition 2.1. Let G be an fppf affine group scheme over S.

(i) We say G is *nice* if it is an extension of a finite étale group scheme, of order prime to the residue characteristics of S, by a group scheme of multiplicative type.

(ii) We say G is *embeddable* if it is a closed subgroup of $\mathrm{GL}_S(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ on S.

Example 2.2. Any torus is nice and embeddable. If S is the spectrum of a field k, then every finite étale group scheme over S of order invertible in k is nice and embeddable.

Remark 2.3. Any nice group scheme G over S is linearly reductive (see [AHR, Rem. 2.2]), i.e., formation of derived global sections on BG is t-exact.

2.2. **The Nisnevich topology.** For a qcqs derived algebraic stack $\mathcal{X}$, denote by $\mathrm{Rep}_{/\mathcal{X}}$ the ∞-category of derived stacks $\mathcal{X}'$ over $\mathcal{X}$ for which the structural morphism $f : \mathcal{X}' \to \mathcal{X}$ is representable and of finite presentation.

Definition 2.4. A *Nisnevich square* over $\mathcal{X}$ is a cartesian square in $\mathrm{Rep}_{/\mathcal{X}}$

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{U} & \xrightarrow{j} & \mathcal{X}' \end{array}$$

where j is an open immersion and p is an étale morphism[8] which induces an isomorphism away from $\mathcal{U}$. The *Nisnevich topology* is the Grothendieck topology on $\mathrm{Rep}_{/\mathcal{X}}$ generated by the following covering families: (a) the empty family, covering the empty stack $\varnothing$; (b) for every Nisnevich square as above, the family $\{j, p\}$ covering $\mathcal{X}$.

Proposition 2.5. *Let $\mathcal{F}$ be a presheaf on the site $\mathrm{Rep}_{/\mathcal{X}}$, with values in an ∞-category $\mathcal{C}$ that admits limits. Then $\mathcal{F}$ satisfies Čech descent for the Nisnevich topology if and only if the following two conditions hold:*

(i) *The object $\mathbf{R}\Gamma(\varnothing, \mathcal{F}) \in \mathcal{C}$ is terminal.*

(ii) *For every Nisnevich square in $\mathrm{Rep}_{/\mathcal{X}}$ as in Definition 2.4, the induced square in $\mathcal{C}$*

$$\begin{array}{ccc} \mathbf{R}\Gamma(\mathcal{X}', \mathcal{F}) & \xrightarrow{j^*} & \mathbf{R}\Gamma(\mathcal{U}, \mathcal{F}) \\ \downarrow{\scriptstyle p^*} & & \downarrow \\ \mathbf{R}\Gamma(\mathcal{V}, \mathcal{F}) & \longrightarrow & \mathbf{R}\Gamma(\mathcal{W}, \mathcal{F}) \end{array}$$

is cartesian.

Proof. Follows from [Kh, Thm. 2.2.7]. □

Remark 2.6. It follows from Proposition 2.5 that the Nisnevich topology on $\mathrm{Rep}_{/\mathcal{X}}$ (as defined in Definition 2.4) coincides with the topology defined in [HK, §2C].

2.3. Scalloped stacks.

Definition 2.7. Let $\mathcal{X}$ be a derived algebraic stack. A *scallop decomposition* $(\mathcal{U}_i, \mathcal{V}_i, u_i)_i$ of $\mathcal{X}$ is a finite filtration by quasi-compact open substacks

$$\varnothing = \mathcal{U}_0 \hookrightarrow \mathcal{U}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{U}_n = \mathcal{X},$$

together with Nisnevich squares

$$\begin{array}{ccc} \mathcal{W}_i & \longrightarrow & \mathcal{V}_i \\ \downarrow & & \downarrow{\scriptstyle u_i} \\ \mathcal{U}_{i-1} & \longrightarrow & \mathcal{U}_i \end{array}$$

where u_i are representable étale morphisms of finite presentation.

[8] necessarily representable

Remark 2.8. Definition 2.7 is a variant of a notion introduced by Lurie (see [Lu3, Def. 2.5.3.1]). In the terminology of *loc. cit.*, a scallop decomposition is as above but where the $\mathcal{V}_i$ are required to be affine schemes. Thus if $\mathcal{X}$ admits a scallop decomposition in Lurie's sense, then it must be a qcqs algebraic space (moreover, this turns out to be a sufficient condition, see [Lu3, 3.4.2.1], [CLO, Thm. 3.1.1]). For our purposes, namely in Definition 2.9 below, it is important to allow $\mathcal{V}_i$ to be a stack.

Definition 2.9. Let $\mathcal{X}$ be a quasi-compact quasi-separated derived algebraic stack.

(i) We say that $\mathcal{X}$ is *fundamental* if it admits an *affine* morphism $\mathcal{X} \to BG$ for some nice embeddable group scheme G over an affine scheme S. That is, $\mathcal{X}$ is the quotient $[X/G]$ of an affine derived scheme X over S with G-action.

(ii) We say that $\mathcal{X}$ is *basic* if it admits a *quasi-affine* morphism $\mathcal{X} \to BG$ for some nice embeddable group scheme G over an affine scheme S. That is, $\mathcal{X}$ is the quotient $[X/G]$ of a quasi-affine derived scheme X over S with G-action.

(iii) We say that $\mathcal{X}$ is *scalloped* if it has separated diagonal and admits a scallop decomposition $(\mathcal{U}_i, \mathcal{V}_i, u_i)_i$ where $\mathcal{V}_i$ are basic.

Remark 2.10. Although we restrict ourselves to derived 1-Artin stacks in this paper for simplicity[9], we note that the above definition admits a natural extension to the world of higher derived stacks. Namely, if 0-*scalloped* stacks are the class defined above, then a higher derived Artin stack $\mathcal{X}$ is n-*scalloped* if it has a scallop decomposition $(\mathcal{U}_i, \mathcal{V}_i, u_i)_i$ where $\mathcal{V}_i$ are $(n-1)$-scalloped and u_i are only required to be n-representable morphisms.

Remark 2.11. There are variants of the above definitions where the class of nice embeddable groups is replaced e.g. by the class of linearly reductive embeddable groups. We expect that most of our results can be further extended from the class of "nicely scalloped" stacks to the class of "linearly scalloped" ones. We plan to address this in a future revision.

2.4. Properties of scalloped stacks.

The next few results collect some of the main properties enjoyed by the class of scalloped stacks.

Theorem 2.12.

(i) *The classes of fundamental, basic, and scalloped derived stacks are each stable under finite disjoint unions.*

(ii) *Let $\mathcal{X}$ be a qcqs derived algebraic stack. If $\mathcal{X}$ has separated diagonal, then the following conditions are equivalent:*

  (a) *$\mathcal{X}$ admits a Nisnevich cover $u : \mathcal{U} \twoheadrightarrow \mathcal{X}$ where $\mathcal{U}$ is fundamental.*

[9] Interested readers will find that many of our arguments, which proceed inductively to reduce to the basic case, would extend without modification.

(b) *$\mathcal{X}$ admits a scallop decomposition $(\mathcal{U}_i, \mathcal{V}_i, u_i)_i$ where the $\mathcal{V}_i$ are basic.*

(c) *$\mathcal{X}$ is scalloped.*

(d) *$\mathcal{X}$ admits a scallop decomposition $(\mathcal{U}_i, \mathcal{V}_i, u_i)_i$ where the $\mathcal{V}_i$ are scalloped.*

(e) *$\mathcal{X}$ has nice stabilizers.*

(iii) *Let $\mathcal{X}$ be a qcqs derived algebraic stack. If $\mathcal{X}$ has affine diagonal, then the following conditions are equivalent:*

(a) *$\mathcal{X}$ is scalloped.*

(b) *$\mathcal{X}$ admits an affine Nisnevich cover $u : \mathcal{U} \twoheadrightarrow \mathcal{X}$ where $\mathcal{U}$ is fundamental.*

Corollary 2.13. *Let $f : \mathcal{X}' \to \mathcal{X}$ be a morphism of qcqs derived algebraic stacks.*

(i) *If $\mathcal{X}$ is basic and f is quasi-affine, then $\mathcal{X}'$ is basic.*

(ii) *If $\mathcal{X}$ is scalloped and f is representable, then $\mathcal{X}'$ is scalloped.*

Proof. The first claim is obvious from the definition. The second follows immediately from the characterization in terms of nice stabilizers (Theorem 2.12), since the stabilizers of $\mathcal{X}'$ will be subgroups of those of $\mathcal{X}$, and the class of nice groups is stable under closed subgroups. □

The following can be regarded as a generalized version of Sumihiro's theorem [Su].

Theorem 2.14. *Let $\mathcal{X} = [X/G]$ be the quotient of a qcqs derived algebraic space X with G-action, where G is a nice group scheme over an affine scheme S. Then we have:*

(i) *$\mathcal{X}$ admits a Nisnevich cover $u : \mathcal{U} \twoheadrightarrow \mathcal{X}$ where $\mathcal{U}$ is of the form $[U/G]$ with U an affine derived scheme over S with G-action. Moreover, if X has affine diagonal, then u is affine.*

(ii) *$\mathcal{X}$ admits a scallop decomposition $(\mathcal{U}_i, \mathcal{V}_i, u_i)_i$ where the $\mathcal{V}_i$ are of the form $[V_i/G]$, for some quasi-affine derived schemes V_i over S with G-action; in particular, $\mathcal{X}$ is scalloped.*

Proof. Since $\mathcal{X}$ has nice stabilizers, Theorem 2.12(ii) implies that there exists a Nisnevich cover $u : \mathcal{U} \twoheadrightarrow \mathcal{X}$ with $\mathcal{U}$ fundamental. By Theorem 2.17, we can refine the cover u such that $\mathcal{U}$ is *affine* over BG, i.e., of the form $[U/G]$ where U is an affine derived scheme over S with G-action. If X has affine diagonal, then $U \to X$ is automatically affine, hence so is $u : \mathcal{U} \to \mathcal{X}$. This shows (i), and (ii) now follows from [HK, Prop. 2.9]. □

2.5. **Tame stacks.** Let us recall a few classes of examples of scalloped stacks:

Example 2.15 (Tame DM stacks)**.** If $\mathcal{X}$ is a tame derived Deligne–Mumford stack with separated diagonal, then its stabilizers are linearly reductive finite étale group schemes (in particular, they are nice). Hence $\mathcal{X}$ is scalloped by Theorem 2.12(ii).

Example 2.16 (Tame Artin stacks)**.** If $\mathcal{X}$ is an algebraic stack with separated diagonal which is tame in the sense of [AOV], then it has nice stabilizers. Hence it is scalloped by Theorem 2.12(ii).

2.6. **Proof of Theorem 2.12.** The proof of Theorem 2.12 will make use of the following result of Alper–Hall–Halpern-Leistner–Rydh.

Theorem 2.17. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of derived algebraic stacks such that $\mathcal{X}$ is fundamental and $\mathcal{Y}$ has affine diagonal. Then there exists a Zariski cover $u : \mathcal{U} \twoheadrightarrow \mathcal{X}$ such that $\mathcal{U}$ is fundamental and the composite $f_0 = f \circ u : \mathcal{U} \to \mathcal{Y}$ is affine.*

Proof. Applying [AHHLR, Thm. 1.7], we get for every point $x \in |\mathcal{X}|$ a Zariski open neighbourhood $\mathcal{U}_x \subseteq \mathcal{X}$ of the residual gerbe $\mathcal{G}_x$ such that $\mathcal{U}_x$ is fundamental and the restriction $f|_{\mathcal{U}_x} : \mathcal{U}_x \to \mathcal{Y}$ is affine. We let $\mathcal{U} := \coprod_x \mathcal{U}_x$ and $u : \mathcal{U} \twoheadrightarrow \mathcal{X}$. □

Proof of Theorem 2.12.

Claim (i): Follows immediately from the definitions.

Claim (ii), (a) ⇒ (b): Follows from [HK, Prop. 2.9] and the fact that open substacks of fundamental stacks are basic.

Claim (ii), (b) ⇒ (c): Obvious.

Claim (ii), (c) ⇒ (a): First note that if $\mathcal{X}$ is basic, then it has affine diagonal and nice stabilizers, so the claim follows by the implication (e) ⇒ (a). If $\mathcal{X}$ is scalloped, then choose a scallop decomposition $(\mathcal{U}_i, \mathcal{V}_i, u_i)_i$ where $\mathcal{V}_i$ are basic. Then $\coprod_i \mathcal{V}_i \twoheadrightarrow \mathcal{X}$ is a Nisnevich cover, and by the basic case we also get an affine Nisnevich cover of the source by a fundamental stack.

Claim (ii), (c) ⇒ (d): Obvious.

Claim (ii), (d) ⇒ (c): By assumption, each $\mathcal{V}_i$ admits a scallop decomposition $(\mathcal{V}^0_{i,j}, \mathcal{V}_{i,j}, u_{i,j})_j$ where $\mathcal{V}_{i,j}$ are basic. Applying [HK, Prop. 2.9] to the Nisnevich covering morphism

$$u : \coprod_{i,j} \mathcal{V}_{i,j} \twoheadrightarrow \mathcal{X},$$

we get a new scallop decomposition $(\mathcal{U}'_k, \mathcal{V}'_k, u'_k)_k$ of $\mathcal{X}$, where $\mathcal{V}'_k$ are by construction open inside $\coprod_{i,j} \mathcal{V}_{i,j}$, hence are basic.

Claim (ii), (a) ⇒ (e): If $u : \mathcal{U} \twoheadrightarrow \mathcal{X}$ is a Nisnevich cover with $\mathcal{U}$ fundamental, then for every point $x \in |\mathcal{X}|$ there exists a point $u \in \mathcal{U}$ lifting x and inducing an isomorphism on stabilizer groups. Since $\mathcal{U}$ has nice stabilizers (as subgroups of nice groups are nice), the claim follows.

Claim (ii), (e) ⇒ (a): This is [AHHLR, Thm. 1.9] (see also [BKRS, Thm. A.1.8]).

Claim (iii): Follows from claim (ii) and Theorem 2.17. □

2.7. **The resolution property.**

Definition 2.18. Let $\mathcal{X}$ be a derived algebraic stack. Denote by $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ the stable ∞-category of quasi-coherent sheaves on $\mathcal{X}$ and by $\mathrm{Qcoh}(\mathcal{X}) \simeq \mathbf{D}_{\mathrm{qc}}(\mathcal{X})^{\heartsuit}$ the full subcategory of discrete quasi-coherent sheaves. We say that $\mathcal{X}$ has the *resolution property* if, for every $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{X})$, there exists a small collection $\{\mathcal{E}_\alpha\}_\alpha$ of finite locally free sheaves and a surjective (on π_0) morphism

$$\bigoplus_\alpha \mathcal{E}_\alpha \twoheadrightarrow \mathcal{F}$$

in $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$.

Remark 2.19. If $\mathcal{X}$ is noetherian, or more generally satisfies the completeness property of [Ry3, Def. 4.2], then the resolution property is equivalent to the condition that for every finitely presented quasi-coherent sheaf $\mathcal{F}$, there exists a surjection $\mathcal{E} \twoheadrightarrow \mathcal{F}$ where $\mathcal{E}$ is finite locally free.

Example 2.20. Let G be an embeddable nice group scheme over an affine scheme S. Then the classifying stack BG has the resolution property. In fact, a resolving set is given by finite locally free G-modules on S. See [AHR, Rem. 2.5].

Proposition 2.21. *Let $\mathcal{X}$ be a basic derived stack. Then $\mathcal{X}$ satisfies the resolution property.*

Proposition 2.21 follows from Example 2.20 and the following lemma.

Lemma 2.22. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-affine morphism of derived algebraic stacks. If $\mathcal{Y}$ has the resolution property, then so does $\mathcal{X}$. In fact, if $\{\mathcal{E}_\alpha\}_\alpha$ is a resolving set for $\mathcal{Y}$, then $\{f^*(\mathcal{E}_\alpha)\}_\alpha$ is a resolving set for $\mathcal{X}$.*

Proof. Follows by the same argument as in the case of classical stacks, see e.g. [HR2, Lem. 7.1]. For any $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{X})$, the resolution property for $\mathcal{Y}$ implies that there exists a surjection $\phi : \bigoplus_\alpha \mathcal{E}_\alpha \twoheadrightarrow f_*(\mathcal{F})$.[10] By derived inverse image, we get a surjection

$$\mathbf{L}f^*(\phi) : \bigoplus_\alpha \mathbf{L}f^*(\mathcal{E}_\alpha) \twoheadrightarrow \mathbf{L}f^* f_*(\mathcal{F}) \twoheadrightarrow \pi_0\,\mathbf{L}f^* f_*(\mathcal{F}) = f^* f_*(\mathcal{F}).$$

[10] For this proof, we adopt the convention that $f^* \coloneqq \pi_0\,\mathbf{L}f^*$ and $f_* \coloneqq \pi_0\,\mathbf{R}f_*$ denote the inverse and direct image functors at the level of Qcoh.

Now consider the composite

$$\bigoplus_\alpha \mathbf{L}f^*(\mathcal{E}_\alpha) \twoheadrightarrow f^*f_*(\mathcal{F}) \twoheadrightarrow \mathcal{F},$$

where the last arrow (induced by the counit) is surjective because f is quasi-affine, and the source is a direct sum of finite locally free sheaves on $\mathcal{X}$. □

2.8. **Compact generation of the derived category.** For a derived algebraic stack $\mathcal{X}$, we write $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ for the stable ∞-category of quasi-coherent complexes on $\mathcal{X}$ (see e.g. [GR, Chap. 3, 1.1.4]). When $\mathcal{X}$ is classical, this is equivalent to the derived ∞-category of $\mathcal{O}_{\mathcal{X}}$-modules with quasi-coherent cohomology (see [AHR, Prop. 1.3]).

Definition 2.23. A derived stack $\mathcal{X}$ is *perfect* if $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ is compactly generated by the full subcategory $\mathbf{D}_{\mathrm{perf}}(\mathcal{X})$ of perfect complexes.

In this subsection we prove:

Theorem 2.24. *Every scalloped derived stack is perfect.*

Remark 2.25. In case $\mathcal{X}$ has affine diagonal, Theorem 2.24 can be proven using the criterion of [HR2, Thm. C], exactly as in the special case of ANS derived stacks (i.e., quasi-compact derived stacks with affine diagonal and nice stabilizers, see [BKRS, Thm. A.3.2]).

Lemma 2.26. *Let $\mathcal{X}$ be a basic derived stack. Then for every cocompact closed subset $Z \subseteq |\mathcal{X}|$, the stable ∞-category $\mathbf{D}_{\mathrm{qc}}(\mathcal{X} \text{ on } Z)$, of quasi-coherent complexes supported on Z, is compactly generated by the full subcategory $\mathbf{D}_{\mathrm{perf}}(\mathcal{X} \text{ on } Z)$ of perfect complexes supported on Z.*

Proof. By [Kh4, Prop. 1.30], it suffices to check that $\mathcal{X}$ is of finite cohomological dimension (as in [Kh4, Ex. 1.15]) and that it satisfies the Thomason condition in the sense of [HR2, Def. 8.1] or [Kh4, Def. 1.29]. The first condition follows from [HR, Thm. 2.1] (cf. [Kh4, Ex. 1.22]) since $\mathcal{X}$ has nice stabilizers. The second follows from [HR2, Prop. 8.4] in view of the following facts:

(i) For any quasi-affine morphism $f : \mathcal{X} \to BG$ with G an embeddable nice group scheme over an affine scheme S, the inverse image functor $f^* : \mathbf{D}_{\mathrm{qc}}(BG) \to \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ is compact[11] and generates its codomain under colimits.

(ii) $\mathcal{X}$ has the resolution property, with a resolving set formed by objects of the form $f^*(\mathcal{E})$, where $\mathcal{E}$ is a finite locally free sheaf on BG (by Lemma 2.22 and Example 2.20).

□

[11] i.e., its right adjoint f_* commutes with colimits

Lemma 2.27. *Let $\mathcal{X}$ be a quasi-separated derived algebraic stack. Then every compact object $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ is perfect. Conversely, if $\mathcal{X}$ is scalloped, then every perfect complex on $\mathcal{X}$ is compact.*

Proof. For the first claim, let $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ be a compact object. It is enough to show that $u^*(\mathcal{F}) \in \mathbf{D}_{\mathrm{qc}}(X)$ is compact for every smooth morphism $u : X \to \mathcal{X}$ with X affine. Since $\mathcal{X}$ is quasi-separated, u is qcqs and representable, so u^* is a compact functor (see e.g. [Kh4, Prop. 1.18, Ex. 1.20]). In particular, $u^*(\mathcal{F})$ is compact and hence also perfect, since X is perfect.

Now suppose $\mathcal{X}$ is scalloped. Let us first note that, given any Nisnevich square

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow p \\ \mathcal{U} & \xrightarrow{j} & \mathcal{X}, \end{array}$$

we have that $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ is compact if and only if $j^*(\mathcal{F}) \in \mathbf{D}_{\mathrm{qc}}(\mathcal{U})$ and $p^*(\mathcal{F}) \in \mathbf{D}_{\mathrm{qc}}(\mathcal{U})$ are compact. The condition is necessary since j^* and p^* are compact functors, and sufficiency follows immediately from the fact that the square of stable ∞-categories

$$\begin{array}{ccc} \mathbf{D}_{\mathrm{qc}}(\mathcal{X}) & \xrightarrow{j^*} & \mathbf{D}_{\mathrm{qc}}(\mathcal{U}) \\ \downarrow p^* & & \downarrow \\ \mathbf{D}_{\mathrm{qc}}(\mathcal{V}) & \longrightarrow & \mathbf{D}_{\mathrm{qc}}(\mathcal{W}) \end{array}$$

is cartesian (see e.g. [BKRS, Thm. 2.2.3]), in view of the definition of compact objects, the fact that filtered colimits of spectra are exact (i.e., commute with finite limits), and that formation of mapping spectra in a stable ∞-category commutes with limits.

Now by induction on the length of a scallop decomposition of $\mathcal{X}$, we may assume that $\mathcal{X}$ is basic. In that case, the claim follows from Lemma 2.26. ☐

Proof of Theorem 2.24. Let $\mathcal{X}$ be a scalloped derived stack. By induction, it will suffice to show the following: suppose given a Nisnevich square

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{j'} & \mathcal{V} \\ \downarrow p' & & \downarrow p \\ \mathcal{U} & \xrightarrow{j} & \mathcal{X}, \end{array}$$

where j is a quasi-compact open immersion and p is a representable étale morphism of finite presentation inducing an isomorphism away from $\mathcal{U}$. We claim that if $\mathcal{U}$ is perfect and $\mathcal{V}$ is basic, then $\mathcal{X}$ is perfect.

Let $\{\mathcal{F}_\alpha\}_\alpha$ be a small set of compact perfect complexes on $\mathcal{U}$ which generate $\mathbf{D}_{\mathrm{qc}}(\mathcal{U})$. By Lemma 2.26 applied to $\mathcal{V}$ and $Z = |\mathcal{V}| \setminus |\mathcal{W}| \subseteq |\mathcal{V}|$, there also exists a small set of compact perfect complexes $\{\mathcal{G}_\beta\}_\beta$ on $\mathcal{V}$ which are supported on Z and which generate $\mathbf{D}_{\mathrm{qc}}(\mathcal{V} \text{ on } Z)$. By Lemma 2.26 and the Thomason–Neeman localization theorem (see e.g. [CDH+, Thm. A.3.11]), the restriction functor

$$\mathbf{D}_{\mathrm{perf}}(\mathcal{V}) \to \mathbf{D}_{\mathrm{perf}}(\mathcal{W})$$

is a Karoubi projection in the sense of [CDH+, Def. A.3.5]. Therefore, by replacing every $\mathcal{F}_\alpha$ by $\mathcal{F}_\alpha \oplus \mathcal{F}_\alpha[1]$ if necessary, we can assume that each $\mathcal{F}_\alpha|_{\mathcal{W}}$ lifts to a perfect complex $\mathcal{F}_{\mathcal{V},\alpha} \in \mathbf{D}_{\mathrm{perf}}(\mathcal{V})$. Using the cartesian square of stable ∞-categories (see e.g. [BKRS, Thm. 2.2.3])

$$\begin{array}{ccc} \mathbf{D}_{\mathrm{perf}}(\mathcal{X}) & \xrightarrow{j^*} & \mathbf{D}_{\mathrm{perf}}(\mathcal{U}) \\ \downarrow{\scriptstyle p^*} & & \downarrow \\ \mathbf{D}_{\mathrm{perf}}(\mathcal{V}) & \longrightarrow & \mathbf{D}_{\mathrm{perf}}(\mathcal{W}), \end{array} \tag{2.28}$$

we construct objects $\mathcal{F}_{\mathcal{X},\alpha} \in \mathbf{D}_{\mathrm{perf}}(\mathcal{X})$ by gluing

$$\mathcal{F}_\alpha \in \mathbf{D}_{\mathrm{perf}}(\mathcal{U}), \quad \mathcal{F}_{\mathcal{V},\alpha} \in \mathbf{D}_{\mathrm{perf}}(\mathcal{V})$$

along some choice of isomorphisms $\mathcal{F}_{\mathcal{V},\alpha}|_{\mathcal{W}} \simeq \mathcal{F}_\alpha|_{\mathcal{W}}$. Similarly, we construct objects $\mathcal{G}_{\mathcal{X},\beta} \in \mathbf{D}_{\mathrm{perf}}(\mathcal{X})$ by gluing

$$0 \in \mathbf{D}_{\mathrm{perf}}(\mathcal{U}), \quad \mathcal{G}_\beta \in \mathbf{D}_{\mathrm{perf}}(\mathcal{V})$$

along the (unique) isomorphism $\mathcal{G}_\beta|_{\mathcal{W}} \simeq 0$. By Lemma 2.27, $\mathcal{F}_{\mathcal{X},\alpha}$ and $\mathcal{G}_{\mathcal{X},\beta}$ are all compact objects of $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$.

