

LECTURES ON ALGEBRAIC STACKS

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| | |
|---|----|
| 0. Overview | 2 |
| 1. ∞ -Categories | 7 |
| 1.1. Simplicial sets | 7 |
| 1.2. Categories as simplicial sets | 8 |
| 1.3. Groupoids and Kan complexes | 10 |
| 1.4. ∞ -Categories as weak Kan complexes | 11 |
| 1.5. The ∞ -category of (∞ -)groupoids | 12 |
| 2. Sheaves and stacks | 14 |
| 2.1. Sheaves | 14 |
| 2.2. Bases of topologies | 16 |
| 3. The stack of quasi-coherent sheaves | 17 |
| 3.1. Cartesian fibrations | 17 |
| 3.2. Descent for quasi-coherent sheaves | 19 |
| 3.3. Quasi-coherent sheaves on stacks | 23 |
| 4. Quotient stacks | 24 |
| 4.1. Groupoid objects and effective epimorphisms | 24 |
| 4.2. Group actions | 26 |
| 4.3. Torsors | 28 |
| 4.4. Quotient stacks | 29 |
| 4.5. Examples of quotient stacks | 31 |
| 5. Algebraic spaces and stacks | 32 |
| 5.1. Algebraic spaces | 32 |
| 5.2. Algebraic stacks | 34 |
| 5.3. Algebraicity of quotient stacks | 34 |
| 5.4. Recognizing algebraic spaces and DM stacks | 35 |
| 5.5. Free and proper actions | 35 |
| 5.6. Stabilizers | 36 |
| 6. The resolution property | 37 |
| 6.1. The resolution property | 37 |
| 6.2. Global quotient stacks | 38 |
| 6.3. Quasi-coherent sheaves on quotient stacks | 38 |
| 6.4. A quasi-affineness criterion | 39 |
| 6.5. Proof of Theorem 6.2.5 | 40 |
| 7. Derived categories | 41 |
| 7.1. Algebraic categories | 41 |
| 7.2. (Connective) nonabelian derived categories | 43 |
| 7.3. Interlude: stable ∞ -categories | 43 |
| 7.4. Nonabelian derived categories | 44 |
| 7.5. Derived functors | 46 |
| 7.6. Animated modules | 47 |

| | | |
|-------|--|----|
| 8. | The cotangent complex | 48 |
| 8.1. | Animated rings | 48 |
| 8.2. | The cotangent complex | 50 |
| 8.3. | The universal property of the cotangent complex | 50 |
| 8.4. | Quasi-coherent and perfect complexes on stacks | 51 |
| 8.5. | Derived stacks | 53 |
| 8.6. | Cotangent complexes of algebraic stacks | 54 |
| 8.7. | Smoothness | 58 |
| 8.8. | Tangent and cotangent bundles | 60 |
| 9. | Deformation theory of perfect complexes | 61 |
| 9.1. | The moduli stack of perfect complexes | 61 |
| 9.2. | The cotangent complex of the moduli of perfect complexes | 62 |
| 10. | Moduli stacks of complexes, sheaves, and bundles | 63 |
| 10.1. | Mapping stacks | 63 |
| 10.2. | Moduli of complexes, sheaves, and bundles | 66 |
| 10.3. | The Artin–Lurie representability theorem | 68 |
| 11. | Cohomology of stacks | 70 |
| 11.1. | Abelian sheaves | 70 |
| 11.2. | Co/homology | 71 |
| 11.3. | Intersection theory | 72 |
| 11.4. | Quotient stacks | 75 |
| 11.5. | Concentration and localization | 76 |
| | References | 76 |

0. OVERVIEW

“Definition” 0.0.1. Given a class of geometric objects, say “gadgets”, a *moduli space* for gadgets is a space whose points correspond to gadgets, modulo some notion of equivalence between gadgets:

$$\text{moduli space of gadgets} = \{\text{gadgets}\}/\text{equivalence}.$$

The idea of moduli theory is to transform questions about gadgets into questions about the moduli space, which we may then try to tackle via, topological, geometric, or cohomological methods.

Example 0.0.2. The complex projective n -space $\mathbf{P}^n(\mathbf{C})$ is the moduli space of lines in \mathbf{C}^{n+1} which pass through the origin. Algebraically, these are 1-dimensional linear subspaces $L \subseteq \mathbf{C}^{n+1}$.

Example 0.0.3. The Grassmannian $\text{Gr}(k, V)$ is the moduli space of k -dimensional linear subspaces of a given vector space V , or equivalently, of $(k - 1)$ -dimensional linear subspaces of the projectivization $\mathbf{P}(V)$.

One can then solve certain problems in enumerative geometry (e.g. “how many lines in $\mathbf{P}^3(\mathbf{C})$ intersect four given general lines?”) by analyzing the cohomology of Grassmannians (see: Schubert calculus).

Example 0.0.4. Given an algebraic variety X , the moduli space of (algebraic) vector bundles on X is the set of vector bundles on X modulo isomorphism. More generally, given an algebraic group G , the moduli space of principal G -bundles is the set of principal G -bundles on X modulo isomorphism.

Example 0.0.5. Given an algebraic variety X , the moduli space of coherent sheaves on X is the set of coherent sheaves on X modulo isomorphism.

In order to get a useful theory of moduli spaces, we will need to refine this naive picture in two ways.

Theorem 0.0.6 (Grothendieck). *A scheme X over a field k is completely determined by its functor of points, i.e., the functor*

$$X : \mathbf{CAlg}_k \rightarrow \mathbf{Set}$$

sending a commutative k -algebra A to the set of A -valued points $X(A)$. (Recall that an A -valued point of X is a morphism of schemes $\mathrm{Spec}(A) \rightarrow X$.)

Thus we may regard a scheme as a family (or fibration) of sets $X(A)$ parametrized by commutative algebras A . That is, a scheme is literally a “scheme” prescribing the A -valued points of some algebro-geometric space.

For example, in order to define complex projective n -space as a scheme, it is not sufficient to specify the set $\mathbf{P}^n(\mathbf{C})$ as above. Instead, we must specify the sets

$\mathbf{P}^n(A) = \{A\text{-linear surjections } A^{n+1} \twoheadrightarrow L \mid L \text{ projective } A\text{-module of rank } 1\}$, for all commutative \mathbf{C} -algebras A , together with the natural maps $\mathbf{P}^n(A) \rightarrow \mathbf{P}^n(A')$ for ring homomorphisms $A \rightarrow A'$.

Secondly, we will need to be smarter about quotients. Let X be a set and $R \subseteq X \times X$ an equivalence relation on X . We may depict this via the diagram

$$\begin{array}{ccc} & \xrightarrow{\mathrm{pr}_1} & \\ R & \xleftarrow{s} & X \\ & \xrightarrow{\mathrm{pr}_2} & \end{array}$$

where pr_i are the projections and $s : X \rightarrow X \times X$ is the diagonal (which factors through R since the relation is reflexive). This diagram defines a *groupoid* $[X/R]$:

- The objects of $[X/R]$ are the elements of X .
- The morphisms of $[X/R]$ are the elements of R .
- The “source” and “target” maps $\mathrm{Mor}[X/R] \rightarrow \mathrm{Obj}[X/R]$ are given by the projections pr_1 and pr_2 .
- The “identity” map $\mathrm{Obj}[X/R] \rightarrow \mathrm{Mor}[X/R]$ (sending an object to its identity morphism) is given by the diagonal s .
- Composition of morphisms is well-defined since the relation is transitive.
- All morphisms are invertible since the relation is symmetric.

Note that the set of connected components $\pi_0[X/R]$ (where objects of $[X/R]$ are identified if and only if they connected by some chain of morphisms) is

canonically isomorphic to the usual set-theoretic quotient X/R . Unlike X/R , the quotient groupoid $[X/R]$ remembers *how* elements are identified.

Similarly, if we have a group G acting on the set X , there is a quotient groupoid $[X/G]$ defined by the diagram

$$G \times X \begin{array}{c} \xrightarrow{\text{pr}_2} \\ \xleftarrow{s} \\ \xrightarrow{\text{act}} \end{array} X$$

where the “source” map is the projection $(g, x) \mapsto x$, the “target” map is the action map $(g, x) \mapsto g \cdot x$, and the “identity” map is $x \mapsto (e, x)$ where $e \in G$ is the neutral element. Whereas the set-theoretic quotient X/G remembers only the binary information of whether two elements $x, y \in X$ belong to the same equivalence class, the groupoid $[X/G]$ contains one isomorphism $x \simeq y$ for every $g \in G$ such that $g \cdot x = y$.

As we will see in this course, it is highly advantageous to allow moduli spaces to have *groupoids* of points rather than sets. Combining these two ideas leads one to the replace our naive definition of moduli space above by the following:

Definition 0.0.7. A *stack* (over a field k) is a functor

$$\mathcal{M} : \text{CAlg}_k \rightarrow \text{Grpd}$$

satisfying certain conditions.

One of our goals will be to prove the following theorem:

Theorem 0.0.8. Let C be a smooth proper curve over k and G an algebraic group over k . Let

$$\mathcal{M}_{\text{Bun}_G(C)} : A \mapsto \text{Bun}_G(C_A)^\simeq$$

be the stack defined by the functor sending a commutative algebra A to the groupoid of principal G -bundles on the scheme $C_A := C \otimes_k A$. Then $\mathcal{M}_{\text{Bun}_G(C)}$ is a smooth algebraic stack.

In particular, taking G to be the general linear group GL_n , we find the same holds for the moduli stack of rank n vector bundles on C . We also have a similar result for the moduli stack $\mathcal{M}_{\text{Coh}(C)}$ of coherent sheaves on C .

Remark 0.0.9. It is important here to work with groupoids rather than sets; the functor sending A to the set $\pi_0 \text{Bun}_G(C)^\simeq$ of isomorphism classes of principal G -bundles on $C \otimes_k A$ is very poorly behaved.

Warning 0.0.10. Making the definition of “stack” above precise is more involved than for schemes, since Grpd is naturally a 2-category (where the 2-morphisms are natural transformations). In other words, in practice we only want to distinguish between groupoids up to *equivalence* rather than isomorphism.

One way to handle this subtlety is to use the language of 2-categories and pseudofunctors. In this course we will instead use the language of ∞ -categories. This language is much more general than that of 2-categories, but

we will see that it has some practical advantages even when all ∞ -categories involved are “2-truncated”. Moreover, the extra generality of ∞ -categories will also be useful to us later in the course when we study concepts like the derived category of (quasi-)coherent sheaves on a stack, and the cotangent complex of a stack.

There are several excellent textbook accounts such as [Lur1, Lur4, Ci] focused on developing the extensive technical machinery necessary to justify the existence of the theory of ∞ -categories. Here we’ll only give a brief and informal introduction, taking well-foundedness of the theory for granted. We’ll focus on understanding how this language is useful in the study of “derived” or “homotopical” objects that are most naturally regarded up to something weaker than isomorphism (such as quasi-isomorphism, equivalence, or weak equivalence). Relevant examples for us will be:

- the singular (co)chain complex of a topological space,
- the cotangent complex of a scheme or stack,
- the derived category of (quasi-)coherent sheaves on a scheme or stack,
- stacks and higher stacks.

Indeed, it is only by working ∞ -categorically that we are able to take advantage of the descent properties satisfied by these objects. For example, we have:

Theorem 0.0.11.

- (i) *The assignment $X \mapsto \mathbf{C}^\bullet(X; \mathbf{Z})$, sending a topological space to its complex of singular cochains, is a sheaf with values in the derived ∞ -category of abelian groups.*
- (ii) *The assignment $X \mapsto \mathbf{D}_{\text{qc}}(X)$, resp. $X \mapsto \mathbf{D}_{\text{coh}}(X)$, sending a scheme to its stable ∞ -category of quasi-coherent (resp. coherent) sheaves, is a sheaf of ∞ -categories.*

This ability to speak about sheaves of derived objects is one of the fundamental features of ∞ -categories and has many far-reaching consequences (of which we shall only see a small glimpse). Note in contrast that $X \mapsto \mathbf{C}^\bullet(X; \mathbf{Z})$ does not satisfy descent when regarded with values in the usual derived category of abelian groups, nor in the category of chain complexes. Likewise, the descent condition fails also if we replace $X \mapsto \mathbf{D}_{\text{qc}}(X)$ by the assignment sending X to the usual derived category (regarded as an ordinary category).

Another key aspect of ∞ -category theory is a very flexible framework for deriving functors. Recall that the usual framework of homological algebra allows us to derive functors between abelian categories. Using the language of ∞ -categories we may even derive constructions like the tangent bundle (or cotangent sheaf); the corresponding derived functor, the derived tangent bundle (or cotangent complex), is naturally defined on the ∞ -category of derived stacks, which is not a derived category in the traditional sense. In fact, the ∞ -categorical approach is useful even for deriving abelian categories like the category of quasi-coherent sheaves on a scheme or stack: by descent

we may reduce to the case of affine schemes, thereby bypassing most of the technicalities in classical approaches.

We will benefit from this flexibility in Lectures 8–10 in our study of the cotangent complex. We will see how to compute the cotangent complexes of the moduli stacks $\mathcal{M}_{\text{Vect}(X)}$, $\mathcal{M}_{\text{Bun}_G(X)}$, and $\mathcal{M}_{\text{Coh}(X)}$ when X is a curve. This will establish the smoothness asserted in Theorem 0.0.8 and will also be used in our proof of algebraicity. In fact, these moduli stacks are algebraic even when X is higher-dimensional. We will give a new proof of this fact by showing that the *derived* versions of these moduli stacks have nice cotangent complexes even over higher-dimensional X . We then appeal to the Artin–Lurie representability criterion to deduce algebraicity. Thus our main goal in this part will be the following fundamental result, refining Theorem 0.0.8 (which unfortunately does not seem to be covered in the standard textbooks [Lur3, TV2, GR]):

Theorem 0.0.12. *Let X be a smooth proper scheme over a field k . Let \mathcal{M} be the derived moduli stack $\mathcal{M}_{\text{Vect}(X)}$ of vector bundles on X , $\mathcal{M}_{\text{Coh}(X)}$ of coherent sheaves on X , or $\mathcal{M}_{\text{Bun}_G(X)}$ of principal G -bundles on X (for an algebraic group G). Then \mathcal{M} is a derived algebraic stack which is “homotopically smooth” in the sense that its cotangent complex $\mathbf{L}_{\mathcal{M}}$ is perfect. In addition, if X is of dimension $\leq d$, then $\mathbf{L}_{\mathcal{M}}$ is of Tor-amplitude $\leq d - 1$. In particular, it is smooth if X is a curve and quasi-smooth if X is a surface.*

Let \mathcal{M}_{cl} denote the classical moduli stack of vector bundles (resp. coherent sheaves, principal G -bundles) on X . There is a surjective closed immersion $\mathcal{M}_{\text{cl}} \hookrightarrow \mathcal{M}$, which is an isomorphism if and only if X is a curve, hence if and only if \mathcal{M} is smooth. As soon as X is of dimension 2 or greater, \mathcal{M}_{cl} is singular with unbounded cotangent complex while \mathcal{M} is still homotopically smooth. In other words, the homotopical smoothness of \mathcal{M} is a property that can be witnessed only through the lens of derived algebraic geometry. This phenomenon, labelled “hidden smoothness” by Kontsevich [Kon], is the source of “virtual” phenomena on \mathcal{M} .

We will conclude the notes with a brief introduction to the cohomological approach to (virtual) intersection theory on stacks following [Kha1]. We will see how ∞ -categorical descent allows us to work effectively with cohomology and Borel–Moore homology of stacks. In particular, this gives a conceptual approach to Kontsevich’s virtual fundamental class and its important properties, such as the virtual torus localization formula (with no need for the usual auxiliary technical hypotheses on global smooth embeddings or global resolutions and such).

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1. ∞ -CATEGORIES

1.1. Simplicial sets. For every integer $n \geq 0$, let $[n]$ denote the finite set $\{0, 1, \dots, n\}$. Let $\mathbf{\Delta}$ denote the category whose objects are the finite sets $[n]$, for all $n \geq 0$, and whose morphisms are order-preserving maps.

Definition 1.1.1. A *simplicial set* is a functor $X : \mathbf{\Delta}^{\text{op}} \rightarrow \text{Set}$, i.e., a contravariant functor on $\mathbf{\Delta}$ with values in the category of sets. A morphism of simplicial sets is a natural transformation of the corresponding functors. The category of simplicial sets is the functor category $\text{SSet} = \text{Fun}(\mathbf{\Delta}^{\text{op}}, \text{Set})$.

In other words, a simplicial set X is a sequence of sets $X_n := X([n])$ together with a collection of maps $\alpha^* : X_n \rightarrow X_m$ for all order-preserving maps $\alpha : [m] \rightarrow [n]$, subject to the identities $\text{id}^* = \text{id}$ and $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$ whenever α and β are composable. Elements of the set X_n are called *n-simplices* of X .

Example 1.1.2. Given a set X , we let $c(X)$ denote the *constant* simplicial set on X . We have $c(X)_n = X$ for all n , and every map $\alpha : [m] \rightarrow [n]$ in $\mathbf{\Delta}$ induces the identity map $\text{id} : X \rightarrow X$. The assignment $X \mapsto c(X)$ defines a canonical functor

$$c : \text{Set} \rightarrow \text{SSet}$$

which is fully faithful.

Notation 1.1.3. Given integers $n \geq 0$ and $0 \leq i \leq n$, we denote by

$$\delta_n^i : [n-1] \rightarrow [n]$$

the injective map that “skips” i , and by

$$\sigma_n^i : [n+1] \rightarrow [n]$$

the surjective map that “doubles” i . Given a simplicial set X , the induced maps

$$d_n^i := X(\delta_n^i) : X_n \rightarrow X_{n-1}$$

are called *face maps* and the induced maps

$$s_n^i := X(\sigma_n^i) : X_n \rightarrow X_{n+1}$$

are called *degeneracy maps*.

Remark 1.1.4. To specify a simplicial set X , it is enough to specify the sets X_n along with face and degeneracy maps satisfying certain relations (which one can read off the simplex category $\mathbf{\Delta}$). We will often depict X by the diagram

$$\cdots \begin{array}{c} \rightrightarrows \\ \rightleftarrows \end{array} X_2 \begin{array}{c} \rightrightarrows \\ \rightleftarrows \end{array} X_1 \rightrightarrows X_0$$

where for simplicity we only draw the face maps.

Example 1.1.5. For every $n \geq 0$, the *standard n -simplex* is a simplicial set Δ^n whose set of k -simplices ($k \geq 0$) is

$$\Delta_k^n = \text{Hom}_{\mathbf{\Delta}}([k], [n]).$$

That is, an k -simplex of Δ^n is an increasing sequence of integers (a_0, \dots, a_k) with $0 \leq a_i \leq a_j \leq n$ for all $i \leq j$. Given a morphism $\alpha : [j] \rightarrow [k]$, the induced map

$$\alpha^* : \Delta_k^n \rightarrow \Delta_j^n$$

sends $([k] \rightarrow [n])$ to the composite $([j] \rightarrow [k] \rightarrow [n])$.

Remark 1.1.6. Let X be a simplicial set. By the Yoneda lemma, the datum of an n -simplex $x \in X_n$ is the same as that of a morphism $x : \Delta^n \rightarrow X$.

Example 1.1.7. For every $n \geq 0$ and $0 \leq k \leq n$, let $\Delta^{n-1} \rightarrow \Delta^n$ denote the map of standard simplices induced by $\delta_n^k : [n-1] \rightarrow [n]$; on i -simplices it sends $([i] \rightarrow [n-1])$ to $([i] \rightarrow [n-1] \rightarrow [n])$. Its image is a simplicial subset

$$\partial^k \Delta^n \subseteq \Delta^n$$

called the *k th face* of the standard n -simplex. The union of $\partial^k \Delta^n$ over k is a simplicial subset

$$\partial \Delta^n \subseteq \Delta^n$$

called the *boundary* of the standard n -simplex.

Example 1.1.8. For every $n \geq 0$ and $0 \leq k \leq n$, the union of the faces $\partial^j \Delta^n$ over $j \neq k$ is a simplicial subset

$$\Lambda_k^n \subseteq \Delta^n$$

called the *k th horn* of the standard n -simplex. In other words, Λ_k^n is the boundary $\partial \Delta^n$ minus the k th face $\partial^k \Delta^n$.

1.2. Categories as simplicial sets.

Construction 1.2.1. Let \mathcal{C} be a category. The *nerve of \mathcal{C}* is a simplicial set $N(\mathcal{C})$ defined as follows. For every $n \geq 0$, we set

$$N(\mathcal{C})_n := \text{Fun}([n], \mathcal{C}).$$

For every $\alpha : [m] \rightarrow [n]$ in $\mathbf{\Delta}$, $\alpha^* : N(\mathcal{C})_n \rightarrow N(\mathcal{C})_m$ is given by composition: $([n] \rightarrow \mathcal{C}) \mapsto ([m] \rightarrow [n] \rightarrow \mathcal{C})$.

Remark 1.2.2. In other words, n -simplices of $N(\mathcal{C})$ are strings

$$c_0 \rightarrow c_1 \rightarrow \dots \rightarrow c_n$$

of morphisms in \mathcal{C} (where c_i are objects of \mathcal{C}). For example, 0-simplices are objects of \mathcal{C} , 1-simplices are morphisms of \mathcal{C} , 2-simplices are diagrams $c_0 \rightarrow c_1 \rightarrow c_2$ in \mathcal{C} , and so on. Informally speaking, the simplicial set $N(\mathcal{C})$ contains all the information about the category \mathcal{C} .

Exercise 1.2.3. The assignment $\mathcal{C} \mapsto N(\mathcal{C})$ determines a fully faithful functor $N : \text{Cat} \rightarrow \text{SSet}$ from the category of categories to the category of simplicial sets. Moreover, it admits a left adjoint $\tau : \text{SSet} \rightarrow \text{Cat}$ (hint: left Kan extension along $\mathbf{\Delta} \rightarrow \text{Cat}$).

Exercise 1.2.3 means that we can think of categories as “special” simplicial sets, or conversely of simplicial sets as “generalized” categories. The second point of view leads to the question: to what extent do simplicial sets admit a category theory?

Definition 1.2.4. Let X be a simplicial set.

- (i) An *object* x of X is a 0-simplex $x \in X_0$, or by Yoneda, a morphism $x : \Delta^0 \rightarrow X$.
- (ii) A *morphism* f in X is a 1-simplex $f \in X_1$, or by Yoneda, a morphism $f : \Delta^1 \rightarrow X$.
- (iii) The *source* (resp. *target*) of a morphism f in X is the image of the corresponding 1-simplex $f \in X_1$ along the face map $d_1^1 : X_1 \rightarrow X_0$ (resp. $d_0^1 : X_1 \rightarrow X_0$). Equivalently, in terms of the corresponding morphism $f : \Delta^1 \rightarrow X$, these are the composites

$$\Delta^0 \rightarrow \Delta^1 \xrightarrow{f} X$$

with the inclusions of the two faces of the standard 1-simplex.

- (iv) For an object x of X , the *identity morphism* $\text{id}_x \in X_1$ is the image of the 0-simplex $x \in X_0$ by the degeneracy map $s_0^0 : X_0 \rightarrow X_1$. Equivalently, in terms of the corresponding morphism $x : \Delta^0 \rightarrow X$, it is the composite

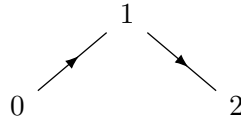
$$\Delta^1 \rightarrow \Delta^0 \xrightarrow{x} X$$

with the (unique) morphism $\Delta^1 \rightarrow \Delta^0$.

Notation 1.2.5. We write $s := d_1^1 : X_1 \rightarrow X_0$ and $t := d_0^1 : X_1 \rightarrow X_0$ for the source and target maps, respectively. Given two objects $x, y \in X_0$, we use the shorthand $f : x \rightarrow y$ to indicate that $f \in X_1$ is a 1-simplex with source $x = s(f)$ and target $y = t(f)$.

A defining feature of morphisms in category theory is that they are *composable*. So to justify the above definitions we need to understand how composition should work in this context.

Remark 1.2.6. Consider the horn Λ_1^2 , which we may depict as “two Δ^1 ’s attached at a Δ^0 ”:



A more precise way of putting this is that there is a cartesian and cocartesian square of simplicial sets

$$\begin{array}{ccc} \Delta^0 & \hookrightarrow & \Delta^1 \\ \downarrow & & \downarrow \\ \Delta^1 & \hookrightarrow & \Lambda_1^2 \end{array}$$

Definition 1.2.7. Let X be a simplicial set.

- (i) A *composable pair of morphisms* in X is a morphism $\Lambda_1^2 \rightarrow X$. This is the same data as that of two morphisms f and g in X such that $t(f) = s(g)$.
- (ii) A *composition* of a composable pair $\sigma : \Lambda_1^2 \rightarrow X$ is a 2-simplex $\tilde{\sigma}$ extending σ . That is, it is a morphism $\tilde{\sigma} : \Delta^2 \rightarrow X$ such that $\tilde{\sigma}|_{\Lambda_1^2} = \sigma$.
- (iii) More generally, given a morphism $\sigma : \Lambda_k^n \rightarrow X$, where $n \geq 2$ and $0 < k < n$, a *composition* of σ is an extension $\tilde{\sigma} : \Delta^n \rightarrow X$.

The question now becomes: in which simplicial sets do compositions exist (uniquely)? If we include the uniqueness requirement, it turns out that this exactly characterizes nerves of categories.

Proposition 1.2.8 (Grothendieck–Segal). *Let X be a simplicial set. Then the following two conditions are equivalent:*

- (i) X belongs to the essential image of the fully faithful functor $N : \text{Cat} \rightarrow \text{SSet}$ (see Exercise 1.2.3).
- (ii) For every $n \geq 2$ and every $0 < k < n$, the map

$$\text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda_k^n, X)$$

given by restriction along the inclusion $\Lambda_k^n \subseteq \Delta^n$, is bijective.

1.3. Groupoids and Kan complexes. In the previous subsection we attempted to set up a “category theory” for simplicial sets, but just ended up recovering usual category theory. Things start to get more interesting if we relax the *uniqueness* condition on composition. Let’s first play with this idea in the context of groupoids (categories in which all morphisms are invertible).

Remark 1.3.1. There is a variant of Proposition 1.2.8 which characterizes *groupoids* \mathcal{C} by the bijectivity of the restriction map

$$\text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda_k^n, X)$$

for all $n \geq 2$ and $0 \leq k \leq n$. The edge cases $k = 0$ and $k = n$ correspond to invertibility of morphisms (rather than composition).

Definition 1.3.2. A simplicial set X is called a *Kan complex* if

$$\text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda_k^n, X)$$

is *surjective* for all $n \neq 0$ and $0 \leq k \leq n$. For example, for a category \mathcal{C} , $N(\mathcal{C})$ is a Kan complex if and only if \mathcal{C} is a groupoid.

Kan complexes (named after Daniel Kan) are like generalized groupoids where compositions (and inverses) exist, but not uniquely. This weak composition still turns out to yield surprisingly good behaviour, at least up to homotopy:

Theorem 1.3.3 (Milnor). *There is an equivalence between the homotopy category¹ of CW complexes and that of Kan complexes, given by the construction $X \mapsto \text{Sing}(X)$. sending a CW complex to its singular simplicial set,*

¹The *homotopy category* of CW complexes is the categorical localization (in the sense of Gabriel–Zisman, see [GZ, Chap. I]) with respect to weak homotopy equivalences. This

whose n -simplices are continuous maps $\Delta_{\text{top}}^n \rightarrow X$ (with Δ_{top}^n the topological standard n -simplex).

In view of Theorem 1.3.3 we can think of objects (0-simplices) of a Kan complex as points in a space, and of morphisms as paths between points. Through this equivalence, we see that up to homotopy, composition in a Kan complex is not that bad: for example, composites not only exist but are unique at least *up to homotopy*. This is encouraging.

1.4. ∞ -Categories as weak Kan complexes. Building on what we have seen so far, our next hope to isolate a class of simplicial sets where composition is well-behaved up to homotopy, but where not all morphisms are required to be invertible.

$$\begin{array}{ccc} \text{groupoid} & \longrightarrow & \text{category} \\ \downarrow & & \downarrow \\ \text{Kan complex} & \longrightarrow & ? \end{array}$$

Definition 1.4.1 (Boardman–Vogt). A simplicial set X is called a *weak Kan complex* (a.k.a. *quasi-category*) if the restriction map

$$\text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda_k^n, X)$$

is *surjective* for all $n \geq 2$ and $0 < k < n$.

Construction 1.4.2. Given a weak Kan complex X and two morphisms $f, g : x \rightarrow y$ in X , a *homotopy* $f \sim g$ is a 2-simplex $\sigma : \Delta^2 \rightarrow X$ of the form

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow \text{id} \\ x & \xrightarrow{g} & y. \end{array}$$

This defines an equivalence relation on the set of morphisms $x \rightarrow y$. The *homotopy category* $\mathbf{h}(X)$ is the category whose set of objects is X_0 and, for $x, y \in X_0$, the set $\text{Hom}_{\mathbf{h}(X)}(x, y)$ is the set of equivalence classes of morphisms $x \rightarrow y$. Since X is a weak Kan complex, we can compose such equivalence classes and check that this gives a well-defined category $\mathbf{h}(X)$. Moreover, one can prove that $\mathbf{h}(X) \simeq \tau(X)$ (where τ is as in Exercise 1.2.3).

Definition 1.4.3. Let X be a weak Kan complex. A morphism $f : x \rightarrow y$ is an *isomorphism* if it is invertible, i.e., if there exists a morphism $g : y \rightarrow x$ and homotopies $f \circ g \sim \text{id}_y$ and $g \circ f \sim \text{id}_x$. One can prove that a morphism is an isomorphism if and only if it induces an isomorphism in $\mathbf{h}(X)$. We say that X is an ∞ -*groupoid* if every morphism in X is an isomorphism.

Theorem 1.4.4 (Joyal). *Let X be a weak Kan complex. Then X is an ∞ -groupoid if and only if X is a Kan complex.*

is analogous to the derived category (which is a categorical localization with respect to quasi-isomorphisms). The homotopy category of Kan complexes is defined similarly; for the definition of homotopy groups and weak homotopy equivalences of Kan complexes, see [GZ, Chap. VI, §3]. See [GZ, Chap. VII, §3] for a proof of Theorem 1.3.3.

Remark 1.4.5. We can think of weak Kan complexes as those simplicial sets which admit compositions up to coherent homotopy. Moreover, after the extensive work of André Joyal and Jacob Lurie, weak Kan complexes do admit a full “category theory”:

- (i) Given weak Kan complexes X and Y , the internal hom $\underline{\text{Hom}}(X, Y)$ behaves like a functor category $\text{Fun}(X, Y)$. Recall that the n -simplices of $\underline{\text{Hom}}(X, Y)$ are maps $\Delta^n \times X \rightarrow Y$.
- (ii) Given a weak Kan complex X and objects x, y of X , there is a Kan complex $\text{Maps}_X(x, y)$ of maps $x \rightarrow y$, defined by the cartesian square

$$\begin{array}{ccc} \text{Maps}_X(x, y) & \longrightarrow & \underline{\text{Hom}}(\Delta^1, X) \\ \downarrow & & \downarrow (s, t) \\ \Delta^0 & \xrightarrow{(x, y)} & X \times X. \end{array}$$

This is a replacement for the Hom-set $\text{Hom}(x, y)$.

This justifies the following definition:

Definition 1.4.6. An ∞ -category is a weak Kan complex.

Remark 1.4.7. The only difference between the two terms is that *weak Kan complex* refers to a specific model (“shadow”) of the platonic notion of ∞ -category; similarly for Kan complexes vs. ∞ -groupoids. In accordance with this point of view, we will simply write \mathcal{C} instead of $N(\mathcal{C})$ when we want to think of an ordinary category \mathcal{C} as an ∞ -category (as opposed to a weak Kan complex). Similarly, we will use letters like \mathcal{C} and \mathcal{D} instead of X and Y when we want to think of them as ∞ -categories, and we will write $\text{Fun}(\mathcal{C}, \mathcal{D})$ instead of $\underline{\text{Hom}}(X, Y)$.

1.5. The ∞ -category of (∞ -)groupoids. Recall that for a category \mathcal{C} we have the weak Kan complex $N(\mathcal{C})$ in which an n -simplex is determined by the following data:

- objects $C_i \in \mathcal{C}$ for $0 \leq i \leq n$;
- morphisms $f_{i,j} : C_i \rightarrow C_j$ for $0 \leq i < j \leq n$;

satisfying the relations $f_{j,k} \circ f_{i,j} = f_{i,k}$ for all $0 \leq i < j < k \leq n$.

Construction 1.5.1. Let \mathcal{C} be a 2-category. The *nerve* (or *Duskin nerve*) of \mathcal{C} is a simplicial set $N^D(\mathcal{C})$. An n -simplex of $N^D(\mathcal{C})$ is determined by the following data:

- objects $C_i \in \mathcal{C}$ for $0 \leq i \leq n$;
- morphisms $f_{i,j} : C_i \rightarrow C_j$ for $0 \leq i < j \leq n$;
- 2-morphisms $f_{j,k} \circ f_{i,j} \Rightarrow f_{i,k}$ for all $0 \leq i < j < k \leq n$;

This data is required to satisfy certain compatibility relations involving $\mu_{i,j,k}$, $\mu_{i,j,l}$, $\mu_{i,k,l}$, and $\mu_{j,k,l}$ (for $0 \leq i < j < k < l \leq n$).

Informally speaking, the difference between $N^D(\mathcal{C})$ and the nerve of the underlying 1-category (where we discard the 2-morphisms) is that the diagrams

$$\begin{array}{ccccc} C_i & \xrightarrow{f_{i,j}} & C_j & \xrightarrow{f_{j,k}} & C_k \\ & & \searrow & \nearrow & \\ & & & & f_{i,k} \end{array}$$

only commute up to the specified natural transformation $\mu_{i,j,k}$ (which need not be invertible).

Theorem 1.5.2 (Duskin). *Let \mathcal{C} be a 2-category. The following conditions are equivalent:*

- (i) \mathcal{C} is a $(2,1)$ -category; i.e., the 2-morphisms of \mathcal{C} are all invertible.
- (ii) $N^D(\mathcal{C})$ is a weak Kan complex.

Notation 1.5.3. Let \mathcal{C} be a category, resp. $(2,1)$ -category. We will write simply \mathcal{C} for the ∞ -category whose underlying weak Kan complex is $N(\mathcal{C})$, resp. $N^D(\mathcal{C})$.

Example 1.5.4. Groupoids naturally form a 2-category whose 2-morphisms are natural transformations. We denote by Grpd the $(2,1)$ -category where we discard the non-invertible 2-morphisms. We also use the same notation for the associated ∞ -category (whose underlying weak Kan complex is $N^D(\text{Grpd})$).

Construction 1.5.5. Consider the (large) simplicial set Kan_\bullet in which an n -simplex is given by the following data:

- Kan complexes K_i for $0 \leq i \leq n$.
- Maps of Kan complexes $f_{i,j} : K_i \rightarrow K_j$ for $0 \leq i \leq j \leq n$.
- A system of “coherent homotopies” up to which $f_{i,j}$ are compatible under composition.

For example, a 2-simplex is a tuple $(X, Y, Z, f, g, h, \sigma)$, where X, Y and Z are Kan complexes, $f : X \rightarrow Y$, $g : Y \rightarrow Z$, $h : X \rightarrow Z$ are maps, and $\sigma \in \underline{\text{Hom}}(X, Z)_1$ with $d_1^0(\sigma) = g \circ f$ and $d_1^1(\sigma) = h$. Here σ is a “homotopy” $g \circ f \simeq h$. The term “coherent” is a shorthand which indicates that not only do we have such homotopies, we are also given higher homotopies between these homotopies (starting from $n \geq 3$), even higher homotopies between those homotopies, and so on.

Remark 1.5.6. The simplicial set Kan_\bullet is a weak Kan complex. In fact, it is an instance of a general construction called the *homotopy coherent nerve* which takes a simplicially enriched category (in this case, that of Kan complexes) as input and yields a weak Kan complex as output. See [Lur4, Tag 00KS].

Remark 1.5.7. There is a map of weak Kan complexes $N(\text{Set}) \rightarrow \text{Kan}_\bullet$ that sends $X \mapsto c(X)$ on 0-simplices. As a functor of ∞ -categories, it is fully faithful with essential image spanned by Kan complexes X that are homotopy equivalent to a constant simplicial set, or equivalently, which satisfy $\pi_i(X) = 0$ for all $i > 0$.

Later on, we will see how the ∞ -category corresponding to the weak Kan complex Kan_\bullet is a very fundamental object called the ∞ -category of *anima*. In practice, we will work with this ∞ -category by manipulating its universal properties and deliberately avoid any considerations involving the simplices of Kan_\bullet .

2. SHEAVES AND STACKS

2.1. Sheaves. Let \mathcal{C} be a site, i.e., a category equipped with a Grothendieck topology τ . Roughly speaking, τ amounts to a notion of *covering sieves* for every object in the category \mathcal{C} . For simplicity, we will assume that τ arises from the following construction.

Construction 2.1.1. Assume that \mathcal{C} admits fibred products and finite coproducts, and satisfies the following conditions:

- (i) For any finite collection of objects $X_i \in \mathcal{C}$, any morphism $f : \coprod_i X_i \rightarrow Y$, and any morphism $Y' \rightarrow Y$, the canonical morphism $\coprod_i X_i \times_Y Y' \rightarrow (\coprod_i X_i) \times_Y Y'$ is invertible.
- (ii) Coproducts are disjoint: for any pair of objects X and Y in \mathcal{C} , the fibred product $X \times_X \coprod_Y Y$ is an initial object in \mathcal{C} .

Let S be a collection of morphisms in \mathcal{C} , which contains all isomorphisms and is stable under composition, and satisfies the following conditions:

- (i) For every morphism $f : X \rightarrow Y$ in S , the base change $f' : X \times_Y Y' \rightarrow Y'$ along any morphism $Y' \rightarrow Y$ in \mathcal{C} belongs to S .
- (ii) For every finite collection of morphisms $(f_i : X_i \rightarrow Y_i)_i$ in S , the induced morphism $\coprod_i X_i \rightarrow \coprod_i Y_i$ belongs to S .

Then there is a Grothendieck topology τ on \mathcal{C} where a sieve on an object $X \in \mathcal{C}$ is covering if and only if it contains a finite collection of morphisms $\{X_i \rightarrow X\}_i$ such that the induced morphism $\coprod_i X_i \rightarrow X$ belongs to S . We refer to this as the Grothendieck topology *generated by* the collection S . See [Lur3, Prop. A.3.2.1].

Example 2.1.2. Let X be a topological space. Then the category $\mathcal{U}(X)$ of opens $U \subseteq X$ (where there is a morphism $U \rightarrow V$ if and only if $U \subseteq V$) admits a Grothendieck topology generated by surjections. In particular, a sieve on $U \subseteq X$ is covering if and only if it contains a finite collection of morphisms $(U_i \hookrightarrow U)_i$ such that $U = \bigcup_i U_i$.

Example 2.1.3. Let X be a scheme. The *small étale site* $X_{\text{ét}}$ is the category of étale morphisms $U \rightarrow X$ (where U is a scheme), with the Grothendieck topology generated by surjections (equivalently, faithfully flat morphisms).

Example 2.1.4. Let X be a scheme. The *big étale site* is the category Sch/X of (arbitrary) morphisms $Y \rightarrow X$ (where Y is a scheme), with the Grothendieck topology generated by étale surjections (equivalently, faithfully

flat and étale morphisms). (We also have the étale topology on the category Sch , which is the special case where $X = \text{Spec}(\mathbf{Z})$.)