It remains to show that the union of the sets $\{\mathcal{F}_{\mathcal{X},\alpha}\}_\alpha$ and $\{\mathcal{G}_{\mathcal{X},\beta}\}_\beta$ generate $\mathbf{D}_{\mathrm{qc}}(\mathcal{X})$. That is, for every object $\mathcal{R} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ which is right orthogonal to the $\mathcal{F}_{\mathcal{X},\alpha}$ and $\mathcal{G}_{\mathcal{X},\beta}$'s, we have $\mathcal{R} \simeq 0$, or equivalently that $\mathcal{R}|_{\mathcal{U}} \simeq 0$ and $\mathcal{R}|_{\mathcal{V}} \simeq 0$.

We begin with the following preliminary observation:

(∗) If $\mathcal{R} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{X})$ is right orthogonal to $\mathcal{G}_{\mathcal{X},\beta}$ for every β, then its restriction $\mathcal{R}|_{\mathcal{V}} \in \mathbf{D}_{\mathrm{qc}}(\mathcal{V})$ belongs to the essential image of $j'_* : \mathbf{D}_{\mathrm{qc}}(\mathcal{W}) \to \mathbf{D}_{\mathrm{qc}}(\mathcal{V})$.

Indeed, it follows from the cartesian square (2.28) and the fact that $\mathcal{G}_{\mathcal{X},\beta}|_{\mathcal{U}} \simeq 0$ that there is an isomorphism of mapping spectra

$$\mathrm{Maps}_{\mathbf{D}_{\mathrm{qc}}(\mathcal{V})}(\mathcal{G}_\beta, \mathcal{R}|_{\mathcal{V}}) \simeq \mathrm{Maps}_{\mathbf{D}_{\mathrm{qc}}(\mathcal{X})}(\mathcal{G}_{\mathcal{X},\beta}, \mathcal{R}) \simeq 0.$$

Therefore, $\mathcal{R}|_{\mathcal{V}}$ is right orthogonal to $\mathbf{D}_{\mathrm{qc}}(\mathcal{V} \text{ on } Z)$. Since the right orthogonal subcategory to the latter is precisely the essential image of $j'_* : \mathbf{D}_{\mathrm{qc}}(\mathcal{W}) \to \mathbf{D}_{\mathrm{qc}}(\mathcal{V})$, (∗) follows.

Let us now show that $\mathcal{R}|_{\mathcal{U}} \simeq 0$. We use the cartesian square (2.28) again to compute $\mathrm{Maps}(\mathcal{F}_{\mathcal{X},\alpha}, \mathcal{R})$. Since $\mathcal{R}|_{\mathcal{V}} \simeq j'_*(\mathcal{R}|_{\mathcal{W}})$ by claim (∗), it follows that the restriction map

$$\begin{aligned} &\mathrm{Maps}_{\mathbf{D}_{\mathrm{qc}}(\mathcal{V})}(\mathcal{F}_{\mathcal{V},\beta}, \mathcal{R}|_{\mathcal{V}}) \\ &\quad \simeq \mathrm{Maps}_{\mathbf{D}_{\mathrm{qc}}(\mathcal{V})}(\mathcal{F}_{\mathcal{V},\beta}, j'_*(\mathcal{R}|_{\mathcal{W}})) \xrightarrow{j'^*} \mathrm{Maps}_{\mathbf{D}_{\mathrm{qc}}(\mathcal{W})}(\mathcal{F}_\beta|_{\mathcal{W}}, \mathcal{R}|_{\mathcal{W}}) \end{aligned} \tag{2.29}$$

is invertible (by adjunction). Thus we get for every α an isomorphism of mapping spectra

$$\mathrm{Maps}_{\mathbf{D}_{\mathrm{qc}}(\mathcal{U})}(\mathcal{F}_\alpha, \mathcal{R}|_{\mathcal{U}}) \simeq \mathrm{Maps}(\mathcal{F}_{\mathcal{X},\alpha}, \mathcal{R}) \simeq 0.$$

Since $\{\mathcal{F}_\alpha\}_\alpha$ generate $\mathbf{D}_{\mathrm{qc}}(\mathcal{U})$, it follows that $\mathcal{R}|_{\mathcal{U}} \simeq 0$.

Finally, we deduce $\mathcal{R}|_{\mathcal{V}} \simeq j'_*(\mathcal{R}|_{\mathcal{W}}) \simeq j'_*(\mathcal{R}|_{\mathcal{U}}|_{\mathcal{W}}) \simeq 0$ by claim $(*)$. This concludes the proof that $\mathcal{R} \simeq 0$. □

3. The unstable homotopy category

3.1. **Construction.** Let $\mathcal{X}$ be a scalloped derived stack. Write $\mathrm{Sm}_{/\mathcal{X}}$ for the full subcategory of $\mathrm{Rep}_{/\mathcal{X}}$ spanned by $\mathcal{X}' \in \mathrm{Rep}_{/\mathcal{X}}$ for which $f : \mathcal{X}' \to \mathcal{X}$ is *smooth* representable of finite presentation.

Definition 3.1. Let $\mathcal{X}$ be a scalloped derived stack. A Sm-*fibred animum* over $\mathcal{X}$ is a presheaf of anima on $\mathrm{Sm}_{/\mathcal{X}}$. A *motivic* Sm-*fibred animum* over $\mathcal{X}$ is a Sm-fibred animum satisfying Nisnevich descent and homotopy invariance. The latter condition means that for every $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$ and every vector bundle $\pi : \mathcal{V} \to \mathcal{X}'$, the map of anima

$$\pi^* : \mathbf{R}\Gamma(\mathcal{X}', \mathcal{F}) \to \mathbf{R}\Gamma(\mathcal{V}, \mathcal{F})$$

is invertible. We write $\mathbf{H}(\mathcal{X})$ for the ∞-category of motivic anima over $\mathcal{X}$.

Lemma 3.2. *Let $\mathcal{F}$ be a* Sm*-fibred animum over $\mathcal{X}$. If the canonical map*

$$\mathbf{R}\Gamma(\mathcal{X}', \mathcal{F}) \to \mathbf{R}\Gamma(\mathcal{X}' \times \mathbf{A}^1, \mathcal{F})$$

is invertible for all $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$, then $\mathcal{F}$ satisfies homotopy invariance. In particular, $\mathcal{F}$ is motivic.

Proof. Recall that there is a canonical strict $\mathbf{A}^1$-homotopy contracting any vector bundle $\mathcal{V} \to \mathcal{X}$ to the zero section, hence $\mathrm{L}_{\mathbf{A}^1}(\mathrm{h}_{\mathcal{X}}(\mathcal{V})) \simeq \mathrm{pt}$. Alternatively, see the argument in the proof of [KrRa, Thm. 5.2]. □

Remark 3.3. The ∞-category $\mathbf{H}(\mathcal{X})$ is an accessible left Bousfield localization of the ∞-category of Sm-fibred anima and in particular is presentable. The localization functor can be computed as the transfinite composite

$$\mathbf{L}(\mathcal{F}) \simeq \varinjlim_{n \geqslant 0} (\mathrm{L}_{\mathbf{A}^1} \circ \mathrm{L}_{\mathrm{Nis}})^{\circ n}(\mathcal{F}), \tag{3.4}$$

where $\mathrm{L}_{\mathrm{Nis}}$ and $\mathrm{L}_{\mathbf{A}^1}$ are the Nisnevich and $\mathbf{A}^1$-localization functors, respectively. Recall that $\mathrm{L}_{\mathbf{A}^1}$ can be computed by the formula

$$\mathbf{R}\Gamma(\mathcal{X}', \mathrm{L}_{\mathbf{A}^1}(\mathcal{F})) = \varinjlim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathbf{R}\Gamma(\mathcal{X}' \times \mathbf{A}^n, \mathcal{F})$$

for every $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$, where $\mathbf{A}^\bullet$ is the cosimplicial affine scheme defined e.g. as in [MV, p. 45].

Example 3.5. Any $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$ represents a motivic Sm-fibred animum

$$\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}') \in \mathbf{H}(\mathcal{X})$$

where $\mathcal{X}' \mapsto \mathrm{h}_{\mathcal{X}}(\mathcal{X}')$ is the Yoneda embedding. When there is no risk of confusion, we will sometimes write simply $\mathcal{X}'$ instead of $\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}')$. These objects are compact, since the conditions of Nisnevich descent and homotopy invariance are stable under filtered colimits.

3.2. Generation.

Proposition 3.6. *Let $\mathcal{X}$ be a scalloped derived stack.*

(i) *The ∞-category $\mathbf{H}(\mathcal{X})$ is generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}')$, where $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$ is basic.*

(ii) *Suppose $\mathcal{X} = [X/G]$, where G is a nice group scheme over an affine scheme S and X is a quasi-compact derived algebraic space over S with G-action. Then $\mathbf{H}(\mathcal{X})$ is generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_{\mathcal{X}}([U/G])$, where U is a* quasi-affine *derived G-scheme, smooth over X.*

(iii) *For every $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$, the object $\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}') \in \mathbf{H}(\mathcal{X})$ is compact.*

Remark 3.7. Let G be a nice group scheme over an affine scheme S and let X be a G-quasi-projective scheme over S. Then it follows from Proposition 3.6(ii) that there is a canonical equivalence

$$\mathbf{H}([X/G]) \simeq \mathbf{H}^G(X),$$

where the right-hand side is the G-equivariant motivic homotopy category of [Ho3]. Note that homotopy invariance for affine bundles is automatic in our setting by [Ho3, Rem. 3.13].

Proof of Proposition 3.6. Let $\mathcal{C}$ be the full subcategory of $\mathbf{H}(\mathcal{X})$ generated under sifted colimits by objects of the form $\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}')$, where $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$ is basic.

Suppose we are given a Nisnevich square in $\mathrm{Sm}_{\mathcal{X}}$

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow p \\ \mathcal{U} & \xrightarrow{j} & \mathcal{X}' \end{array}$$

where j is an open immersion and p is an étale morphism inducing an isomorphism away from $\mathcal{U}$. Then by definition, the canonical map

$$\mathrm{h}_{\mathcal{X}}(\mathcal{U}) \bigsqcup_{\mathrm{h}_{\mathcal{X}}(\mathcal{W})} \mathrm{h}_{\mathcal{X}}(\mathcal{V}) \to \mathrm{h}_{\mathcal{X}}(\mathcal{X}')$$

is a Nisnevich-local equivalence. Thus if $\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{U}) \in \mathcal{C}$ and $\mathcal{V}$ (and hence $\mathcal{W}$) is basic, then also $\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}') \in \mathcal{C}$. By induction and Theorem 2.12(ii), it follows that for every $\mathcal{X}' \in \mathrm{Sm}_{\mathcal{X}}$, we have $\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}') \in \mathcal{C}$. Finally, it now follows from [Lu, Lem. 5.5.8.14] that $\mathcal{C} = \mathbf{H}(\mathcal{X})$.

The second statement follows similarly by Theorem 2.14. For the last one, note that $\mathbf{L}$ preserves compact objects because the full subcategory of motivic Sm-fibred anima is closed under filtered colimits. □

3.3. **Functoriality.** We record the basic functorialities.

Proposition 3.8. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of scalloped derived stacks. Then there exists a canonical functor*

$$f^* : \mathbf{H}(\mathcal{Y}) \to \mathbf{H}(\mathcal{X})$$

satisfying the following properties.

(i) *f^* commutes with colimits, hence in particular admits a right adjoint*
$$f_* : \mathbf{H}(\mathcal{X}) \to \mathbf{H}(\mathcal{Y}).$$
(ii) *For any $\mathcal{V} \in \mathrm{Sm}_{/\mathcal{Y}}$ and $n \geqslant 0$, there is a canonical isomorphism*
$$f^*\big(\mathbf{L}\mathrm{h}_{\mathcal{Y}}(\mathcal{V})\big) \simeq \mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{V} \underset{\mathcal{Y}}{\times} \mathcal{X}).$$
(iii) *f^* is compact, i.e., its right adjoint f_* preserves filtered colimits.*
(iv) *f^* is symmetric monoidal.*

Proof. Compare [Kh5, Prop. 1.22]. □

Proposition 3.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a* smooth *representable morphism of scalloped derived stacks. Then the inverse image functor f^* admits a left adjoint*

$$f_\sharp : \mathbf{H}(\mathcal{X}) \to \mathbf{H}(\mathcal{Y}).$$

This functor is characterized uniquely by commutativity with colimits and the formula

$$f_\sharp\big(\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{U})\big) \simeq \mathrm{h}_{\mathcal{Y}}(\mathcal{U})$$

for any $\mathcal{U} \in \mathrm{Sm}_{/\mathcal{X}}$. Moreover, it is $\mathbf{H}(\mathcal{Y})$-linear; in particular, we have the projection formula

$$f_\sharp(\mathcal{F}) \otimes \mathcal{G} \simeq f_\sharp(\mathcal{F} \otimes f^*(\mathcal{G}))$$

for every $\mathcal{F} \in \mathbf{H}(\mathcal{X})$ and $\mathcal{G} \in \mathbf{H}(\mathcal{Y})$.

Proof. Compare [Kh5, Prop. 1.23]. □

Proposition 3.10 (Smooth base change)**.** *Suppose given a cartesian square of scalloped derived stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{q} & \mathcal{Y}' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{p} & \mathcal{Y} \end{array}$$

where p and q are smooth representable. Then there are canonical isomorphisms

$$q_\sharp f^* \to g^* p_\sharp,$$
$$p^* g_* \to f_* q^*.$$

Proof. Compare [Kh5, Prop. 1.26]. □

Proposition 3.11. *Let $(u_\alpha : \mathcal{X}_\alpha \to \mathcal{X})_\alpha$ be a Nisnevich covering family of a scalloped derived stack $\mathcal{X}$. Then the family of functors*

$$u_\alpha^* : \mathbf{H}(\mathcal{X}) \to \mathbf{H}(\mathcal{X}_\alpha)$$

is jointly conservative as α varies.

Proof. Compare e.g. [Kh, Prop. 2.5.7]. □

3.4. **Exactness of i_*.** Recall from Proposition 3.8 that the direct image functor commutes with filtered colimits. For closed immersions, it commutes with almost all colimits:

Proposition 3.12. *Let $i : \mathcal{X} \to \mathcal{Y}$ be a closed immersion of scalloped derived stacks. Then the direct image functor*

$$i_* : \mathbf{H}(\mathcal{X}) \to \mathbf{H}(\mathcal{Y})$$

commutes with contractible[12] *colimits.*

Proposition 3.12 follows from the following result of Alper–Hall–Halpern-Leistner–Rydh:

Theorem 3.13. *Let $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed immersion of scalloped derived stacks. Let $f_0 : \mathcal{Z}' \to \mathcal{Z}$ be a smooth (resp. étale) representable morphism. Then, up to passing to a Nisnevich cover of $\mathcal{Z}'$, there exists a smooth (resp. étale) representable morphism $f : \mathcal{X}' \to \mathcal{X}$ with $\mathcal{X}'$ fundamental and a cartesian square*

$$\begin{array}{ccc} \mathcal{Z}' & \longrightarrow & \mathcal{X}' \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{i} & \mathcal{X}. \end{array}$$

Moreover, if $\mathcal{X}$ has affine diagonal then f can be taken to be affine.

Proof. Note that the claim is Nisnevich-local both on $\mathcal{Z}'$ and $\mathcal{X}$. Therefore, we may assume $\mathcal{Z}'$ and $\mathcal{X}$ (and hence $\mathcal{Z}$) are fundamental (Theorem 2.12(ii)). In that case, the claim follows by combining Theorems 6.3 and 1.7 in [AHHLR]. □

Proof of Proposition 3.12. This follows from Proposition 3.6 and Theorem 3.13, exactly as in the proof of [Kh, Thm. 3.1.1]. □

3.5. **Derived invariance.**

Theorem 3.14. *Let $\mathcal{X}$ be a scalloped derived stack and let $i : \mathcal{X}_{\mathrm{cl}} \to \mathcal{X}$ denote the inclusion of the classical truncation. Then the functor $i^* : \mathbf{H}(\mathcal{X}) \to \mathbf{H}(\mathcal{X}_{\mathrm{cl}})$ is an equivalence.*

[12] An ∞-category is contractible if the ∞-groupoid/animum obtained by formally adjoining inverses to all its morphisms is contractible.

Construction 3.15. Let $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$ with structural morphism $p : \mathcal{X}' \to \mathcal{X}$. Let $t : \mathcal{X}_{\mathrm{cl}} \to \mathcal{X}'_{\mathrm{cl}}$ be a section of the classical truncation $p_{\mathrm{cl}} : \mathcal{X}'_{\mathrm{cl}} \to \mathcal{X}_{\mathrm{cl}}$. Consider the fibred animum

$$\mathrm{h}_{\mathcal{X}}(\mathcal{X}', t)$$

of "t-trivialized maps": its sections over $\mathcal{X}'' \in \mathrm{Sm}_{/\mathcal{X}}$ are pairs (f, α) with $f : \mathcal{X}'' \to \mathcal{X}'$ an $\mathcal{X}$-morphism and α a "t-trivialization" of f, i.e., a commutative triangle

$$\begin{array}{ccc} \mathcal{X}''_{\mathrm{cl}} & \xrightarrow{f_{\mathrm{cl}}} & \mathcal{X}'_{\mathrm{cl}}. \\ & \searrow \quad \nearrow_{t} & \\ & \mathcal{X}_{\mathrm{cl}} & \end{array}$$

More precisely, $\mathbf{R}\Gamma(\mathcal{X}'', \mathrm{h}_{\mathcal{X}}(\mathcal{X}', t))$ is the homotopy fibre of the canonical map

$$\mathrm{Maps}_{\mathcal{X}}(\mathcal{X}'', \mathcal{X}') \to \mathrm{Maps}_{\mathcal{X}_{\mathrm{cl}}}(\mathcal{X}''_{\mathrm{cl}}, \mathcal{X}'_{\mathrm{cl}})$$

at the point defined by the morphism $\mathcal{X}''_{\mathrm{cl}} \to \mathcal{X}_{\mathrm{cl}} \overset{t}{\hookrightarrow} \mathcal{X}'_{\mathrm{cl}}$. See [Kh, Constr. 4.1.3].

Proposition 3.16. *Let $(\mathcal{X}', t)$ be as in Construction 3.15. Then, Nisnevich-locally on $\mathcal{X}'$, the fibred animum $\mathrm{h}_{\mathcal{X}}(\mathcal{X}', t)$ is motivically contractible, i.e.,*

$$\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}', t) \simeq \mathrm{pt}$$

in $\mathbf{H}(\mathcal{X})$.

The proof of Proposition 3.16 will require a few geometric preliminaries.

Lemma 3.17. *Let $p : \mathcal{Y} \to \mathcal{X}$ be a smooth representable morphism of derived stacks where $\mathcal{X}$ is fundamental. Suppose given a commutative triangle as on the left-hand side below:*

$$\begin{array}{ccc} \mathcal{Z}_0 & \xrightarrow{i_0} & \mathcal{Y}_{\mathrm{cl}} \\ & {}_{q_0}\searrow & \downarrow{\scriptstyle p_{\mathrm{cl}}} \\ & & \mathcal{X}_{\mathrm{cl}} \end{array} \qquad \begin{array}{ccc} \mathcal{Z} & \overset{i}{\dashrightarrow} & \mathcal{Y} \\ & {}_{q}\searrow & \downarrow{\scriptstyle p} \\ & & \mathcal{X}, \end{array}$$

where i_0 is a closed immersion and q_0 is smooth (resp. étale). Then there exists, Nisnevich-locally on $\mathcal{Y}$, a quasi-smooth closed immersion $i : \mathcal{Z} \to \mathcal{Y}$ lifting i_0 such that the composite $q : \mathcal{Z} \to \mathcal{Y} \to \mathcal{X}$ is a smooth (resp. étale) morphism lifting q_0.

Proof. Let $\mathcal{I}$ denote the ideal of definition of $i_0 : \mathcal{Z}_0 \hookrightarrow \mathcal{Y}_{\mathrm{cl}}$. Since q_0 and p_{cl} are smooth, i_0 is quasi-smooth and hence has finite locally free conormal sheaf $\mathcal{N}_{i_0}$. By [Ho3, Erratum, Thm. B], there exists an affine Nisnevich cover $\mathcal{Y}'_0 \twoheadrightarrow \mathcal{Y}_{\mathrm{cl}}$ and a finite locally free $\mathcal{E}_0$ on $\mathcal{Y}'_0$ lifting $\mathcal{N}_{i_0}$. By derived invariance of the étale site and [BKRS, Lem. A.2.6], we can lift this data to an affine Nisnevich cover $\mathcal{Y}' \twoheadrightarrow \mathcal{Y}$ and a finite locally free $\mathcal{E}$ on $\mathcal{Y}'$. As the statement is Nisnevich-local on $\mathcal{Y}$, we may replace $\mathcal{Y}$ by $\mathcal{Y}'$ and thereby assume that there exists a finite locally free $\mathcal{E}$ on $\mathcal{Y}$ such that $i_0^* k^*(\mathcal{E}) \simeq \mathcal{N}_{i_0}$, where $k : \mathcal{Y}_{\mathrm{cl}} \to \mathcal{Y}$ denotes the inclusion of the classical truncation.

Consider the unit morphism $\phi : \mathcal{E} \to t_* i_{0,*} i_0^* k^*(\mathcal{N}_{i_0}) \simeq k_* i_{0,*}(\mathcal{N}_{i_0})$. Since $\mathcal{E}$ is projective on $\mathcal{Y}$ by [BKRS, Prop. A.3.4], ϕ lifts along the π_0-surjection $\mathcal{I} \twoheadrightarrow \mathcal{I}/\mathcal{I}^2 \simeq k_* i_{0,*}(\mathcal{N}_{i_0})$ to a morphism

$$\psi : \mathcal{E} \to \mathcal{I} \subseteq \mathcal{O}_{\mathcal{Y}}.$$

Up to passing to a Zariski cover of $\mathcal{Y}_{\mathrm{cl}}$, the Nakayama lemma implies that $\mathcal{Z}_0$ is the zero locus of $\pi_0(\psi) : \mathcal{E}_0 \to \mathcal{O}_{\mathcal{Y}_{\mathrm{cl}}}$. In other words, the square

$$\begin{array}{ccc} \mathcal{Z}_0 & \xrightarrow{i_0} & \mathcal{Y}_0 \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle \psi_0} \\ \mathcal{Y}_0 & \xrightarrow{0} & \mathbf{V}_{\mathcal{Y}_{\mathrm{cl}}}(\mathcal{E}_{\mathrm{cl}}) \end{array}$$

is (classically) cartesian. We define $\mathcal{Z}$ as the derived zero locus of ψ, so that there is a homotopy cartesian square

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{i} & \mathcal{Y} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \psi} \\ \mathcal{Y} & \xrightarrow{0} & \mathbf{V}_{\mathcal{Y}}(\mathcal{E}). \end{array}$$

In particular, $\mathcal{Z}_{\mathrm{cl}} \simeq \mathcal{Z}_0$ and i lifts i_0. It remains only to show that $q := p \circ i : \mathcal{Z} \to \mathcal{X}$ is smooth.

Recall that smoothness can be checked after derived base change to the classical truncation of the target (e.g. [Lu3, Cor. 11.2.2.8]). Thus it will suffice to show that the following squares are homotopy cartesian:

$$\begin{array}{ccccc} \mathcal{Z}_0 & \xrightarrow{i_0} & \mathcal{Y}_{\mathrm{cl}} & \xrightarrow{p_{\mathrm{cl}}} & \mathcal{X}_{\mathrm{cl}} \\ \downarrow & & \downarrow{\scriptstyle k} & & \downarrow \\ \mathcal{Z} & \xrightarrow{i} & \mathcal{Y} & \xrightarrow{p} & \mathcal{X}. \end{array}$$

The right-hand square is homotopy cartesian because p is smooth, and the left-hand square is classically cartesian by construction. We claim the latter square is in fact homotopy cartesian. For this it is enough to show that the morphism $\mathcal{Z} \to \widetilde{\mathcal{Z}} := \mathcal{Z} \times^{\mathbf{R}}_{\mathcal{Y}} \mathcal{Y}_{\mathrm{cl}}$ is étale. But in the transitivity triangle

$$\mathcal{L}_{\widetilde{\mathcal{Z}}/\mathcal{Y}_{\mathrm{cl}}}|_{\mathcal{Z}_0} \to \mathcal{L}_{\mathcal{Z}_0/\mathcal{Y}_{\mathrm{cl}}} \to \mathcal{L}_{\mathcal{Z}_0/\widetilde{\mathcal{Z}}},$$

the first map is identified with the canonical isomorphism $\mathcal{L}_{\widetilde{\mathcal{Z}}/\mathcal{Y}_{\mathrm{cl}}}|_{\mathcal{Z}_0} \simeq \mathcal{E}|_{\mathcal{Z}_0}[1] \simeq \mathcal{N}_{i_0}[1] \simeq \mathcal{L}_{\mathcal{Z}_0/\mathcal{Y}_{\mathrm{cl}}}$, so the relative cotangent complex $\mathcal{L}_{\mathcal{Z}_0/\widetilde{\mathcal{Z}}}$ vanishes. □

Lemma 3.18. *Let $p : \mathcal{Y} \to \mathcal{X}$ be a smooth affine morphism of fundamental derived stacks. Then for any section $s : \mathcal{X} \to \mathcal{Y}$, there exists a morphism $f : \mathcal{Y} \to N_{\mathcal{X}/\mathcal{Y}}$, étale on the image of s, and a homotopy cartesian square*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{s} & \mathcal{Y} \\ \| & & \downarrow{\scriptstyle f} \\ \mathcal{X} & \xrightarrow{0} & N_{\mathcal{X}/\mathcal{Y}}. \end{array}$$

Here $N_{\mathcal{X}/\mathcal{Y}} = \mathbf{V}_{\mathcal{X}}(\mathcal{N}_{\mathcal{X}/\mathcal{Y}})$ is the normal bundle, total space of the conormal sheaf $\mathcal{N}_{\mathcal{X}/\mathcal{Y}} = \mathcal{L}_{\mathcal{X}/\mathcal{Y}}[-1]$.

Proof. Since s is a section of p, we have $\mathcal{N}_{\mathcal{X}/\mathcal{Y}} \simeq \mathcal{L}_{\mathcal{Y}/\mathcal{X}}|\mathcal{X}$. Since p is smooth, the latter is finite locally free and we have

$$\pi_0(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) \simeq \mathcal{N}_{\mathcal{X}/\mathcal{Y}}|_{\mathcal{X}_{\mathrm{cl}}} \simeq \mathcal{N}_{\mathcal{X}_{\mathrm{cl}}/\mathcal{Y}_{\mathrm{cl}}} \simeq \mathcal{I}/\mathcal{I}^2,$$

where $\mathcal{I}$ is the ideal sheaf of the closed immersion $s_{\mathrm{cl}} : \mathcal{X}_{\mathrm{cl}} \hookrightarrow \mathcal{Y}_{\mathrm{cl}}$. In particular, there is a canonical $\mathcal{O}_{\mathcal{X}_{\mathrm{cl}}}$-module surjection $p_{\mathrm{cl},*}(\mathcal{I}) \twoheadrightarrow \pi_0(\mathcal{N}_{\mathcal{X}/\mathcal{Y}})$. Since $\pi_0(\mathcal{N}_{\mathcal{X}/\mathcal{Y}})$ is projective (see [Ho3, Lem. 2.17] or [BKRS, Prop. A.3.4]), this admits a splitting and the resulting morphism $\phi_0 : \pi_0(\mathcal{N}_{\mathcal{X}/\mathcal{Y}}) \to p_{\mathrm{cl},*}(\mathcal{I}) \subseteq p_{\mathrm{cl},*}(\mathcal{O}_{\mathcal{Y}_{\mathrm{cl}}})$ lifts from $\mathcal{X}_{\mathrm{cl}}$ to $\mathcal{X}$ by [BKRS, Lem. A.2.6]:

$$\phi : \mathcal{N}_{\mathcal{X}/\mathcal{Y}} \to p_*(\mathcal{O}_{\mathcal{Y}}).$$

This determines a morphism $f : \mathcal{Y} \to N_{\mathcal{X}/\mathcal{Y}}$ (recall that p is affine) which fits in a commutative square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{s} & \mathcal{Y} \\ \| & & \downarrow{\scriptstyle f} \\ \mathcal{X} & \xrightarrow{0} & N_{\mathcal{X}/\mathcal{Y}}. \end{array}$$

which is cartesian on classical truncations by construction. To show that the square is homotopy cartesian, it will suffice to show that the induced morphism $\mathcal{X} \to \widehat{\mathcal{X}} := \mathcal{X} \times_{N_{\mathcal{X}/\mathcal{Y}}} \mathcal{Y}$ is étale. For this consider the exact triangle

$$\mathcal{L}_{\mathcal{T}/\mathcal{Y}}|\mathcal{X} \to \mathcal{L}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{L}_{\mathcal{X}/\mathcal{T}}$$

where the first arrow is identified with the canonical isomorphism

$$\mathcal{L}_{\mathcal{T}/\mathcal{Y}}|\mathcal{X} \simeq \mathcal{L}_{\mathcal{X}/N_{\mathcal{X}/\mathcal{Y}}} \simeq^1 \mathcal{N}_{\mathcal{X}/\mathcal{Y}}[1] \simeq \mathcal{L}_{\mathcal{X}/\mathcal{Y}}.$$

Finally, the fact that f is étale on the image of s follows from the isomorphism $s^*\mathcal{L}_f \simeq \mathcal{L}_{\mathcal{X}/\mathcal{X}} \simeq 0$, which comes from the homotopy cartesianness of the above square. ☐

We return to the proof of Proposition 3.16.

Proof of Proposition 3.16. By Lemma 3.17, we may pass to a Nisnevich cover of $\mathcal{X}'$ (if necessary) to assume that t lifts to a section $s : \mathcal{X} \to \mathcal{X}'$ of $p : \mathcal{X}' \to \mathcal{X}$. By Lemma 3.18 there exists a homotopy cartesian square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{s} & \mathcal{X}' \\ \| & & \downarrow{\scriptstyle f} \\ \mathcal{X} & \xrightarrow{0} & N_{\mathcal{X}/\mathcal{X}'} \end{array}$$

where f is étale on a Zariski neighbourhood $\mathcal{X}'_0 \subseteq \mathcal{X}'$ of s. This gives rise to canonical isomorphisms of motivic anima

$$\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}', s) \simeq \mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}'_0, s) \simeq \mathbf{L}\mathrm{h}_{\mathcal{X}}(N_{\mathcal{X}/\mathcal{X}'}, 0)$$

by [Kh, Lem. 4.2.6] (whose proof extends to our setting without modification). Now note that, for any vector bundle $\mathcal{V} \to \mathcal{X}$, the animum $\mathrm{h}_{\mathcal{X}}(\mathcal{V}, 0)$ is contracted to the zero section $0 : \mathcal{X} \to \mathcal{V}$ via the canonical $\mathbf{A}^1$-action on $\mathcal{V}$ by scaling (see [Kh, Lem. 4.2.5]). ☐

Proof of Theorem 3.14. By Theorem 2.12(ii) and Proposition 3.11, we are immediately reduced to the case where $\mathcal{X}$ is fundamental. A formal consequence of Propositions 3.12 and 3.16 is that the unit

$$\mathrm{id} \to i_* i^*$$

is invertible, i.e., that i^* is fully faithful; see [Kh, §4.3]. Thus it will suffice to show that it also generates its codomain $\mathbf{H}(\mathcal{X}_{\mathrm{cl}})$ under colimits. This follows immediately from Proposition 3.6 and Theorem 3.13, which says that every $\mathcal{X}'_0 \in \mathrm{Sm}_{/\mathcal{X}_{\mathrm{cl}}}$ lifts (up to Nisnevich-localizing on $\mathcal{X}'_0$) to some $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$. ☐

3.6. **Localization.**

Theorem 3.19. *Suppose given a complementary closed-open pair*

$$\mathcal{Z} \xhookrightarrow{i} \mathcal{X} \xhookleftarrow{j} \mathcal{U}$$

of scalloped derived stacks. Then the commutative square of endofunctors of $\mathbf{H}(\mathcal{X})$

$$\begin{array}{ccc} j_\sharp j^* & \xrightarrow{\text{counit}} & \mathrm{id} \\ \Big\downarrow{\scriptstyle\text{unit}} & & \Big\downarrow{\scriptstyle\text{unit}} \\ j_\sharp j^* i_* i^* & \xrightarrow{\text{counit}} & i_* i^*, \end{array}$$

where $j_\sharp j^* i_* i^*$ is contractible by the smooth base change formula (Proposition 3.10), is homotopy cocartesian.

Proof. By Theorem 3.14 we may assume that $\mathcal{X}$ and $\mathcal{Z}$ are classical. By Theorem 2.12(ii), Proposition 3.11 and Proposition 3.10, we may assume that $\mathcal{X}$ is fundamental. Now the statement is [Ho3, Thm. 4.18], modulo the equivalence mentioned in Remark 3.7. ☐

4. The stable homotopy category

4.1. **Thom anima.** The stabilization $\mathbf{SH}(\mathcal{X})$ of $\mathbf{H}(\mathcal{X})$ is defined, at least Nisnevich-locally on $\mathcal{X}$, by adjoining $\otimes$-inverses of certain pointed objects in $\mathbf{H}(\mathcal{X})$.

Definition 4.1 (Thom anima)**.** Let $\mathcal{X}$ be a scalloped derived stack. For any finite locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, write $\mathcal{V} = \mathbf{V}_{\mathcal{X}}(\mathcal{E})$ for its total space and $\mathcal{V} \smallsetminus \mathcal{X}$ for the complement of the zero section. The *Thom animum* of $\mathcal{E}$ is the pointed motivic animum

$$\mathrm{Th}_{\mathcal{X}}(\mathcal{E}) \coloneqq \mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{V}) / \mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{V} \smallsetminus \mathcal{X}),$$

i.e., the cofibre of the inclusion $\mathcal{V} \smallsetminus \mathcal{X} \hookrightarrow \mathcal{V}$ taken in the ∞-category $\mathbf{H}(\mathcal{X})$. Note that this is compact by Proposition 3.6(iii).

Example 4.2. The Thom animum of the free sheaf of rank one is

$$\mathrm{Th}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}) \simeq \Sigma_{S^1}\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}\times\mathbf{A}^1\smallsetminus\{0\}) \simeq \mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathbf{P}^1\times\mathcal{X}) \in \mathbf{H}(\mathcal{X})_\bullet,$$

where Σ_{S^1} denotes topological suspension, $\mathbf{A}^1\smallsetminus\{0\}$ is pointed at 1, and $\mathbf{P}^1$ is pointed at ∞.

Remark 4.3. Given an exact triangle $\mathcal{E}'\to\mathcal{E}\to\mathcal{E}''$ of finite locally free sheaves on $\mathcal{X}$, there is a canonical isomorphism

$$\mathrm{Th}_{\mathcal{X}}(\mathcal{E}) \simeq \mathrm{Th}_{\mathcal{X}}(\mathcal{E}') \wedge \mathrm{Th}_{\mathcal{X}}(\mathcal{E}'')$$

in $\mathbf{H}(\mathcal{X})_\bullet$. This follows by the arguments of [CD, 2.4.10] or [Ho3, §3.5].

4.2. Characterization.

The stable motivic homotopy category $\mathbf{SH}(\mathcal{X})$, as a functor in $\mathcal{X}$, will be characterized uniquely as follows.

Theorem 4.4. *For every scalloped derived stack $\mathcal{X}$, there exists a symmetric monoidal colimit-preserving functor*

$$\Sigma^\infty : \mathbf{H}(\mathcal{X})_\bullet \to \mathbf{SH}(\mathcal{X})$$

satisfying the following properties:

(i) *For every scalloped derived stack $\mathcal{X}$ and every finite locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, the object $\Sigma^\infty\mathrm{Th}_{\mathcal{X}}(\mathcal{E})\in\mathbf{SH}(\mathcal{X})$ is $\otimes$-invertible. Moreover, the assignment $\mathcal{E}\mapsto\Sigma^\infty\mathrm{Th}_{\mathcal{X}}(\mathcal{E})$ induces a canonical map of $\mathcal{E}_\infty$-groups*

$$\mathrm{Th}_{\mathcal{X}} : \mathrm{K}(\mathcal{X}) \to \mathrm{Pic}(\mathbf{SH}(\mathcal{X})),$$

from the algebraic K-theory animum of perfect complexes on $\mathcal{X}$ to the Picard animum of $\otimes$-invertible objects in $\mathbf{SH}(\mathcal{X})$.

(ii) *For every morphism $f:\mathcal{X}\to\mathcal{Y}$ of scalloped derived stacks, there is a symmetric monoidal colimit-preserving functor*

$$f^* : \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X})$$

which commutes with Σ^∞ and Th.

(iii) *The assignments*

$$\mathcal{X}\mapsto\mathbf{SH}(\mathcal{X}),\quad f\mapsto f^*$$

of (ii) determine a presheaf $\mathbf{SH}^*$ with values in the ∞-category of symmetric monoidal presentable ∞-categories and symmetric monoidal colimit-preserving functors, which satisfies Nisnevich descent.

(iv) *For every* representable *morphism $f:\mathcal{X}\to\mathcal{Y}$ of scalloped derived stacks, there is a canonical isomorphism of $\mathbf{SH}(\mathcal{Y})$-modules*

$$\mathbf{H}(\mathcal{X})_\bullet \otimes_{\mathbf{H}(\mathcal{Y})_\bullet} \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X}).$$

(v) *If $\mathcal{X}=BG$ is the classifying stack of an embeddable nice group scheme G over an affine scheme S, then*

$$\Sigma^\infty : \mathbf{H}(\mathcal{X})_\bullet \to \mathbf{SH}(\mathcal{X})$$

is the universal *symmetric monoidal colimit-preserving functor which* $\otimes$*-inverts the Thom anima* $\mathrm{Th}_{\mathcal{X}}(\mathcal{E})$ *of all finite locally free sheaves* $\mathcal{E}$ *on* $\mathcal{X}$ *(i.e., finite representations of* G *over* S*).*

Remark 4.5. The proof of Theorem 4.4 will in fact construct morphisms of presheaves

$$\Sigma^\infty : \mathbf{H}^*_\bullet \to \mathbf{SH}^*,$$
$$\mathrm{Th} : \mathrm{K} \to \mathrm{Pic}(\mathbf{SH}^*).$$

Moreover, $\Sigma^\infty : \mathbf{H}^*_\bullet \to \mathbf{SH}^*$ will be the unique morphism of presheaves out of $\mathbf{H}^*_\bullet$, on the site of scalloped derived stacks, satisfying properties (iii), (iv) and (v).

Notation 4.6. We will write $\Omega^\infty : \mathbf{SH}(\mathcal{X}) \to \mathbf{H}(\mathcal{X})_\bullet$ for the right adjoint of Σ^∞. For a morphism $f : \mathcal{X} \to \mathcal{Y}$, we will write $f_* : \mathbf{SH}(\mathcal{X}) \to \mathbf{SH}(\mathcal{Y})$ for the right adjoint of f^*.

Remark 4.7. Let G be an embeddable nice group scheme over an affine scheme S, and X a qcqs derived algebraic space over S with G-action. Then $f : [X/G] \to BG$ is a representable morphism of scalloped derived stacks (Theorem 2.14). Combining claims (v) and (iv) of Theorem 4.4, we find that $\mathbf{SH}([X/G])$ is obtained from $\mathbf{H}([X/G])_\bullet$ by formally adjoining $\otimes$-inverses of the objects

$$f^* \mathrm{Th}_{BG}(\mathcal{E}) \simeq \mathrm{Th}_{[X/G]}(f^*(\mathcal{E}))$$

where $\mathcal{E}$ ranges over finite locally free sheaves on BG.

Remark 4.8. In the situation of Remark 4.7, it follows from Remarks 3.7 and 4.7 that there is a canonical equivalence of symmetric monoidal stable ∞-categories

$$\mathbf{SH}([X/G]) \simeq \mathbf{SH}^G(X)$$

where the right-hand side is Hoyois's G-equivariant stable motivic homotopy category (see [Ho3, §6]), when the latter is defined (i.e., when X is a G-quasi-projective scheme). For general X, the left-hand side can be taken as the definition of $\mathbf{SH}^G(X)$.

4.3. **Functoriality.** Before proceeding to the construction of $\mathbf{SH}^*$, let us also record its functorial properties:

Theorem 4.9. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a morphism of scalloped derived stacks.*

(i) *If* f *is smooth representable, then* $f^* : \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X})$ *satisfies the following properties:*

   (a) Left adjoint. *It admits a left adjoint* $f_\sharp$ *which commutes with* Σ^∞*. In particular,* f^* *commutes with* Ω^∞*.*

   (b) Smooth base change. *The functor* $f_\sharp$ *commutes with arbitrary* $*$*-inverse image. Equivalently,* f^* *commutes with arbitrary* $*$*-direct image.*

(c) Smooth projection formula. *The functor* $f_\sharp$ *is a morphism of* $\mathbf{SH}(\mathcal{Y})$*-modules.*

(ii) *If* f *is a closed immersion, then* $f_* : \mathbf{SH}(\mathcal{X}) \to \mathbf{SH}(\mathcal{Y})$ *satisfies the following properties:*

(a) Closed base change. *It commutes with arbitrary* $*$*-inverse image.*

(b) Smooth-closed base change. *It commutes with* $\sharp$*-direct image by smooth representable morphisms.*

(c) Closed projection formula. *It is a morphism of* $\mathbf{SH}(\mathcal{Y})$*-modules.*

(d) Localization. *It is fully faithful. If the complementary open immersion* $j : \mathcal{U} \to \mathcal{X}$ *is quasi-compact, then the essential image of* f_* *is spanned by the kernel of* $j^* : \mathbf{SH}(\mathcal{X}) \to \mathbf{SH}(\mathcal{U})$*. (For example,* f_* *is an equivalence if* f *is surjective.)*

(iii) *The functor* $f^* : \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X})$ *is compact, i.e., its right adjoint* f_* *commutes with colimits.*

4.4. **The basic case.** We will construct $\mathbf{SH}(\mathcal{X})$ in increasingly greater generality, starting with the basic case:

Construction 4.10. Let $\mathcal{X}$ be a basic derived stack.

(i) Consider the ∞-category $\mathbf{H}(\mathcal{X})_\bullet$ of pointed motivic anima, with the symmetric monoidal structure given by smash product. Let $\mathcal{T}_\mathcal{X}$ denote the (small) set of objects

$$\mathrm{Th}_\mathcal{X}(\mathcal{E}) \in \mathbf{H}(\mathcal{X})_\bullet$$

where $\mathcal{E}$ ranges over all finite locally free sheaves on $\mathcal{X}$. Now formally adjoin to $\mathbf{H}(\mathcal{X})_\bullet$ the $\otimes$-inverse of every object in $\mathcal{T}_\mathcal{X}$ (in the sense of [Ro], [Ho3, §6.1]) to get the symmetric monoidal presentable stable ∞-category

$$\mathbf{SH}(\mathcal{X}) = \mathbf{H}(\mathcal{X})_\bullet[\mathcal{T}_\mathcal{X}^{\otimes -1}]$$

together with the canonical symmetric monoidal colimit-preserving functor $\Sigma^\infty : \mathbf{H}(\mathcal{X})_\bullet \to \mathbf{SH}(\mathcal{X})$.[13]

(ii) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of basic derived stacks. Note that the functor $f^* : \mathbf{H}(\mathcal{Y})_\bullet \to \mathbf{H}(\mathcal{X})_\bullet$ preserves Thom anima, i.e.,

$$f^*(\mathrm{Th}_\mathcal{Y}(\mathcal{E})) = \mathrm{Th}_\mathcal{X}(f^*(\mathcal{E}))$$

for every finite locally free $\mathcal{E}$ over $\mathcal{Y}$. By universal properties it follows that there is a unique extension of f^* to a symmetric monoidal colimit-preserving

$$f^* : \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X})$$

such that f^* commutes with Σ^∞ and f_* commutes with Ω^∞.

[13]Note that the objects in $\mathcal{T}_\mathcal{X}$ are *symmetric* in the sense of [Ro]. Indeed, this follows by functoriality from the case of BG, for any nice embeddable group scheme G over an affine S, which is a special case of [Ho3, Lem. 6.3].

This defines a morphism of presheaves $\Sigma^\infty : \mathbf{H}^*_\bullet \to \mathbf{SH}^*$ on the site of basic derived stacks.

Remark 4.11. For a basic derived stack $\mathcal{X}$, the assignment

$$\mathcal{E} \mapsto \Sigma^\infty \operatorname{Th}_{\mathcal{X}}(\mathcal{E})$$

defines by construction a functor from the ∞-groupoid of finite locally free sheaves on $\mathcal{X}$ to the Picard $\mathcal{E}_\infty$-group of $\otimes$-invertible objects in $\mathbf{SH}(\mathcal{X})$. Moreover, since it sends direct sums to tensor products (Remark 4.3), it is a map of $\mathcal{E}_\infty$-monoids. Since the target is group-complete, the map factors through the group completion of the source, which is the algebraic K-theory animum of $\mathcal{X}$ (since basic stacks have the resolution property, see Proposition 2.21). Thus we have a map of presheaves

$$\operatorname{Th} : \mathrm{K} \to \operatorname{Pic}(\mathbf{SH}^*)$$

on the site of basic derived stacks.

Lemma 4.12. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of basic derived stacks. Then the morphism of $\mathbf{SH}(\mathcal{X})$-modules*

$$\mathbf{H}(\mathcal{X})_\bullet \otimes_{\mathbf{H}(\mathcal{Y})_\bullet} \mathbf{SH}(\mathcal{Y}) \to \mathbf{SH}(\mathcal{X})$$

is an equivalence.

Proof. First assume that the morphism f is quasi-affine. In this case, for every finite locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, the proof of Lemma 2.22 shows that one can find a finite locally free sheaf $\mathcal{E}'$ on $\mathcal{Y}$ and a surjection

$$f^*(\mathcal{E}') \twoheadrightarrow \mathcal{E}.$$

If $\mathcal{K}$ denotes its kernel, then we get a canonical isomorphism (Remark 4.3)

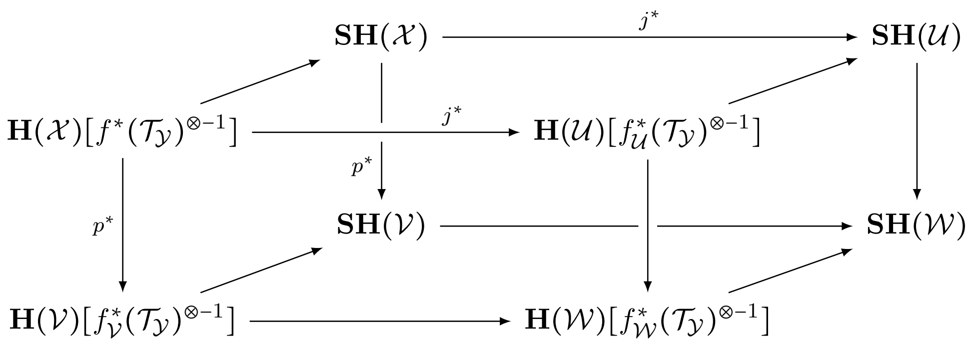
$$\operatorname{Th}_{\mathcal{X}}(f^*\mathcal{E}') \simeq \operatorname{Th}_{\mathcal{X}}(\mathcal{E}) \wedge \operatorname{Th}_{\mathcal{X}}(\mathcal{K})$$

in $\mathbf{H}(\mathcal{X})_\bullet$. Thus $\operatorname{Th}_{\mathcal{X}}(\mathcal{E})$ is invertible in $\mathbf{H}(\mathcal{X})_\bullet[f^*(\mathcal{T}_{\mathcal{Y}})^{\otimes -1}]$.

Next consider the case of a general representable morphism. Since $\mathcal{X}$ has affine diagonal, we can apply Theorem 2.14(ii) to get a scallop decomposition $(\mathcal{U}_i, \mathcal{V}_i, u_i)$, where $\mathcal{V}_i$ are quasi-affine over $\mathcal{Y}$. Hence by induction and the quasi-affine case above, it will suffice to show that for any Nisnevich square

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow p \\ \mathcal{U} & \xrightarrow{j} & \mathcal{X} \end{array}$$

where j is an open immersion and p is an étale morphism inducing an isomorphism away from $\mathcal{U}$, we have that if the claim holds for $f_{\mathcal{U}} = f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{Y}$, $f_{\mathcal{V}} = f|_{\mathcal{V}}$ and $f_{\mathcal{W}} = f|_{\mathcal{W}}$, then it also holds for f. Indeed, this follows by consideration of the commutative cube of ∞-categories



where the front and back faces are both cartesian (by Nisnevich descent). $\square$

Lemma 4.13. *Let $\mathcal{X}$ be a basic derived stack.*

(i) *For every $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$ and every $\alpha \in \mathrm{K}(\mathcal{X})$, the object*
$$\Sigma^\infty_+(\mathcal{X}') \otimes \mathrm{Th}_{\mathcal{X}}(\alpha) \in \mathbf{SH}(\mathcal{X})$$
is compact.

(ii) *Choose a quasi-affine morphism $f : \mathcal{X} \to BG$ where G is an embeddable nice group scheme over an affine scheme S. Then $\mathbf{SH}(\mathcal{X})$ is compactly generated by the set of objects*
$$\Sigma^\infty_+(\mathcal{X}') \otimes \mathrm{Th}_{\mathcal{X}}(f^*(\mathcal{E}))^{\otimes -1},$$
where $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$, and $\mathcal{E}$ is a finite locally free sheaf on BG.

Proof. It is a formal consequence of Proposition 3.6 and general facts about $\otimes$-inversion of objects (see e.g. the proof of [Ho3, Prop. 6.4]) that objects of the form $\Sigma^\infty_+(\mathcal{X}') \otimes \mathrm{Th}_{\mathcal{X}}(\mathcal{E})^{\otimes -1}$, where $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$ and $\mathcal{E}$ is a finite locally free sheaf on $\mathcal{X}$, generate $\mathbf{SH}(\mathcal{X})$ under sifted colimits. The second claim follows by combining this with the canonical equivalence $\mathbf{H}(\mathcal{X})_\bullet[f^*(\mathcal{T}_{BG})^{\otimes -1}] \simeq \mathbf{SH}(\mathcal{X})$ (Lemma 4.12).

Recall that the object $\mathrm{Th}_{\mathcal{X}}(\mathcal{E}) \in \mathbf{H}(\mathcal{X})$ is compact for every finite locally free $\mathcal{E}$ on $\mathcal{X}$ (Example 4.1). By construction of $\mathbf{SH}(\mathcal{X})$, this implies that Σ^∞ is a compact functor. The functor $- \otimes \mathrm{Th}_{\mathcal{X}}(\alpha) : \mathbf{SH}(\mathcal{X}) \to \mathbf{SH}(\mathcal{X})$ is also compact, for every $\alpha \in \mathrm{K}(\mathcal{X})$, since it is an equivalence. Thus the claim follows from the fact that the object $\mathbf{L}\mathrm{h}_{\mathcal{X}}(\mathcal{X}') \in \mathbf{H}(\mathcal{X})$ is compact for every $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$ (Proposition 3.6(iii)). $\square$

Remark 4.14. At this point, we can already prove Theorems 4.4 and 4.9 on the site of basic derived stacks. Indeed, Theorem 4.4(v) holds by construction, (i) by Remark 4.11, and (iv) by Lemma 4.12. The last property (iii) is a standard consequence of the properties asserted in Theorem 4.9 (cf. [Ho3, Prop. 6.24]).