Definition 2.1.5. Let \mathcal{V} be an ∞ -category. Let F be a diagram in \mathcal{V} indexed by an ∞ -category I , i.e., a functor of ∞ -categories $F : I \rightarrow \mathcal{C}$. Suppose given an object $V \in \mathcal{V}$ and a natural transformation $\alpha : V_{\text{cst}} \rightarrow F$ where V_{cst} denotes the constant diagram $(i \in I) \mapsto (V \in \mathcal{V})$. We say that the pair (V, α) in \mathcal{V} *exhibits V as the limit of F* if for every object $V' \in \mathcal{V}$ the induced functor of mapping ∞ -groupoids

$$\text{Maps}_{\mathcal{V}}(V', V) \rightarrow \text{Maps}_{\text{Fun}(I, \mathcal{V})}(V'_{\text{cst}}, F),$$

sending $(V' \rightarrow V) \mapsto (V'_{\text{cst}} \rightarrow V_{\text{cst}} \rightarrow F)$, is invertible. In this case, we will write

$$V \simeq \varprojlim_{i \in I} F_i := \varprojlim(F).$$

Definition 2.1.6. Let \mathcal{V} be an ∞ -category. A *presheaf* on \mathcal{C} with values in \mathcal{V} is a functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$. A presheaf F is a *sheaf* if it satisfies the following conditions:

- (i) F sends finite coproducts in \mathcal{C} to products in \mathcal{V} . In other words, for every finite collection $(X_i)_i$ of objects of \mathcal{C} , the canonical morphism

$$F\left(\coprod_i X_i\right) \rightarrow \prod_i F(X_i)$$

is invertible.

- (ii) For every morphism $f : U \rightarrow X$ in S , let U_{\bullet} denote the Čech nerve of f , i.e., the simplicial object

$$\cdots \rightrightarrows U \times_U U \times_U U \rightrightarrows U \times_U U \rightrightarrows U$$

whose n th term is the $(n+1)$ -fold fibre power $U \times_X \cdots \times_X U$. Then the canonical map

$$F(X) \rightarrow \varprojlim_{[n] \in \Delta} F(U_n)$$

is invertible. In other words, the diagram

$$F(X) \rightarrow F(U) \rightrightarrows F(U \times_U U) \rightrightarrows F(U \times_U U \times_U U) \rightrightarrows \cdots$$

exhibits $F(X)$ as the limit of $F(U_{\bullet})$.

Notation 2.1.7. We denote by $\text{Shv}(\mathcal{C}; \mathcal{V})$ the full subcategory of $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V})$ spanned by sheaves.

When \mathcal{V} is a 1-category (resp. $(2,1)$ -category), this is equivalent to the usual sheaf condition:

Proposition 2.1.8. *Let $V^{\bullet} : \Delta \rightarrow \mathcal{V}$ be a cosimplicial diagram in an ∞ -category \mathcal{V} . If \mathcal{V} is equivalent to a 1-category, then the limit of V^{\bullet} is identified with the equalizer of $V^0 \rightrightarrows V^1$:*

$$\varprojlim_{[n] \in \Delta} V^n \simeq \varprojlim(V^0 \rightrightarrows V^1).$$

Similarly, if \mathcal{V} is equivalent to a $(2, 1)$ -category, then it is identified with the 2-limit:

$$\lim_{\leftarrow [n] \in \Delta} V^n \simeq 2\text{-}\lim(V^0 \rightrightarrows V^1 \rightrightarrows V^2).$$

Remark 2.1.9. More generally, say \mathcal{V} is an $(n, 1)$ -category if all mapping ∞ -groupoids $\text{Maps}_{\mathcal{V}}(V, V')$ are $(n-1)$ -truncated (have trivial higher homotopy groups $\pi_i \text{Maps}_{\mathcal{V}}(V, V') = 0$ for $i \geq n$). In this case the limit of V^\bullet is isomorphic to the limit of the restriction $V|_{\Delta_{\leq n}}$ to the full subcategory of Δ spanned by the objects $[0], [1], \dots, [n]$. (This follows from a variant of Quillen's Theorem A, because the inclusion $\Delta_{\leq n} \hookrightarrow \Delta$ is an n -final functor, i.e. the category $\Delta_{\leq n} \times_{\Delta} \Delta_{/[m]}$ has n -connected nerve for every $[m] \in \Delta$.) Note also that a limit over $\Delta_{\leq 1}$ is (by an easy finality argument) isomorphic to the limit over the subcategory where the morphism $[1] \rightarrow [0]$ is discarded; i.e., it is the equalizer of the two parallel arrows $V^0 \rightrightarrows V^1$.

Definition 2.1.10. A *stack* is a sheaf of groupoids on the category of schemes (with the étale topology).

In other words, a stack \mathcal{X} is a functor

$$\mathcal{X} : \text{Sch}^{\text{op}} \rightarrow \text{Grpd}$$

to the $(2, 1)$ -category of groupoids such that for every finite collection of étale morphisms $(U_i \rightarrow U)_i$ which is jointly surjective, the diagram

$$\mathcal{X}(U) \rightarrow \prod_i \mathcal{X}(U_i) \rightrightarrows \prod_{i,j} \mathcal{X}(U_i \times_U U_j) \rightrightarrows \prod_{i,j,k} \mathcal{X}(U_i \times_U U_j \times_U U_k)$$

is a limit diagram in the $(2, 1)$ -category of groupoids.

2.2. Bases of topologies. Denote by $\text{Aff} \subseteq \text{Sch}$ the full subcategory spanned by affine schemes. Note that the étale topology on Sch restricts to Aff , and is still generated in the sense of Construction 2.1.1 by étale surjections.

Theorem 2.2.1. *Let \mathcal{V} be an ∞ -category admitting limits. The canonical functor*

$$\text{Fun}(\text{Sch}^{\text{op}}, \mathcal{V}) \rightarrow \text{Fun}(\text{Aff}^{\text{op}}, \mathcal{V}),$$

given by restriction along the inclusion $\text{Aff} \subseteq \text{Sch}$, restricts to an equivalence of ∞ -categories

$$\text{Shv}(\text{Sch}; \mathcal{V}) \rightarrow \text{Shv}(\text{Aff}; \mathcal{V}).$$

Example 2.2.2. Stacks can be defined as sheaves of groupoids on Aff .

Theorem 2.2.1 is a special case of a following more general result.

Definition 2.2.3. Let \mathcal{C} be a site with Grothendieck topology generated by a class of morphisms S as in Construction 2.1.1. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under fibred products and finite coproducts, and regard it with the Grothendieck topology generated by $S \cap \mathcal{C}_0$ (the subclass of morphisms in S whose source and target belong to \mathcal{C}_0). We say that \mathcal{C}_0 is a *basis* for \mathcal{C} if for every object $X \in \mathcal{C}$ there exists a collection of morphisms $(Y_i \rightarrow X)_i$ such that $Y_i \in \mathcal{C}_0$, the coproduct $\coprod_i Y_i$ exists in \mathcal{C} , and $\coprod_i Y_i \rightarrow X$ belongs to S .

Theorem 2.2.4. *In the situation of Definition 2.2.3, the functor*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \rightarrow \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{V})$$

restricts to an equivalence

$$\mathrm{Shv}(\mathcal{C}; \mathcal{V}) \rightarrow \mathrm{Shv}(\mathcal{C}_0; \mathcal{V})$$

for all ∞ -categories \mathcal{V} admitting limits.

We can moreover give a more precise version of Theorem 2.2.4.

Definition 2.2.5. Let $i : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ be a fully faithful functor of categories. Let $F_0 : \mathcal{C}_0^{\mathrm{op}} \rightarrow \mathcal{V}$ be a presheaf with values in an ∞ -category \mathcal{V} admitting limits. The *right Kan extension* of F_0 , denoted²

$$F := \mathrm{RKE}_{\mathcal{C}_0 \rightarrow \mathcal{C}}(F_0) := i_*(F_0)$$

is the unique limit-preserving functor $F : \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{V}$ which restricts to F_0 . Explicitly, it is given by the formula

$$F(X) \simeq \varprojlim_{(Y, f)} F(Y)$$

where the limit is taken over the category of pairs (Y, f) where $Y \in \mathcal{C}_0$ and $f : i(Y) \rightarrow X$ is a morphism in \mathcal{C} (and morphisms $(Y', f') \rightarrow (Y, f)$ are morphisms $Y' \rightarrow Y$ in \mathcal{C}_0 which are compatible with f and f').

Theorem 2.2.6. *In the situation of Definition 2.2.3, let $F : \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{V}$ be a \mathcal{V} -valued presheaf on \mathcal{C} where \mathcal{V} is an ∞ -category with limits. Then F is a sheaf if and only if the following conditions hold:*

- (i) $F_0 := F|_{\mathcal{C}_0}$ is a sheaf on \mathcal{C}_0 .
- (ii) F is the right Kan extension of F_0 along $\mathcal{C}_0 \rightarrow \mathcal{C}$.

See [Aok, Cor. A.8] for a proof.

3. THE STACK OF QUASI-COHERENT SHEAVES

3.1. Cartesian fibrations. Given a scheme X , quasi-coherent sheaves on X form a category $\mathrm{QCoh}(X)$. For any morphism $f : X \rightarrow Y$, we have the adjoint pair of functors

$$f^* : \mathrm{QCoh}(Y) \rightleftarrows \mathrm{QCoh}(X) : f_*$$

where the left adjoint f^* is inverse image and the right adjoint f_* is direct image. Note that for a pair of composable morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Z) & \xrightarrow{g^*} & \mathrm{QCoh}(Y) \\ & \searrow^{(g \circ f)^*} & \swarrow_{f^*} \\ & & \mathrm{QCoh}(X) \end{array}$$

²The explanation for the notation $i_*(F_0)$ is that the assignment $F_0 \mapsto F$ actually determines a right adjoint i_* to the restriction functor $i^* : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \rightarrow \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{V})$.

does not commute in the 1-category of categories. Instead, there is an invertible natural transformation

$$(g \circ f)^* \rightarrow f^* \circ g^*$$

up to which it commutes. Together with this extra piece of data, the diagram above does determine a commutative diagram in the ∞ -category Grpd .

In particular, the assignment $X \mapsto \text{QCoh}(X)$ cannot be assembled into a functor $\text{Sch}^{\text{op}} \rightarrow \text{Cat}'$ into the 1-category Cat' of categories, but only into a functor

$$\text{QCoh} : \text{Sch}^{\text{op}} \rightarrow \text{Cat}$$

into the $(2,1)$ -category³ (or equivalently, ∞ -category) of categories. Still, a precise construction of this functor requires more than just the data of the invertible natural transformation above for all pairs of morphisms f and g ; for example, we need to require compatibilities between this data whenever we have three composable morphisms. This is somewhat messy, so we will prefer to take the following alternative perspective.

Definition 3.1.1. Let $\pi : \mathcal{E} \rightarrow \mathcal{C}$ be a functor of categories. Let $f : C \rightarrow D$ be a morphism in \mathcal{C} and $\tilde{D} \in \mathcal{E}$ a lift of D (so that $\pi(\tilde{D}) = D$). Let $\tilde{f} : \tilde{C} \rightarrow \tilde{D}$ be a lift of f , i.e. $\pi(\tilde{C}) = C$ and $\pi(\tilde{f}) = f$. We say that \tilde{f} is π -cartesian if for every $E \in \mathcal{E}$ we require that the canonical map

$$\text{Hom}_{\mathcal{E}}(E, \tilde{C}) \rightarrow \text{Hom}_{\mathcal{E}}(E, \tilde{D}) \times_{\text{Hom}_{\mathcal{C}}(\pi(E), D)} \text{Hom}_{\mathcal{E}}(\pi(E), \pi(\tilde{C}))$$

is bijective. Informally speaking, \tilde{f} is terminal among all lifts of f with target \tilde{D} .

Definition 3.1.2. Let $\pi : \mathcal{E} \rightarrow \mathcal{C}$ be a functor of categories. We say that π is a *cartesian fibration* if for every $E \in \mathcal{E}$, every $C \in \mathcal{C}$, and every morphism $f : C \rightarrow \pi(E)$ in \mathcal{C} , there exists a lift $\tilde{C} \in \mathcal{E}$ of C and a π -cartesian morphism $\tilde{f} : \tilde{C} \rightarrow E$ lifting f .

Example 3.1.3. Let QCoh_{Sch} denote the category of pairs (X, \mathcal{F}) where $X \in \text{Sch}$ and $\mathcal{F} \in \text{QCoh}(X)$. A morphism $(X', \mathcal{F}') \rightarrow (X, \mathcal{F})$ is a morphism $f : X' \rightarrow X$ together with a morphism $\phi : f^* \mathcal{F} \rightarrow \mathcal{F}'$ in $\text{QCoh}(X')$. Given morphisms $(f, \phi) : (X', \mathcal{F}') \rightarrow (X, \mathcal{F})$ and $(g, \psi) : (X'', \mathcal{F}'') \rightarrow (X', \mathcal{F}')$, the composite is defined by

$$f \circ g : X'' \rightarrow X' \rightarrow X$$

and

$$g^* f^* \mathcal{F} \xrightarrow{g^* \phi} g^* \mathcal{F}' \xrightarrow{\psi} \mathcal{F}''.$$

Then the projection $(X, \mathcal{F}) \mapsto X$ determines a functor

$$\text{QCoh}_{\text{Sch}} \rightarrow \text{Sch}$$

which is a cartesian fibration.

³Recall that by convention, we identify $(2,1)$ -categories \mathcal{C} with the corresponding ∞ -category (whose underlying weak Kan complex is the Duskin nerve $N^D(\mathcal{C})$).

Construction 3.1.4. Let \mathcal{C} be a fixed category. We denote by $\text{Cart}(\mathcal{C})$ the $(2, 1)$ -category whose objects are cartesian fibrations $\pi : \mathcal{E} \rightarrow \mathcal{C}$, whose 1-morphisms $(\pi' : \mathcal{E}' \rightarrow \mathcal{C}) \rightarrow (\pi : \mathcal{E} \rightarrow \mathcal{C})$ are morphisms $f : \mathcal{E}' \rightarrow \mathcal{E}$ that are compatible with π and π' , and whose 2-morphisms $f \Rightarrow g$ (where f, g are morphisms $\pi' \rightarrow \pi$) are invertible natural transformations $\theta : f \Rightarrow g$ such that for all $E' \in \mathcal{E}'$, $\pi : \mathcal{E} \rightarrow \mathcal{C}$ sends

$$\theta_{E'} : f(E') \rightarrow g(E')$$

to the identity of $\pi(f(E')) = \pi(g(E'))$.

Theorem 3.1.5 (Grothendieck). *For every category \mathcal{C} , there is an equivalence of ∞ -categories*

$$\text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}) \rightarrow \text{Cart}(\mathcal{C}).$$

Definition 3.1.6. Given $F : \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$, the corresponding cartesian fibration is called the *unstraightening* of F . Given a cartesian fibration $\pi : \mathcal{E} \rightarrow \mathcal{C}$, the *straightening* of π is the (essentially unique) presheaf of categories whose unstraightening is π .

Remark 3.1.7. Lurie proved a generalization of Theorem 3.1.5, where \mathcal{C} is allowed to be an ∞ -category and Cat is replaced by the ∞ -category of ∞ -categories.

Definition 3.1.8. We let $\text{QCoh} : \text{Sch}^{\text{op}} \rightarrow \text{Cat}$ denote the unstraightening of the cartesian fibration $\text{QCoh}_{\text{Sch}} \rightarrow \text{Sch}$. We let $\text{QCoh}^{\simeq} : \text{Sch}^{\text{op}} \rightarrow \text{Grpd}$ denote the presheaf of groupoids given by sending

$$X \mapsto \text{QCoh}(X)^{\simeq}$$

where $(-)^{\simeq}$ indicates that we discard all non-invertible morphisms.

3.2. Descent for quasi-coherent sheaves.

Theorem 3.2.1. *The presheaf of categories $\text{QCoh} : \text{Sch}^{\text{op}} \rightarrow \text{Cat}$ satisfies étale descent.*

In fact, we will see that it even satisfies descent for the *fpqc*⁴ topology on Sch , which is the Grothendieck topology generated by faithfully flat quasi-compact morphisms.

Corollary 3.2.2. *The presheaf of groupoids $\text{QCoh}^{\simeq} : \text{Sch}^{\text{op}} \rightarrow \text{Grpd}$ satisfies descent. In particular, it is a stack.*

Proof. The functor $\text{Cat} \rightarrow \text{Grpd}$ sending $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$ preserves limits. In fact, it is right adjoint to the inclusion $\text{Grpd} \hookrightarrow \text{Cat}$. Thus the sheaf condition for QCoh^{\simeq} follows from that of QCoh . \square

To prove Theorem 3.2.1 we will apply a general descent criterion.

⁴This stands for “fidèlement plat et quasi-compact” in French.

Definition 3.2.3. Given a cosimplicial diagram $X^\bullet : \Delta \rightarrow \mathcal{C}$ in an ∞ -category \mathcal{C} , we refer to its limit as the *totalization* of X^\bullet , and write

$$\mathrm{Tot}(X^\bullet) := \varprojlim_{\Delta} X^\bullet.$$

Definition 3.2.4. Let Δ_+ denote the category whose objects are the finite sets $[n] = \{0, 1, \dots, n\}$ for all $n \geq -1$, where $[-1] = \emptyset$ by convention, and whose morphisms are order-preserving maps. An *augmented cosimplicial diagram* in any ∞ -category \mathcal{C} is a functor $X^\bullet : \Delta_+ \rightarrow \mathcal{C}$. We will depict X^\bullet by the diagram

$$X^{-1} \rightarrow X^0 \rightrightarrows X^1 \rightrightarrows X^2 \rightrightarrows \dots$$

We will say this is a *limit diagram* if the induced morphism of cosimplicial diagrams

$$X_{\mathrm{cst}}^{-1} \rightarrow X^\bullet|_{\Delta}$$

exhibits X^{-1} as the limit (totalization) of $X^\bullet|_{\Delta}$.

Definition 3.2.5. Let $\Delta_{-\infty}$ denote the category whose objects are the finite sets $[n]$ for all $n \geq -1$, and whose morphisms $[m] \rightarrow [n]$ are order-preserving maps $[m] \cup \{-\infty\} \rightarrow [n] \cup \{-\infty\}$ which preserve $-\infty$. A *splitting* of a cosimplicial diagram $X^\bullet : \Delta \rightarrow \mathcal{C}$ is an extension to a functor $X^\bullet : \Delta_{-\infty} \rightarrow \mathcal{C}$. A simplicial diagram X^\bullet is *split* if it admits a splitting. In this case, $X^\bullet|_{\Delta_+}$ is a limit diagram, i.e. we have $\mathrm{Tot}(X^\bullet|_{\Delta}) \simeq X^{-1}$.

Theorem 3.2.6 (Descent criterion). *Let $\mathcal{C}^\bullet : \Delta_+ \rightarrow \mathrm{Cat}_\infty$ ⁵ be an augmented cosimplicial diagram of ∞ -categories, which we depict as follows:*

$$\mathcal{C}^{-1} \xrightarrow{F} \mathcal{C}^0 \rightrightarrows \mathcal{C}^1 \rightrightarrows \mathcal{C}^2 \rightrightarrows \dots \quad (3.2.7)$$

Suppose the following conditions hold:

- (i) *The functor $F : \mathcal{C}^{-1} \rightarrow \mathcal{C}^0$ is conservative.*
- (ii) *For every morphism $\alpha : [m] \rightarrow [n]$ in Δ_+ , let $\beta : [m+1] \rightarrow [n+1]$ denote the unique morphism which commutes with $\delta^0 : [m] \rightarrow [m+1]$ and $\delta^0 : [n] \rightarrow [n+1]$, and consider the commutative square*

$$\begin{array}{ccc} \mathcal{C}^m & \xrightarrow{d^0} & \mathcal{C}^{m+1} \\ \downarrow \alpha & & \downarrow \beta \\ \mathcal{C}^n & \xrightarrow{d^0} & \mathcal{C}^{n+1}. \end{array}$$

Then the horizontal arrows admit right adjoints $d^{0,R}$ which also commute with the vertical arrows; more precisely, the natural transformation

$$\alpha \circ d^{0,R} \xrightarrow{\mathrm{unit}} d^{0,R} \circ d^0 \circ \alpha \circ d^{0,R} \simeq d^{0,R} \circ \beta \circ d^0 \circ d^{0,R} \xrightarrow{\mathrm{counit}} d^{0,R} \circ \beta$$

is invertible.