Theorem 4.9(iii) follows from Corollary 4.13. Theorem 4.9(i) follows immediately from the analogous unstable statements (Propositions 3.9, 3.10, 3.11) by extending scalars along Σ^∞ (in view of Theorem 4.4). For example, for smooth base change we argue as follows. By the unstable statement we have the commutative square

$$\begin{array}{ccc} \mathbf{H}(\mathcal{X}) & \xrightarrow{p_\sharp} & \mathbf{H}(\mathcal{Y}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} \\ \mathbf{H}(\mathcal{X}') & \xrightarrow{q_\sharp} & \mathbf{H}(\mathcal{Y}'). \end{array}$$

Then by Lemma 4.12, extension of scalars along Σ^∞ gives the upper commutative square in the diagram

$$\begin{array}{ccc} \mathbf{SH}(\mathcal{X}) & \xrightarrow{p_\sharp} & \mathbf{SH}(\mathcal{Y}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} \\ \mathbf{H}(\mathcal{X}') \otimes_{\mathbf{H}(\mathcal{X})} \mathbf{SH}(\mathcal{X}) & \xrightarrow{q_\sharp} & \mathbf{H}(\mathcal{Y}') \otimes_{\mathbf{H}(\mathcal{X})} \mathbf{SH}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \mathbf{H}(\mathcal{X}') \otimes_{\mathbf{H}(\mathcal{X}')} \mathbf{SH}(\mathcal{X}') & \longrightarrow & \mathbf{H}(\mathcal{Y}') \otimes_{\mathbf{H}(\mathcal{X}')} \mathbf{SH}(\mathcal{X}') \\ \| & & \| \\ \mathbf{SH}(\mathcal{X}') & \xrightarrow{q_\sharp} & \mathbf{SH}(\mathcal{Y}') \end{array}$$

where the middle square commutes tautologically and the lower right-hand isomorphism is Lemma 4.12.

In Theorem 4.9(ii), localization is deduced similarly from the unstable statement (Theorem 3.19, compare the proof of [Kh5, Thm. 1.36]). The other properties are immediate consequences of localization, just as in [Kh5, Lems 2.19, 2.20, 2.21].

4.5. **Proof of Theorem 4.4.** We now extend Construction 4.10 to general scalloped stacks.

Construction 4.15. Let $\mathcal{X}$ be a scalloped derived stack. We define the ∞-category $\mathbf{SH}(\mathcal{X})$ by the formula

$$\mathbf{SH}(\mathcal{X}) = \varprojlim_{(\mathcal{U}, \mathcal{U} \to \mathcal{X})} \mathbf{SH}(\mathcal{U})$$

where the limit is taken over the ∞-category of pairs $(\mathcal{U}, u)$ where $\mathcal{U}$ is basic and $u : \mathcal{U} \to \mathcal{X}$ is a representable étale morphism.

To show that this determines a presheaf $\mathbf{SH}^*$ satisfying Nisnevich descent, we will show that this construction is right Kan extended from the subcategory of basic derived stacks.

Lemma 4.16. *Let $\mathcal{C}$ be the ∞-category of scalloped derived stacks and $\mathcal{C}_0$ the full subcategory of basic derived stacks. Regard $\mathcal{C}$ and $\mathcal{C}_0$ as sites with the representable Nisnevich topology. Then restriction along the inclusion $i : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ induces an equivalence from the ∞-category of sheaves on $\mathcal{C}$ to the ∞-category of sheaves on $\mathcal{C}_0$. In particular, every sheaf on $\mathcal{C}$ is right Kan extended from $\mathcal{C}_0$.*

Proof. Note that i is both topologically continuous and cocontinuous, so that restriction i^* and its right adjoint i_* (right Kan extension) preserve sheaves. Its left adjoint $\mathbf{L}_{\mathrm{Nis}} i_!$ is Nisnevich-localized left Kan extension. Since i is fully faithful, it is clear that $\mathbf{L}_{\mathrm{Nis}} i_!$ is fully faithful. Hence the claim follows from the fact that $\mathbf{L}_{\mathrm{Nis}} i_!$ generates the ∞-category of sheaves on $\mathcal{C}$ under colimits, which is proven exactly as in Proposition 3.6. □

Remark 4.17. Let $\mathbf{SH}'^*$ denote the right Kan extension of the presheaf $\mathbf{SH}^*$, defined on the site of basic derived stacks (Construction 4.10), to the site of scalloped derived stacks. By Lemma 4.16, $\mathbf{SH}'^*$ satisfies Nisnevich descent. Recall that for any scalloped $\mathcal{X}$, $\mathbf{SH}'(\mathcal{X})$ can be computed by the same limit as in Construction 4.15, except taken over the ∞-category of pairs $(\mathcal{U}, u)$ where $\mathcal{U}$ is basic and $u : \mathcal{U} \to \mathcal{X}$ is *any* morphism. We claim that the canonical functor $\mathbf{SH}'(\mathcal{X}) \to \mathbf{SH}(\mathcal{X})$ is an equivalence. This is clear when $\mathcal{X}$ is basic. In general, $\mathcal{X}$ admits a scallop decomposition $(\mathcal{U}_i, \mathcal{V}_i, u_i)_i$ of $\mathcal{X}$ where $\mathcal{V}_i$ are basic (Theorem 2.12(ii)). By induction, it will therefore suffice to show that if $\mathcal{X}$ is covered by a Nisnevich square

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{U} & \xrightarrow{\;j\;} & \mathcal{X} \end{array}$$

such that the claim is true for $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$, then it is also true for $\mathcal{X}$. By Nisnevich descent, we have a canonical equivalence

$$\mathbf{SH}'(\mathcal{X}) \simeq \mathbf{SH}'(\mathcal{U}) \underset{\mathbf{SH}'(\mathcal{W})}{\times} \mathbf{SH}'(\mathcal{V}) \simeq \mathbf{SH}(\mathcal{U}) \underset{\mathbf{SH}(\mathcal{W})}{\times} \mathbf{SH}(\mathcal{V}).$$

Since each of the maps $\mathcal{U} \to \mathcal{X}$, $\mathcal{V} \to \mathcal{X}$ and $\mathcal{W} \to \mathcal{X}$ are representable and étale, there is a projection functor $\mathbf{SH}(\mathcal{X}) \to \mathbf{SH}'(\mathcal{X})$, which one checks is inverse to the canonical one.

Construction 4.18.

(i) It follows from Remark 4.17 that Construction 4.15 lifts to a presheaf $\mathbf{SH}^*$, given by

$$\mathcal{X} \mapsto \mathbf{SH}(\mathcal{X}), \quad f \mapsto f^*,$$

on the site of scalloped derived stacks. This presheaf takes values in the ∞-category of symmetric monoidal presentable stable ∞-categories and symmetric monoidal colimit-preserving functors (since the forgetful functor from the latter to the ∞-category of large ∞-categories is limit-preserving).

(ii) Since $\mathbf{H}^*_\bullet$ is also right Kan extended from basic stacks (by Lemma 4.16), it follows that $\Sigma^\infty : \mathbf{H}^*_\bullet \to \mathbf{SH}^*$ also admits a unique extension to the site of scalloped derived stacks.

(iii) Similarly, right Kan extension[14] produces an extension of the map $\mathrm{Th} : \mathrm{K} \to \mathrm{Pic}(\mathbf{SH}^*)$ (Remark 4.11) to the site of scalloped derived stacks.

[14]Algebraic K-theory satisfies Nisnevich descent on scalloped derived stacks (see Theorem 10.2), hence is also right Kan extended from fundamentals.

4.6. Proof of Theorem 4.9(i).

4.6.1. *Left adjoint (a).* Note that, over the site of fundamental derived stacks and smooth representable morphisms, $\mathbf{SH}^*$ takes values in the ∞-category of presentable ∞-categories and *right adjoint* functors (by Theorem 4.9(i), which we have already proven in the fundamental case). It follows from [Lu, Thm. 5.5.3.18] that the same holds for the extension of $\mathbf{SH}^*$ to the site of scalloped derived stacks, since the transition functors in the limit in Construction 4.15 are in particular smooth representable.

It follows that, for every smooth representable morphism $p : \mathcal{X} \to \mathcal{Y}$ of scalloped derived stacks p^* admits a left adjoint

$$p_\sharp : \mathbf{SH}(\mathcal{X}) \to \mathbf{SH}(\mathcal{Y}).$$

It follows from the construction that it commutes with Σ^∞.

4.6.2. *Smooth base change (b).* Let $p : \mathcal{X} \to \mathcal{Y}$ be a smooth representable morphism of scalloped derived stacks. For any morphism of scalloped derived stacks $g : \mathcal{Y}' \to \mathcal{Y}$, form the base change square

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{q} & \mathcal{Y}' \\ \downarrow f & & \downarrow g \\ \mathcal{X} & \xrightarrow{p} & \mathcal{Y}. \end{array}$$

The claim is that the exchange transformation

$$\mathrm{Ex}^*_\sharp : q_\sharp f^* \to g^* p_\sharp,$$

is invertible. For the proof, we will require the following lemma.

Lemma 4.19. *Let the notation be as above. Suppose that $\mathcal{X}$ is covered by a Nisnevich square*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \downarrow & \searrow^{w} & \downarrow v \\ \mathcal{U} & \xrightarrow{u} & \mathcal{X} \end{array}$$

such that $(p \circ u)_\sharp$, $(p \circ v)_\sharp$, and $(p \circ w)_\sharp$ satisfy base change against g^*. Then $p_\sharp$ satisfies base change against g^*.

Proof. The assumption is that the exchange transformation

$$\mathrm{Ex}^*_\sharp : q_{\mathcal{U}',\sharp} f^*_{\mathcal{U}} \to g^* p_{\mathcal{U},\sharp}$$

associated to the composite square

$$\begin{array}{ccccc} \mathcal{U}' & \xrightarrow{u'} & \mathcal{X}' & \xrightarrow{q} & \mathcal{Y}' \\ \downarrow f_{\mathcal{U}} & & \downarrow f & & \downarrow g \\ \mathcal{U} & \xrightarrow{u} & \mathcal{X} & \xrightarrow{p} & \mathcal{Y}, \end{array}$$

is invertible, where $\mathcal{U}' = \mathcal{U} \times_{\mathcal{Y}} \mathcal{Y}'$, $f_{\mathcal{U}} : \mathcal{U}' \to \mathcal{U}$ is the base change of f, and $q_{\mathcal{U}'} : \mathcal{U}' \to \mathcal{Y}'$ is the base change of $p_{\mathcal{U}}$. Similarly for $\mathrm{Ex}^*_\sharp : q_{\mathcal{V}',\sharp} f^*_{\mathcal{V}} \to g^* p_{\mathcal{V},\sharp}$ and $\mathrm{Ex}^*_\sharp : q_{\mathcal{W}',\sharp} f^*_{\mathcal{W}} \to g^* p_{\mathcal{W},\sharp}$.

To show that the exchange transformation $\mathrm{Ex}^*_\sharp : q_\sharp f^* \to g^* p_\sharp$ is invertible, combine the above isomorphisms with the descent isomorphisms,

$$p_\sharp \simeq p_{\mathcal{U},\sharp} u^* \underset{p_{\mathcal{W},\sharp} w^*}{\sqcup} p_{\mathcal{V},\sharp} v^*$$
$$q_\sharp \simeq q_{\mathcal{U}',\sharp} u'^* \underset{q_{\mathcal{W}',\sharp} w'^*}{\sqcup} q_{\mathcal{V}',\sharp} v'^*,$$

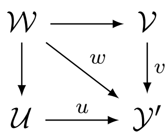
cf. Proposition 5.10. □

We now return to the proof of smooth base change.

Case 0. If $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}'$ and $\mathcal{Y}'$ are basic, then the claim is Remark 4.14.

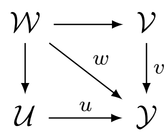
Case 1. Assume $\mathcal{Y}$ and $\mathcal{Y}'$ are basic and $\mathcal{X}$ and $\mathcal{X}'$ are scalloped. Since $\mathcal{X}$ is representable over $\mathcal{Y}$, we have $\mathcal{X} = [X/G]$ where $\mathcal{Y} = [Y/G]$, G is an embeddable nice group scheme over an affine scheme S, Y is a quasi-affine derived G-scheme, and X is a quasi-compact derived algebraic space over Y with G-action. Then by Theorem 2.14(ii) and Lemma 4.19, we may reduce to the case where $\mathcal{X} = [U/G]$ where U is quasi-affine with G-action. In this case $p : \mathcal{X} \to \mathcal{Y}$ is quasi-affine since $U \to X$ is quasi-affine and $\mathcal{Y}$ has the resolution property (Proposition 2.21).

Case 2. Assume $\mathcal{Y}$ is basic and $\mathcal{X}$, $\mathcal{X}'$, $\mathcal{Y}'$ are scalloped. By Theorem 2.12(ii) and Case 1 it will suffice to show that if we have a Nisnevich square



such that the claim holds after replacing $g : \mathcal{Y}' \to \mathcal{Y}$ by any of the composites $g \circ u$, $g \circ v$, or $g \circ w$, then it also holds for g. This follows immediately from Nisnevich descent and [GR, Pt. I, Chap. 1, Lem. 2.6.4].

Case 3. Let $\mathcal{X}$, $\mathcal{X}'$, $\mathcal{Y}$, and $\mathcal{Y}'$ be scalloped. By Theorem 2.12(ii) and Case 2 it will suffice to show that if we have a Nisnevich square



such that the claim holds after base changing $p : \mathcal{X} \to \mathcal{Y}$ (and hence the whole square) along any of u, v, or w, then it also holds for p itself. This follows immediately from Nisnevich descent and [GR, Pt. I, Chap. 1, Lem. 2.6.4].

4.6.3. *Smooth projection formula (c).* By adjunction, $p_\sharp$ inherits from p^* a canonical structure of colax morphism of $\mathbf{SH}(\mathcal{Y})$-modules. In particular, there are canonical morphisms

$$p_\sharp(\mathcal{F} \otimes f^*(\mathcal{G})) \to p_\sharp(\mathcal{F}) \otimes \mathcal{G}$$

for every object $\mathcal{F} \in \mathbf{SH}(\mathcal{X})$ and $\mathcal{G} \in \mathbf{SH}(\mathcal{Y})$, which we claim are invertible. This claim is Nisnevich-local on $\mathcal{Y}$, in view of smooth base change (b).

Arguing as in Lemma 4.19, we also see that it is local on $\mathcal{X}$. Hence we can reduce to the case where $\mathcal{X}$ and $\mathcal{Y}$ are basic, proven in Remark 4.14.

4.7. **Proof of Theorem 4.9(ii).** As in (i), we can argue Nisnevich-locally on the target. When $\mathcal{Y}$ is basic, then so is $\mathcal{X}$, and we are reduced to the situation of Remark 4.14.

4.8. **Proof of Theorem 4.9(iii).** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of scalloped derived stacks. The claim is that for every filtered diagram $(\mathcal{F}_\alpha)_\alpha$ of objects of $\mathbf{SH}(\mathcal{X})$, the canonical morphism in $\mathbf{SH}(\mathcal{Y})$

$$\varinjlim_\alpha f_*(\mathcal{F}_\alpha) \to f_*\big(\varinjlim_\alpha \mathcal{F}_\alpha\big)$$

is invertible. If $v : \mathcal{Y}_0 \twoheadrightarrow \mathcal{Y}$ is a representable Nisnevich cover by a fundamental derived stack $\mathcal{V}$, then since v^* commutes with colimits and with f_* (Theorem 4.9(i)), we may reduce to the case where $\mathcal{Y} = \mathcal{Y}_0$ is fundamental.

The claim is also local on $\mathcal{X}$ in the sense that if

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \downarrow & \overset{w}{\searrow} & \downarrow v \\ \mathcal{U} & \xrightarrow{u} & \mathcal{X} \end{array}$$

is a Nisnevich square such that the claim holds for $f_U = f \circ u$, $f_V = f \circ v$ and $f_W = f \circ w$ in place of f, then it also holds for f. Indeed by Nisnevich descent (see Proposition 5.10), there is a canonical isomorphism

$$f_* \simeq f_{U,*}u^* \underset{f_{W,*}w^*}{\times} f_{V,*}v^*,$$

and filtered colimits commute with finite limits in $\mathbf{SH}(\mathcal{Y})$.

Thus by induction (on the length of an appropriate scallop decomposition of $\mathcal{X}$), we eventually reduce to the case where $\mathcal{X}$ and $\mathcal{Y}$ are both basic, which was proven in Remark 4.14.

5. Axiomatization

We now make a brief interlude to give an axiomatic description of the results proven so far.

5.1. $(*, \sharp, \otimes)$-formalisms.

Notation 5.1.

(i) Given a presheaf of ∞-categories $\mathbf{D}^*$ on the ∞-category of scalloped derived stacks, we will write

$$\mathbf{D}(\mathcal{X}) \coloneqq \mathbf{D}^*(\mathcal{X})$$

for every scalloped derived stack $\mathcal{X}$. For every morphism $f : \mathcal{X} \to \mathcal{Y}$, we denote by

$$f^* \coloneqq \mathbf{D}^*(f) : \mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X})$$

the functor of *inverse image* along f.

(ii) If $\mathbf{D}^*$ takes values in *presentable* ∞-categories and colimit-preserving functors, then we say simply that $\mathbf{D}^*$ is a *presheaf of presentable ∞-categories*. In this case, every inverse image functor f^* admits a right adjoint f_* called *direct image* along f.

(iii) If $\mathbf{D}^*$ moreover factors through the ∞-category of *symmetric monoidal* presentable ∞-categories, then we say that $\mathbf{D}^*$ is a *presheaf of symmetric monoidal presentable ∞-categories*. We write

$$\otimes : \mathbf{D}(\mathcal{X}) \otimes \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X})$$

for the monoidal product and $\mathbf{1}_{\mathcal{X}} \in \mathbf{D}(\mathcal{X})$ for the monoidal unit over any $\mathcal{X}$. Since $\otimes$ commutes with colimits in each argument (recall our conventions), it admits as right adjoint an internal hom bifunctor

$$\underline{\mathrm{Hom}} : \mathbf{D}(\mathcal{X})^{\mathrm{op}} \times \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X}).$$

Definition 5.2. A $(*, \sharp, \otimes)$-*formalism* (on scalloped derived stacks) is a presheaf $\mathbf{D}^*$ of symmetric monoidal presentable ∞-categories on the site of scalloped derived stacks satisfying the following properties.

(i) For every smooth representable morphism $f : \mathcal{X} \to \mathcal{Y}$, the inverse image functor f^* admits a left adjoint

$$f_\sharp : \mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X}).$$

(Cf. Theorem 4.9(i)(a).)

(ii) The $\sharp$-direct image functors satisfy base change against arbitrary $*$-inverse image. (Cf. Theorem 4.9(i)(b).)

(iii) The $\sharp$-direct image functors satisfy the projection formula. That is, $f_\sharp : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y})$ is a morphism of $\mathbf{D}(\mathcal{Y})$-modules, where $\mathbf{D}(\mathcal{X})$ is regarded as a $\mathbf{D}(\mathcal{Y})$-module via the symmetric monoidal functor f^*. (Cf. Theorem 4.9(i)(c).)

(iv) *Additivity*. For any finite family $(\mathcal{X}_\alpha)_\alpha$ of scalloped derived stacks, the canonical functor

$$\mathbf{D}\big(\coprod_\alpha \mathcal{X}_\alpha\big) \to \prod_\alpha \mathbf{D}(\mathcal{X}_\alpha)$$

is an equivalence.

Remark 5.3. In practice, it is useful to consider the variant of Definition 5.2 where the site of scalloped derived stacks is replaced with the site of scalloped derived stacks over some base algebraic space S, or some nice full subcategory thereof.

Definition 5.4 (Thom twist)**.** Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism. Let $\mathcal{E}$ be a finite locally free sheaf on a scalloped derived stack $\mathcal{X}$. The *Thom twist* $\langle \mathcal{E} \rangle$ is the endofunctor of $\mathbf{D}(\mathcal{X})$ given by

$$\mathcal{F} \mapsto \mathcal{F}\langle \mathcal{E} \rangle := p_\sharp 0_*(\mathcal{F}),$$

where p is the total space of $\mathcal{E}$ and 0 is the zero section.

5.2. **The Voevodsky conditions.** The following key conditions were singled out by Voevodsky in the case of schemes (see [Vo2, §2, 1.2.1]).

Definition 5.5. Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism. We say that $\mathbf{D}^*$ *satisfies the Voevodsky conditions* if the following all hold:

(i) *Homotopy invariance.* For every vector bundle $p : \mathcal{V} \to \mathcal{X}$, the unit map
$$\mathrm{id} \to p_* p^*$$
is invertible.

(ii) *Localization.* For every complementary closed-open pair
$$\mathcal{Z} \xrightarrow{i} \mathcal{X} \xleftarrow{j} \mathcal{U}$$
of scalloped derived stacks, the functor i_* is fully faithful with essential image spanned by objects in the kernel of j^*.

(iii) *Thom stability.* For every finite locally free sheaf $\mathcal{E}$ on a scalloped derived stack $\mathcal{X}$, the endofunctor $\langle \mathcal{E} \rangle : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X})$ is an equivalence.

Remark 5.6. The base change formula for $\sharp$-direct image implies that for any complementary closed-open pair
$$\mathcal{Z} \xrightarrow{i} \mathcal{X} \xleftarrow{j} \mathcal{U},$$
we have the identities
$$j^* j_\sharp = \mathrm{id}, \quad j^* j_* = \mathrm{id}, \quad j^* i_* \simeq 0, \quad i^* j_\sharp \simeq 0. \tag{5.7}$$
The localization property can be reformulated as exactness of the triangle
$$j_\sharp j^* \xrightarrow{\text{counit}} \mathrm{id} \xrightarrow{\text{unit}} i_* i^*,$$
or by passing to right adjoints,
$$i_* i^! \simeq i_! i^! \xrightarrow{\text{counit}} \mathrm{id} \xrightarrow{\text{unit}} j_* j^*.$$

5.3. **Constructible separation.** Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism satisfying the Voevodsky conditions. The following is an immediate consequence of the localization property (see e.g. [Kh5, 2.11, 2.13]).

Proposition 5.8. *For any constructible covering*[15] $(j_\alpha : \mathcal{X}_\alpha \to \mathcal{X})_\alpha$ *of a scalloped derived stack* $\mathcal{X}$*, the inverse image functors*
$$j_\alpha^* : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X}_\alpha)$$
are jointly conservative.

Corollary 5.9 (Nil invariance)**.** *Let* $i : \mathcal{X}' \to \mathcal{X}$ *be a surjective closed immersion of scalloped derived stacks. Then the pair of adjoint functors*
$$i^* : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X}'), \quad i_* : \mathbf{D}(\mathcal{X}') \to \mathbf{D}(\mathcal{X})$$

[15] i.e., a family that generates a covering for the constructible topology

is an equivalence of ∞-categories. In particular, for every scalloped derived stack $\mathcal{X}$, there are canonical equivalences

$$\mathbf{D}(\mathcal{X}) \simeq \mathbf{D}(\mathcal{X}_{\mathrm{cl}}) \simeq \mathbf{D}(\mathcal{X}_{\mathrm{cl,red}})$$

where $\mathcal{X}_{\mathrm{cl}}$ is the classical truncation and $\mathcal{X}_{\mathrm{cl,red}}$ is its reduction.

5.4. **Nisnevich descent.** Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism satisfying the Voevodsky conditions. We have the following (see e.g. [Kh5, 4.26, 4.52]):

Proposition 5.10 (Étale excision)**.** *Let $f : \mathcal{X}' \to \mathcal{X}$ be a representable étale morphism which induces an isomorphism away from a quasi-compact open immersion $j : \mathcal{U} \to \mathcal{X}$. Then the commutative square*

$$\begin{array}{ccc} \mathrm{id} & \longrightarrow & j_*j^* \\ \downarrow & & \downarrow \\ f_*f^* & \longrightarrow & g_*g^*, \end{array}$$

is homotopy cartesian, where $g : f^{-1}(\mathcal{U}) \to \mathcal{X}$.

Proposition 5.11 (Nisnevich descent)**.** *The presheaf of ∞-categories $\mathbf{D}^*$ satisfies Nisnevich descent on the ∞-category of scalloped derived stacks. In particular, for any scalloped derived stack $\mathcal{X}$ and Nisnevich covering family $(f_\alpha : \mathcal{X}_\alpha \to \mathcal{X})_\alpha$, the inverse image functors*

$$f_\alpha^* : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X}_\alpha)$$

are jointly conservative.

5.5. **The example of $\mathbf{SH}^*$.**

Example 5.12. Consider the presheaf $\mathbf{SH}^*$, given by $\mathcal{X} \mapsto \mathbf{SH}(\mathcal{X})$, $f \mapsto f^*$ (see Theorem 4.4). This is a $(*, \sharp, \otimes)$-formalism satisfying the Voevodsky conditions:

(i) The existence of $\sharp$-direct images for smooth representable morphisms, satisfying the base change and projection formula, was proven in Theorem 4.9(i).

(ii) The additivity property follows from Nisnevich descent (Theorem 4.4(iii)).

(iii) For homotopy invariance, use Nisnevich descent to reduce to the case of $\mathcal{X}$ fundamental, in which case it holds by construction.

(iv) For Thom stability, observe that by localization (Theorem 4.9(ii)(d)) and the smooth and closed projection formulas (Theorem 4.9(i)(c) and (ii)(c)), there are canonical isomorphisms of functors

$$\langle \mathcal{E} \rangle \simeq (-) \otimes \Sigma^\infty \mathrm{Th}_{\mathcal{X}}(\mathcal{E})$$

for every finite locally free $\mathcal{E}$ on $\mathcal{X}$. These are equivalences by Theorem 4.4(i).

This example is universal in the following sense:

Proposition 5.13. *Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on scalloped derived stacks satisfying the Voevodsky conditions. Then there exists a unique system of colimit-preserving functors*

$$R_{\mathcal{X}} : \mathbf{SH}(\mathcal{X}) \to \mathbf{D}(\mathcal{X})$$

for every scalloped derived stack $\mathcal{X}$, which commute with $\sharp$-direct images (along smooth representable morphisms), inverse images (along arbitrary morphisms), tensor products, and arbitrary Thom twists.

Proof. By Nisnevich descent (Proposition 5.11), both $\mathbf{SH}^*$ and $\mathbf{D}^*$ are right Kan extended from the subcategory of fundamental stacks. For fundamental stacks, this holds by Remark 4.7 and the construction of $\mathbf{H}^*$; compare [Kh5, Rem. 2.14]. □

Example 5.14 (Étale motivic spectra)**.** The étale-local stable motivic homotopy category $\mathbf{SH}^*_{\text{ét}}$ defines a $(*, \sharp, \otimes)$-formalism satisfying the Voevodsky conditions. In fact, this formalism extends to the site of all derived algebraic stacks (not necessarily scalloped). More generally, any constructible ∞-category (as in [Kh5]) satisfying étale descent on the site of schemes or algebraic spaces admits a canonical extension to the site of algebraic stacks. Moreover, $\sharp$-direct image also exists for non-representable smooth morphisms. See [Kh3, App. A] and [LZ2], and also Example 12.2.

Corollary 5.15. *Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on scalloped derived stacks satisfying the Voevodsky conditions. Then for every scalloped derived stack $\mathcal{X}$, the assignment $\mathcal{E} \mapsto \langle \mathcal{E} \rangle$ extends from finite locally frees to a map of anima*

$$\mathrm{K}(\mathcal{X}) \to \mathrm{Aut}_{\mathbf{D}(\mathcal{X})}(\mathbf{D}(\mathcal{X})) \simeq \mathrm{Pic}(\mathbf{D}(\mathcal{X})),$$

to the animum of $\mathbf{D}(\mathcal{X})$-linear autoequivalences of $\mathbf{D}(\mathcal{X})$, i.e., the Picard animum of $\otimes$-invertible objects in $\mathbf{D}(\mathcal{X})$.

Proof. In fact, we have the canonical map of presheaves

$$\mathrm{K} \xrightarrow{\mathrm{Th}} \mathrm{Pic}(\mathbf{SH}^*) \xrightarrow{\mathrm{Pic}(R)} \mathrm{Pic}(\mathbf{D}^*)$$

where the first map is as in Remark 4.5 and the second is induced by the (symmetric monoidal) realization $R : \mathbf{SH}^* \to \mathbf{D}^*$ of Proposition 5.13. □

Notation 5.16. In the situation of Corollary 5.15, every K-theory class $\alpha \in \mathrm{K}(\mathcal{X})$ induces a canonical auto-equivalence of $\mathbf{D}(\mathcal{X})$ which we denote by

$$\langle \alpha \rangle : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X})$$

and continue to call the *Thom twist* by α.

6. Proper base change

6.1. **Statement.** For the duration of this section, we fix a $(*, \sharp, \otimes)$-formalism $\mathbf{D}^*$ satisfying the Voevodsky conditions on the site of scalloped derived stacks. The following theorem summarizes our results about direct image along proper representable morphisms:

Theorem 6.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper representable morphism of scalloped derived stacks. Then we have:*

(i) Proper base change. *For any commutative square of scalloped derived stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{Y}' \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

which is cartesian on classical truncations, there is a canonical isomorphism

$$\mathrm{Ex}^*_* : v^* f_* \to g_* u^*$$

of functors $\mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y}')$.

(ii) Smooth-proper base change. *For any cartesian square*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{Y}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array}$$

where p and q smooth, there is a canonical isomorphism

$$\mathrm{Ex}_{\sharp,*} : q_\sharp g_* \to f_* p_\sharp$$

of functors $\mathbf{D}(\mathcal{X}') \to \mathbf{D}(\mathcal{Y})$.

(iii) Atiyah duality. *If f is moreover* smooth, *then the canonical morphism of functors $\mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y})$ (see Construction 6.7)*

$$\varepsilon_f : f_\sharp\langle\mathcal{L}_f\rangle \to f_*$$

is invertible. In particular, $f_*\langle-\mathcal{L}_f\rangle$ is left adjoint to f^*.

(iv) Proper excision. *If $f : \mathcal{X} \to \mathcal{Y}$ is an isomorphism away from a closed substack $\mathcal{Z} \subseteq \mathcal{Y}$, then the commutative square*

$$\begin{array}{ccc} \mathrm{id} & \longrightarrow & i_* i^* \\ \downarrow & & \downarrow \\ f_* f^* & \longrightarrow & g_* g^* \end{array}$$

is homotopy cartesian in $\mathbf{D}(\mathcal{Y})$, where $i : \mathcal{Z} \to \mathcal{Y}$ is the inclusion and $g : f^{-1}(\mathcal{Z}) \subseteq \mathcal{X} \to \mathcal{Y}$ is the induced morphism.

6.2. **Cdh descent.** Before proceeding, let us mention a reformulation (Remark 6.4) of proper excision as a descent statement for the $*$-direct image functor.

Definition 6.2.

(i) The *proper cdh topology* on the site of scalloped derived stacks is the Grothendieck topology associated to the pretopology generated by the following covering families: (a) the empty family, covering the empty stack $\varnothing$; (b) for every scalloped derived stack $\mathcal{X}$ and every representable proper morphism $f : \mathcal{X}' \to \mathcal{X}$ inducing an isomorphism away from a closed immersion $i : \mathcal{Z} \hookrightarrow \mathcal{X}$, the family $\{i, f\}$ covering $\mathcal{X}$.

(ii) The *cdh topology* on the site of scalloped derived stacks is the union of the Nisnevich (Definition 2.4) and proper cdh topologies.

Remark 6.3. In the definition of the proper cdh topology, it suffices to take only families $\{i, f\}$ where f is a *projective* morphism inducing an isomorphism away from i. This follows from [HK, Cor. 2.4].

Remark 6.4. Fix a scalloped derived stack $\mathcal{Y}$ and a coefficient $\mathcal{F} \in \mathbf{D}(\mathcal{Y})$, and let $\mathcal{C}$ be (a full subcategory of) the ∞-category of scalloped derived stacks over $\mathcal{Y}$. Proper excision, for all proper representable morphisms $f : \mathcal{X}' \to \mathcal{X}$ in $\mathcal{C}$, is equivalent to the assertion that the presheaf on $\mathcal{C}$ given by $(h : \mathcal{X} \to \mathcal{Y}) \mapsto h_* h^*(\mathcal{F})$ satisfies descent for the proper cdh topology. This follows from [Kh, Thm. 2.2.7].

Corollary 6.5. *Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism satisfying the Voevodsky conditions. Then the presheaf of ∞-categories $\mathbf{D}^*$ satisfies cdh descent.*

Proof. Nisnevich descent is Proposition 5.11. Proper cdh descent follows from proper base change (Theorem 6.1) and proper cdh descent for $f_* f^*$ (Remark 6.4); see [Kh5, Thm. 2.52] for details. □

6.3. **Relative purity.** We begin with a construction of the morphism which is asserted to be invertible in Theorem 6.1(iii). This will require a preliminary result:

Theorem 6.6 (Relative purity)**.** *Let $i : \mathcal{X} \to \mathcal{Y}$ be a closed immersion of scalloped derived stacks which are smooth and representable over a scalloped derived stack $\mathcal{S}$. If $\mathcal{N}_{\mathcal{X}/\mathcal{Y}}$ denotes the conormal sheaf of i, and $p : \mathcal{X} \to \mathcal{S}$ and $q : \mathcal{Y} \to \mathcal{S}$ denote the structural morphisms, then there is a canonical isomorphism*

$$q_\sharp i_* \simeq p_\sharp \langle \mathcal{N}_{\mathcal{X}/\mathcal{Y}} \rangle$$

of functors $\mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{S})$.

Proof. For any $i : \mathcal{X} \to \mathcal{Y}$ as in the statement, set $P_{\mathcal{S}}(\mathcal{X}, \mathcal{Y}) := q_\sharp i_*$. Then we have

$$P_{\mathcal{S}}(\mathcal{X}, N_{\mathcal{X}/\mathcal{Y}}) = p_\sharp \pi_\sharp s_* \simeq p_\sharp \langle \mathcal{N}_{\mathcal{X}/\mathcal{Y}} \rangle,$$

where $\pi : N_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is the projection of the normal bundle (i.e., the total space of $\mathcal{N}_{\mathcal{X}/\mathcal{Y}}$) and $s : \mathcal{X} \to N_{\mathcal{X}/\mathcal{Y}}$ is the zero section. The deformation to the normal bundle $D_{\mathcal{X}/\mathcal{Y}}$ (see [KhRy, Thm. 4.1.13]) is a scalloped derived stack (as $D_{\mathcal{X}/\mathcal{Y}} \to \mathcal{Y}$ is representable), smooth over $\mathcal{S}$ and equipped with canonical morphisms of pairs (i.e., homotopy cartesian squares)

$$(\mathcal{X}, \mathcal{Y}) \to (\mathcal{X} \times \mathbf{A}^1, D_{\mathcal{X}/\mathcal{Y}}) \leftarrow (\mathcal{X}, N_{\mathcal{X}/\mathcal{Y}})$$

given by the inclusions of the fibres over 0 and 1 of $D_{\mathcal{X}/\mathcal{Y}} \to \mathbf{A}^1$. It will suffice to show that the induced morphism (cf. [CD, 2.4.32])

$$P_{\mathcal{S}}(\mathcal{X},\mathcal{Y}) \to P_{\mathcal{S}}(\mathcal{X}\times\mathbf{A}^1, D_{\mathcal{X}/\mathcal{Y}}) \circ \mathrm{pr}^* \leftarrow P_{\mathcal{S}}(\mathcal{X}, N_{\mathcal{X}/\mathcal{Y}})$$

is invertible, where $\mathrm{pr} : \mathcal{X}\times\mathbf{A}^1 \to \mathcal{X}$ is the projection. By Nisnevich descent (Proposition 5.11) and derived invariance (Corollary 5.9) we may reduce to the case where $\mathcal{Y}$ (and hence $\mathcal{X}$) is fundamental. In that case the result is proven as in [Ho3, Prop. 5.7]. □

Construction 6.7. Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth proper representable morphism of scalloped derived stacks. Its diagonal is a closed immersion $\Delta_f : \mathcal{X} \to \mathcal{X}\times_{\mathcal{Y}}\mathcal{X}$ (since f is separated and representable) whose conormal sheaf is canonically identified with the cotangent complex $\mathcal{L}_f$. The exchange transformation $\mathrm{Ex}_{\sharp,*}$ associated to the square

$$\begin{CD} \mathcal{X}\times_{\mathcal{Y}}\mathcal{X} @>{\mathrm{pr}_1}>> \mathcal{X} \\ @V{\mathrm{pr}_2}VV @VV{f}V \\ \mathcal{X} @>{f}>> \mathcal{Y} \end{CD}$$

gives rise to a natural transformation

$$\varepsilon_f : f_\sharp = f_\sharp \mathrm{pr}_{2,*}\Delta_{f,*} \xrightarrow{\mathrm{Ex}_{\sharp,*}} f_*\mathrm{pr}_{1,\sharp}\Delta_{f,*} \simeq f_*\langle\mathcal{L}_f\rangle$$

where the isomorphism is relative purity (Theorem 6.6). By the smooth base change formula, formation of ε_f commutes with smooth representable inverse image.

6.4. **Reductions.** We discuss some of the relationships between the various assertions in Theorem 6.1.

Lemma 6.8. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth proper representable morphism.*

(i) *If $\mathbf{D}^*$ satisfies Atiyah duality (Theorem 6.1(iii)) for any base change of f, then it also satisfies proper base change and smooth-proper base change (Theorem 6.1(i)-(ii)) for f.*
(ii) *If $\mathbf{D}^*$ satisfies smooth-proper base change (Theorem 6.1(ii)) for f, then it also satisfies Atiyah duality for f (Theorem 6.1(iii)).*

Proof. Immediate consequence of the definition of ε, see [Kh5, Lem. 2.28, 2.29]. □

Lemma 6.9. *Let $\mathcal{S}$ be a scalloped derived stack. For any finite locally free sheaf $\mathcal{E}$ on $\mathcal{S}$, let $f : \mathbf{P}(\mathcal{E}) \to \mathcal{S}$ denote the associated projective bundle. Then $\mathbf{D}^*$ satisfies Atiyah duality for f.*

Proof. Set $\mathcal{X} := \mathbf{P}(\mathcal{E})$ to simplify the notation. By Nisnevich descent (Proposition 5.11) and the fact that formation of ε_f commutes with inverse image by representable étale morphisms, we may reduce to the case where $\mathcal{S}$ is fundamental. We may also assume $\mathcal{S}$ is classical by derived invariance (Corollary 5.9). In this case the Pontryagin–Thom collapse map

$\eta_f : \mathbf{1}_{\mathcal{S}} \to f_\sharp(\mathbf{1}_{\mathcal{X}}\langle -\mathcal{L}_f\rangle)$ in $\mathbf{SH}(\mathcal{S})$, constructed in [Ho3, §5.3], induces by Proposition 5.13 a morphism of the same form in $\mathbf{D}(\mathcal{S})$, which in turn induces by the smooth projection formula a natural transformation

$$\eta_f : \mathrm{id}_{\mathbf{D}(\mathcal{S})} \to f_\sharp f^*\langle -\mathcal{L}_f\rangle.$$

To show that $\varepsilon_f : f_* \to f_\sharp\langle \mathcal{L}_f\rangle$ is invertible it will suffice to show that its left transpose $\varepsilon'_f : f^* f_\sharp\langle -\mathcal{L}_f\rangle \to \mathrm{id}$ and η_f are the counit and unit of an adjunction $(f^*, f_\sharp\langle -\mathcal{L}_f\rangle)$. The verification of the triangle identities reduces to showing that the composite

$$f^* \xrightarrow{\eta_f} f^* f_\sharp\langle -\mathcal{L}_f\rangle f^* \xrightarrow{\varepsilon'_f} f^*$$

induces the identity when evaluated on the unit $\mathbf{1}_{\mathcal{S}}$, as in the beginning of the proof of [Ho3, Thm. 5.22]. Again by Proposition 5.13 it will suffice to show this in the case $\mathbf{D}^* = \mathbf{SH}^*$, which is done in [Ho3, Thm. 6.9]. ☐

Lemma 6.10. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper representable morphism of scalloped derived stacks. If $\mathbf{D}^*$ satisfies proper base change for f (Theorem 6.1(i)), then it also satisfies proper excision for f (Theorem 6.1(iv)).*

Proof. Let $i : \mathcal{Z} \hookrightarrow \mathcal{Y}$ be a closed immersion, with complementary open immersion $j : \mathcal{U} \to \mathcal{X}$, such that f is an isomorphism over $\mathcal{U}$. By Proposition 5.8 it will suffice to show the square in question is homotopy cartesian after applying either i^* or j^*. The i^* case follows easily from the localization property and proper base change (Theorem 6.1(i)). The j^* case follows immediately from the smooth base change formula. ☐

6.5. **Proof of Theorem 6.1, projective case.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a projective morphism between scalloped derived stacks.

Let us prove the proper base change formula (i) and smooth-proper base change formula (ii) for f. By definition, f factors through a closed immersion into a projective bundle over $\mathcal{Y}$. For the closed immersion the claims are Theorem 4.9(ii). For the projective bundle the claims follow from Lemmas 6.8 and 6.9. The claims for f then follow immediately.

Atiyah duality (iii) and proper excision (iv) then follow in view of Lemmas 6.8 and 6.10.

6.6. **Proof of Theorem 6.1, general case.** Combining Remarks 6.3 and 6.4 with the projective case of Theorem 6.1 (proven in Subsect. 6.5), we have both proper cdh descent and proper excision (iv) for all proper representable morphisms.

For proper base change (i) and smooth-proper base change (ii), we can use descent along a proper cdh cover of the following type to reduce to the case of a projective morphism.

Theorem 6.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a separated representable morphism of finite type between quasi-compact quasi-separated derived algebraic stacks. Assume that $\mathcal{X}$ is noetherian or that $\mathcal{Y}$ is scalloped. Then there exists* *a proper cdh cover* $g : \widetilde{\mathcal{X}} \to \mathcal{X}$ *such that the composite* $f \circ g : \widetilde{\mathcal{X}} \to \mathcal{Y}$ *is quasi-projective.*

Proof. The inclusion of the classical truncation is a proper cdh cover, so we may assume that $\mathcal{X}$ and $\mathcal{Y}$ are classical.

By the generalization of Chow's lemma proven in [Ry4], we can find a projective morphism $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ which is an isomorphism over a non-empty open $\mathcal{U} \subseteq \mathcal{X}$, such that $f \circ \pi : \widetilde{\mathcal{X}} \to \mathcal{X} \to \mathcal{Y}$ is quasi-projective. Then the family $\{\pi, i\}$, where $i : \mathcal{Z} \to \mathcal{X}$ is any closed immersion complementary to j, generates a proper cdh cover of $\mathcal{X}$. Thus if $\mathcal{X}$ is noetherian, then the claim follows by noetherian induction.

Otherwise, if $\mathcal{Y}$ is scalloped then by Theorem 2.12(ii) it is of global type in the sense of *op. cit.*, as is $\mathcal{X}$ by Corollary 2.13. Thus we may apply noetherian approximation in the form of [Ry3, Thm. D] to the morphism $\mathcal{X} \to \mathcal{Y}$. The conclusion is that f factors through an affine morphism $\mathcal{X} \to \mathcal{X}_0$ and a separated representable morphism $\mathcal{X}_0 \to \mathcal{Y}$ which is of finite *presentation*. Since proper cdh covers are stable under base change, as are quasi-projective morphisms, we may replace $\mathcal{X}$ by $\mathcal{X}_0$ and thereby assume that f is of finite presentation.

By another application of [Ry3, Thm. D] to the morphism $\mathcal{Y} \to \mathrm{Spec}(\mathbf{Z})$ we find an affine morphism $\mathcal{Y} \to \mathcal{Y}_0$ such that $\mathcal{Y}_0$ is of finite type over $\mathrm{Spec}(\mathbf{Z})$ (hence noetherian). Since f is of finite presentation, we can moreover choose this approximation such that f descends to a separated representable morphism of finite presentation $f_0 : \mathcal{X}_0 \to \mathcal{Y}_0$. By the noetherian case we have a proper cdh cover $\widetilde{\mathcal{X}_0} \to \mathcal{X}_0$ such that $\widetilde{\mathcal{X}_0} \to \mathcal{Y}_0$ is quasi-projective. Base changing from $\mathcal{X}_0$ to $\mathcal{X}$ now yields the desired proper cdh cover. □

Finally, Atiyah duality (iii) again follows from Lemma 6.8.

7. The !-operations

7.1. **Statement.** Let $\mathbf{D}^*$ be a $(*, \sharp, \otimes)$-formalism on the ∞-category of scalloped derived stacks. We assume $\mathbf{D}^*$ satisfies the Voevodsky conditions.

Theorem 7.1. *For any representable morphism of finite type* $f : \mathcal{X} \to \mathcal{Y}$ *in* $\mathcal{C}$*, there exists a pair of adjoint functors*

$$f_! : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y}), \quad f^! : \mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X}),$$

and a natural transformation $\alpha_f : f_! \to f_*$*, satisfying the following conditions:*

(i) *There are canonical isomorphisms* $f_! \simeq f_\sharp$ *and* $f^! \simeq f^*$ *if* f *is an open immersion.*

(ii) *The natural transformation* $\alpha_f : f_! \to f_*$ *is invertible if* f *is proper.*

(iii) *The functor $f_!$ satisfies the base change formula. That is, for any commutative square of scalloped derived stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{Y}' \\ \downarrow u & & \downarrow v \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

which is cartesian on classical truncations, the canonical morphisms of functors $\mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y}')$

$$\mathrm{Ex}^*_! : v^* f_! \to g_! u^*,$$
$$\mathrm{Ex}^!_* : u_* g^! \to f^! v_*$$

are invertible.

(iv) *The functor $f_!$ satisfies the projection formula. That is, $f_! : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{Y})$ is a morphism of $\mathbf{D}(\mathcal{Y})$-module ∞-categories, where $\mathbf{D}(\mathcal{X})$ is regarded as a $\mathbf{D}(\mathcal{Y})$-module via the symmetric monoidal functor $f^* : \mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X})$. In particular, the canonical morphisms*

$$\mathcal{F} \otimes f_!(\mathcal{G}) \to f_!(f^*(\mathcal{F}) \otimes \mathcal{G}),$$
$$\underline{\mathrm{Hom}}(f^*(\mathcal{F}), f^!(\mathcal{F}')) \to f^!(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}')),$$
$$f_*(\underline{\mathrm{Hom}}(\mathcal{F}, f^!(\mathcal{G}))) \to \underline{\mathrm{Hom}}(f_!(\mathcal{F}), \mathcal{G})$$

are invertible for all $\mathcal{F}, \mathcal{F}' \in \mathbf{D}(\mathcal{X})$ and $\mathcal{G} \in \mathbf{D}(\mathcal{Y})$.

Moreover, the assignment $f \mapsto f_!$ (resp. $f \mapsto f^!$) extends to a functor $\mathbf{D}_!$ (resp. to a contravariant functor $\mathbf{D}^!$) from the ∞-category of scalloped derived stacks to the ∞-category of presentable ∞-categories and left-adjoint functors (resp. right-adjoint functors).

Remark 7.2. Thom twists (Notation 5.16) commute with each of the six operations. That is, for every morphism $f : \mathcal{X} \to \mathcal{Y}$ and $\alpha \in \mathrm{K}(\mathcal{Y})$, we have canonical isomorphisms of functors

$$f^* \circ \langle \alpha \rangle \simeq \langle f^*\alpha \rangle \circ f^*, \quad f_* \circ \langle f^*\alpha \rangle \simeq \langle \alpha \rangle \circ f_*,$$
$$f_! \circ \langle f^*\alpha \rangle \simeq \langle \alpha \rangle \circ f_!, \quad f^! \circ \langle \alpha \rangle \simeq \langle f^*\alpha \rangle \circ f^!. \tag{7.3}$$

Indeed, it suffices by Proposition 5.11 to check this on fundamental stacks, in which case we can assume that $\mathcal{E}$ is locally free (since fundamental stacks have the resolution property, see Proposition 2.21). In that case the claim is an easy consequence of various base change formulas. In view of these formulas, we will often abuse notation by writing e.g. $f_* \circ \langle \alpha \rangle$ instead of $\langle \alpha \rangle \circ f_*$ when α lives on the target.

7.2. **Compactifications.** The construction of the !-operations will be done, at least locally, by compactifying.

Definition 7.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of scalloped derived stacks. We say that f is *compactifiable* if there exists a factorization

$$f : \mathcal{X} \xrightarrow{j} \overline{\mathcal{X}} \xrightarrow{g} \mathcal{Y}$$

where j is an open immersion and g is proper representable. Note that if f is compactifiable, then it is separated of finite type.

Remark 7.5. Compactifiability can be checked on classical truncations. Indeed, if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism and $\mathcal{X}_{\mathrm{cl}} \xrightarrow{j_0} \overline{\mathcal{X}}_{\mathrm{cl}} \xrightarrow{g_0} \mathcal{Y}_{\mathrm{cl}}$ is a compactification of f_{cl}, then

$$\mathcal{X} \to \overline{\mathcal{X}}_{\mathrm{cl}} \sqcup_{\mathcal{X}_{\mathrm{cl}}} \mathcal{X} \to \mathcal{Y}$$

is a compactification of f. See the proof of [GR, Pt. II, Chap. 5, 2.1.6].

Example 7.6. Any affine morphism of derived stacks is compactifiable.

Example 7.7. If $\mathcal{Y}$ is Deligne–Mumford, or at least has quasi-finite diagonal, then a representable morphism $f : \mathcal{X} \to \mathcal{Y}$ is compactifiable if and only if it is separated and of finite type (see [Ry2, Thm. B]).

Remark 7.8. For any representable morphism $f : \mathcal{X} \to \mathcal{Y}$, the ∞-category of compactifications of f is either empty or contractible. In the case of classical stacks this follows from [SGA4, Exp. XVII, Prop. 3.2.6(ii)]. The derived case follows by the argument of [GR, Pt. II, Chap. 5, 2.1.6].

Remark 7.9. Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of finite type between scalloped derived stacks. Then there exists a commutative square

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f_0} & \mathcal{V} \\ \downarrow u & & \downarrow v \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where u and v are representable Nisnevich covers and f_0 is affine of finite type. (Indeed, choose v such that $\mathcal{V}$ is fundamental by Theorem 2.12(iii) and then use Theorem 2.14(i) to choose a Nisnevich cover $\mathcal{U} \twoheadrightarrow \mathcal{X} \times^{\mathbf{R}}_{\mathcal{Y}} \mathcal{V}$ such that $f_0 : \mathcal{U} \to \mathcal{V}$ is affine.) In particular, every such f is "locally compactifiable" (on the source and target, in the representable Nisnevich topology). Furthermore, note that if $\mathcal{X}$ and $\mathcal{Y}$ have affine diagonal, then by Theorem 2.12(iii) and the last part of Theorem 2.14(i), we can take u and v to be affine.

7.3. **Proof of Theorem 7.1.** For compactifiable morphisms, the claims follow from Theorem 6.1, Remark 7.8, and the general machinery of [LZ] or [GR, Chap. 8, Thm. 6.1.5] (cf. [LZ, Thm. 9.4.8], [LZ2, Eqn. (3.8)], [GR, Chap. 5, Thm. 3.4.3], [Kh5, Thm. 2.34]). Then one extends to all representable morphisms of finite type by Remark 7.9 and [LZ2, Thm. 4.1.8].

7.4. **Constructible separation.** We can formulate an analogue of Proposition 5.8 using !-inverse image:

Proposition 7.10. *For any constructible covering family $(j_\alpha : \mathcal{X}_\alpha \to \mathcal{X})_\alpha$ of scalloped derived stacks, the inverse image functors*

$$j_\alpha^! : \mathbf{D}(\mathcal{X}) \to \mathbf{D}(\mathcal{X}_\alpha)$$

are jointly conservative.

Proof. We reduce to the case of a closed-open pair (i, j) and use the exact triangle
$$i_*i^! \xrightarrow{\text{counit}} \mathrm{id} \xrightarrow{\text{unit}} j_*j^*$$
from Remark 5.6 (recall that $j^* \simeq j^!$). □

7.5. Purity.

Theorem 7.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable smooth morphism of scalloped derived stacks. Assume that f is compactifiable or that $\mathcal{X}$ and $\mathcal{Y}$ have affine diagonal. Then there is a canonical isomorphism*
$$\mathrm{pur}_f : f^! \simeq f^*\langle \mathcal{L}_{\mathcal{X}/\mathcal{Y}} \rangle$$
of functors $\mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X})$. Equivalently, by adjunction, there is a canonical isomorphism $f_! \simeq f_\sharp\langle -\mathcal{L}_{\mathcal{X}/\mathcal{Y}} \rangle$.