⁵Here the ∞ -category Cat_∞ of ∞ -categories may be defined just as we defined the ∞ -category of ∞ -groupoids, replacing Kan complexes with weak Kan complexes.

- (iii) The functor $F : \mathcal{C}^{-1} \rightarrow \mathcal{C}^0$ preserves totalizations of F -split simplicial diagrams in \mathcal{C}^{-1} . That is, for every cosimplicial diagram X^\bullet in \mathcal{C}^{-1} whose image $F(X^\bullet)$ is split, the canonical map $F(\mathrm{Tot}(X^\bullet)) \rightarrow \mathrm{Tot}(F(X^\bullet))$ is invertible.

Then the induced functor

$$\mathcal{C}^{-1} \rightarrow \mathrm{Tot}(\mathcal{C}^\bullet |_{\Delta})$$

is an equivalence of ∞ -categories. That is, (3.2.7) is a limit diagram.

This result is a corollary of the monadicity theorem of Barr–Beck, generalized to ∞ -categories by Lurie. See [Lur3, Cor. 4.7.5.3].

Consider the functor

$$\mathrm{CRing} \rightarrow \mathrm{Cat}, \quad R \mapsto \mathrm{Mod}_R$$

sending a commutative ring R to the category Mod_R of R -modules, and a ring homomorphism $\phi : R \rightarrow S$ to the extension of scalars functor

$$\phi^* := (-) \otimes_R S : \mathrm{Mod}_R \rightarrow \mathrm{Mod}_S.$$

It can be defined using Theorem 3.1.5, or alternatively is the restriction of $\mathrm{QCoh} : \mathrm{Sch}^{\mathrm{op}} \rightarrow \mathrm{Cat}$ to affine schemes (under the equivalence $\mathrm{Aff}^{\mathrm{op}} \simeq \mathrm{CRing}$).

Theorem 3.2.8. *The functor $R \mapsto \mathrm{Mod}_R$ satisfies descent for the flat topology, i.e. the Grothendieck topology on CRing generated by faithfully flat ring homomorphisms.*

Proof. Let $\phi : R \rightarrow S$ be a faithfully flat ring homomorphism. We need to show that the augmented cosimplicial diagram

$$\mathrm{Mod}_R \xrightarrow{\phi^*} \mathrm{Mod}_S \rightrightarrows \mathrm{Mod}_{S \otimes_R S} \rightrightarrows \cdots$$

is a limit diagram, to which end we apply the criterion of Theorem 3.2.6:

- (i) Conservativity of the functor ϕ^* is a consequence of the assumption that ϕ is faithfully flat.
- (ii) For every morphism $\alpha : [m] \rightarrow [n]$ in Δ_+ , consider the commutative square

$$\begin{array}{ccc} \mathrm{Mod}_{S^{\otimes m+1}} & \xrightarrow{d^{0,*}} & \mathrm{Mod}_{S^{\otimes m+2}} \\ \downarrow \alpha & & \downarrow \beta \\ \mathrm{Mod}_{S^{\otimes n+1}} & \xrightarrow{d^{0,*}} & \mathrm{Mod}_{S^{\otimes n+2}}, \end{array}$$

where the tensor powers are taken over R and the horizontal arrows are extension of scalars along $d^0 : S^{\otimes m+1} \rightarrow S^{\otimes m+2}$. These functors are left adjoint to the restriction of scalars functors d_*^0 . The vertical arrows are flat (as base changes of flat homomorphisms), and hence commute with d_*^0 by the flat base change formula.

- (iii) Let M^\bullet be a cosimplicial diagram in Mod_R such that the cosimplicial diagram $\phi^*(M^\bullet)$ in Mod_S is split. The claim is that the S -module homomorphism

$$\phi^*(\text{Tot}(M^\bullet)) \rightarrow \text{Tot}(\phi^*(M^\bullet))$$

is invertible. Since Mod_R and Mod_S are 1-categories, these totalizations can be computed as equalizers (Proposition 2.1.8). Since ϕ^* is an exact functor, it preserves equalizers.

We also need to show that the functor $R \mapsto \text{Mod}_R$ preserves finite products:

- The category of modules over the zero ring is equivalent to the trivial category.
- For any pair of commutative rings R_1 and R_2 , the category of modules over $R_1 \times R_2$ is equivalent to the product of categories $\text{Mod}_{R_1} \times \text{Mod}_{R_2}$.

It follows that $R \mapsto \text{Mod}_R$ is a sheaf for the flat topology. \square

The proof of Theorem 3.2.1 is almost the same, using the following lemma:

Lemma 3.2.9. *Let $f : X \rightarrow Y$ be a morphism of schemes. If f is faithfully flat and quasi-compact, then the inverse image functor $f^* : \text{QCoh}(Y) \rightarrow \text{QCoh}(X)$ is conservative.*

Proof. Let $(V_j \rightarrow X)_j$ be a (possibly infinite) family of affine opens $V_j \subseteq Y$ covering Y . For each j , let $V'_j := V_j \times_Y X = f^{-1}(V_j) \subseteq X$. Since f is quasi-compact, V'_j is quasi-compact. Thus there exists a finite family $(U_{i,j} \rightarrow V'_j)_i$ where $U_{i,j} \subseteq V'_j$ are affine opens covering V'_j . We have the commutative square

$$\begin{array}{ccc} \coprod_i U_{i,j} & \xrightarrow{f_j} & V_j \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $f_j : \coprod_i U_{i,j} \rightarrow V'_j \rightarrow V_j$ is a faithfully flat of affine schemes, since faithfully flat morphisms are stable under composition and base change.

Now let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of quasi-coherent sheaves on Y , whose inverse image $f^*(\phi) : f^*(\mathcal{F}) \rightarrow f^*(\mathcal{G})$ is invertible. We need to show that ϕ is itself invertible. It will clearly suffice to show that each restriction $\phi|_{V_j}$ is invertible. Since f_j is a faithfully flat morphism between affines, we know (by definition) that f_j^* is conservative (since it corresponds to extension of scalars of a module along a faithfully flat ring homomorphism), so it will moreover suffice to show that ϕ is invertible after inverse image along $\coprod_i U_{i,j} \rightarrow V_j \subseteq Y$. But the latter factors through f , hence is invertible by assumption. \square

Proof of Theorem 3.2.1. Same as the proof of Theorem 3.2.8, except Lemma 3.2.9 is used to check the first condition of Theorem 3.2.6. \square

3.3. Quasi-coherent sheaves on stacks. Since the full subcategory $\text{Aff} \subseteq \text{Sch}$ is a basis with respect to the Zariski topology, we have by Theorem 2.2.6 the following further consequence of Theorem 3.2.1:

Corollary 3.3.1. *The presheaf of categories $\text{QCoh} : \text{Sch}^{\text{op}} \rightarrow \text{Cat}$ is right Kan extended from its restriction to Aff .*

Under the equivalence $\text{Aff}^{\text{op}} \simeq \text{CRing}$, $\text{QCoh}|_{\text{Aff}}$ is identified with the functor $\text{CRing} \rightarrow \text{Cat}$ sending $R \mapsto \text{Mod}_R$. Thus we have:

Corollary 3.3.2. *For every scheme X , there is a canonical equivalence of categories*

$$\text{QCoh}(X) \simeq \varprojlim_{(R,x)} \text{Mod}_R$$

where the limit is taken over the category of pairs (R, x) where $R \in \text{CRing}$ and $x \in X(R)$ is an R -point, where morphisms $(R, x) \rightarrow (R', x')$ are ring homomorphisms $R \rightarrow R'$ such that $X(R) \rightarrow X(R')$ sends x to x' .

Definition 3.3.3. Let \mathcal{X} be a stack. We define the category $\text{QCoh}(\mathcal{X})$ of quasi-coherent sheaves on \mathcal{X} as the limit

$$\text{QCoh}(\mathcal{X}) := \varprojlim_{(R,x)} \text{Mod}_R$$

over the category of pairs (R, x) where $R \in \text{CRing}$ and $x \in \mathcal{X}(R)$ is an R -point. More precisely, we define

$$\text{QCoh} : \text{Stk}^{\text{op}} \rightarrow \text{Cat}$$

as the right Kan extension of the presheaf $\text{Spec}(R) \mapsto \text{Mod}_R$ along the inclusion $\text{Aff} \hookrightarrow \text{Stk}$, where Stk is the ∞ -category of stacks.

Remark 3.3.4. By definition, a quasi-coherent sheaf \mathcal{F} on a stack \mathcal{X} amounts to the following data:

- (i) For every commutative ring R and every R -point $x \in \mathcal{X}(R)$, an R -module $\mathcal{F}(x)$.
- (ii) For every ring homomorphism $R \rightarrow R'$, R -point $x \in \mathcal{X}(R)$, and R' -point $x' \in \mathcal{X}(R')$ such that $\mathcal{X}(R) \rightarrow \mathcal{X}(R')$ sends $x \mapsto x'$, an R' -module isomorphism

$$\alpha_{x,x'} : \mathcal{F}(x) \otimes_R R' \simeq \mathcal{F}(x').$$

This data is subject to the following cocycle condition: for every pair of ring homomorphisms $R \rightarrow R'$ and $R' \rightarrow R''$, R -point $x \in \mathcal{X}(R)$, R' -point $x' \in \mathcal{X}(R')$, and R'' -point $x'' \in \mathcal{X}(R'')$ such that $\mathcal{X}(R) \rightarrow \mathcal{X}(R')$ sends $x \mapsto x'$ and $\mathcal{X}(R') \rightarrow \mathcal{X}(R'')$ sends $x' \mapsto x''$, there is a commutative diagram

$$\begin{array}{ccc} (\mathcal{F}(x) \otimes_R R') \otimes_{R'} R'' & \xrightarrow{\alpha_{x,x'}} & \mathcal{F}(x') \otimes_{R'} R'' & \xrightarrow{\alpha_{x',x''}} & \mathcal{F}(x'') \\ \parallel & & & & \parallel \\ \mathcal{F}(x) \otimes_R R'' & \xrightarrow{\alpha_{x,x''}} & & & \mathcal{F}(x''). \end{array}$$

4. QUOTIENT STACKS

Let G be a group. As we discussed in Sect. 0, a G -action on a set X may be encoded by the diagram

$$G \times X \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} X$$

where the two rightwards arrows are the action map $(g, x) \mapsto g \cdot x$ and the projection $(g, x) \mapsto x$, and the leftward arrow is the section $x \mapsto (e, x)$ where $e \in G$ is the neutral element. The colimit of this diagram is the (set-theoretic) quotient X/G . If we instead take the colimit in the ∞ -category Grpd , call it the *quotient groupoid* $[X/G]$, then we may recover the above diagram by taking the fibred product of the canonical functor $X \rightarrow [X/G]$ along itself: there is a canonical isomorphism in Grpd

$$X \times_{[X/G]} X \simeq G \times X$$

under which the two projections of $X \times_{[X/G]} X$ are identified with the action and projection maps $G \times X \rightarrow X$, and the diagonal $X \rightarrow X \times_{[X/G]} X$ is identified with the section $(e, \text{id}) : X \rightarrow G \times X$. Informally speaking, the quotient groupoid $[X/G]$ remembers everything about the G -action on X .

This story may be generalized to the case where X is a groupoid or even ∞ -groupoid. In that setting, the diagram above must be replaced by a simplicial diagram

$$\cdots \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} G \times G \times X \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} G \times X \rightrightarrows X.$$

We let $\mathcal{X} = [X/G]$ denote the the geometric realization, i.e. the colimit taken in ∞ -groupoids. Then we can recover the above diagram as the Čech nerve of the quotient map $X \twoheadrightarrow \mathcal{X}$. This leads to a bijective correspondence between *groupoid objects* acting on X (certain simplicial diagrams) and maps $X \twoheadrightarrow \mathcal{X}$ which are surjective on π_0 , where one direction is formation of the geometric realization, and the other is formation of the Čech nerve.

We begin by briefly recalling how this works in the greater generality where ∞ -groupoids are replaced by objects of any “ ∞ -topos”, such as the ∞ -topos of ∞ -stacks, i.e., étale sheaves of ∞ -groupoids on Sch .

4.1. Groupoid objects and effective epimorphisms.

Definition 4.1.1. A *groupoid object* in an ∞ -category \mathcal{C} is a simplicial diagram $U_\bullet : \Delta^{\text{op}} \rightarrow \mathcal{C}$ such that for all $[n] \in \Delta$ and all partitions $[n] = S \cup S'$ into subsets S and S' such that $S \cap S'$ consists of a single element $m \in [n]$, the square

$$\begin{array}{ccc} U_n = U_\bullet([n]) & \longrightarrow & U_\bullet(S) \\ \downarrow & & \downarrow \\ U_\bullet(S') & \longrightarrow & U_\bullet(\{m\}) = U_0 \end{array}$$

is cartesian. A *morphism* of groupoid objects is a morphism of simplicial diagrams (i.e., a natural transformation).

Remark 4.1.2. Given a groupoid object U_\bullet , we think of $U_0 \in \mathcal{C}$ as the object of “objects” of U_\bullet and $U_1 \in \mathcal{C}$ as the object of “morphisms” of U_\bullet . The above condition for “ordered”⁶ partitions such as $[2] = \{0, 1\} \cup \{1, 2\}$ means, informally speaking, that we may fill all inner horns: for example, the morphism $(d_2^0, d_2^2) : U_2 \rightarrow U_1 \times_{U_0} U_1$ is invertible, where the two morphisms $U_1 \rightarrow U_0$ are “source” and “target”, so that there is a “composition” morphism

$$U_1 \times_{U_0} U_1 \simeq U_2 \xrightarrow{d_2^1} U_1.$$

Similarly, for “unordered” partitions such as $[2] = \{0, 1\} \cup \{0, 2\}$ or $[2] = \{0, 2\} \cup \{1, 2\}$, the condition allows us to fill outer horns (and thus choose inverses to “morphisms” in U_\bullet).

Example 4.1.3. Given a 1-groupoid \mathcal{G} , its nerve $\mathcal{G}_\bullet := N(\mathcal{G}) \in \mathbf{SSet}$ defines a groupoid object in the 1-category \mathbf{Set} . In fact, the assignment $\mathcal{G} \mapsto \mathcal{G}_\bullet$ determines an equivalence from the 1-category of 1-groupoids to the ∞ -category of groupoid objects in \mathbf{Set} . We may also regard \mathcal{G}_\bullet as a groupoid object in \mathbf{Grpd}_∞ (via the fully faithful functor $\mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Set}) \rightarrow \mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Grpd}_\infty)$ induced by the embedding $\mathbf{Set} \rightarrow \mathbf{Grpd}_\infty$ of sets as discrete ∞ -groupoids).

Example 4.1.4. Given a group G , we may consider the 1-groupoid with a single object $*$ and endomorphism group $\text{End}(*) = G$, with composition law defined by the multiplication law of G . The corresponding groupoid object G_\bullet is a simplicial diagram of the form

$$\cdots \rightrightarrows G \times G \rightrightarrows G \rightrightarrows \text{pt},$$

where the face maps are induced by the group multiplication $m : G \times G \rightarrow G$ and the degeneracy maps are induced by the neutral element $e : \text{pt} \rightarrow G$.

Example 4.1.5. Let $f : U \rightarrow X$ be a morphism in \mathcal{C} such that the iterated fibred products $U \times_X \cdots \times_X U$ all exist in \mathcal{C} . The Čech nerve U_\bullet of f is the simplicial diagram

$$\cdots \rightrightarrows U \times_U U \times_X U \rightrightarrows U \times_X U \rightrightarrows U.$$

More precisely, let $\Delta_{\leq 0, +} \subseteq \Delta_+$ denote the full subcategory spanned by the objects $[0]$ and $[-1]$. Then the morphism f determines a diagram $\Delta_{\leq 0, +}^{\text{op}} \rightarrow \mathcal{C}$, whose right Kan extension is an augmented simplicial diagram $U_\bullet^+ : \Delta_+^{\text{op}} \rightarrow \mathcal{C}$. The Čech nerve of f is its restriction $U_\bullet := U_\bullet^+|_{\Delta}$. This is always a groupoid object in \mathcal{C} (see [Lur1, Prop. 6.1.2.11]).

Notation 4.1.6. Given a simplicial diagram $U_\bullet : \Delta^{\text{op}} \rightarrow \mathcal{C}$, we will also refer to its colimit as its *geometric realization*, and denote it by

$$|U_\bullet| := \varinjlim_{[n] \in \Delta} U_n \in \mathcal{C}$$

when it exists.

⁶i.e., partitions $[n] = S \cup S'$ such that $s \leq s'$ for all $s \in S$ and $s' \in S'$

Definition 4.1.7. A morphism $f : U \rightarrow X$ in \mathcal{C} is an *effective epimorphism* if the iterated fibred products $U \times_X \cdots \times_X U$ all exist in \mathcal{C} , and the augmented simplicial diagram

$$\cdots \rightrightarrows U \times_U U \times_X U \rightrightarrows U \times_X U \rightrightarrows U \rightarrow X$$

is a colimit diagram, i.e., exhibits X as the geometric realization of the Čech nerve U_\bullet .

We now specialize to the ∞ -category of ∞ -stacks.

Notation 4.1.8. Let Stk_∞ denote the ∞ -category of ∞ -stacks, i.e., étale sheaves of ∞ -groupoids $\mathcal{X} : \text{Sch}^{\text{op}} \rightarrow \text{Grpd}_\infty$. Any stack can be regarded as an ∞ -stack, via the fully faithful functor $\text{Stk} \hookrightarrow \text{Stk}_\infty$ induced by the inclusion $\text{Grpd} \hookrightarrow \text{Grpd}_\infty$.

The following two results hold more generally for sheaves of ∞ -groupoids on any site (see [Lur1, Thm. 6.1.0.6, Cor. 6.2.3.5]):

Theorem 4.1.9 (Lurie).

- (i) Let U_\bullet be a groupoid object in Stk_∞ . Then the canonical morphism $U_0 \rightarrow |U_\bullet|$ is an effective epimorphism in Stk_∞ .
- (ii) The assignment sending U_\bullet to the morphism $U_0 \rightarrow |U_\bullet|$ determines an equivalence from the ∞ -category of groupoid objects in Stk_∞ to the ∞ -category of effective epimorphisms in Stk_∞ . Its inverse is the functor forming the Čech nerve.

It will also be useful to have the following more explicit description of effective epimorphisms:

Proposition 4.1.10. A morphism $f : U \rightarrow X$ is an effective epimorphism in Stk_∞ if and only if it is étale-locally surjective, i.e., for every commutative ring R and every R -point $x \in X(R)$, there exists a faithfully flat étale homomorphism $R \rightarrow R'$ such that the image of x in $X(R')$ belongs to the essential image of the functor $f(R') : U(R') \rightarrow X(R')$.

Example 4.1.11. A smooth morphism of schemes $f : X \rightarrow Y$ is surjective if and only if it is an effective epimorphism.

4.2. Group actions.

Definition 4.2.1. A *group object* in an ∞ -category \mathcal{C} is a groupoid object G_\bullet such that $G_0 \in \mathcal{C}$ is a terminal object. Given a group object G_\bullet , we will often abuse notation by saying that $G := G_1 \in \mathcal{C}$ is a group object.

Example 4.2.2. For any group G , the corresponding groupoid object G_\bullet in Set (Example 4.1.4) is a group object. It may also be regarded as a group object in Grpd , which we still denote by G_\bullet . Similarly, if $G : \text{Sch}^{\text{op}} \rightarrow \text{Grp}$ is an (étale) sheaf of groups (such as represented by a group scheme), then it determines a group object in $\text{Shv}(\text{Sch}; \text{Set})$ and hence also in Stk (or Stk_∞).