The following is a corollary, but will in fact feature in our proof of Theorem 7.11.

Corollary 7.12. *Suppose given a commutative square of scalloped derived stacks*
$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{Y}' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$
which is cartesian on classical truncations, where f is representable of finite type and p and q are representable and smooth. Consider the natural transformation
$$\mathrm{Ex}^{*!} : p^*f^! \xrightarrow{\text{unit}} g^!g_!p^*f^! \simeq g^!q^*f_!f^! \xrightarrow{\text{counit}} g^!q^*,$$
where the isomorphism comes from the base change formula (Theorem 7.1(iii)). Assume either that f is compactifiable or that $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}'$ and $\mathcal{Y}'$ have affine diagonal. Then $\mathrm{Ex}^{*!}$ is invertible.

Before proving Corollary 7.12, we record a couple weaker variants of Theorem 7.11 which it implies.

Corollary 7.13. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable étale morphism of scalloped derived stacks. Assume that f is compactifiable or that $\mathcal{X}$ and $\mathcal{Y}$ have affine diagonal. Then there is a canonical isomorphism $f^! \simeq f^*$.*

Proof. Since the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an open immersion, we have $\Delta^! \simeq \Delta^*$ (Theorem 7.1(i)). Applying Corollary 7.12 to the homotopy cartesian square
$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{\mathrm{pr}_2} & \mathcal{X} \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle f} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array}$$

we get the invertible natural transformation

$$f^! = \Delta^* \mathrm{pr}_1^* f^! \xrightarrow{\mathrm{Ex}^{*!}} \Delta^* \mathrm{pr}_2^! f^* \simeq \Delta^! \mathrm{pr}_2^! f^* = f^*.$$

□

Corollary 7.14. *Let $f : \mathcal{X} \to \mathcal{Y}$ be an unramified morphism of derived stacks which are representable and smooth over a scalloped derived stack $\mathcal{S}$. Assume that f is compactifiable or that $\mathcal{X}$ and $\mathcal{Y}$ have affine diagonal. Then there is a canonical isomorphism*

$$f^! q^* \simeq p^* \langle \mathcal{L}_{\mathcal{X}/\mathcal{Y}} \rangle,$$

where $p : \mathcal{X} \to \mathcal{S}$ and $q : \mathcal{Y} \to \mathcal{S}$ are the structural morphisms and $\mathcal{L}_{\mathcal{X}/\mathcal{Y}}$ is the relative cotangent complex of f.

Proof. If f is a closed immersion, then this follows from Theorem 6.6 by transposition. In general, there exists by [Ry, Thm. (1.2)] a canonical global factorization of f through a closed immersion i and a representable étale morphism g:

$$\mathcal{X} \xrightarrow{i} \mathcal{X}' \xrightarrow{g} \mathcal{Y}.$$

Let $p' = q \circ g : \mathcal{X}' \to \mathcal{S}$ denote the structural morphism. Combining the closed immersion case and Corollary 7.13, we get a canonical isomorphism

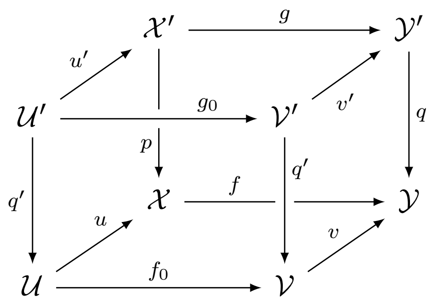
$$f^! q^* = i^! g^! q^* \simeq i^! g^* q^* = i^! (p')^* \simeq p^* \langle -\mathcal{N}_{X/X'} \rangle \simeq p^* \langle \mathcal{L}_{X/Y} \rangle,$$

where the identification $-\mathcal{N}_{\mathcal{X}/\mathcal{X}'} \simeq \mathcal{L}_{\mathcal{X}/\mathcal{Y}}$ in $\mathrm{K}(\mathcal{X})$ is induced by the isomorphism of perfect complexes $\mathcal{N}_{\mathcal{X}/\mathcal{X}'}[1] = \mathcal{L}_{\mathcal{X}/\mathcal{X}'} \simeq \mathcal{L}_{\mathcal{X}/\mathcal{Y}}$ induced by the étale morphism g. □

We now return to the proof of Corollary 7.12.

Proof of Corollary 7.12. If f is proper and representable, then $\mathrm{Ex}^{*!}$ is the right transpose of the smooth-proper base change isomorphism (hence is an isomorphism). If f is an open immersion, then invertibility of $\mathrm{Ex}^{*!}$ is clear from Theorem 7.1(i). This shows the claim when f is compactifiable. We also get Corollary 7.13 for compactifiable étale morphisms.

For the case of general f (but where the stacks have affine diagonal), choose an affine morphism of finite type $f_0 : \mathcal{U} \to \mathcal{V}$ and a commutative square as in Remark 7.9. Its base change along $q : \mathcal{Y}' \to \mathcal{Y}$ defines the cartesian cube:



By Proposition 5.11 it will suffice to show that the morphism

$$\mathrm{Ex}^{*!} : u'^* p^* f^! \to u'^* g^! q^*$$

is invertible. Since u and v are affine, and hence compactifiable (Example 7.6), there are canonical isomorphisms $u^* \simeq u^!$ and $v^* \simeq v^!$ by above. Under these identifications the above morphism is identified with

$$\mathrm{Ex}^{*!} : p'^* f_0^! v^* \to g_0^! q'^* v^*,$$

which is invertible by the compactifiable case applied to the front face (since f_0 is affine). ☐

Finally, now that Corollary 7.12 (and hence Corollary 7.14) is available to us, we are in position to prove Theorem 7.11.

Proof of Theorem 7.11. Applying Corollary 7.12 to the homotopy cartesian square

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{\mathrm{pr}_1} & \mathcal{X} \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle f} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

yields a canonical isomorphism $\mathrm{Ex}^{*!} : \mathrm{pr}_1^* f^! \simeq \mathrm{pr}_2^! f^*$. Since f is representable and smooth, the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is unramified with cotangent complex $\mathcal{L}_\Delta \simeq \mathcal{L}_f[1]$. Applying $\Delta^!$ and using the relative purity isomorphism $\Delta^! \mathrm{pr}_1^* \simeq \langle -\mathcal{L}_f \rangle$ (Corollary 7.14), we get the canonical isomorphism

$$f^! \langle -\mathcal{L}_f \rangle \simeq \Delta^! \mathrm{pr}_1^* f^! \xrightarrow{\mathrm{Ex}^{*!}} \Delta^! \mathrm{pr}_2^! f^* \simeq f^*.$$

The purity isomorphism $\mathrm{pur}_f : f^! \simeq f^* \langle \mathcal{L}_f \rangle$ is obtained by Thom twisting by $\mathcal{L}_f$. ☐

7.6. **Descent.**

Corollary 7.15. *Let $\mathcal{C}$ be the ∞-category of scalloped derived stacks and representable morphisms of finite type. Then the presheaf of ∞-categories $\mathbf{D}^!$ on $\mathcal{C}$ satisfies cdh descent. Similarly, $\mathbf{D}_!$ satisfies cdh co-descent on $\mathcal{C}$ when regarded as a co-presheaf with values in the ∞-category of presentable ∞-categories and left-adjoint functors.*

Proof. Follows from Theorem 7.1, see [Ho3, Prop. 6.24] or [Kh5, Thm. 2.52]. ☐

8. The Euler and Gysin transformations

We fix a $(*, \sharp, \otimes)$-formalism $\mathbf{D}^*$ satisfying the Voevodsky conditions, so that $\mathbf{D}^*$ extends to a formalism of six operations by Theorem 7.1.

8.1. Euler transformation.

Construction 8.1. Let $\mathcal{X}$ be a scalloped derived stack. Given a finite locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, let p be the projection of its total space and 0 the zero section. The *Euler transformation* associated to $\mathcal{E}$, denoted

$$\mathrm{eul}_{\mathcal{E}} : \mathrm{id} \to \langle \mathcal{E} \rangle,$$

is the composite

$$\mathrm{id} \simeq p_! p^! \xrightarrow{\text{unit}} p_! 0_! 0^* p^! \simeq \langle \mathcal{E} \rangle$$

where the first isomorphism is homotopy invariance and the second is purity (Theorem 7.11).

Lemma 8.2. *Let the notation be as in Construction 8.1. Suppose $\mathcal{E}$ admits a surjective cosection $s : \mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathcal{X}}$. Then s induces a null-homotopy $\mathrm{eul}_{\mathcal{E}} \simeq 0$.*

Proof. Write $\mathcal{V} = \mathbf{V}_{\mathcal{X}}(\mathcal{E})$ for the total space. Note that s corresponds to a nowhere zero section of $s : \mathcal{X} \to \mathcal{V}$, i.e., it factors through the complement $\mathcal{V} \smallsetminus 0$ of the zero section. Let $q : \mathcal{V} \smallsetminus 0 \to \mathcal{X}$ denote the projection. The localization triangle

$$q_! q^! \xrightarrow{\text{counit}} p_! p^! \xrightarrow{\text{unit}} p_! 0_! 0^* p^!$$

is isomorphic to

$$q_! q^! \xrightarrow{\text{counit}} \mathrm{id} \xrightarrow{\mathrm{eul}_{\mathcal{E}}} \langle \mathcal{E} \rangle.$$

Since $s : \mathcal{X} \to \mathcal{V} \smallsetminus 0$ is a section of q, the counit $s_! s^! \to \mathrm{id}$ induces a natural transformation $\mathrm{id} \to q_! q^!$ splitting this triangle, and hence a null-homotopy of $\mathrm{eul}_{\mathcal{E}}$. □

8.2. Gysin transformation.

The Gysin transformation of [DJK, 4.3.1][16] extends immediately to our setting. We will need the following technical hypothesis on our morphisms.

Definition 8.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of scalloped derived stacks. We say that f is *representably smoothable* if it admits a global factorization

$$\mathcal{X} \xrightarrow{i} \mathcal{A} \xrightarrow{p} \mathcal{Y}$$

where p is smooth representable and i is unramified representable.

Recall that a representable morphism of derived stacks is *quasi-smooth* if it is locally of finite presentation and the relative cotangent complex is of Tor-amplitude $[0,1]$ (with homological grading), see e.g. [KhRy].

Theorem 8.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth, representably smoothable morphism of scalloped derived stacks with affine diagonal. Then there exists a natural transformation*

$$\mathrm{gys}_{\mathcal{X}/\mathcal{Y}} := \mathrm{gys}_f : f^* \langle \mathcal{L}_f \rangle \to f^!$$

of functors $\mathbf{D}(\mathcal{Y}) \to \mathbf{D}(\mathcal{X})$, satisfying the following properties.

[16] called the *purity* transformation in *op. cit.*

(i) *If f is smooth, then* gys_f *is the purity isomorphism of Theorem 7.11.*

(ii) *If f is a closed immersion and $\mathcal{X}$ and $\mathcal{Y}$ are smooth representable over a base $\mathcal{S}$, then $\mathrm{gys}_f * q^*$ (where $*$ denotes horizontal composition) is canonically identified with the relative purity isomorphism (Corollary 7.14)*

$$p^*\langle \mathcal{L}_{\mathcal{X}/\mathcal{Y}}\rangle \simeq f^! q^*,$$

where $p : \mathcal{X} \to \mathcal{S}$ and $q : \mathcal{Y} \to \mathcal{S}$ are the structural morphisms.

Remark 8.5. By adjunction, the Gysin transformation can be rewritten as a trace or cotrace:

$$\mathrm{tr}_f : \mathrm{id} \to f_* f^! \langle -\mathcal{L}_f\rangle$$
$$\mathrm{cotr}_f : f_! f^* \langle \mathcal{L}_f\rangle \to \mathrm{id},$$

cf. [SGA4, Exp. XVIII, §3.2].

Remark 8.6. The Gysin transformation is functorial in f, up to homotopy, and also enjoys a base change property for homotopy cartesian squares. See [DJK, Prop. 2.5.4] for the precise formulation.

8.3. **Proof of Theorem 8.4.** We only briefly sketch the construction, as is it is the same as in the proofs of [DJK, Thms. 4.1.4, 4.3.1].

Construction 8.7. Assume first that f is unramified. Consider the deformation to the normal bundle [KhRy, Thm. 4.1.13], which fits in the following diagram of homotopy cartesian squares:

$$\begin{array}{ccccc}
\mathcal{X} & \longrightarrow & \mathcal{X} \times \mathbf{A}^1 & \longleftarrow & \mathcal{X} \times \mathbf{G}_m \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \widehat{f}} & & \downarrow{\scriptstyle f\times \mathrm{id}} \\
N_{\mathcal{X}/\mathcal{Y}} & \xrightarrow{\widehat{\imath}} & D_{\mathcal{X}/\mathcal{Y}} & \xleftarrow{\widehat{\jmath}} & \mathcal{Y} \times \mathbf{G}_m \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle t} & & \| \\
\mathcal{Y} & \longrightarrow & \mathcal{Y} \times \mathbf{A}^1 & \longleftarrow & \mathcal{Y} \times \mathbf{G}_m
\end{array}$$

where the left-hand side is the fibre over 0 and the right-hand side is the complement. The morphism u is the projection $\pi : N_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ followed by $f : \mathcal{X} \to \mathcal{Y}$.

The boundary map for the localization triangle associated to the closed-open pair $(\widehat{\imath}, \widehat{\jmath})$ yields a natural transformation

$$\partial : q_* q^! [-1] \to u_* u^!,$$

where $q : \mathcal{Y} \times \mathbf{G}_m \to \mathcal{Y}$ is the projection. Using the unit section $1 : \mathcal{Y} \to \mathcal{Y} \times \mathbf{G}_m$ to split q, we get a canonical isomorphism $q_* q^! \simeq \mathrm{id}[1] \oplus \mathrm{id}(1)[2]$ and thus, by including as the first component, a natural transformation

$$\mathrm{sp}_{\mathcal{X}/\mathcal{Y}} \coloneqq \mathrm{sp}_f : \mathrm{id} \to u_* u^!. \tag{8.8}$$

Now by homotopy invariance and purity (Theorem 7.11), we have a canonical isomorphism

$$\pi_*\pi^! \simeq \pi_*\pi^*\langle \mathcal{L}_{\mathcal{X}/\mathcal{Y}} \rangle \simeq \langle \mathcal{L}_{\mathcal{X}/\mathcal{Y}} \rangle$$

using the canonical identification $\mathcal{L}_{\mathcal{X}/\mathcal{Y}} \simeq -\mathcal{N}_{\mathcal{X}/\mathcal{Y}}$ in $\mathrm{K}(\mathcal{X})$. In particular, we get

$$u_*u^! \simeq f_*\pi_*\pi^!f^! \simeq f_*f^!\langle \mathcal{L}_{\mathcal{X}/\mathcal{Y}} \rangle.$$

The Gysin transformation for f, or rather the trace, is then

$$\mathrm{tr}_f : \mathrm{id} \xrightarrow{\mathrm{sp}_f} u_*u^! \simeq f_*f^!\langle \mathcal{L}_{\mathcal{X}/\mathcal{Y}} \rangle. \tag{8.9}$$

The compatibility with relative purity (Corollary 7.14) is proven as in [DJK, Lem. 3.2.15].

Finally, for the general case, choose a factorization $f = p \circ i$ as in Definition 8.3 and define gys_f to be the composite

$$\mathrm{id} \xrightarrow{\mathrm{gys}_p} p_*p^!\langle \mathcal{L}_p \rangle \xrightarrow{\mathrm{gys}_i} i_*p_*p^!i^!\langle \mathcal{L}_p + \mathcal{L}_i \rangle \simeq f_*f^!\langle \mathcal{L}_f \rangle,$$

where gys_p is the purity isomorphism (Theorem 7.11). One checks this is independent of the choice up to homotopy just as in [DJK, Thm. 3.3.2].

8.4. **Self-intersection formula.** Let us also record the following formulation of the self-intersection formula, proven the same way as [DJK, Cor. 4.2.3], which for a closed immersion relates the Gysin transformation with the Euler transformation of its conormal sheaf.

Proposition 8.10. *Let $i : \mathcal{X} \to \mathcal{Y}$ be a quasi-smooth closed immersion of scalloped derived stacks. Then there is a commutative diagram*

$$\begin{array}{ccccc} i^*\langle -\mathcal{N}_{\mathcal{X}/\mathcal{Y}} \rangle & & \xrightarrow{\quad i^*\mathrm{eul}_{\mathcal{N}_{\mathcal{X}/\mathcal{Y}}} \quad} & & i^* \\ \Vert & & & & \Vert \\ i^*\langle \mathcal{L}_{\mathcal{X}/\mathcal{Y}} \rangle & \xrightarrow{\mathrm{gys}_{\mathcal{X}/\mathcal{Y}}} & i^! & \xrightarrow{\mathrm{Ex}^{*!}} & i^*. \end{array}$$

Here $\mathrm{Ex}^{*!} : i^! \to i^*$ is the exchange transformation (Corollary 7.12) associated to the self-intersection square

$$\begin{array}{ccc} \mathcal{X} & = & \mathcal{X} \\ \Vert & & \downarrow{\scriptstyle i} \\ \mathcal{X} & \xrightarrow{i} & \mathcal{Y}. \end{array}$$

9. Cohomology and Borel–Moore homology theories

9.1. **Definitions.** We fix a $(*, \sharp, \otimes)$-formalism $\mathbf{D}^*$ on scalloped derived stacks satisfying the Voevodsky conditions. Recall that by Theorem 7.1, $\mathbf{D}^*$ extends to a formalism of six operations.

Definition 9.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of finite type between scalloped derived stacks. The *relative Borel–Moore homology spectrum* with coefficients in a sheaf $\mathcal{F} \in \mathbf{D}(\mathcal{Y})$, twisted by a class $\alpha \in \mathrm{K}(\mathcal{X})$, is the following mapping spectrum[17].

$$\mathbf{R}\Gamma_\alpha(\mathcal{X}/\mathcal{Y}, \mathcal{F}) \coloneqq \mathrm{Maps}_{\mathbf{D}(\mathcal{Y})}(\mathbf{1}_\mathcal{Y}, f_* f^!(\mathcal{F})\langle -\alpha \rangle)$$

(For $\alpha = 0$, we omit the index.) Its 0-th homotopy group will be denoted

$$\mathrm{H}_\alpha(\mathcal{X}/\mathcal{Y}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{D}(\mathcal{Y})}(\mathbf{1}_\mathcal{Y}\langle \alpha \rangle, f_* f^!(\mathcal{F})).$$

Example 9.2. Let X be a derived algebraic space of finite type over an affine scheme S, with an action of a nice group scheme G over S. The relative Borel–Moore homology spectrum with coefficients in $\mathcal{F} \in \mathbf{D}(BG)$,

$$\mathbf{R}\Gamma_*([X/G]/BG, \mathcal{F}),$$

can be regarded as a (genuine) G-equivariant Borel–Moore homology spectrum for X over S.

Example 9.3. For any scalloped derived algebraic stack $\mathcal{X}$, the *cohomology spectrum* with coefficients in $\mathcal{F} \in \mathbf{D}(\mathcal{X})$, twisted by $\alpha \in \mathrm{K}(\mathcal{X})$, is

$$\mathbf{R}\Gamma^\alpha(\mathcal{X}, \mathcal{F}) \coloneqq \mathrm{Maps}_{\mathbf{D}(\mathcal{X})}(\mathbf{1}_\mathcal{X}, \mathcal{F}\langle \alpha \rangle).$$

This is the relative Borel–Moore homology of the identity, with opposite grading. Its 0-th homotopy group is the cohomology group

$$\mathrm{H}^\alpha(\mathcal{X}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{D}(\mathcal{X})}(\mathbf{1}_\mathcal{X}, \mathcal{F}\langle \alpha \rangle).$$

9.2. **Operations.** Just as in [DJK, §2] and [Kh3, §2.2], we immediately get the following structure on Borel–Moore homology from the formalism of six operations. This structure is subject to the same type of compatibilities as in Fulton and MacPherson's formalism of bivariant theories [FM, Sect. 2.2], see also [Kh3, §2.3].

Notation 9.4. We fix a base $\mathcal{S}$, a scalloped derived stack, and a coefficient $\mathcal{F} \in \mathbf{D}(\mathcal{S})$. We denote by $\mathcal{C}_{/\mathcal{S}}$ for the ∞-category of scalloped derived stacks $\mathcal{X}$ over $\mathcal{S}$, and $\mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$ for the full subcategory spanned by $\mathcal{X} \in \mathcal{C}_{/\mathcal{S}}$ for which $\mathcal{X} \to \mathcal{S}$ is representable of finite type.

9.2.1. *Direct image.* If $f : \mathcal{X} \to \mathcal{Y}$ is a proper morphism in $\mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$, then there are direct image maps

$$f_* : \mathbf{R}\Gamma_\alpha(\mathcal{X}/\mathcal{S}, \mathcal{F}) \to \mathbf{R}\Gamma_\alpha(\mathcal{Y}/\mathcal{S}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{Y})$.

If $\mathcal{X}, \mathcal{Y} \in \mathcal{C}_{/\mathcal{S}}$ have affine diagonal and $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-smooth, proper, representably smoothable morphism, then there are also Gysin maps in cohomology

$$f_! : \mathbf{R}\Gamma^{\alpha + \mathcal{L}_{\mathcal{X}/\mathcal{Y}}}(\mathcal{X}, \mathcal{F}) \to \mathbf{R}\Gamma^\alpha(\mathcal{Y}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{Y})$.

[17] Recall our convention from Remark 7.2.

9.2.2. *Inverse image.* If $f : \mathcal{X} \to \mathcal{Y}$ is a representable morphism in $\mathcal{C}_{/\mathcal{S}}$, then there are inverse image maps

$$f^* : \mathbf{R}\Gamma^{\alpha}(\mathcal{Y}, \mathcal{F}) \to \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{Y})$.

If $\mathcal{X}, \mathcal{Y} \in \mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$ have affine diagonal and $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-smooth, representably smoothable morphism, then there are also Gysin maps in Borel–Moore homology

$$f^! : \mathbf{R}\Gamma_{\alpha}(\mathcal{Y}/\mathcal{S}, \mathcal{F}) \to \mathbf{R}\Gamma_{\alpha+\mathcal{L}_{\mathcal{X}/\mathcal{Y}}}(\mathcal{X}/\mathcal{S}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{Y})$.

9.2.3. *Change of base.* For any commutative square of scalloped derived stacks

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{T} \\ \downarrow & \Delta & \downarrow f \\ \mathcal{X} & \longrightarrow & \mathcal{S} \end{array}$$

which is cartesian on classical truncations, where $\mathcal{X} \to \mathcal{S}$ and $\mathcal{Y} \to \mathcal{T}$ are representable of finite type, there are maps

$$f^*_{\Delta} : \mathbf{R}\Gamma_{\alpha}(\mathcal{X}/\mathcal{S}, \mathcal{F}) \to \mathbf{R}\Gamma_{\alpha}(\mathcal{Y}/\mathcal{T}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{X})$.

9.2.4. *Euler class.* Assume that $\mathcal{F}$ is unital, i.e., it admits a unit map $\eta : \mathbf{1} \to \mathcal{F}$. For any finite locally free sheaf $\mathcal{E}$ on $\mathcal{X} \in \mathcal{C}_{/\mathcal{S}}$, there is an Euler class

$$e(\mathcal{E}) \in \mathbf{R}\Gamma^{\mathcal{E}}(\mathcal{X}, \mathcal{F}).$$

9.2.5. *Composition product.* Assume that $\mathcal{F}$ is multiplicative, i.e., it admits a multiplication map $\mu : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$. Given representable of finite type morphisms $\mathcal{X} \to \mathcal{T}$ and $\mathcal{T} \to \mathcal{S}$ between scalloped derived stacks, there is a pairing

$$\circ : \mathbf{R}\Gamma_{\alpha}(\mathcal{X}/\mathcal{T}, \mathcal{F}) \otimes \mathbf{R}\Gamma_{\beta}(\mathcal{T}/\mathcal{S}, \mathcal{F}) \to \mathbf{R}\Gamma_{\alpha+\beta}(\mathcal{X}/\mathcal{S}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{X})$, $\beta \in \mathrm{K}(\mathcal{T})$.

Special cases of the composition product are cap and cup products:

9.2.6. *Cap product.* Given $\mathcal{X} \in \mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$, there is a pairing

$$\cap : \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}) \otimes \mathbf{R}\Gamma_{\beta}(\mathcal{X}/\mathcal{S}, \mathcal{F}) \to \mathbf{R}\Gamma_{\beta-\alpha}(\mathcal{X}/\mathcal{S}, \mathcal{F}) \tag{9.5}$$

for every $\alpha, \beta \in \mathrm{K}(\mathcal{X})$.

9.2.7. *Cup product.* Given $\mathcal{X} \in \mathcal{C}_{/\mathcal{S}}$, there is a pairing

$$\cup : \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}) \otimes \mathbf{R}\Gamma^{\beta}(\mathcal{X}, \mathcal{F}) \to \mathbf{R}\Gamma^{\alpha+\beta}(\mathcal{X}, \mathcal{F}) \tag{9.6}$$

for every $\alpha, \beta \in \mathrm{K}(\mathcal{X})$.

Remark 9.7. For every $\mathcal{X} \in \mathcal{C}_{/\mathcal{S}}$ and $\mathcal{F} \in \mathbf{D}(\mathcal{S})$, the cohomology spectra of $\mathcal{X}$ can be assembled into a $\mathrm{K}_0(\mathcal{Y})$-graded spectrum

$$\mathbf{R}\Gamma^{*}(\mathcal{X}, \mathcal{F}) \coloneqq \bigoplus_{\alpha \in \mathrm{K}_0(\mathcal{Y})} \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}),$$

with graded ring structure coming from the cup product. For every morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$, the Borel–Moore homology spectra can be assembled into a $\mathrm{K}_0(\mathcal{Y})$-graded spectrum

$$\mathbf{R}\Gamma_{*}(\mathcal{X}/\mathcal{Y}, \mathcal{F}) \coloneqq \bigoplus_{\alpha \in \mathrm{K}_0(\mathcal{Y})} \mathbf{R}\Gamma_{\alpha}(\mathcal{X}/\mathcal{Y}, \mathcal{F}),$$

which becomes a graded module over $\mathbf{R}\Gamma^{*}(\mathcal{X}, \mathcal{F})$ via cap product. We can also collapse these into $\mathbf{Z}$-gradings, where the homogeneous components of degree $r \in \mathbf{Z}$ are

$$\bigoplus_{\mathrm{rk}(\alpha)=r} \mathbf{R}\Gamma^{\alpha}(\mathcal{X}, \mathcal{F}), \qquad \bigoplus_{\mathrm{rk}(\alpha)=r} \mathbf{R}\Gamma_{\alpha}(\mathcal{X}/\mathcal{Y}, \mathcal{F}),$$

respectively.