Definition 4.2.3. Let G_\bullet be a group object in an ∞ -category \mathcal{C} . An *action* of G_\bullet on an object $U \in \mathcal{C}$ is a groupoid object U_\bullet with an isomorphism $U_0 \simeq U$ and a morphism of simplicial diagrams $U_\bullet \rightarrow G_\bullet$ such that for every morphism $[m] \rightarrow [n]$ in Δ sending $0 \mapsto 0$, the square

$$\begin{array}{ccc} U_n & \longrightarrow & U_m \\ \downarrow & & \downarrow \\ G_n & \longrightarrow & G_m \end{array}$$

is cartesian.

Remark 4.2.4. In the definition above, it is equivalent to require that for every $[n] \in \Delta$, the square

$$\begin{array}{ccc} U_n & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ G_n & \longrightarrow & G_0, \end{array}$$

where the horizontal arrows are induced by the morphism $[0] \rightarrow [n]$ in Δ sending $0 \mapsto 0$, is cartesian. In particular, there is for every $[n] \in \Delta$ a canonical isomorphism $U_n \simeq G_n \times_{G_0} U_0 \simeq G^{\times n} \times U$. Thus the simplicial diagram U_\bullet is of the form

$$\cdots \rightrightarrows G \times G \times U \rightrightarrows G \times U \rightrightarrows U.$$

Moreover, the face map $d^1 : G \times U \rightarrow U$ is the projection $(g, u) \mapsto u$, since the square

$$\begin{array}{ccc} U_1 & \xrightarrow{d^1} & U_0 \\ \downarrow & & \downarrow \\ G_1 & \xrightarrow{d^1} & G_0 \end{array}$$

is cartesian by assumption. Similarly, the degeneracy map $s^0 : U \rightarrow G \times U$ is the section (e', id) where $e' : U \rightarrow \text{pt} \rightarrow G$ with pt the terminal object and $e = s^0 : \text{pt} \rightarrow G$ the “neutral element” (part of the data of the group object G_\bullet).

Notation 4.2.5. Let U_\bullet be an action of a group object G_\bullet on $U \in \mathcal{C}$. By Remark 4.2.4, the face map $d^0 : U_1 \rightarrow U_0$ determines an isomorphism

$$\text{act} : G \times U \rightarrow U$$

which we call the *action morphism*. Informally speaking, we may summarize the data of the action U_\bullet as an action morphism $G \times U \rightarrow U$ which satisfies the analogues of the usual axioms of a group action *up to coherent homotopy*.

Definition 4.2.6. We say that an action U_\bullet is *trivial* if for every morphism $[m] \rightarrow [n]$ the square

$$\begin{array}{ccc} U_n & \xrightarrow{d^0} & U_m \\ \downarrow & & \downarrow \\ G_n & \xrightarrow{d^0} & G_m \end{array}$$

is cartesian. In particular, the action morphism $G \times U \rightarrow U$ is in this case also the projection. For any object U , the trivial action on U is defined by the groupoid $U_{\bullet}^{\text{triv}} := G_{\bullet} \times_{G_0} U$.

Notation 4.2.7. We will often abuse language by saying “let G be a group object acting on U ” instead of “let U_{\bullet} be an action of a group object G_{\bullet} on U ”. If we need to speak of U_{\bullet} , we will refer to it as the *action groupoid* (or *quotient groupoid*). If G acts on U and V (meaning that there are actions U_{\bullet} and V_{\bullet} with $U_0 \simeq U$ and $V_0 \simeq V$), a G -equivariant morphism $f : U \rightarrow V$ is a morphism of simplicial diagrams $f_{\bullet} : U_{\bullet} \rightarrow V_{\bullet}$ fitting in a commutative diagram as follows.

$$\begin{array}{ccc} U_{\bullet} & \xrightarrow{f_{\bullet}} & V_{\bullet} \\ & \searrow & \swarrow \\ & G_{\bullet} & \end{array}$$

4.3. Torsors.

Definition 4.3.1. Let X be a stack and let G be a group object in Stk . A G -torsor over X is a G -equivariant morphism $\pi : U \rightarrow X$ where U is a stack with an action of G and X is regarded with trivial G -action, satisfying the following conditions:

- (i) $\pi : U \rightarrow X$ is an effective epimorphism;
- (ii) the canonical morphism of simplicial diagrams $U_{\bullet} \rightarrow C_{\bullet}$ is invertible, where U_{\bullet} is the action groupoid and C_{\bullet} denotes the Čech nerve of $\pi : U \rightarrow X$.

Remark 4.3.2. It is immediate from the definition that if $\pi : U \rightarrow X$ is a G -torsor, then we may identify X as the geometric realization of the action groupoid U_{\bullet} . That is, there is a colimit diagram

$$\cdots \rightrightarrows G \times G \times U \rightrightarrows G \times U \rightrightarrows U \rightarrow X.$$

Remark 4.3.3. Condition (ii) in Definition 4.3.1 implies that the square

$$\begin{array}{ccc} U_1 & \xrightarrow{d^0} & U_0 \\ \downarrow d^1 & & \downarrow \\ U_0 & \longrightarrow & X \end{array}$$

is cartesian, i.e., that the canonical morphism

$$(\text{act}, \text{pr}) : G \times U \rightarrow U \times_X U$$

is invertible. In fact, one can show that this is equivalent to (ii).

Remark 4.3.4. We will also consider the variant of Definition 4.3.1 over a fixed base scheme S . This amounts to looking at presheaves on Sch_S instead of Sch ; for example, a stack $X : \text{Sch}^{\text{op}} \rightarrow \text{Grpd}$ over S is the same data as a sheaf $X : \text{Sch}_S^{\text{op}} \rightarrow \text{Grpd}$. Below, we will leave S implicit and simply denote it by “pt”.

Example 4.3.5. The trivial G -torsor is the projection $\pi : G \times X \rightarrow X$. More precisely:

- (i) The G -action on $G \times X$ is given by the groupoid object U_\bullet :

$$\cdots \rightrightarrows G \times G \times G \times X \rightrightarrows G \times G \times X \rightrightarrows G \times X$$

where the face maps are $d^0 = m \times \text{id}$ (where $m : G \times G \rightarrow G$ is the multiplication of the group) and $d^1 = \text{pr}_1$. In other words, G acts by multiplication on the first component and trivially on the second.

- (ii) The G -equivariant map $\pi : G \times X \rightarrow X$ is given by the morphism of simplicial diagrams $U_\bullet \rightarrow X^{\text{triv}} = G_\bullet \times X$:

$$\begin{array}{ccc} \cdots \rightrightarrows & G \times G \times X & \rightrightarrows & G \times X \\ & \downarrow & & \downarrow \\ \cdots \rightrightarrows & G \times X & \rightrightarrows & X \end{array}$$

which on level n is given by projecting onto the last $n+1$ components of $G^{n+1} \times X$.

- (iii) Note that $G \rightarrow \text{pt}$ is an effective epimorphism, since it admits a section $e : \text{pt} \rightarrow G$. Hence its base change $\pi : G \times X \rightarrow X$ is an effective epimorphism.

- (iv) The square

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \text{id}} & G \times X \\ \downarrow \text{pr}_{2,3} & & \downarrow \text{pr}_2 \\ G \times X & \xrightarrow{\text{pr}_2} & X \end{array}$$

is cartesian, hence U_\bullet is identified with the Čech nerve of π by Remark 4.3.3. Indeed, this square is obtained from the cartesian square

$$\begin{array}{ccc} G \times G & \xrightarrow{\text{pr}_1} & G \\ \downarrow \text{pr}_2 & & \downarrow \\ G & \longrightarrow & \text{pt} \end{array}$$

by precomposing with the inverse of the morphism $(m, \text{pr}_2) : G \times G \rightarrow G \times X$ (which is invertible since G is a group).

Definition 4.3.6. We say that a G -torsor $\pi : U \rightarrow X$ is *trivial* if U is G -equivariantly isomorphic to $G \times X$ over X . Note that this is equivalent to the existence of a section $s : X \rightarrow U$ (so that $\pi \circ s \simeq \text{id}$), since such s gives rise to an isomorphism

$$G \times X \xrightarrow{\text{id} \times s} G \times U \xrightarrow{\text{act}} U.$$

4.4. Quotient stacks.

Definition 4.4.1. Let G a group stack over a base scheme S (i.e., a group object in Stk). Let U be a stack over S with G -action. The *quotient stack*

$[U/G]$ is defined as the geometric realization of the action groupoid U_\bullet , i.e.:

$$[U/G] := |U_\bullet| = \varinjlim_{[n] \in \Delta} U_n,$$

where the colimit is taken in Stk . In particular, there is a colimit diagram

$$\cdots \rightrightarrows G \times G \times U \rightrightarrows G \times U \rightrightarrows U \rightarrow [U/G]$$

in Stk .

Remark 4.4.2. Let $|U_\bullet|^{\text{PreStk}}$ denote the geometric realization taken in the category of prestacks (i.e., in the category $\text{Fun}(\text{Sch}_S^{\text{op}}, \text{Grpd})$). Then $|U_\bullet|^{\text{PreStk}}$ is given by the functor of points sending $T \in \text{Sch}_S$ to the geometric realization $|U_\bullet(T)|$ of the simplicial diagram of groupoids $U_\bullet(T)$. This is rarely a stack, and there is a canonical morphism of prestacks

$$|U_\bullet|^{\text{PreStk}} \rightarrow |U_\bullet| = [U/G]$$

which exhibits $[U/G]$ as the sheafification.

Remark 4.4.3. By construction, the quotient morphism $p: U \rightarrow [U/G]$ is a G -torsor.

We can compute the functor of points of $[U/G]$ as follows:

Theorem 4.4.4. *For every S -scheme T , there is a (functorial) equivalence of groupoids between $[U/G](T)$ and the groupoid of diagrams*

$$T \xleftarrow{\pi} Y \xrightarrow{f} U$$

where Y is an S -stack with G -action, $f: Y \rightarrow U$ is a G -equivariant S -morphism, and π is a G -torsor over Y . An isomorphism $(Y, f, \pi) \rightarrow (Y', f', \pi')$ is a G -equivariant isomorphism $\phi: Y \rightarrow Y'$ together with commutative squares expressing the compatibility of ϕ with f and f' , and with π and π' .

Proof. By the Yoneda lemma, $[U/G](T)$ is the groupoid of morphisms $T \rightarrow [U/G]$. Such a morphism gives rise by base change to a cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{f} & U \\ \downarrow \pi & & \downarrow p \\ T & \longrightarrow & [U/G] \end{array}$$

where the vertical arrows are G -torsors. The morphism f is G -equivariant, since it lifts to a morphism of simplicial diagrams

$$U_\bullet \times_{[U/G]} T \rightarrow U_\bullet$$

by base change, where U_\bullet is the action groupoid of the G -action on U .

Conversely, given a G -torsor $\pi: Y \rightarrow T$, with G -action on Y given by an action groupoid Y_\bullet , and a G -equivariant morphism $f: Y \rightarrow U$ given by a morphism of simplicial diagrams $Y_\bullet \rightarrow U_\bullet$, we may recover $T \rightarrow [U/G]$ as the geometric realization

$$T \simeq [Y/G] = |Y_\bullet| \rightarrow |U_\bullet| = [U/G]$$

by Theorem 4.1.9. \square

Example 4.4.5. The *classifying stack* of G is defined as the quotient stack

$$BG := B_S(G) := [S/G]$$

with respect to the trivial G -action on S . By Theorem 4.4.4, every G -torsor $\pi : Y \rightarrow T$ for an S -scheme T is classified by a morphism $T \rightarrow BG$ fitting into a cartesian square

$$\begin{array}{ccc} Y & \longrightarrow & S \\ \downarrow \pi & & \downarrow p \\ T & \longrightarrow & BG. \end{array}$$

Informally speaking, the quotient morphism $p : S \rightarrow BG$ is the universal G -torsor.

Remark 4.4.6. In particular, the G -torsor $p : U \rightarrow [U/G]$ is classified by the morphism

$$[U/G] \rightarrow BG$$

induced by the G -equivariant morphism $U \rightarrow S$. That is, there is a cartesian square

$$\begin{array}{ccc} U & \longrightarrow & S \\ \downarrow & & \downarrow \\ [U/G] & \longrightarrow & BG. \end{array}$$

More generally:

Lemma 4.4.7. *Let G be a group stack over a scheme S . Let $f : X \rightarrow Y$ be a G -equivariant morphism of stacks over S . Then the induced morphism $g : [X/G] \rightarrow [Y/G]$ fits in a cartesian square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ [X/G] & \xrightarrow{g} & [Y/G] \end{array}$$

where the vertical arrows are the quotient morphisms.

Proof. Exercise. \square

4.5. Examples of quotient stacks.

Example 4.5.1. Let $G = \mathrm{GL}_n$ be the general linear group scheme over S . Let X be an S -scheme. For every locally free sheaf \mathcal{E} on X of rank n , there exists a GL_n -torsor

$$\pi : \mathrm{Isom}_X(\mathcal{O}_X^n, \mathcal{E}) \rightarrow X$$

whose total space represents the sheaf $\mathrm{Sch}_{/X}^{\mathrm{op}} \rightarrow \mathrm{Set}$ which sends $t : T \rightarrow X$ to the set of isomorphisms $\mathcal{O}_T^n \simeq t^* \mathcal{E}$. The assignment $\mathcal{E} \mapsto \mathrm{Isom}_X(\mathcal{O}_X^n, \mathcal{E})$ determines an equivalence between the groupoid of locally free sheaves on X of rank n and the groupoid of GL_n -torsors over X . In particular, morphisms $X \rightarrow \mathrm{BGL}_n$ may be identified with vector bundles over X of rank n .

Example 4.5.2. Consider the quotient stack $[\mathbf{A}^1/\mathbf{G}_m]$ with respect to the weight 1 scaling action of the multiplicative group scheme $\mathbf{G}_m = \mathrm{GL}_1$ on the affine line \mathbf{A}^1 . Then the groupoid of morphisms $T \rightarrow [\mathbf{A}^1/\mathbf{G}_m]$ is equivalent to the groupoid of *generalized divisors* on T , i.e., pairs (\mathcal{L}, s) where \mathcal{L} is a locally free sheaf of rank 1 on T and $s : \mathcal{O}_T \rightarrow \mathcal{L}$ is a section. Indeed, there is a cartesian square in Stk of the form

$$\begin{array}{ccc} \mathbf{A}^1 & \longrightarrow & \mathrm{GDiv} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{BG}_m \end{array}$$

where the right-hand vertical arrow is the forgetful map $(\mathcal{L}, s) \mapsto \mathcal{L}$ (or rather, the \mathbf{G}_m -torsor corresponding to \mathcal{L}); the lower horizontal arrow is the quotient map (which classifies the trivial line bundle); and the upper horizontal arrow sends a T -point of \mathbf{A}^1 given by $f \in \Gamma(T, \mathcal{O}_T)$ to the generalized divisor $(\mathcal{O}_T, f : \mathcal{O}_T \rightarrow \mathcal{O}_T) \in \mathrm{GDiv}(T)$. Since the lower horizontal arrow is a \mathbf{G}_m -torsor, it follows that the upper horizontal arrow is as well. In particular, it exhibits GDiv as the quotient stack $[\mathbf{A}^1/\mathbf{G}_m]$ as claimed.

5. ALGEBRAIC SPACES AND STACKS

5.1. Algebraic spaces.

Definition 5.1.1. A morphism $f : X \rightarrow Y$ in an ∞ -category \mathcal{C} is a *monomorphism* if the square

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \parallel & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian. In other words, the fibred product $X \times_Y X$ exists and the diagonal $\Delta_f : X \rightarrow X \times_Y X$ is invertible.

Example 5.1.2. A functor of groupoids $F : \mathcal{C} \rightarrow \mathcal{D}$ is a monomorphism in the ∞ -category Grpd if it is fully faithful. Equivalently, the induced map on sets of connected components $\pi_0(F) : \pi_0(\mathcal{C}) \rightarrow \pi_0(\mathcal{D})$ is injective, and for every object $X \in \mathcal{C}$ the map of automorphism groups $\mathrm{Aut}_{\mathcal{C}}(X) \rightarrow \mathrm{Aut}_{\mathcal{D}}(F(X))$ is bijective. More generally, a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of ∞ -groupoids is a monomorphism in the ∞ -category Grpd_{∞} if and only if it is fully faithful, or equivalently injective on π_0 and bijective on $\pi_i(\mathcal{C}, X) \rightarrow \pi_i(\mathcal{D}, F(X))$ for all $i > 0$ and objects $X \in \mathcal{C}$.

Remark 5.1.3. Let X be an ∞ -groupoid. If its diagonal $\Delta : X \rightarrow X \times X$ is a monomorphism, then X is discrete (i.e., $X \simeq \pi_0(X)$). Indeed, the ∞ -groupoid of isomorphisms between two objects $x, y \in X$ is given by the fibred product

$$\begin{array}{ccc} \mathrm{Maps}_X(x, y) & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ \mathrm{pt} & \xrightarrow{(x, y)} & X \times X \end{array}$$

Hence $\text{Maps}_X(x, y)$ is either empty or contractible (isomorphic in Grpd_∞ to pt) for all x, y . The converse also holds: for X discrete, the diagonal is always a monomorphism.

Example 5.1.4. A morphism of stacks $f : X \rightarrow Y$ is a monomorphism if and only if $f(T) : X(T) \rightarrow Y(T)$ is a monomorphism of groupoids for all schemes T . Thus if X is a stack with monomorphic diagonal, X takes values in sets (by Remark 5.1.3).

Definition 5.1.5. A morphism $f : X \rightarrow Y$ of stacks is *schematic* (or *representable by schemes*) if for every scheme V and every morphism $v : V \rightarrow Y$, the fibred product $X \times_Y V$ is a scheme.

Lemma 5.1.6. *Let X be a stack. The following conditions are equivalent:*

- (i) *The diagonal $\Delta : X \rightarrow X \times X$ is schematic.*
- (ii) *For every pair of morphisms $u : U \rightarrow X$ and $v : V \rightarrow X$, where U and V are schemes, the fibred product $U \times_X V$ is a scheme.*
- (iii) *Every morphism $f : U \rightarrow X$, where U is a scheme, is schematic.*

Proof. We have (i) \Rightarrow (ii) by the cartesian square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ U \times V & \xrightarrow{u \times v} & X \times X, \end{array}$$

and (ii) = (iii) by definition. Suppose condition (iii) holds and let $f = (u, v) : U \rightarrow X \times X$ be a morphism. Then we have the cartesian square

$$\begin{array}{ccccc} U \times_{X \times X} X & \longrightarrow & U \times_X U & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \Delta \\ U & \xrightarrow{\Delta_U} & U \times U & \xrightarrow{u \times v} & X \times X, \end{array}$$

where the lower horizontal composite arrow is f . Now $U \times_X U$ is a scheme by (iii), so $U \times_{X \times X} X$ is a scheme since Δ_U is schematic (as a morphism between schemes). As U and f vary, this shows (i). \square

Definition 5.1.7. Let X be a stack. We say that X is an *algebraic space* if it satisfies the following conditions:

- (i) The diagonal $\Delta : X \rightarrow X \times X$ is schematic and a monomorphism.
- (ii) There exists a scheme U and a morphism $U \twoheadrightarrow X$ which is étale and surjective, i.e. for every scheme V and every morphism $V \rightarrow X$ the base change $U \times_X V \rightarrow V$ is an étale surjection (where $U \times_X V$ is a scheme by Lemma 5.1.6).

Note that X takes values in sets (Example 5.1.4), i.e., an algebraic space is equivalently a sheaf of sets $X : \text{Sch}^{\text{op}} \rightarrow \text{Set}$ with schematic diagonal and an étale atlas as in (ii).

5.2. Algebraic stacks.

Definition 5.2.1. A morphism $f : X \rightarrow Y$ of stacks is *representable* if for every scheme⁷ V and every morphism $v : V \rightarrow Y$, the fibred product $X \times_Y V$ is an algebraic space.

Definition 5.2.2. A stack X is *algebraic* if it satisfies the following conditions:

- (i) The diagonal $\Delta : X \rightarrow X \times X$ is representable.
- (ii) There exists a scheme U and a morphism $U \twoheadrightarrow X$ which is smooth and surjective, i.e. for every scheme V and every morphism $V \rightarrow X$ the base change $U \times_X V \rightarrow V$ is a smooth surjection (where $U \times_X V$ is an algebraic space by the analogue of Lemma 5.1.6).

We say that X is *Deligne–Mumford* if in (ii), *smooth* is replaced by *étale*.

Remark 5.2.3. We call the smooth surjection $U \twoheadrightarrow X$ in (ii) an *atlas* for X .

5.3. Algebraicity of quotient stacks.

Theorem 5.3.1. *Let G be a smooth group algebraic space over a scheme S . Let U be a stack with G -action. If U is algebraic, then so is the quotient stack $[U/G]$.*

Proof. Let us show that the diagonal of $[U/G]$ is representable. Given a scheme T and a morphism $(t, t') : T \rightarrow [U/G] \times [U/G]$, the claim is that the fibred product $T \times_{[U/G]} T \simeq T \times_{[U/G] \times [U/G]} [U/G]$ is an algebraic space. Since the quotient morphism $p : U \twoheadrightarrow [U/G]$ is an effective epimorphism, there exists by Proposition 4.1.10 a scheme T' and an étale surjection $T' \twoheadrightarrow T$ such that the composite $T' \twoheadrightarrow T \rightarrow [U/G]$ factors through $p : U \twoheadrightarrow [U/G]$. Thus the composite of the cartesian squares

$$\begin{array}{ccccc} T' \times_{[U/G]} T & \longrightarrow & T \times_{[U/G]} T & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow t' \\ T' & \longrightarrow & T & \xrightarrow{t} & [U/G]. \end{array}$$

is identified with the composite of the cartesian squares

$$\begin{array}{ccccc} T' \times_{[U/G]} T & \longrightarrow & Y & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow t' \\ T' & \longrightarrow & U & \xrightarrow{p} & [U/G] \end{array}$$

where $Y \twoheadrightarrow T$ is the G -torsor classified by t' . Since U is algebraic (hence has representable diagonal) and T' and Y are algebraic spaces (the latter because G is), it follows by the analogue of Lemma 5.1.6 that $T' \times_{[U/G]} T$ is an algebraic space. It follows then that $T \times_{[U/G]} T$ is also an algebraic space.

⁷equivalently, algebraic space

It remains to show the existence of an atlas for $[U/G]$. Since $G \rightarrow S$ is a smooth surjection, it follows that the quotient morphism $p : U \rightarrow [U/G]$ is a smooth surjection. (Indeed, given a scheme V and a morphism $v : V \rightarrow [U/G]$, the base change of p along v is the G -torsor classified by v , which is smooth and surjective because $G \rightarrow S$ is.) Since U is algebraic, there exists a scheme U' and a smooth surjection $U' \rightarrow U$. Then the composite $U' \rightarrow U \rightarrow [U/G]$ is smooth and surjective. \square

Remark 5.3.2. Since $p : U \rightarrow [U/G]$ is a G -torsor, the square

$$\begin{array}{ccc} G \times U & \xrightarrow{\text{pr}} & U \\ \downarrow \text{act} & & \downarrow p \\ U & \xrightarrow{p} & [U/G] \end{array}$$

is cartesian (Remark 4.3.3). Equivalently, the diagonal of $[U/G]$ fits into the cartesian square

$$\begin{array}{ccc} G \times U & \xrightarrow{(\text{act}, \text{pr})} & U \times U \\ \downarrow & & \downarrow p \times p \\ [U/G] & \xrightarrow{\Delta} & [U/G] \times [U/G] \end{array}$$

5.4. Recognizing algebraic spaces and DM stacks.

Theorem 5.4.1. *Let X be an algebraic stack.*

- (i) *X is a Deligne–Mumford stack if and only if its diagonal is unramified.*
- (ii) *X is an algebraic space if and only if its diagonal is a monomorphism.*

See [LMB, Thm. 8.1, Cor. 8.1.1].

Corollary 5.4.2. *Let X be a stack whose diagonal $\Delta_X : X \rightarrow X \times X$ is a monomorphism (equivalently, $X : \text{Sch}^{\text{op}} \rightarrow \text{Grpd}$ takes values in Set). Then the following conditions are equivalent:*

- (i) *X is an algebraic space.*
- (ii) *The diagonal Δ_X is representable, and there exists an algebraic space U and an étale surjection $U \rightarrow X$.*
- (iii) *There exists an algebraic space U and a representable étale surjection $U \rightarrow X$.*

5.5. Free and proper actions.

Proposition 5.5.1. *Let G be a smooth group algebraic space over a scheme S . Let U be an algebraic space with G -action. Then the quotient stack $[U/G]$ is an algebraic space if and only if the action of G on U is free, i.e., the morphism*

$$(\text{act}, \text{pr}) : G \times U \rightarrow U \times U$$

is a monomorphism.

Proof. By Theorem 5.3.1, $[U/G]$ is an algebraic stack. Thus by Theorem 5.4.1 it will suffice to show that its diagonal is a monomorphism. This may be tested after base change along a smooth surjection. By Remark 5.3.2, the base change along $p \times p : U \times U \rightarrow [U/G] \times [U/G]$ is precisely the morphism (act, pr) in the statement. \square

Warning 5.5.2. The analogue of Proposition 5.5.1 for schemes is false: there exist schemes X of finite type over a field (which may be taken separated or even proper) and groups G (which may be taken finite discrete) acting on X such that the quotient stack $[X/G]$ is not representable by a scheme.

Similarly, we have:

Proposition 5.5.3. *Let G be a smooth group algebraic space over a scheme S . Let U be an algebraic space with G -action. Then the quotient stack $[U/G]$ is separated, i.e. has proper diagonal, if and only if the action of G on U is proper, i.e., the morphism*

$$(\text{act}, \text{pr}) : G \times U \rightarrow U \times U$$

is proper.

Remark 5.5.4. In Proposition 5.5.3, suppose that G is affine and U has affine diagonal (e.g. U is separated), so that the quotient stack $[U/G]$ has affine diagonal. It follows that the action is proper if and only if $[U/G]$ has *finite* diagonal. If S has all residue fields of characteristic zero, then this implies that $[U/G]$ has unramified diagonal, hence is Deligne–Mumford by Theorem 5.4.1.

5.6. Stabilizers.

Definition 5.6.1. Let X be a stack and $x : \text{Spec}(R) \rightarrow X$ an R -valued point for some $R \in \text{CRing}$. The *stabilizer* of X at the point x is a group stack $\text{St}_X(x)$ over $\text{Spec}(R)$ defined by the cartesian square:

$$\begin{array}{ccc} \text{St}_X(x) & \longrightarrow & \text{Spec}(R) \\ \downarrow & & \downarrow x \\ \text{Spec}(R) & \xrightarrow{x} & X. \end{array}$$

Equivalently, it fits in the cartesian square

$$\begin{array}{ccc} \text{St}_X(x) & \longrightarrow & X \\ \downarrow & & \downarrow \Delta_x \\ \text{Spec}(R) \times \text{Spec}(R) & \xrightarrow{x \times x} & X \times X. \end{array}$$

Its groupoid of R' -points, for an R -algebra R' , is thus the group object in Grpd of automorphisms of (the image of) x in the groupoid $X(R')$.

Remark 5.6.2. If X has representable (resp. schematic, affine) diagonal, then its stabilizers are all algebraic spaces (resp. schemes, affine schemes).

Example 5.6.3. Let U be an algebraic space with an action of a group algebraic space G . The stabilizers of the quotient stack $X = [U/G]$ are precisely the stabilizers of the G -action. More precisely, given an R -point $u \in U(R)$ (where $R \in \text{CRing}$), it follows from Remark 5.3.2 that the group algebraic space $\text{St}_{[U/G]}(u)$ fits in the cartesian square

$$\begin{array}{ccc} \text{St}_{[U/G]}(u) & \longrightarrow & G \times \text{Spec}(R) \\ \downarrow & & \downarrow \text{act}(-, u) \\ \text{Spec}(R) & \xrightarrow{u} & U. \end{array}$$

where $\text{act}(-, u)$ is the composite of $\text{id} \times u : G \times \text{Spec}(R) \rightarrow G \times U$ and $\text{act} : G \times U \rightarrow U$.

By Theorem 5.4.1 we have:

Corollary 5.6.4. *Let X be an algebraic stack. Then X is an algebraic space if and only if for every $R \in \text{CRing}$ and every R -valued point $x : \text{Spec}(R) \rightarrow X$ the stabilizer $\text{St}_X(x)$ is trivial.*

Remark 5.6.5. In Corollary 5.6.4, it turns out that it is moreover sufficient to require that the stabilizer at every *geometric* point is trivial. See [Con, Thm. 2.2.5(1)].

6. THE RESOLUTION PROPERTY

6.1. The resolution property.

Definition 6.1.1. Let \mathcal{F} be a quasi-coherent sheaf on an algebraic stack X . We say that \mathcal{F} is *coherent* if there exists a scheme U and a smooth surjection $p : U \rightarrow X$ such that $p^*(\mathcal{F})$ is a coherent sheaf on U . We say that \mathcal{F} is *locally free* of rank n if there exists a scheme U , a smooth surjection $p : U \rightarrow X$, and an isomorphism $p^*\mathcal{F} \simeq \mathcal{O}_U^{\oplus n}$.

Definition 6.1.2. We say that an algebraic stack X has the *resolution property* if there exists a collection $(\mathcal{G}_\alpha)_\alpha$ of locally free sheaves of finite rank on X such that for every quasi-coherent sheaf \mathcal{F} on X there exists a surjection

$$\bigoplus_{\alpha} \mathcal{G}_\alpha \twoheadrightarrow \mathcal{F}$$

of quasi-coherent sheaves on X .

Remark 6.1.3. If X is quasi-compact and quasi-separated, then it admits the resolution property if and only if for every finite type quasi-coherent sheaf \mathcal{F} , there exists a finite locally free sheaf \mathcal{E} and a surjection $\mathcal{E} \twoheadrightarrow \mathcal{F}$. This follows from the fact that every quasi-coherent sheaf on X can be written as a filtered colimit of quasi-coherent sheaves of finite type (see [Ryd2]).

Example 6.1.4. Let X be a scheme and consider the following conditions:

- (i) X is quasi-affine.
- (ii) X is quasi-projective.

- (iii) X admits an ample family of line bundles.
- (iv) X has the resolution property.

We obviously have (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).

Remark 6.1.5. It is possible to show that the resolution property implies that X has affine diagonal (for example, this follows from Theorem 6.2.5 below, or see [SP, Tag 0F8C]). Conversely, there is apparently no known example of a scheme X of finite type over a field which has affine diagonal but does not admit the resolution property.

6.2. Global quotient stacks. It turns out that, under very mild assumptions on X , the resolution property is equivalent to the existence of a global quotient presentation.

Definition 6.2.1. Let X be an algebraic stack. We say that X is *quasi-compact* if there exists a quasi-compact scheme U and a smooth surjection $U \twoheadrightarrow X$. We say that X is *quasi-separated* if it has quasi-compact quasi-separated diagonal.

Definition 6.2.2. Let X be an algebraic stack. We say that X has *affine stabilizers* if for every geometric point x of X , the stabilizer $\mathrm{St}_X(x)$ is affine. For example, if X has affine diagonal, then it has affine stabilizers.

Definition/Proposition 6.2.3. *We say that X is a global quotient stack if satisfies the following equivalent conditions:*

- (i) *There exists a quasi-affine scheme U with an action of GL_n (for some $n \geq 0$), and an isomorphism $X \simeq [U/\mathrm{GL}_n]$.*
- (ii) *There exists a GL_n -torsor (for some $n \geq 0$) $U \twoheadrightarrow X$ where U is a quasi-affine scheme.*
- (iii) *There exists a quasi-affine morphism $X \rightarrow \mathrm{BGL}_n$ (for some $n \geq 0$).*

Remark 6.2.4. Note that any global quotient stack is quasi-compact quasi-separated with affine stabilizers.

Theorem 6.2.5 (Totaro, Gross). *Let X be an algebraic stack. Then X has the resolution property if and only if it is a global quotient stack.*

6.3. Quasi-coherent sheaves on quotient stacks. Let us begin by recording the following tautological fact:

Proposition 6.3.1. *Let $F : \mathrm{Sch}^{\mathrm{op}} \rightarrow \mathcal{V}$ be an étale sheaf where \mathcal{V} is a small ∞ -category admitting limits, and denote by F^+ its right Kan extension along the Yoneda embedding $\mathrm{Sch} \hookrightarrow \mathrm{Stk}_\infty = \mathrm{Shv}(\mathrm{Sch}; \mathrm{Grpd}_\infty)$. Then for every effective epimorphism $f : X \rightarrow Y$ in Stk_∞ , the canonical morphism*

$$F(Y) \rightarrow \mathrm{Tot}(F(X_\bullet))$$

is invertible where X_\bullet denotes the Čech nerve of f .

Proof. If $f : X \rightarrow Y$ is an effective epimorphism, then the canonical morphism $|X_\bullet| \rightarrow Y$ is invertible. Recall that F^+ is by definition the unique functor

$\mathrm{Stk}_\infty^{\mathrm{op}} \rightarrow \mathcal{V}$ extending F which preserves limits, i.e., sends colimits in Stk_∞ to limits in \mathcal{V} . The claim follows. \square

Corollary 6.3.2. *Let X be an algebraic stack. Then for every scheme U and smooth surjection $p: U \twoheadrightarrow X$, the canonical functor*

$$\mathrm{QCoh}(X) \rightarrow \mathrm{Tot}(\mathrm{QCoh}(U_\bullet))$$

is an equivalence, where U_\bullet is the Čech nerve of p .

Corollary 6.3.3. *Let G be a group stack over a base scheme S acting on a stack X over S . Then the canonical functor*

$$\mathrm{QCoh}([X/G]) \rightarrow \mathrm{Tot}(\mathrm{QCoh}(X_\bullet))$$

is an equivalence, where $X_\bullet = [\cdots \rightrightarrows G \times X \rightrightarrows X]$ is the action groupoid.

Definition 6.3.4. Let G be a group stack over a scheme S , and let X be a stack over S with G -action. We define

$$\mathrm{QCoh}^G(X) := \mathrm{Tot}(\mathrm{QCoh}(X_\bullet)).$$

Note that its objects, which we call *G -equivariant quasi-coherent sheaves* on X , are quasi-coherent sheaves \mathcal{F} on X together with a (specified) isomorphism $\mathrm{act}^* \mathcal{F} \simeq \mathrm{pr}^* \mathcal{F}$ on $G \times X$, satisfying a cocycle condition on $G \times G \times X$. Thus Corollary 6.3.3 says that quasi-coherent sheaves on $[X/G]$ are G -equivariant quasi-coherent sheaves on X .

Theorem 6.3.5 (Thomason). *Every global quotient stack admits the resolution property.*

Proof. Let U be a quasi-affine scheme with an action of $G = \mathrm{GL}_n$. Since the structure sheaf \mathcal{O}_U is obviously G -equivariant, it defines a locally free sheaf on $[U/G]$. Moreover, \mathcal{O}_U is ample (since U is quasi-affine). In this situation, Thomason proved that every G -equivariant coherent sheaf on U admits a surjection from a G -equivariant locally free sheaf (see [Tho, Lem. 2.4, Lem. 2.6]). This immediately implies that $[U/G]$ has the resolution property, translating via Definition 6.3.4 and using Remark 6.1.3. \square

6.4. A quasi-affineness criterion.

Definition 6.4.1. Let $f: X \rightarrow Y$ be a morphism of algebraic stacks. Let $(\mathcal{E}_\alpha)_\alpha$ be a collection of coherent sheaves on X .

- (i) We say that $(\mathcal{E}_\alpha)_\alpha$ is *generating over Y* , or *f -generating*, if for all $\mathcal{E} \in \mathrm{QCoh}(X)$ there exists an $\mathcal{F} \in \mathrm{QCoh}(Y)$ and a surjection

$$\bigoplus_\alpha \mathcal{E}_\alpha \otimes f^*(\mathcal{F}) \twoheadrightarrow \mathcal{E}.$$

- (ii) We say that $(\mathcal{E}_\alpha)_\alpha$ is *universally generating over Y* (or *universally f -generating*) if the above property holds after base change to any affine scheme (or equivalently, after base change to any algebraic stack).

Remark 6.4.2. If Y is affine, then $(\mathcal{G}_\alpha)_\alpha \subseteq \mathrm{QCoh}(X)$ is generating over Y if and only if for every $\mathcal{F} \in \mathrm{QCoh}(X)$ there exists a surjection

$$\bigoplus_{\alpha} \mathcal{G}_\alpha \twoheadrightarrow \mathcal{F}.$$

Theorem 6.4.3 (Gross). *Let $f : X \rightarrow Y$ be a morphism of qcqs algebraic stacks. Then f is quasi-affine if and only if it satisfies the following properties:*

- (i) *The morphism f has affine stabilizers. That is, for every affine scheme V and every morphism $v : V \rightarrow Y$, the fibre $X \times_Y V$ has affine stabilizers (at geometric points).*
- (ii) *The collection (\mathcal{O}_X) is universally generating over Y .*

See [Gro, Prop. 3.1].

6.5. Proof of Theorem 6.2.5. Let X be an algebraic stack with the resolution property. Let $(\mathcal{G}_\alpha)_\alpha$ be a collection of locally free sheaves of ranks n_α on X , and denote by $(F_\alpha)_\alpha$ the corresponding GL_{n_α} -torsors (Example 4.5.1). For every finite subset of indices I , the product (fibred over X)

$$F_I = \prod_{\alpha \in I} F_\alpha$$

defines a $G_I := \prod_{\alpha \in I} \mathrm{GL}_{n_\alpha}$ -torsor over X , over which F_α trivializes for each $\alpha \in I$. Note that $F_I \rightarrow X$ is an affine morphism (since G_I is affine). The limit $F = \varprojlim_I F_I$ over the cofiltered system of finite subsets I is affine over X and in particular algebraic, since the transition maps are affine.

By Theorem 6.4.3 we deduce that F is a quasi-affine scheme, i.e., that $F \rightarrow \mathrm{Spec}(\mathbf{Z})$ is quasi-affine:

- (a) F has affine stabilizers. Indeed, it is affine over X , which has affine stabilizers.
- (b) \mathcal{O}_F is universally generating (over $\mathrm{Spec}(\mathbf{Z})$). Indeed, since $F \rightarrow X$ is affine, the fact that $(\mathcal{G}_\alpha)_\alpha \subseteq \mathrm{QCoh}(X)$ is universally generating over Y implies that $(\mathcal{G}_\alpha|_F)_\alpha \subseteq \mathrm{QCoh}(F)$ is universally generating over $\mathrm{Spec}(\mathbf{Z})$. But $\mathcal{G}_\alpha|_F \simeq \mathcal{O}_F$ for every α by construction of F .

Since $\mathrm{Spec}(\mathbf{Z})$ is quasi-compact, it follows that I may be chosen sufficiently large such that $F_I \rightarrow \mathrm{Spec}(\mathbf{Z})$ is already quasi-affine (see [Ryd1, Thm. C]).

The G_I -torsor $F_I \rightarrow \mathrm{Spec}(\mathbf{Z})$ is classified by a morphism $X \rightarrow BG_I$, which is quasi-affine because its base change along the quotient map $\mathrm{Spec}(\mathbf{Z}) \twoheadrightarrow BG_I$ (which is $F_I \rightarrow \mathrm{Spec}(\mathbf{Z})$) is quasi-affine. The diagonal embedding of G_I into GL_n , where $n = \sum_\alpha n_\alpha$, has quotient GL_n/G_I representable by an affine scheme (the ‘‘Stiefel manifold’’). The induced morphism of classifying stacks $BG_I \rightarrow \mathrm{BGL}_n$ is therefore quasi-affine, since its base change along $\mathrm{Spec}(\mathbf{Z}) \twoheadrightarrow \mathrm{BGL}_n$ is GL_n/G_I . We have thus constructed a quasi-affine morphism $X \rightarrow \mathrm{BGL}_n$, hence X is a global quotient by Proposition 6.2.3.