9.3. **Properties.** Let the notation be as in Notation 9.4. The following properties follow immediately from the results of Sect. 5, just as in [DJK].

Proposition 9.8 (Localization). *Given a complementary closed-open pair*

$$\mathcal{Z} \xhookrightarrow{i} \mathcal{X} \xhookleftarrow{j} \mathcal{U}$$

in $\mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$, there is an exact triangle

$$\mathbf{R}\Gamma_{\alpha}(\mathcal{Z}/\mathcal{S}, \mathcal{F}) \xrightarrow{i_*} \mathbf{R}\Gamma_{\alpha}(\mathcal{X}/\mathcal{S}, \mathcal{F}) \xrightarrow{j^!} \mathbf{R}\Gamma_{\alpha}(\mathcal{U}/\mathcal{S}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{X})$.

Proposition 9.9 (Derived invariance). *For every $\mathcal{X} \in \mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$, we have:*

(i) *Change of base along the inclusion of the classical truncation $\mathcal{S}_{\mathrm{cl}} \to \mathcal{S}$ induces isomorphisms*

$$\mathbf{R}\Gamma_{\alpha}(\mathcal{X}/\mathcal{S}, \mathcal{F}) \to \mathbf{R}\Gamma_{\alpha}(\mathcal{X} \overset{\mathbf{R}}{\underset{\mathcal{S}}{\times}} \mathcal{S}_{\mathrm{cl}}/\mathcal{S}_{\mathrm{cl}}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{X})$.

(ii) *Direct image along the inclusion of the classical truncation $i_{\mathcal{X}} : \mathcal{X}_{\mathrm{cl}} \to \mathcal{X}$ induces an isomorphisms*

$$i_{\mathcal{X},*} : \mathbf{R}\Gamma_{\alpha}(\mathcal{X}_{\mathrm{cl}}/\mathcal{S}, \mathcal{F}) \to \mathbf{R}\Gamma_{\alpha}(\mathcal{X}/\mathcal{S}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{X})$.

Moreover, both statements also hold with $\mathcal{X}_{\mathrm{cl}}$ replaced by the reduction $\mathcal{X}_{\mathrm{cl,red}}$.

Proposition 9.10 (Thom isomorphism)**.** *Let $\mathcal{X} \in \mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$ and $\mathcal{E}$ a finite locally free sheaf on $\mathcal{X}$ with total space $\pi : \mathcal{V} \to \mathcal{X}$. Then the Gysin map*

$$\pi^! : \mathbf{R}\Gamma_\alpha(\mathcal{X}/\mathcal{S}, \mathcal{F}) \to \mathbf{R}\Gamma_{\alpha+\mathcal{E}}(\mathcal{V}/\mathcal{S}, \mathcal{F}),$$

is an isomorphism for every $\alpha \in \mathrm{K}(\mathcal{X})$.

9.4. **Fundamental classes and Poincaré duality.** Let the notation be as in 9.4, and assume that $\mathcal{F}$ is unital.

Definition 9.11. Let $\mathcal{X} \in \mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$ with affine diagonal. If $\mathcal{X}$ is smooth over $\mathcal{S}$, or more generally quasi-smooth and representably smoothable, then there is a relative *fundamental class*

$$[\mathcal{X}/\mathcal{S}] \in \mathbf{R}\Gamma_{\mathcal{L}_{\mathcal{X}/\mathcal{S}}}(\mathcal{X}/\mathcal{S}, \mathcal{F})$$

defined as the image of the unit by the Gysin map

$$f^! : \mathbf{R}\Gamma(\mathcal{S}/\mathcal{S}, \mathcal{F}) \to \mathbf{R}\Gamma_{\mathcal{L}_{\mathcal{X}/\mathcal{S}}}(\mathcal{X}/\mathcal{S}, \mathcal{F}).$$

Since the Gysin transformation is invertible for smooth morphisms (see Theorems 8.4 and 7.11), we have:

Proposition 9.12 (Poincaré duality)**.** *Let $\mathcal{X} \in \mathcal{C}^{\mathrm{rep}}_{/\mathcal{S}}$ with affine diagonal. If $\mathcal{X}$ is smooth over $\mathcal{S}$, then cap product with the fundamental class $[\mathcal{X}/\mathcal{S}]$ induces isomorphisms*

$$\mathbf{R}\Gamma^\alpha(\mathcal{X}, \mathcal{F}) \xrightarrow{\cap[\mathcal{X}/\mathcal{S}]} \mathbf{R}\Gamma_{-\alpha+\mathcal{L}_{\mathcal{X}/\mathcal{S}}}(\mathcal{X}/\mathcal{S}, \mathcal{F})$$

for every $\alpha \in \mathrm{K}(\mathcal{X})$.

10. Examples

10.1. Homotopy invariant K-theory.

Notation 10.1. Given a scalloped derived stack $\mathcal{X}$, we let $\mathrm{K}^{\mathrm{B}}(\mathcal{X})$ denote the Bass–Thomason–Trobaugh K-theory spectrum of the stable ∞-category of perfect complexes on $\mathcal{X}$ (see [Kh4, Def. 2.5], [CK, Sect. 4]). By construction, its infinite loop animum $\Omega^\infty(\mathrm{K}^{\mathrm{B}}(\mathcal{X}))$ is the K-theory animum $\mathrm{K}(\mathcal{X})$.

We have the following extension of the celebrated result of Thomason–Trobaugh [TT]:

Theorem 10.2. *The assignment $\mathcal{X} \mapsto \mathrm{K}^{\mathrm{B}}(\mathcal{X})$ determines a Nisnevich sheaf of spectra on the site of scalloped derived stacks.*

Proof. This follows from Theorem 2.24 and [Kh4, Thm. 2.12 and Rem. 2.14]. □

Construction 10.3. Let $\mathcal{X}$ be a scalloped derived stack. Restricting the presheaf $\mathcal{X}' \mapsto \mathrm{K}^{\mathrm{B}}(\mathcal{X}')$ to the site $\mathrm{Sm}_{/\mathcal{X}}$ (notation as in Subsect. 3.1) and applying the (exact) $\mathbf{A}^1$-localization functor, we get a motivic Sm-fibred S^1-spectrum $\mathrm{KH}_{\mathcal{X}}$ (with $\mathcal{E}_\infty$-ring structure). The homotopy invariant K-theory spectrum $\mathrm{KH}(\mathcal{X})$ is given by its global sections:

$$\mathrm{KH}(\mathcal{X}) = \mathbf{R}\Gamma(\mathcal{X}, \mathrm{KH}_{\mathcal{X}}) \simeq \varinjlim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{K}(\mathcal{X} \times \mathbf{A}^n).$$

This definition agrees with [HK, §4C] and [KrRa, §5A] in case $\mathcal{X}$ is classical and with [Kh2, 5.4.1] in case $\mathcal{X}$ is a derived algebraic space. If $\mathcal{X}$ is regular, then $\mathrm{KH}(\mathcal{X}) \simeq \mathrm{K}^{\mathrm{B}}(\mathcal{X}) \simeq \mathrm{K}(\mathcal{X}) \simeq \mathrm{G}(\mathcal{X})$, where $\mathrm{K}(\mathcal{X})$ is the (connective) K-theory spectrum of perfect complexes on $\mathcal{X}$ and $\mathrm{G}(\mathcal{X})$ is the (connective) K-theory spectrum of coherent sheaves on $\mathcal{X}$ (see [Kh4, Thm. 3.5]).

Remark 10.4. The motivic S^1-spectrum $\mathrm{KH}_{\mathcal{X}}$ is stable under representable $*$-inverse image. Indeed, the proof over classical stacks in [Ho4, Prop. 4.6] generalizes in view of [BKRS, Prop. A.2.5].

Remark 10.5. Combining Remark 10.4 with Theorem 3.14, we deduce that for every scalloped derived stack $\mathcal{X}$, the canonical map

$$i^* : \mathrm{KH}(\mathcal{X}) \to \mathrm{KH}(\mathcal{X}_{\mathrm{cl}}),$$

where i is the inclusion of the classical truncation, is invertible. This gives a proof of Corollary F which is independent of our stable results such as proper base change (Theorem 6.1).

Remark 10.6. Using the cdh descent criterion of [Kh2, Thm. E, Rem. 5.11(c)], we can give a direct proof of Corollary G by following [Kh2, 5.3.4]. The new input in our setting is Remark 10.4 and the localization theorem for $\mathbf{H}^*$ (Theorem 3.19), which together imply closed descent (cf. [Kh2, Ex. 5.9]).

We have the following stable representability result:

Theorem 10.7. *For every scalloped derived stack $\mathcal{X}$, there exists a canonical motivic $\mathcal{E}_\infty$-ring spectrum $\mathrm{KGL}_{\mathcal{X}} \in \mathbf{SH}(\mathcal{X})$ satisfying the following properties:*

(i) *For every $\mathcal{X}' \in \mathrm{Sm}_{/\mathcal{X}}$, there are functorial isomorphisms of spectra*
$$\mathrm{KH}(\mathcal{X}') \simeq \mathbf{R}\Gamma(\mathcal{X}', \mathrm{KGL}_{\mathcal{X}}).$$

(ii) *For every finite locally free sheaf $\mathcal{E}$ on $\mathcal{X}$, there is a canonical Bott periodicity isomorphism*
$$\mathrm{KGL}_{\mathcal{X}}\langle\mathcal{E}\rangle \simeq \mathrm{KGL}_{\mathcal{X}}$$
in $\mathbf{SH}(\mathcal{X})$.

(iii) *For any representable morphism $f : \mathcal{X} \to \mathcal{Y}$, there is a canonical isomorphism*
$$f^*(\mathrm{KGL}_{\mathcal{Y}}) \simeq \mathrm{KGL}_{\mathcal{X}}$$
in $\mathbf{SH}(\mathcal{X})$.

Proof. We follow the proof of [Ho4, Thm. 1.7], which proves the result in the case of certain (classical) global quotient stacks, and "N-quasi-projective" morphisms between them.

If $\mathcal{X}$ is fundamental, it follows from the description of $\mathbf{SH}(\mathcal{X})$ in Remark 4.7, [Ho4, Prop. 3.2], and the projective bundle formula, that there is a unique Bott-periodic delooping $\mathrm{KGL}_{\mathcal{X}} \in \mathbf{SH}(\mathcal{X})$ of $\mathrm{KH}_{\mathcal{X}} \in \mathbf{H}(\mathcal{X})$. In particular, (i) and (ii) hold by construction and (iii) follows from Remark 10.4. See the discussion around Def. 5.1 in [Ho4].

For general $\mathcal{X}$, the claim now follows from Remark 10.4 and Nisnevich descent (cf. Lemma 4.16). More precisely, there exists a unique motivic $\mathcal{E}_\infty$-ring spectrum $\mathrm{KGL}_{\mathcal{X}} \in \mathbf{SH}(\mathcal{X})$ with the property that

$$u^*(\mathrm{KGL}_{\mathcal{X}}) \simeq \mathrm{KGL}_{\mathcal{U}}$$

for every fundamental derived stack $\mathcal{U}$ and every affine Nisnevich covering $u : \mathcal{U} \twoheadrightarrow \mathcal{X}$. See the comments on Thm. 1.7 in [Ho4, p. 24]. □

Remark 10.8. One can similarly construct a motivic spectrum $\mathrm{KQ}_{\mathcal{X}} \in \mathbf{SH}(\mathcal{X})$ representing hermitian K-theory, see [PW] or [HJNY, §6], at least assuming that 2 is invertible on $\mathcal{X}$ (although see [HJNY, Rem. 6.3]).

10.2. **Algebraic cobordism.** Following Voevodsky [Vo, §6.3], we can use our formalism to introduce a theory of algebraic cobordism for stacks.

The following definition generalizes the one in [BH, §16] in the case of schemes.

Construction 10.9. Given a scalloped derived stack $\mathcal{X}$, we define $\mathrm{MGL}_{\mathcal{X}} \in \mathbf{SH}(\mathcal{X})$ as the colimit

$$\mathrm{MGL}_{\mathcal{X}} = \varinjlim_{(\mathcal{U},\alpha)} f_\sharp(\mathbf{1}_{\mathcal{U}}\langle\alpha\rangle)$$

over the ∞-category[18] of pairs $(\mathcal{U},\alpha)$ with $f : \mathcal{U} \to \mathcal{X}$ smooth representable and $\alpha \in \mathrm{K}(\mathcal{U})$ a K-theory class of virtual rank 0. By construction, $\mathrm{MGL}_{\mathcal{X}}$ admits a canonical (homotopy coherent) *orientation* in the sense that there is a homotopy coherent system of Thom isomorphisms

$$\mathrm{MGL}_{\mathcal{X}}\langle\mathcal{E}\rangle \simeq \mathrm{MGL}_{\mathcal{X}}(r)[2r]$$

for every locally free sheaf $\mathcal{E}$ of rank r on $\mathcal{X}$. See [BH, Prop. 16.28, Ex. 16.30].

Proposition 10.10. *For every representable morphism $f : \mathcal{X} \to \mathcal{Y}$ of scalloped derived stacks, there is a canonical isomorphism*

$$f^*(\mathrm{MGL}_{\mathcal{Y}}) \simeq \mathrm{MGL}_{\mathcal{X}}$$

in $\mathbf{SH}(\mathcal{X})$.

Proof. Let $\mathrm{K}^\circ_{\mathcal{X}}$ denote the presheaf on $\mathrm{Sm}_{/\mathcal{X}}$ sending $\mathcal{X}'$ to the rank 0 component of the K-theory animum $\mathrm{K}(\mathcal{X}')$. Then the canonical morphism

$$f^*(\mathrm{K}^\circ_{\mathcal{Y}}) \to \mathrm{K}^\circ_{\mathcal{X}}$$

is a Nisnevich-local equivalence by [Ho4, Cor. 2.9] (cf. the discussion near the end of [Ho4, §5]). Then the claim follows by construction of MGL. □

[18] i.e., the "total space" of the cartesian fibration associated to the presheaf sending $\mathcal{U} \in \mathrm{Sm}_{/\mathcal{X}}$ to the virtual rank 0 part of $\mathrm{K}(\mathcal{U})$

Remark 10.11. It follows from [Ho4, Cor. 2.10] that, when $\mathcal{X}$ is a quotient stack, $\mathrm{MGL}_{\mathcal{X}}$ can be described in terms of tautological bundles over "infinite Grassmannians", similarly to Voevodsky's original definition in the case of schemes [Vo, §6.3]. Compare [BH, Thm. 16.13].

Remark 10.12. Following [BH, Ex. 16.22] one can similarly construct a motivic spectrum $\mathrm{MSL}_{\mathcal{X}} \in \mathbf{SH}(\mathcal{X})$ representing special linear algebraic cobordism.

10.3. **Motivic cohomology.** We construct a motivic cohomology spectrum for scalloped derived stacks following the framed description given in [Ho5, Thm. 21]. We begin with the following natural generalization of [EHKSY, Def. 2.3.4].

Definition 10.13. Let $\mathcal{X}$ be a scalloped derived stack and $\mathcal{X}', \mathcal{X}'' \in \mathrm{Sm}_{/\mathcal{X}}$. A *framed correspondence* from $\mathcal{X}'$ to $\mathcal{X}''$ is a diagram

$$\begin{array}{ccc} & \mathcal{Z} & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ \mathcal{X}' & & \mathcal{X}'' \end{array}$$

where f is a finite quasi-smooth morphism and g is representable, together with an isomorphism $\mathcal{L}_f \simeq 0$ in the ∞-groupoid $\mathrm{K}(\mathcal{Z})$. (Compare [EHKSY, Def. 2.3.4].)

Construction 10.14. Given a scalloped derived stack $\mathcal{X}$, we let $\mathrm{Sm}^{\mathrm{fr}}_{/\mathcal{X}}$ denote the ∞-category whose objects are those of $\mathrm{Sm}_{/\mathcal{X}}$ and morphisms are framed correspondences (defined as in [EHKSY, §4]). The ∞-categories $\mathbf{H}_{\mathrm{fr}}(\mathcal{X})$ and $\mathbf{SH}_{\mathrm{fr}}(\mathcal{X})$ of framed motivic spectra over $\mathcal{X}$ are defined by repeating the constructions of $\mathbf{H}(\mathcal{X})$ and $\mathbf{SH}(\mathcal{X})$ over $\mathrm{Sm}^{\mathrm{fr}}_{/\mathcal{X}}$ (the conditions of Nisnevich descent and $\mathbf{A}^1$-invariance being imposed on the restrictions to $\mathrm{Sm}_{/\mathcal{X}}$). As $\mathcal{X}$ varies, this defines a $(*, \sharp, \otimes)$-formalism which satisfies homotopy invariance and Thom stability by construction, and should also satisfy localization (the proof in the case of schemes [Ho5, Thm. 8] likely generalizes).

Remark 10.15. A framed correspondence as above acts on cohomology with coefficients in $\mathcal{F} \in \mathbf{D}(\mathcal{X})$, for any $(*, \sharp, \otimes)$-formalism $\mathbf{D}$:

$$\mathbf{R}\Gamma(\mathcal{X}'', \mathcal{F}) \xrightarrow{g^*} \mathbf{R}\Gamma(\mathcal{Z}, \mathcal{F}) \simeq \mathbf{R}\Gamma^{\mathcal{L}_f}(\mathcal{Z}, \mathcal{F}) \xrightarrow{f_!} \mathbf{R}\Gamma(\mathcal{X}', \mathcal{F}).$$

This observation implies that the canonical morphism $R : \mathbf{SH}^* \to \mathbf{D}^*$ (Proposition 5.13) admits a canonical factorization through $\mathbf{SH}^*_{\mathrm{fr}}$.[19] On the restriction to (derived) schemes this factorization is *unique*, i.e., that the morphism $\mathbf{SH}^* \to \mathbf{SH}^*_{\mathrm{fr}}$ ("free transfers") is invertible (see [Ho5, Thm. 18]). This should generalize to scalloped stacks, but the necessary analysis of the geometry of framed correspondences over stacks will not be undertaken here.

[19] This argument requires a *homotopy coherent* action of framed correspondences, which we do not construct here.

Construction 10.16. Consider the constant sheaf $\mathbf{Z}_{\mathcal{X}}$ on $\mathrm{Sm}_{/\mathcal{X}}$, with its canonical framed transfers (see [Ho5, §4]). We may regard it as an object of the unstable category $\mathbf{H}_{\mathrm{fr}}(\mathcal{X})$, form the framed infinite suspension $\Sigma^\infty_{\mathrm{fr}}(\mathbf{Z}_{\mathcal{X}}) \in \mathbf{SH}_{\mathrm{fr}}(\mathcal{X})$, and forget transfers to get a motivic $\mathcal{E}_\infty$-ring spectrum

$$\mathbf{Z}_{\mathcal{X}} \in \mathbf{SH}(\mathcal{X})$$

that we call the (integral) *motivic cohomology spectrum* over $\mathcal{X}$. In the same manner, we get an A-linear motivic cohomology spectrum $A_{\mathcal{X}} \in \mathbf{SH}(\mathcal{X})$ for every abelian group A. The argument of [Ho5, Lem. 20] should generalize to show that this construction is stable under representable $*$-inverse image.

Remark 10.17. Note that the *definition* of $\mathbf{Z}_{\mathcal{X}}$ is unconditional on the above conjectures on framed correspondences over stacks.

Remark 10.18. If the description of MGL in [EHKSY3, Thm. 3.4.1] is extended to stacks, then we also get a canonical $\mathcal{E}_\infty$-ring morphism

$$\mathrm{MGL}_{\mathcal{X}} \to \mathbf{Z}_{\mathcal{X}}$$

for every scalloped derived stack $\mathcal{X}$.

Remark 10.19. One can similarly construct a motivic spectrum $\widetilde{\mathbf{Z}}_{\mathcal{X}} \in \mathbf{SH}(\mathcal{X})$ representing Milnor–Witt motivic cohomology, following the framed construction in [HJNY, Thm. 7.3].

11. Fixed point localization

In this section we prove Theorem C. We fix the following notation.

Notation 11.1.

(i) Let S be a connected noetherian affine base scheme. Let $T = \mathbf{G}_{m,S}^{\times l}$ be a split torus over S of dimension $l \geqslant 0$.

(ii) Given a motivic $\mathcal{E}_\infty$-spectrum $\mathcal{F} \in \mathbf{SH}(BT)$, we consider a certain localization[20] of the $\mathbf{Z}$-graded cohomology ring spectrum (see Remark 9.7),

$$\mathbf{R}\Gamma^*(BT, \mathcal{F})_{\mathrm{loc}} := \mathcal{S}^{-1}\,\mathbf{R}\Gamma^*(BT, \mathcal{F})$$

at a set $\mathcal{S}$ of homogeneous elements of degree 1. Namely, let $L = [\mathbf{A}^1_S/\mathbf{G}_{m,S}]$ denote the tautological line bundle on $B\mathbf{G}_{m,S}$, where $\mathbf{G}_{m,S}$ acts on $\mathbf{A}^1_S$ by scaling with weight 1, and let $\mathrm{pr}_i : T \to \mathbf{G}_{m,S}$ denote the ith projection. Then $\mathcal{S}$ is the multiplicative closure of the set of Euler classes

$$\mathrm{pr}_i^* e(L^{\otimes n}) \in \pi_0\,\mathbf{R}\Gamma^{L^{\otimes n}}(BT, \mathcal{F})$$

for $n \geqslant 1$ and $1 \leqslant i \leqslant l$.

(iii) Given an $\mathbf{R}\Gamma^*(BT, \mathcal{F})$-module M, we also set

$$M_{\mathrm{loc}} := M \otimes^{\mathbf{L}}_{\mathbf{R}\Gamma^*(BT,\mathcal{F})} \mathbf{R}\Gamma^*(BT, \mathcal{F})_{\mathrm{loc}}$$

[20] in the sense of $\mathcal{E}_\infty$-ring spectra, see e.g. [Lu2, §7.2.3]

for the extension of scalars.

Theorem 11.2 (Concentration). *Let $i : Z \to X$ be a closed immersion of T-equivariant derived algebraic spaces of finite type over S, such that T acts without fixed points on the complement $X \smallsetminus Z$. Then for every motivic spectrum $\mathcal{F} \in \mathbf{SH}(BT)$, the $\mathbf{R}\Gamma^*(BT, \mathcal{F})$-module map*

$$i_* : \mathbf{R}\Gamma_*([Z/T]/BT, \mathcal{F}) \to \mathbf{R}\Gamma_*([X/T]/BT, \mathcal{F})$$

induces an isomorphism of $\mathbf{R}\Gamma^*(BT, \mathcal{F})_{\mathrm{loc}}$-modules

$$i_* : \mathbf{R}\Gamma_*([Z/T]/BT, \mathcal{F})_{\mathrm{loc}} \simeq \mathbf{R}\Gamma_*([X/T]/BT, \mathcal{F})_{\mathrm{loc}}.$$

Corollary 11.3. *Let X be a T-equivariant derived algebraic space, separated of finite type over S. If $i : X^T \to X$ is the inclusion of the locus of fixed points*

$$X^T := \mathrm{Maps}_{BT}(BT, X),$$

then for every motivic spectrum $\mathcal{F} \in \mathbf{SH}(BT)$, there is an isomorphism of $\mathbf{R}\Gamma^*(BT, \mathcal{F})_{\mathrm{loc}}$-modules

$$i_* : \mathbf{R}\Gamma_*(X \times BT/BT, \mathcal{F})_{\mathrm{loc}} \simeq \mathbf{R}\Gamma_*([X/T]/BT, \mathcal{F})_{\mathrm{loc}}.$$

Remark 11.4. In the situation of Theorem C, where S is the spectrum of a field and $T = \mathbf{G}_{m,S}$, the separation hypothesis in Corollary 11.3 can be dropped in view of [Dr, Prop. 1.2.2].

Remark 11.5. Theorem 11.2 and Corollary 11.3 hold more generally, with the same proof, for any coefficient $\mathcal{F} \in \mathbf{D}(BT)$, where $\mathbf{D}^*$ is as in Subsect. 7.1.

Proof of Theorem 11.2. We may assume that X is classical and reduced by Proposition 9.9. By Nisnevich descent and Theorem 2.14, we may also assume that X is separated. By the localization triangle (Theorem 4.9(ii)(d)), it will suffice to show that if T acts without fixed points on the whole of X, then

$$\mathbf{R}\Gamma_*([X/T]/BT, \mathcal{F})_{\mathrm{loc}} \simeq 0.$$

Since T then acts without fixed points on every T-invariant proper closed subspace $Y \subsetneq X$ as well, it will suffice by noetherian induction (on the quotient stack $[X/T]$), and the localization triangle again, to show that this claim holds after replacing X by some nonempty T-invariant open subspace.

By Thomason's generic slice theorem (see [Th, Thm. 4.10, Rem. 4.11]), there exists a nonempty open $U \subseteq X$, a proper diagonalizable subgroup $T' \subsetneq T$, and a T'-equivariant algebraic space V such that $[U/T] \simeq [V/T']$. Therefore, the $\mathbf{R}\Gamma^*(BT, \mathcal{F})$-module structure on $\mathbf{R}\Gamma_*([U/T]/BT, \mathcal{F})$ is obtained by restriction of scalars from the $\mathbf{R}\Gamma^*(BT', \mathcal{F})$-module structure on $\mathbf{R}\Gamma_*([V/T']/BT, \mathcal{F})$. Then we have

$$\mathbf{R}\Gamma_*([U/T]/BT, \mathcal{F})_{\mathrm{loc}} \simeq \mathbf{R}\Gamma_*([U/T]/BT, \mathcal{F}) \otimes_{\mathbf{R}\Gamma^*(BT', \mathcal{F})} \mathbf{R}\Gamma^*(BT', \mathcal{F})_{\mathrm{loc}},$$

so it will suffice to show that $\mathbf{R}\Gamma^*(BT', \mathcal{F})_{\mathrm{loc}}$ vanishes.