7. DERIVED CATEGORIES

7.1. Algebraic categories.

Definition 7.1.1. Let \mathcal{C} be a category. We say \mathcal{C} is *algebraic* if there exists an essentially small full subcategory $\mathcal{F}_{\mathcal{C}} \subseteq \mathcal{C}$ closed under finite coproducts, and an equivalence

$$\mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathrm{Set}) \rightarrow \mathcal{C},$$

which restricts along the Yoneda embedding $\mathcal{F}_{\mathcal{C}} \hookrightarrow \mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathrm{Set})$ to the inclusion $\mathcal{F}_{\mathcal{C}} \hookrightarrow \mathcal{C}$. The subscript Π indicates the full subcategory of finite product-preserving functors.

Definition 7.1.2. Let \mathcal{C} be a category.

- (i) An object $X \in \mathcal{C}$ is *compact* if and only if $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \mathrm{Set}$ preserves filtered colimits.
- (ii) A compact object $X \in \mathcal{C}$ is *projective* if and only if $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \mathrm{Set}$ preserves reflexive coequalizers.

The category \mathcal{C} is algebraic if and only if the condition of Definition 7.1.1 holds with $\mathcal{F}_{\mathcal{C}}$ the full subcategory of compact projective objects. When not otherwise specified, $\mathcal{F}_{\mathcal{C}}$ will denote this subcategory by default.

Example 7.1.3. The category of sets is algebraic: we have

$$\mathrm{Set} \simeq \mathrm{Fun}_{\Pi}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Set}),$$

where $\mathrm{Fin} \subseteq \mathrm{Set}$ is the full subcategory of finite sets. Certainly any set X represents a product-preserving functor $\mathrm{Fin}^{\mathrm{op}} \rightarrow \mathrm{Set}$ sending $Y \mapsto \mathrm{Hom}(Y, X)$. This gives a functor $\mathrm{Set} \rightarrow \mathrm{Fun}_{\Pi}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Set})$. Conversely, given a product-preserving functor $F : \mathrm{Fin}^{\mathrm{op}} \rightarrow \mathrm{Set}$, we get the following data:

- (i) Sets $F_0 = F(\emptyset)$, $F_1 = F(\{1\})$, $F_2 = F(\{1, 2\})$, \dots
- (ii) For every map of finite sets $\{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, m\}$, an induced map $F_m \rightarrow F_n$.
- (iii) Canonical isomorphisms $F_0 \simeq \mathrm{pt} := \{*\}$, and $F_n \simeq (F_1)^{\times n}$ for all n .

The assignment $F \mapsto F_1$ defines a functor from the RHS to Set , and it is easy to check that these two functors define an equivalence of categories. Moreover, the compact projective objects in Set are exactly the finite sets.

Example 7.1.4. The category Ab of abelian groups is algebraic:

$$\mathrm{Ab} \simeq \mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathrm{Ab}}^{\mathrm{op}}, \mathrm{Set}) \tag{7.1.5}$$

where $\mathcal{F}_{\mathrm{Ab}}$ is the category of finitely generated free abelian groups. Moreover, the compact projective objects are exactly those that belong to $\mathcal{F}_{\mathrm{Ab}}$.

Note that $\mathcal{F}_{\mathrm{Ab}}$ is equivalent to the category whose objects are natural numbers, morphisms are matrices of integers

$$\mathrm{Hom}(n, m) = \mathrm{Hom}_{\mathrm{Ab}}(\mathbf{Z}^{\oplus n}, \mathbf{Z}^{\oplus m}) = \mathrm{Mat}_{m \times n}(\mathbf{Z}),$$

and the composition rule is given by matrix multiplication.

An object of the RHS of (7.1.5) amounts to the data of sets F_0, F_1, F_2, \dots with isomorphisms $F_n \simeq (F_1)^{\times n}$ for all n , and for every matrix $\phi \in \text{Mat}_{m \times n}(\mathbf{Z})$, a map $F_\phi : F_m \rightarrow F_n$. This data is subject to identities imposed by functoriality. From this we can derive an *underlying set* $M := F_1 \in \text{Set}$ equipped with various *operations* encoded by matrices. For example, every $(n \times 1)$ -matrix

$$\phi = \begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix}$$

encodes the operation $F_\phi : M^{\times n} \rightarrow M$ of “forming the linear combination with coefficients a_i ”:

$$(x_1, \dots, x_n) \mapsto \sum_i a_i x_i.$$

In particular:

- (i) There is a (commutative) *addition map* $M \times M \rightarrow M$, $(x_1, x_2) \mapsto x_1 + x_2$, encoded by the matrix

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- (ii) There is a zero element $0 \in M$, encoded as the map $0 : \text{pt} \rightarrow M$ coming from the empty (0×1) -matrix.

- (iii) There is an *additive inverse map* $M \rightarrow M$, $x_1 \mapsto -x_1$, encoded by the (1×1) -matrix (-1) .

Essentially, this is just a very redundant way of specifying an abelian group.

Example 7.1.6. More generally, for every commutative ring R the category Mod_R of R -modules is algebraic:

$$\text{Mod}_R \simeq \text{Fun}_{\Pi}(\mathcal{F}_R^{\text{op}}, \text{Set})$$

where \mathcal{F}_R is the category of finitely generated free R -modules. The compact projective objects are the finitely generated *projective* R -modules (and not, in general, just the finitely generated free ones).

If \mathcal{C} is algebraic, then it is the free completion of $\mathcal{F}_{\mathcal{C}}$ by filtered colimits and reflexive coequalizers.

Proposition 7.1.7. *Let \mathcal{C} be an algebraic category. For every category \mathcal{D} admitting filtered colimits and reflexive coequalizers, the canonical functor*

$$\text{Fun}'(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{D}),$$

from the category of functors $\mathcal{C} \rightarrow \mathcal{D}$ that preserve filtered colimits and reflexive coequalizers, is an equivalence.

Reminder 7.1.8. A *reflexive pair* in a category \mathcal{C} is a diagram

$$x \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{s} \\ \xrightarrow{g} \end{array} y,$$

where $f \circ s = g \circ s$, i.e., s is a common section of f and g . *Reflexive coequalizers* are colimits of reflexive pairs. Note that reflexive pairs are nothing else than diagrams indexed by the full subcategory $\Delta_{\leq 1}^{\text{op}} \subseteq \Delta^{\text{op}}$ spanned by $[0]$ and $[1]$.

7.2. (Connective) nonabelian derived categories. In an algebraic category \mathcal{C} , every object is built out of objects of $\mathcal{F}_{\mathcal{C}}$ under filtered colimits and reflexive coequalizers. The idea of Quillen's nonabelian derived category is to build an ∞ -category out of $\mathcal{F}_{\mathcal{C}}$ by replacing reflexive coequalizers by geometric realizations. Following [CS], we will refer to this construction as the *animation* of \mathcal{C} .

Definition 7.2.1. Let \mathcal{C} be an algebraic category. Given an ∞ -category \mathcal{C}^+ admitting colimits, we say that a fully faithful functor $\mathcal{F}_{\mathcal{C}} \hookrightarrow \mathcal{C}^+$ exhibits \mathcal{C}^+ as an *animation* of \mathcal{C} if it satisfies the following universal property: for every ∞ -category \mathcal{D} admitting filtered colimits and geometric realizations, restriction induces an equivalence

$$\text{Fun}'(\mathcal{C}^+, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{D})$$

from the ∞ -category of functors $\mathcal{C}^+ \rightarrow \mathcal{D}$ that preserve filtered colimits and geometric realizations. Informally speaking, \mathcal{C}^+ is freely generated by $\mathcal{F}_{\mathcal{C}}$ under filtered colimits and geometric realizations.

We will denote by $\text{Anim}(\mathcal{C})$ the animation of an algebraic category \mathcal{C} . An *animated object* of \mathcal{C} is an object of $\text{Anim}(\mathcal{C})$.

Theorem 7.2.2 (Quillen, Lurie).

- (i) *The functor $\text{Fin} \hookrightarrow \text{Set} \hookrightarrow \text{Grpd}_{\infty}$ (Remark 1.5.7) exhibits Grpd_{∞} as an animation of Set .*
- (ii) *For any algebraic category \mathcal{C} , the Yoneda embedding*

$$\mathcal{F}_{\mathcal{C}} \hookrightarrow \text{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\text{op}}, \text{Grpd}_{\infty})$$

exhibits its target as an animation of \mathcal{C} .

Proof. See [Lur1, Prop. 5.5.8.15]. □

Notation 7.2.3. We denote by Ani the animation $\text{Anim}(\text{Set}) \simeq \text{Grpd}_{\infty}$ of the category of sets, and will refer to its objects as *anima*. (In other words, “animum” is a synonym for ∞ -groupoid, but reflects that we want to think of anima as “derived sets” rather than a generalization of groupoids.)

7.3. Interlude: stable ∞ -categories.

Definition 7.3.1. Let \mathcal{C} be an ∞ -category with a zero object 0 (i.e., an object which is both terminal and initial). Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} . The *cofibre* of f , denoted $\text{Cofib}(f)$, is the colimit of the diagram

$$0 \leftarrow X \xrightarrow{f} Y.$$

The *fibre* of f , denoted $\text{Fib}(f)$, is the limit of the diagram

$$0 \rightarrow Y \xleftarrow{f} X.$$

Definition 7.3.2. Let \mathcal{C} be an ∞ -category with zero object 0 . The *suspension* of an object $X \in \mathcal{C}$ is the object

$$\Sigma(X) := \text{Cofib}(X \rightarrow 0).$$

The *loop space* of X is the object

$$\Omega(X) := \text{Fib}(0 \rightarrow X).$$

Definition/Proposition 7.3.3. Let \mathcal{C} be an ∞ -category. We say that \mathcal{C} is *stable* if it satisfies the following equivalent conditions:

- (i) \mathcal{C} admits finite limits and a zero object (an object which is both terminal and initial), and the loop space functor $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence.
- (ii) \mathcal{C} admits finite colimits and a zero object, and the suspension functor $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence.
- (iii) \mathcal{C} admits finite limits, finite colimits, and a zero object; and any commutative square is cartesian if and only if it is cocartesian.

Example 7.3.4. The ∞ -category Spt is stable.

Definition 7.3.5. Let \mathcal{C} be a stable ∞ -category. An *exact triangle* in \mathcal{C} is a cartesian (hence also cocartesian) square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object. (Such an exact triangle gives rise to a canonical isomorphism $\text{Cofib}(g) \rightarrow Z$.) We will usually depict it by the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

but note that a null-homotopy $g \circ f \simeq 0$ must be specified as part of the data of an exact triangle.

Remark 7.3.6. The homotopy category $h(\mathcal{C})$ of a stable ∞ -category admits a canonical triangulated structure, where the exact/distinguished triangles are those coming from exact triangles in \mathcal{C} (via the canonical functor $\mathcal{C} \rightarrow h(\mathcal{C})$).

7.4. Nonabelian derived categories.

Proposition 7.4.1. Let \mathcal{C} be an algebraic category. If \mathcal{C} is additive, then the forgetful functor

$$\text{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\text{op}}, \text{Ab}) \rightarrow \text{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\text{op}}, \text{Set}) \simeq \mathcal{C}$$

is an equivalence.

Indeed, every finite product-preserving functor $X : \mathcal{F}_{\mathcal{C}}^{\text{op}} \rightarrow \text{Ab}$ automatically takes values in abelian groups. We will see an analogue of this statement for the animation. First we need to introduce the analogue of abelian groups.

Definition 7.4.2.

(i) A *pointed animus* is an animus X equipped with a morphism $\text{pt} \rightarrow X$.

(ii) A *spectrum* is a sequence of pointed anima

$$X = (X_0, X_1, X_2, \dots)$$

together with isomorphisms

$$X_n \simeq \Omega(X_{n+1})$$

for all $n \geq 0$.

(iii) A spectrum X is *connective* if the n th component X_n is n -connective for every $n \geq 0$ (i.e., if $\pi_i(X_n) = 0$ for all $i < n$).

Informally speaking, we say that a spectrum X is an *infinite delooping* of X_0 , or that it gives X_0 the structure of an *infinite loop space*.

Notation 7.4.3. The ∞ -categories Spt and $\text{Spt}_{\geq 0}$ of spectra and connective spectra, respectively, can be defined as the limits of the following towers in the ∞ -category of ∞ -categories:

$$\begin{aligned} \dots &\xrightarrow{\Omega} \text{Ani}_\bullet \xrightarrow{\Omega} \text{Ani}_\bullet \xrightarrow{\Omega} \text{Ani}_\bullet, \\ \dots &\xrightarrow{\Omega} (\text{Ani}_\bullet)_{\geq 2} \xrightarrow{\Omega} (\text{Ani}_\bullet)_{\geq 1} \xrightarrow{\Omega} (\text{Ani}_\bullet)_{\geq 0}, \end{aligned}$$

where Ani_\bullet denotes the ∞ -category of pointed anima, and the subscript $\geq n$ indicates the full subcategory of n -connective pointed anima. We have projections

$$\Omega^{\infty-n} : \text{Spt} \rightarrow \text{Ani}_\bullet,$$

sending $(X_0, X_1, \dots) \mapsto X_n$.

In the following important theorem, \mathcal{E}_∞ -groups are the analogues of abelian groups in the derived/animated world. Informally speaking, an \mathcal{E}_∞ -group structure on an animus is an infinite collection of group structures, all commuting with each other up to coherent homotopy (cf. the Eckmann–Hilton argument which implies that if a set admits two commuting group laws, both agree and are commutative).

Theorem 7.4.4 (Infinite loop space machine).

(i) Let X be a spectrum. Then the infinite loop space $\Omega^\infty(X)$ admits an \mathcal{E}_∞ -group structure. More precisely, $\Omega^\infty : \text{Spt} \rightarrow \text{Ani}_\bullet$ lifts to a functor

$$\Omega^\infty : \text{Spt} \rightarrow \{\mathcal{E}_\infty\text{-groups}\}.$$

(ii) Restricted to the full subcategory of connective spectra, this defines an equivalence of ∞ -categories

$$\text{Spt}_{\geq 0} \simeq \{\mathcal{E}_\infty\text{-groups}\}.$$

In particular, any \mathcal{E}_∞ -group structure on a pointed animus X gives rise to a unique infinite delooping $B^\infty X$. Here $B^\infty X$ is a connective spectrum such that $\Omega^\infty B^\infty X \simeq X$.

Proposition 7.4.5. *Let \mathcal{C} be an additive algebraic category. Then there is a canonical equivalence*

$$\mathrm{Anim}(\mathcal{C}) \simeq \mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathrm{Spt}_{\geq 0}).$$

In particular, there exists a canonical fully faithful embedding

$$\Sigma^{\infty} : \mathrm{Anim}(\mathcal{C}) \hookrightarrow \mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathrm{Spt})$$

whose target is a stable ∞ -category, and whose essential image is closed under colimits and extensions.

Definition 7.4.6. The *derived ∞ -category* of \mathcal{C} is the stable ∞ -category

$$\mathrm{D}(\mathcal{C}) := \mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathrm{Spt}).$$

We also let $\mathrm{D}(\mathcal{C})_{\geq 0}$ denote the essential image of the fully faithful embedding $\mathrm{Anim}(\mathcal{C}) \hookrightarrow \mathrm{D}(\mathcal{C})$. We will say that an object $X \in \mathrm{D}(\mathcal{C})$ is *connective* if it lies in $\mathrm{D}(\mathcal{C})_{\leq 0}$.

7.5. Derived functors.

Construction 7.5.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor of algebraic categories. Its restriction to $\mathcal{F}_{\mathcal{C}}$ gives rise by the universal property of animation to an essentially unique functor $\mathbf{L}F : \mathrm{Anim}(\mathcal{C}) \rightarrow \mathrm{Anim}(\mathcal{D})$ such that:

- (i) $\mathbf{L}F$ preserves filtered colimits and geometric realizations.
- (ii) There is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{C}} & \xrightarrow{F|_{\mathcal{F}_{\mathcal{C}}}} & \mathcal{D} \hookrightarrow \mathrm{Anim}(\mathcal{D}) \\ \downarrow & \nearrow \mathbf{L}F & \\ \mathrm{Anim}(\mathcal{C}) & & \end{array}$$

Moreover, $\mathbf{L}F$ preserves finite coproducts (and hence all colimits) if and only if F preserves finite coproducts.

Remark 7.5.2. We have canonical isomorphisms

$$\pi_0 \mathbf{L}F(X) \simeq F(\pi_0(X)),$$

functorial in $X \in \mathrm{Anim}(\mathcal{C})$, where π_0 is left adjoint to the fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$. Indeed, both functors agree by definition after restriction to $\mathcal{F}_{\mathcal{C}} \subseteq \mathcal{C}$ (since π_0 is identity on discrete objects), so this follows by universal property of $\mathrm{Anim}(\mathcal{C})$.

Proposition 7.5.3. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ be functors of algebraic categories which preserve filtered colimits and reflexive coequalizers. Then there is a canonical equivalence*

$$\mathbf{L}G \circ \mathbf{L}F \simeq \mathbf{L}(G \circ F)$$

under either of the following conditions:

- (i) F sends $\mathcal{F}_{\mathcal{C}}$ to $\mathcal{F}_{\mathcal{D}}$. (More generally, it is enough that for every $X \in \mathcal{F}_{\mathcal{C}}$, $F(X) \in \mathcal{D}$ is a filtered colimit of objects in $\mathcal{F}_{\mathcal{D}}$.)

- (ii) $\mathbf{L}G : \mathbf{Anim}(\mathcal{D}) \rightarrow \mathbf{Anim}(\mathcal{E})$ preserves discrete objects. (More generally, it is enough that for every $X \in \mathcal{F}_{\mathcal{E}}$, $\mathbf{L}G(F(X)) \in \mathbf{Anim}(\mathcal{E})$ is discrete).

Proof. In either case, one easily verifies that $\mathbf{L}G \circ \mathbf{L}F$ satisfies the universal property of $\mathbf{L}(G \circ F)$. \square

By universal properties of stabilization we have:

Proposition 7.5.4. *Let \mathcal{E} and \mathcal{D} be additive algebraic categories and $F : \mathbf{Anim}(\mathcal{E}) \rightarrow \mathbf{Anim}(\mathcal{D})$ a functor.*

- (i) *If F commutes with Ω then it extends uniquely to a functor*

$$F : \mathbf{D}(\mathcal{E}) \rightarrow \mathbf{D}(\mathcal{D})$$

such that $\Omega^{\infty-n} \circ F \simeq F \circ \Omega^{\infty-n}$ for all $n \geq 0$. Informally speaking, it sends $(X_0, X_1, \dots) \mapsto (F(X_0), F(X_1), \dots)$.

- (ii) *If F preserves colimits and zero objects, then it extends uniquely to a colimit-preserving functor*

$$F : \mathbf{D}(\mathcal{E}) \rightarrow \mathbf{D}(\mathcal{D})$$

such that $F \circ \Sigma^{\infty-n} \simeq \Sigma^{\infty-n} \circ F$ for all $n \geq 0$. Here $\Sigma^{\infty-n} : \mathbf{Anim}(-) \rightarrow \mathbf{D}(-)$ is left adjoint to $\Omega^{\infty-n}$.

7.6. Animated modules. Let A be a ring⁸ and consider the algebraic category \mathbf{Mod}_A of A -modules (Example 7.1.6). Let's specialize the animation construction to this case.

Definition 7.6.1. Let $\mathcal{F}_A \subseteq \mathbf{Mod}_A$ denote the full subcategory of finitely generated free A -modules. An *animated A -module* is a finite product-preserving functor $M : \mathcal{F}_A^{\mathrm{op}} \rightarrow \mathbf{Ani}$. We write

$$\mathbf{D}(A)_{\geq 0} := \mathbf{Anim}(\mathbf{Mod}_A)$$

for the ∞ -category of animated A -modules. (Cohomologically: $\mathbf{D}(A)^{\leq 0}$.)