For simplicity, assume $T = \mathbf{G}_{m,S}$ and $T' = \mu_{n,S}$ ($n \geqslant 1$); since $T' \subsetneq T$ is a proper inclusion and S is connected, the argument readily generalizes using [SGA3, Exp. VIII, 1.4, 3.2] (as in the proof of [Th2, Prop. 1.2]). Note that the line bundle $L^{\otimes n}$ on BT, which corresponds to the $\mathbf{G}_{m,S}$-equivariant line bundle $\mathbf{A}^1_S$ where $\mathbf{G}_{m,S}$ acts with weight n, restricts to the *trivial* line bundle on $\mu_{n,S}$. Therefore, by Lemma 8.2 its Euler class $e(L^{\otimes n})$ restricts to 0 on BT'. Hence 0 is a unit in $\mathbf{R}\Gamma^*(BT',\mathcal{F})_{\mathrm{loc}}$. □

Remark 11.6. One can show that the classes $t_{i,n} = \mathrm{pr}_i^* e(L^{\otimes n})$ are usually nonzero. Indeed, rationalization and étale localization gives a canonical map $\mathcal{F} \to \mathcal{F}_{\mathbf{Q},\text{ét}}$. Using, say, the additive formal group law to orient $\mathcal{F}_{\mathbf{Q},\text{ét}}$, this determines a homomorphism of $\mathbf{Z}$-graded ring spectra

$$\mathbf{R}\Gamma^*(BT,\mathcal{F}) \to \mathbf{R}\Gamma^*(BT,\mathcal{F}_{\mathbf{Q},\text{ét}}),$$

which in degree r is the map

$$\bigoplus_{\mathrm{rk}(\alpha)=r} \mathbf{R}\Gamma^\alpha(BT,\mathcal{F}) \to \bigoplus_{\mathrm{rk}(\alpha)=r} \mathbf{R}\Gamma^\alpha(BT,\mathcal{F}_{\mathbf{Q},\text{ét}}) \xrightarrow{\text{fold}} \mathbf{R}\Gamma^r(BT,\mathcal{F}_{\mathbf{Q},\text{ét}}).$$

Under the Thom isomorphism

$$\mathbf{R}\Gamma^{L^{\otimes n}}(BT,\mathcal{F}_{\mathbf{Q},\text{ét}}) \simeq \mathbf{R}\Gamma^1(BT,\mathcal{F}_{\mathbf{Q},\text{ét}}),$$

the element $t_{i,n}$ maps to $\mathrm{pr}_i^* c_1(L^{\otimes n}) = n{\cdot}t_i$, where $t_i = \mathrm{pr}_i^* c_1(L) \in \mathrm{H}^1(BT,\mathcal{F}_{\mathbf{Q},\text{ét}})$ is nonzero in the polynomial ring (of characteristic zero)

$$\mathrm{H}^*(BT,\mathcal{F}_{\mathbf{Q},\text{ét}}) \simeq \mathrm{H}^*(S,\mathcal{F}_{\mathbf{Q},\text{ét}})[t_1,\ldots,t_l],$$

see [MV, §4, Prop. 3.7], as long as $\mathrm{H}^*(S,\mathcal{F}_{\mathbf{Q},\text{ét}})$ is nonzero.

Remark 11.7. For unoriented examples such as Milnor–Witt motivic cohomology, hermitian K-theory, or special linear algebraic cobordism, Theorem 11.2 is probably not very satisfactory. At least for the limit-extended theories (see Sect. 12), the Witt cohomology of $B\mathbf{G}_m$ is trivial, so that Euler classes in these theories should have no "Witt contribution". We thank Marc Levine for explaining this to us.

12. Limit extensions

12.1. **Limit-extended categories.** We begin with a construction of *limit extensions* of categories of coefficients from algebraic spaces to stacks, and a proof of Theorem D.[21] Throughout the section, we implicitly assume all (derived) algebraic spaces and stacks are quasi-compact and quasi-separated. Let S be a derived algebraic space and $\mathbf{D}^*$ a $(*,\sharp,\otimes)$-formalism satisfying the Voevodsky conditions over derived algebraic spaces over S (see [Kh5, §2]).

Construction 12.1. Let $\mathcal{X}$ be a derived algebraic stack over S. We define

$$\mathbf{D}_\triangleleft(\mathcal{X}) = \varprojlim_{(T,t)} \mathbf{D}(T)$$

where the limit is taken over the ∞-category $\mathrm{Lis}_{\mathcal{X}}$ of pairs (T,t) where T is a derived algebraic space and $t : T \to \mathcal{X}$ is a smooth morphism. Note that $\mathbf{D}_\triangleleft(\mathcal{X}) \simeq \mathbf{D}(\mathcal{X})$ if $\mathcal{X} = X$ is a derived algebraic *space.*

[21] The idea to consider such a generalization originally arose in unpublished work of Marc Hoyois with the first author.

Example 12.2. If $\mathbf{D}^*$ has étale descent (on algebraic spaces), then this coincides with the construction of [Kh3, App. A].

Remark 12.3. The limit in Construction 12.1 can also be taken over the full subcategory $\mathrm{Lis}_{\mathcal{X}}^{\mathrm{aff}}$ of $\mathrm{Lis}_{\mathcal{X}}$ spanned by (T,t) with T *affine*. More precisely, the canonical functor

$$\varprojlim_{(T,t)\in\mathrm{Lis}_{\mathcal{X}}} \mathbf{D}(T) \to \varprojlim_{(T,t)\in\mathrm{Lis}_{\mathcal{X}}^{\mathrm{aff}}} \mathbf{D}(T) \tag{12.4}$$

is an equivalence. This follows from Nisnevich descent for $\mathbf{D}$ (over algebraic spaces). Indeed, we can write the source as

$$\varprojlim_{(T,t)\in\mathrm{Lis}_{\mathcal{X}}} \varprojlim_{(S\text{ aff}, s:S\to T\text{ ét})} \mathbf{D}(S) \simeq \varprojlim_{(T,t,S\text{ aff}, s:S\to T\text{ ét})} \mathbf{D}(S).$$

The forgetful functor from the right-hand indexing category to $\mathrm{Lis}_{\mathcal{X}}^{\mathrm{aff}}$ (which remembers only S and $S\to T\to\mathcal{X}$) induces a functor to this category from the target of (12.4), which one checks is inverse to (12.4).

Remark 12.5. The discussion of [Kh3, App. A] can be adapted to show that on the limit-extended categories, the following operations extend: $\otimes$ and $\underline{\mathrm{Hom}}$, f^* and f_* for arbitrary morphisms, $f_!$ and $f^!$ for finite type morphisms, and $\langle\alpha\rangle$ for K-theory classes α. We also have the base change formula, the isomorphism $\alpha_f : f_! \simeq f_*$ for f proper representable, the purity isomorphism $\mathrm{pur}_f : f^! \simeq f^*\langle\mathcal{L}_f\rangle$ for a smooth morphism, homotopy invariance for vector bundles, and the localization triangles for complementary closed/open pairs. The only nontrivial part is the exceptional functoriality, which we will not use here.

Remark 12.6. From the categorical point of view, it seems that the main difference between the genuine theory $\mathbf{SH}$ and the limit extension $\mathbf{SH}_\triangleleft$ is that the latter admits $\sharp$-direct images (and the !-operations) even for non-representable morphisms. It would be interesting to know whether the canonical natural transformation $\mathbf{SH}\to\mathbf{SH}_\triangleleft$ can be universally characterized by this property.

12.2. **Cohomology.**

Definition 12.7. For every derived algebraic stack $\mathcal{X}$ over S, the cohomology spectrum with coefficients in $\mathcal{F}\in\mathbf{D}(\mathcal{X})$, twisted by $\alpha\in\mathrm{K}(\mathcal{X})$, is defined by the formula

$$\mathbf{R}\Gamma_\triangleleft^\alpha(\mathcal{X},\mathcal{F}) = \mathrm{Maps}_{\mathbf{D}_\triangleleft(\mathcal{X})}(\mathbf{1},\mathcal{F}\langle\alpha\rangle).$$

The 0-th homotopy groups will be denoted

$$\mathrm{H}_\triangleleft^\alpha(\mathcal{X},\mathcal{F}) = \mathrm{Hom}_{\mathbf{D}_\triangleleft(\mathcal{X})}(\mathbf{1},\mathcal{F}\langle\alpha\rangle).$$

We have inverse images along arbitrary morphisms and Gysin direct images along proper smooth representable morphisms.

Remark 12.8. For any derived algebraic stack $\mathcal{X}$, $\mathcal{F} \in \mathbf{D}_{\triangleleft}(\mathcal{X})$, and $\alpha \in \mathrm{K}(\mathcal{X})$, the cohomology spectrum is by construction the homotopy limit

$$\mathbf{R}\Gamma^{\alpha}_{\triangleleft}(\mathcal{X},\mathcal{F}) \simeq \varprojlim_{(T,t)} \mathbf{R}\Gamma^{\alpha}_{\triangleleft}(T,\mathcal{F})$$

over $(T,t) \in \mathrm{Lis}_{\mathcal{X}}$.

12.3. **The Borel construction.** The following result shows that, for quotient stacks, limit-extended cohomology theories can be computed via Totaro's algebraic version of the Borel construction.

Let S be the spectrum of a perfect field k. (For non-perfect fields the result will also follow, up to inverting the characteristic exponent, in view of [EK].)

Theorem 12.9. *Let G be an fppf group scheme over S. Suppose given a tower*

$$\mathcal{V}_0 \hookrightarrow \mathcal{V}_1 \hookrightarrow \mathcal{V}_2 \hookrightarrow \cdots$$

of inclusions of vector bundles over BG, together with closed substacks $\mathcal{W}_i \subseteq \mathcal{V}_i$, such that for each i we have:

(i) *The open complement $U_i = \mathcal{V}_i \smallsetminus \mathcal{W}_i$ is representable (by an algebraic space).*
(ii) *There is an inclusion $U_i \subseteq U_{i+1}$.*
(iii) *There is a strict inequality $\mathrm{codim}_{\mathcal{V}_i}(\mathcal{W}_i) < \mathrm{codim}_{\mathcal{V}_{i+1}}(\mathcal{W}_{i+1})$.*

Then for any motivic spectrum $\mathcal{F} \in \mathbf{SH}(S)$ and any $\alpha \in \mathrm{K}(S)$, there is a canonical isomorphism

$$\mathbf{R}\Gamma^{\alpha}_{\triangleleft}(BG,\mathcal{F}) \simeq \varprojlim_{i} \mathbf{R}\Gamma^{\alpha}(U_i,\mathcal{F}).$$

More generally, if $\mathcal{X} = [X/G]$ is the quotient of a smooth algebraic space X with G-action, we have

$$\mathbf{R}\Gamma^{\alpha}_{\triangleleft}(\mathcal{X},\mathcal{F}) \simeq \varprojlim_{i} \mathbf{R}\Gamma^{\alpha}(\mathcal{X} \underset{BG}{\times} U_i,\mathcal{F}).$$

Remark 12.10. If G is a smooth embeddable group scheme over S, then there exists a choice of $(\mathcal{V}_i,\mathcal{W}_i)_i$ as in Theorem 12.9 by [Tot, Rem. 1.4].

Remark 12.11. In the situation of Theorem 12.9, the canonical maps

$$\mathrm{H}^{\alpha}_{\triangleleft}(\mathcal{X},\Lambda) \simeq \pi_0 \varprojlim_{i} \mathbf{R}\Gamma^{\alpha}(\mathcal{X} \underset{BG}{\times} U_i,\Lambda) \twoheadrightarrow \varprojlim_{i} \mathrm{H}^{\alpha}(\mathcal{X} \underset{BG}{\times} U_i,\Lambda)$$

are always surjective by the Milnor exact sequence. More generally for every $s \in \mathbf{Z}$ we have surjections

$$\pi_s \mathbf{R}\Gamma^{\alpha}_{\triangleleft}(\mathcal{X},\Lambda) \twoheadrightarrow \varprojlim_{i} \pi_s \mathbf{R}\Gamma^{\alpha}(\mathcal{X} \underset{BG}{\times} U_i,\Lambda).$$

In the case of motivic cohomology, we will show (see Remark 12.12) that these are in fact bijective.

12.4. **Proof of Theorem 12.9 for motivic cohomology.** Let Λ be a commutative ring in which the characteristic exponent of the field k is invertible. In this subsection, we will give a proof of Theorem 12.9 in the special case of Λ-linear motivic cohomology. This will be independent of the proof of the general statement proven in the next subsection, but we decided to also include this argument due to its comparative simplicity.

Proof. Let $\pi_i : \mathcal{X} \times_{BG} \mathcal{V}_i \to \mathcal{X}$ denote the projections and $j_i : \mathcal{X} \times_{BG} U_i \to \mathcal{X} \times_{BG} \mathcal{V}_i$ the inclusions. The inverse image maps

$$\mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{X}, \Lambda) \xrightarrow{\pi_i^*} \mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{X} \underset{BG}{\times} \mathcal{V}_i, \Lambda) \xrightarrow{j_i^*} \mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{X} \underset{BG}{\times} U_i, \Lambda),$$

where

$$\mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{X} \underset{BG}{\times} U_i, \Lambda) \simeq \mathbf{R}\Gamma^n(\mathcal{X} \underset{BG}{\times} U_i, \Lambda)$$

since U_i is an algebraic space, induce a canonical map

$$\mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{X}, \Lambda) \to \varprojlim_i \mathbf{R}\Gamma^n(\mathcal{X} \underset{BG}{\times} U_i, \Lambda).$$

By homotopy invariance, π_i^* is invertible for every i, so it will suffice to show that j_i^* is invertible for $i \gg 0$. More precisely, we will show that this holds for all i such that $\mathrm{codim}_{\mathcal{V}_i}(\mathcal{W}_i) > n$.

By construction of the limit-extended theory and cofinality, it is enough to prove the claim with $\mathcal{X}$ replaced by T, for any $(T,t) \in \mathrm{Lis}^{\mathrm{aff}}_{\mathcal{X}}$. By the localization triangle and Poincaré duality, the fibre of the map j_i^* is the Borel–Moore homology spectrum

$$\mathbf{R}\Gamma_{d-n}(T \underset{BG}{\times} \mathcal{W}_i/S, \Lambda),$$

where $d = \dim(T \times_{BG} \mathcal{V}_i)$. But this spectrum vanishes as soon as $d - n > \dim(T \times_{BG} \mathcal{W}_i)$, i.e., whenever

$$\mathrm{codim}_{T \times_{BG} \mathcal{V}_i}(T \underset{BG}{\times} \mathcal{W}_i) = \mathrm{codim}_{\mathcal{V}_i}(\mathcal{W}_i)$$

is strictly larger than n. (This follows from the comparison with the Bloch cycle complex; see [MWV, Prop. 19.18] and [CD2, Cor. 8.12], and note that $T \times_{BG} \mathcal{W}_i$ is affine since T is.) □

Remark 12.12. Our proof also shows that the canonical homomorphisms (Remark 12.11)

$$\pi_s \mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{X}, \Lambda) \twoheadrightarrow \varprojlim_i \pi_s \mathbf{R}\Gamma^n(\mathcal{X} \underset{BG}{\times} U_i, \Lambda)$$

are bijective for all $n, s \in \mathbf{Z}$. Indeed, these are the limits over i of the restriction maps

$$\pi_s \mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{X}, \Lambda) \simeq \pi_s \mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{X} \underset{BG}{\times} \mathcal{V}_i, \Lambda) \xrightarrow{j_i^*} \pi_s \mathbf{R}\Gamma^n(\mathcal{X} \underset{BG}{\times} U_i, \Lambda)$$

which we showed were invertible for $i \gg 0$. (The first isomorphism is homotopy invariance for the vector bundle $\mathcal{X} \times_{BG} \mathcal{V}_i \to \mathcal{X}$.)

12.5. **Proof of Theorem 12.9 in general.** We will deduce the general case of Theorem 12.9 from a stronger comparison at the level of stable motivic homotopy types.

Notation 12.13. Let $\mathcal{X}$ be a smooth algebraic stack of finite presentation over a qcqs algebraic space S. Then we write

$$M_S^{\triangleleft}(\mathcal{X}) := f_{\sharp}(\mathbf{1}_{\mathcal{X}}) \in \mathbf{SH}(S),$$

where $f_{\sharp} : \mathbf{SH}_{\triangleleft}(\mathcal{X}) \to \mathbf{SH}_{\triangleleft}(S) \simeq \mathbf{SH}(S)$ is $\sharp$-direct image along the structural morphism $f : \mathcal{X} \to S$. Note that if $\mathcal{X} = X$ is an algebraic space, then $M_S(X) \simeq \Sigma^{\infty}_{+}\mathbf{L}\mathrm{h}_S(X)$. In general it is computed as

$$M_S^{\triangleleft}(\mathcal{X}) \simeq \varinjlim_{(T,t)\in \mathrm{Lis}_{\mathcal{X}}} M_S^{\triangleleft}(T).$$

The following can be regarded as a comparison of the limit-extended motivic stable homotopy type of a quotient stack with its Morel–Voevodsky motivic stable homotopy type (see [MV, §4.2] and [Kri2, §3]).

Theorem 12.14. *Let G be an fppf group scheme over S. Let $(\mathcal{V}_i)_i$, $(\mathcal{W}_i)_i$ and $(U_i)_i$ be as in Theorem 12.9. Let $\mathcal{X} = [X/G]$ be the quotient of a smooth algebraic space X of finite presentation over S with G-action. Then there is a canonical isomorphism*

$$M_S^{\triangleleft}(\mathcal{X}) \simeq \varinjlim_i M_S(\mathcal{X} \underset{BG}{\times} U_i)$$

in $\mathbf{SH}(S)$.

Proof. As in the proof of Theorem 12.9, the morphism

$$\varinjlim_i M_S(\mathcal{X} \underset{BG}{\times} U_i) \to M_S^{\triangleleft}(\mathcal{X})$$

is induced by the canonical morphisms

$$M_S^{\triangleleft}(\mathcal{X} \underset{BG}{\times} U_i) \to M_S^{\triangleleft}(\mathcal{X} \underset{BG}{\times} \mathcal{V}_i) \simeq M_S^{\triangleleft}(\mathcal{X})$$

for every i. These can be described as the colimits over $(T,t) \in \mathrm{Lis}^{\mathrm{aff}}_{\mathcal{X}}$ over the analogous morphisms

$$M_S(T \underset{BG}{\times} U_i) \to M_S(T \underset{BG}{\times} \mathcal{V}_i) \simeq M_S(T).$$

We claim that for every (T,t), the colimit over i of the cofibres

$$K_i := M_S(T \underset{BG}{\times} \mathcal{V}_i)/M_S(T \underset{BG}{\times} U_i)$$

vanishes. In fact, we will show that each K_i is c_i-connective for the homotopy t-structure on $\mathbf{SH}(S)$, where c_i is the codimension of $\mathcal{W}_i$ in $\mathcal{V}_i$. This will imply (see [Ho2, Cor. 2.4]) that for every $Y \in \mathrm{Sm}_{/S}$ and $r, s \in \mathbf{Z}$, we have

$$\mathrm{Hom}_{\mathbf{SH}(S)}(M_S(Y)(r)[s], K_i) = 0$$

for i large enough that $c_i > s - r + \dim(Y)$. Since the objects $M_S(Y)(r)[s]$ form a set of compact generators of $\mathbf{SH}(S)$, it will follow that $\varinjlim_i K_i = 0$ as claimed.

Since k is perfect, each scheme $T \times_{BG} \mathcal{W}_i$ can be stratified by smooth closed subschemes. By the localization triangle and relative purity, it follows that each K_i is contained in the full subcategory of $\mathbf{SH}(S)$ generated under colimits and extensions by objects of the form $M_S(W)\langle\mathcal{E}\rangle$ for $W \in \mathrm{Sm}_{/S}$ and $\mathcal{E}$ a locally free sheaf on W of rank $\geqslant c_i$. It will thus suffice to show that every such $M_S(W)\langle\mathcal{E}\rangle$ is c_i-connective, since this property is preserved under colimits and extensions. But this holds by [Ho2, Lem. 3.1]. □

12.6. **Equivariant Chow groups, cobordism and K-theory.** Let S be the spectrum of a perfect field k, G be a smooth embeddable group scheme over S, and X a smooth G-quasi-projective S-scheme. Theorem 12.9 yields the following comparisons.

Example 12.15. The limit-extended motivic cohomology of $[X/G]$

$$\mathbf{R}\Gamma^n_{\triangleleft}([X/G], \Lambda)$$

is computed for every $n \in \mathbf{Z}$ by Λ-linear Bloch cycle complexes of the Borel construction. Here Λ is any commutative ring in which the characteristic exponent of k is invertible. This follows from the comparison of the motivic complexes and Bloch cycle complexes of schemes (see [MWV, Lect. 19]). Moreover, by Remark 12.12 we moreover get for all $n, s \in \mathbf{Z}$ canonical isomorphisms

$$\pi_s \, \mathbf{R}\Gamma^n_{\triangleleft}([X/G], \Lambda) \simeq \mathrm{A}^G_{d-n}(X, s) \otimes \Lambda,$$

if X is of pure dimension d, where on the right-hand side are the G-equivariant higher Chow groups of X as defined by Edidin–Graham [EG]. In particular,

$$\mathrm{H}^n_{\triangleleft}([X/G], \Lambda) \simeq \mathrm{A}^G_{d-n}(X) \otimes \Lambda.$$

Example 12.16. Applying Theorem 12.9 to the algebraic cobordism spectrum MGL, we find that the limit-extended algebraic cobordism of $[X/G]$

$$\mathbf{R}\Gamma^n_{\triangleleft}([X/G], \mathrm{MGL})$$

can be computed via Voevodsky's algebraic cobordism [Vo, §6.3] of the Borel construction. If k is of characteristic zero, then it follows from [Le2, Ho2] that there are surjections

$$\mathrm{H}^n_{\triangleleft}([X/G], \mathrm{MGL}) \twoheadrightarrow \varprojlim_i \Omega^n([X/G] \underset{BG}{\times} U_i)$$

where the notation is as in Theorem 12.9. The right-hand side here has been considered in [HML] and [Kri, Thm. 6.1]. We note that these theories are not known to satisfy several fundamental properties such as the right-exact localization sequence[22] (see however [Kri, Cor. 6.2] for the special case of sections of projective morphisms). Therefore, limit-extended cobordism can be viewed as a well-behaved replacement for the latter theories which does admit the right-exact localization sequence. Moreover, using the higher groups it is also extends to the left. Indeed, we have:

[22]The contrary is claimed in [HML, Thm. 20]. However, as brought to our attention by M. Levine (who attributes the observation to A. Merkurjev) and H. Park, the proof relies on the (false) assertion that the limit of a right-exact sequence of a projective system of abelian groups, the $\mathbf{R}^1 \varprojlim$ terms each of which vanish, is still right-exact.

Proposition 12.17. *Let $i : Z \to X$ and $j : U \to X$ be a complementary closed-open pair of smooth G-equivariant k-schemes. Write $\mathcal{X} = [X/G]$, $\mathcal{Z} = [Z/G]$, and $\mathcal{U} = [U/G]$ for the quotient stacks. Then for every $n \in \mathbf{Z}$ there is a long-exact sequence*

$$\cdots \xrightarrow{\partial} \pi_s \mathbf{R}\Gamma^{n-c}_{\triangleleft}(\mathcal{Z}, \mathrm{MGL}) \xrightarrow{i_!} \pi_s \mathbf{R}\Gamma^{n}_{\triangleleft}(\mathcal{X}, \mathrm{MGL}) \xrightarrow{j^*} \pi_s \mathbf{R}\Gamma^{n}_{\triangleleft}(\mathcal{U}, \mathrm{MGL}) \xrightarrow{\partial}$$

$$\cdots \xrightarrow{\partial} \mathrm{H}^{n-c}_{\triangleleft}(\mathcal{Z}, \mathrm{MGL}) \xrightarrow{i_!} \mathrm{H}^{n}_{\triangleleft}(\mathcal{X}, \mathrm{MGL}) \xrightarrow{j^*} \mathrm{H}^{n}_{\triangleleft}(\mathcal{U}, \mathrm{MGL}) \to 0$$

where $c = \mathrm{codim}_X(Z)$.

Proof. In view of the localization triangle and relative purity for $\mathbf{SH}_{\triangleleft}$, we only need to demonstrate right-exactness of the last row. In other words, it is enough to show that $\pi_{-1} \mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{Z}, \mathrm{MGL}) = 0$ for every n. For this we use the description in terms of the Borel construction (Theorem 12.9). By the Milnor exact sequence and the fact that the Mittag–Leffler condition holds for the projective system $\{\pi_0 \mathbf{R}\Gamma^n(\mathcal{Z} \times_{BG} U_i, \mathrm{MGL})\}_i$ (see [HML, Lem. 18]), we have

$$\pi_{-1} \mathbf{R}\Gamma^n_{\triangleleft}(\mathcal{Z}, \mathrm{MGL}) \simeq \varprojlim_i \pi_{-1} \mathbf{R}\Gamma^n(\mathcal{Z} \underset{BG}{\times} U_i, \mathrm{MGL}).$$

We can choose U_i to be quasi-projective as in Remark 12.10, so the claim follows from the corresponding vanishing on smooth quasi-projective schemes (see e.g. [Ho]). □

Example 12.18. There is a canonical ring homomorphism[23]

$$\mathrm{K}_0([X/G]) \to \mathrm{H}^0_{\triangleleft}([X/G], \mathrm{KGL})$$

which however is *not* an isomorphism. In fact, [Kri2, Thm. 9.10] (combined with Theorem 12.9) shows that it exhibits the target as the completion of the source along the augmentation ideal.

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