Remark 7.6.2. Let us unpack the definition a bit. An animated A -module M amounts to the following data:

- (i) For every integer $n \geq 0$, an animus $M_n \in \mathbf{Ani}$.
- (ii) For every A -linear map $\phi : A^{\oplus n} \rightarrow A^{\oplus m}$ (or $\phi \in \mathbf{Mat}_{m \times n}(A)$), a map of anima $M_\phi : M_m \rightarrow M_n$.
- (iii) For every two A -linear maps $\phi : A^{\oplus n} \rightarrow A^{\oplus m}$ and $\psi : A^{\oplus m} \rightarrow A^{\oplus l}$, a homotopy $h_{\phi, \psi} : M_\phi \circ M_\psi \simeq M_{\psi \circ \phi}$ of maps $M_l \rightarrow M_n$.
- (iv) For every three A -linear maps ϕ, ψ, ω , a tetrahedron-shaped diagram expressing a “higher” homotopy between the homotopies $h_{\phi, \psi}$, $h_{\psi, \omega}$, $h_{\phi, \omega \circ \psi}$, and $h_{\psi \circ \phi, \omega}$.
- (v) ...

⁸From now on, all our rings be assumed commutative.

This data is subject to the condition that the canonical map $M_n \rightarrow (M_1)^{\times n}$ is invertible for every $n \geq 0$ (in particular, $M_0 \simeq \text{pt}$). We summarize (iii) and (iv) by saying that the maps M_ϕ are functorial *up to coherent homotopy*.

In particular, this data encodes:

- (i) The underlying animus $M^\circ := M_1 \in \text{Ani}$.
- (ii) Operations $(M^\circ)^{\times n} \rightarrow M^\circ$ on M° , for every $\phi \in \text{Mat}_{n \times 1}(A)$. In particular, an addition operation $\text{add} : M^\circ \times M^\circ \rightarrow M^\circ$.
- (iii) An action of A on M° , i.e., a map $A \rightarrow \text{End}(M^\circ)$ given by

$$A \simeq \text{Mat}_{1 \times 1}(A) = \text{Hom}_{\mathcal{F}_A}(1, 1) \xrightarrow{M} \text{Maps}_{\text{Ani}}(M_1, M_1) = \text{End}(M^\circ).$$

The endomorphism induced by $a \in A$ is the operation encoded by the matrix $a \in \text{Mat}_{1 \times 1}(A)$.

- (iv) Associativity up to coherent homotopy. For example, given three points $x, y, z \in M$ (i.e., maps of anima $\text{pt} \rightarrow M^\circ$) we have a homotopy

$$\text{add}(\text{add}(x, y), z) \simeq \text{add}(x, \text{add}(y, z)).$$

Diagrammatically,

$$\begin{array}{ccc} M^\circ \times M^\circ \times M^\circ & \xrightarrow{\text{add} \times \text{id}} & M^\circ \times M^\circ \\ \downarrow \text{id} \times \text{add} & & \downarrow \text{add} \\ M^\circ \times M^\circ & \xrightarrow{\text{add}} & M^\circ. \end{array}$$

Informally speaking, we can think of an animated A -module as an animus equipped with a homotopy coherent A -module structure.

8. THE COTANGENT COMPLEX

Given a morphism of schemes $f : X \rightarrow Y$, recall the quasi-coherent sheaf of relative algebraic Kähler differentials

$$\Omega_{X/Y} \in \text{QCoh}(X).$$

Given another morphism $g : Y \rightarrow Z$, there is a right-exact sequence

$$f^* \Omega_{Y/Z} \rightarrow \Omega_{X/Z} \rightarrow \Omega_{X/Y} \rightarrow 0.$$

In this lecture we will use the formalism of the previous lecture to define the “derived functor of $\Omega_{-/}$ ”, which will allow us to extend the above sequence to the left.

8.1. Animated rings. Given a commutative ring R , we denote by CAlg_R the category of commutative R -algebras.

Proposition 8.1.1. *Denote by $\text{Poly}_R \subseteq \text{CAlg}_R$ the full subcategory spanned by finitely generated polynomial algebras $R[T_1, \dots, T_m]$, $m \geq 0$. The inclusion $\text{Poly}_R \hookrightarrow \text{CAlg}_R$ extends to an equivalence*

$$\text{Fun}_{\Pi}(\text{Poly}_R^{\text{op}}, \text{Set}) \rightarrow \text{CAlg}_R.$$

In particular, the category CAlg_R is algebraic.

Definition 8.1.2. Denote by $\mathrm{ACAlg}_R := \mathrm{Anim}(\mathrm{CAlg}_R)$ the animation of CAlg_R . Objects of ACAlg_R will be called *animated (commutative) R -algebras*. When $R = \mathbf{Z}$, we abbreviate $\mathrm{ACRing} := \mathrm{ACAlg}_{\mathbf{Z}}$ and call its objects *animated (commutative) rings*.

Remark 8.1.3. Note that CAlg_R is not additive: for $A, B \in \mathrm{CAlg}_R$, the coproduct is the tensor product $A \otimes_R B$, which is usually distinct from the product $A \times B$.

Notation 8.1.4. Let R be a commutative ring. We denote by $\mathrm{CAlgMod}_R$ the category of pairs (A, M) where $A \in \mathrm{CAlg}_R$ and $M \in \mathrm{Mod}_A$; a morphism $(A, M) \rightarrow (A', M')$ is an R -algebra homomorphism $A \rightarrow A'$ together with an A' -module homomorphism $M \otimes_A A' \rightarrow M'$.

Definition/Proposition 8.1.5. Let R be a commutative ring.

- (i) Denote by $\mathrm{CAlgMod}_R^{\mathrm{free}} \subseteq \mathrm{CAlgMod}_R$ the full subcategory spanned by pairs (A, M) where $A = R[T_1, \dots, T_m]$, $m \geq 0$, and $M = A^{\oplus n}$, $n \geq 0$. The inclusion $\mathrm{CAlgMod}_R^{\mathrm{free}} \hookrightarrow \mathrm{CAlgMod}_R$ extends to an equivalence

$$\mathrm{Fun}_{\Pi}(\mathrm{CAlgMod}_R^{\mathrm{free,op}}, \mathrm{Set}) \rightarrow \mathrm{CAlgMod}_R.$$

In particular, the category $\mathrm{CAlgMod}_R$ is algebraic.

- (ii) Consider the canonical functor

$$\pi : \mathrm{Anim}(\mathrm{CAlgMod}_R) \rightarrow \mathrm{Anim}(\mathrm{CAlg}_R) = \mathrm{ACAlg}_R$$

induced by $\mathrm{CAlgMod}_R \rightarrow \mathrm{CAlg}_R$, $(A, M) \mapsto A$. Then π is a co-cartesian fibration. Given an animated ring A , the ∞ -category of animated A -modules is the fibre AMod_A over $A \in \mathrm{CAlg}_R$.

- (iii) Let $\mathcal{F}_A \subseteq \mathrm{AMod}_A$ denote the full subcategory spanned by finite direct sums $A^{\oplus n}$, $n \geq 0$. Then there are canonical equivalences

$$\mathrm{AMod}_A \simeq \mathrm{Fun}_{\Pi}(\mathcal{F}_A^{\mathrm{op}}, \mathrm{Ani}) \simeq \mathrm{Fun}_{\Pi}(\mathcal{F}_A^{\mathrm{op}}, \mathrm{Spt}_{\geq 0}).$$

- (iv) The ∞ -category of derived A -modules is

$$\mathrm{D}(A) := \mathrm{Fun}_{\Pi}(\mathcal{F}_A^{\mathrm{op}}, \mathrm{Spt}).$$

We write $\mathrm{D}(A)_{\geq 0}$ for the essential image of the fully faithful functor $\mathrm{AMod}_A \hookrightarrow \mathrm{D}(A)$.

Definition/Proposition 8.1.6. Let $A \rightarrow B$ be a morphism of animated rings. We say that it is *flat* if it satisfies the following equivalent conditions:

- (i) The functor $(-) \otimes_A^{\mathrm{L}} B : \mathrm{D}(A)_{\geq 0} \rightarrow \mathrm{D}(B)_{\geq 0}$ is left-exact, i.e., it preserves discrete objects.
- (ii) The induced ring homomorphism $\pi_0(A) \rightarrow \pi_0(B)$ is flat, and the canonical homomorphisms

$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_i(B)$$

are bijective for all $i \geq 0$.

Definition 8.1.7. We say that $A \rightarrow B$ is *étale*, resp. *smooth*, if it is flat and the induced ring homomorphism $\pi_0(A) \rightarrow \pi_0(B)$ is étale, resp. smooth (in the sense of ordinary commutative algebra).

8.2. The cotangent complex.

Construction 8.2.1. Let R be a commutative ring. Consider the functor $\mathrm{CAlg}_R \rightarrow \mathrm{CAlgMod}_R$ sending $A \mapsto (A, \Omega_{A/R})$. Its restriction to $\mathrm{CAlg}_R^{\mathrm{free}}$ extends uniquely to a sifted colimit-preserving functor

$$\mathrm{ACAlg}_R = \mathrm{Anim}(\mathrm{CAlg}_R) \rightarrow \mathrm{Anim}(\mathrm{CAlgMod}_R)$$

such that the diagram

$$\begin{array}{ccc} \mathrm{ACAlg}_R & \xrightarrow{\quad} & \mathrm{Anim}(\mathrm{CAlgMod}_R) \\ & \searrow & \swarrow \pi \\ & \mathrm{ACAlg}_R & \end{array}$$

commutes. The image of $A \in \mathrm{Anim}(\mathrm{CAlg}_R)$ may be regarded as a pair $(A, \mathbf{L}\Omega_{A/R})$ where $\mathbf{L}\Omega_{A/R} \in \mathrm{Anim}(\mathrm{Mod}_A)$.

Definition 8.2.2. Given $A \in \mathrm{ACAlg}_R$, the (relative) *cotangent complex* is $\mathbf{L}_{A/R} := \mathbf{L}\Omega_{A/R}$, which we regard as an object of $\mathrm{D}(A)_{\geq 0}$, i.e., as a connective derived A -module.

By construction, we have $\pi_0 \mathbf{L}_{A/R} \simeq \Omega_{\pi_0(A)/\pi_0(R)}$. We also have the following further properties:

Proposition 8.2.3. (i) *Let $A \rightarrow B$ be a morphism in $\mathrm{Anim}(\mathrm{CAlg}_R)$. Then there is an exact triangle*

$$\mathbf{L}_{A/R} \otimes_A^{\mathbf{L}} B \rightarrow \mathbf{L}_{B/R} \rightarrow \mathbf{L}_{B/A}$$

in $\mathrm{D}(B)$.

(ii) *For every cocartesian square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

in $\mathrm{Anim}(\mathrm{CAlg}_R)$ (exhibiting B' as the derived tensor product $A' \otimes_A^{\mathbf{L}} B$), there is a canonical isomorphism

$$\mathbf{L}_{B/A} \otimes_B^{\mathbf{L}} B' \simeq \mathbf{L}_{B'/A'}.$$

8.3. The universal property of the cotangent complex.

Definition 8.3.1. Let R be a commutative ring. Consider the functor

$$\mathrm{CAlgMod}_R \rightarrow \mathrm{CAlg}_R$$

sending a pair (A, M) to the trivial square-zero extension $A \oplus M$, with multiplication $(a, m) \cdot (a', m') = (aa', a'm + am')$. This gives rise to a functor

$$\mathrm{Anim}(\mathrm{CAlgMod}_R) \rightarrow \mathrm{Anim}(\mathrm{CAlg}_R) = \mathrm{ACAlg}_R$$

on animations. We still denote the image of (A, M) by $A \oplus M$ and call it the *trivial square-zero extension* of A by $M \in \mathrm{D}(A)_{\geq 0}$.

Definition 8.3.2. Let $R \in \mathbf{CRing}$, $A \in \mathbf{ACAlg}_R$, and $M \in \mathbf{AMod}_R$ an animated R -module. The canonical morphism $A \oplus M \rightarrow A$, given informally by $(a, m) \mapsto a$, induces a map

$$\mathbf{Maps}_{\mathbf{ACAlg}_R}(A, A \oplus M) \rightarrow \mathbf{Maps}_{\mathbf{ACAlg}_R}(A, A).$$

The animun $\mathbf{Der}_R(A, M)$ of R -linear derivations of A with values in M is the homotopy fibre of this map at the point id_A . In other words, a point of $\mathbf{Der}_R(A, M)$ amounts to a dashed arrow making the below diagram commute:

$$\begin{array}{ccc} & R & \\ & \swarrow & \searrow \\ A & \overset{\text{---}}{\longrightarrow} & A \oplus M \\ & \searrow & \swarrow \\ & A & \end{array}$$

The *trivial derivation* is defined informally by $a \mapsto (a, 0)$.

Theorem 8.3.3. Let A be an animated R -algebra. Then the cotangent complex $\mathbf{L}_{A/R}$ corepresents the functor $\mathbf{D}(A)_{\geq 0} \rightarrow \mathbf{Ani}$, $M \mapsto \mathbf{Der}_R(A, M)$. That is, there are canonical isomorphisms

$$\mathbf{Maps}_{\mathbf{D}(A)}(\mathbf{L}_{A/R}, M) \simeq \mathbf{Der}_R(A, M)$$

functorial in $M \in \mathbf{D}(A)_{\geq 0}$.

Proof. If $A = R[T_1, \dots, T_m]$ is a polynomial algebra, we have

$$\mathbf{L}_{A/R} \simeq \Omega_{A/R}$$

which is free on m generators. In general, write A as a geometric realization of a simplicial diagram of filtered systems of polynomial algebras. \square

8.4. Quasi-coherent and perfect complexes on stacks.

Definition 8.4.1. We extend the flat topology on $\mathbf{CRing}^{\mathrm{op}}$ to $\mathbf{ACRing}^{\mathrm{op}}$ as follows: we say that a morphism $A \rightarrow B$ in \mathbf{ACRing} is *faithfully flat* if it is flat in the sense of Definition 8.1.6, and the induced ring homomorphism $\pi_0(A) \rightarrow \pi_0(B)$ is faithfully flat.

Theorem 8.4.2 (Lurie, Toën). *The functor*

$$\mathbf{ACRing} \rightarrow \mathbf{Cat}_{\infty}, \quad R \mapsto \mathbf{D}(R)$$

satisfies descent for the flat topology.

Proof. Similar to the proof of Theorem 3.2.8. \square

Corollary 8.4.3. *The functor*

$$\mathbf{CRing} \rightarrow \mathbf{Cat}_{\infty}, \quad R \mapsto \mathbf{D}(R)$$

satisfies descent for the flat topology.

Definition 8.4.4. Let X be a stack. The stable ∞ -category of quasi-coherent sheaves $D_{\text{qc}}(X)$ is the limit

$$D_{\text{qc}}(X) := \varprojlim_{(R,x)} D(R)$$

over the category of pairs (R, x) where $R \in \text{CRing}$ and $x \in X(R)$. More precisely,

$$D_{\text{qc}} : \text{Stk}^{\text{op}} \rightarrow \text{Cat}_{\infty}$$

is the right Kan extension of the presheaf $\text{Spec}(R) \mapsto D(R)$ along the inclusion $\text{Aff} \hookrightarrow \text{Stk}$. We refer to objects of $D_{\text{qc}}(X)$ as *quasi-coherent complexes*.

Example 8.4.5. If $X = \text{Spec}(R)$ is affine, then we have the tautological equivalence

$$\mathbf{R}\Gamma(X, -) : D_{\text{qc}}(X) \rightarrow D(R).$$

Example 8.4.6. Let G be a group stack over a scheme S , and let X be a stack over S with G -action. The ∞ -category of G -equivariant quasi-coherent complexes on X is defined as the totalization

$$D_{\text{qc}}^G(X) := \text{Tot}(D_{\text{qc}}(X_{\bullet}))$$

where $X_{\bullet} = [\cdots \rightrightarrows G \times X \rightrightarrows X]$ is the action groupoid. Informally speaking, its objects are quasi-coherent complexes on X equipped with a homotopy coherent G -action. By Proposition 6.3.1 there is a canonical equivalence

$$D_{\text{qc}}([X/G]) \simeq D_{\text{qc}}^G(X).$$

Proposition 8.4.7. *Let X be a stack. If X is algebraic (resp. Deligne–Mumford, schematic), then the limit in Definition 8.4.4 can be taken only over pairs (R, x) such that $x : \text{Spec}(R) \rightarrow X$ is smooth (resp. étale, resp. an open immersion).*

Proof. □

Definition 8.4.8.

- (i) Let R be a commutative ring. We say that a derived R -module $M \in D(R)$ is *perfect* if it is contained in the full subcategory of $D(R)$ generated by R under finite colimits, finite limits, and direct summands.
- (ii) Let X be a stack. We say that a quasi-coherent complex $\mathcal{F} \in D_{\text{qc}}(X)$ is *perfect* if for every (R, x) the derived R -module $\mathbf{R}\Gamma(\text{Spec}(R), x^*\mathcal{F})$ is perfect.

We write $D_{\text{perf}}(X) \subseteq D_{\text{qc}}(X)$ for the full subcategory spanned by perfect complexes, so that

$$D_{\text{perf}}(X) \simeq \varprojlim_{(R,x)} D_{\text{perf}}(R)$$

where $D_{\text{perf}}(R)$ is the ∞ -category of perfect derived R -modules.

Corollary 8.4.9. *Let $p : U \twoheadrightarrow X$ be an effective epimorphism. Then $\mathcal{F} \in D_{\text{qc}}(X)$ is perfect if and only if $\mathbf{L}p^*(\mathcal{F}) \in D_{\text{qc}}(U)$ is perfect.*

Proof. Follows from Proposition 6.3.1. \square

Corollary 8.4.10. *Let U be a stack with G -action, and denote by $p : U \rightarrow [U/G]$ the quotient. Then a quasi-coherent complex $\mathcal{F} \in D_{\text{qc}}([U/G]) \simeq D_{\text{qc}}^G(U)$ is perfect if and only if the underlying quasi-coherent complex $p^*\mathcal{F} \in D_{\text{qc}}(U)$ is perfect.*

8.5. Derived stacks.

Definition 8.5.1. A *derived stack* is a functor $X : \text{ACRing} \rightarrow \text{Grpd}_\infty$ satisfying étale descent. We denote the ∞ -category of derived stacks by $\text{DStk} = \text{Shv}(\text{ACRing}^{\text{op}}; \text{Grpd}_\infty)$.

Example 8.5.2. An *affine derived scheme* is the functor $\text{Spec}(A) : \text{ACRing} \rightarrow \text{Grpd}_\infty$, $B \mapsto \text{Maps}(A, B)$, corepresented by an animated ring $A \in \text{ACRing}$.

Definition 8.5.3. Given a derived stack X , the restriction of the functor $X : \text{ACRing} \rightarrow \text{Grpd}_\infty$ along $\text{CRing} \hookrightarrow \text{ACRing}$ is called the *classical truncation* of X and is denoted X_{cl} . If the ∞ -groupoid $X(R)$ is 1-truncated for every $R \in \text{CRing}$, then $X_{\text{cl}} : \text{CRing} \rightarrow \text{Grpd}$ is a stack. For example, this will be the case if X is a derived algebraic stack (see Definition 8.5.7).

Notation 8.5.4. We distinguish limits in Stk and DStk by writing $\mathbf{R}\varprojlim$ instead of \varprojlim for the latter. For example, given a diagram of stacks $X \rightarrow Z \leftarrow Y$, the usual fibre product is the classical truncation of the derived one:

$$(X \times_Z^{\mathbf{R}} Y)_{\text{cl}} \simeq X \times_Z Y,$$

while $X \times_Z^{\mathbf{R}} Y \simeq X \times_Z Y$ if and only if X and Y are Tor-independent over Z .

Definition 8.5.5. Given a derived stack X , we define the ∞ -category $D_{\text{qc}}(X)$ by a further right Kan extension of $D_{\text{qc}} : \text{Stk}^{\text{op}} \rightarrow \text{Cat}_\infty$ along the fully faithful functor $\text{Stk} \hookrightarrow \text{DStk}$ (induced by the fully faithful functor $\text{CRing} \hookrightarrow \text{ACRing}$). Thus for any derived stack X we have

$$D_{\text{qc}}(X) \simeq \varprojlim_{(A,x)} D(A),$$

where the limit is now taken over the ∞ -category of pairs $(A \in \text{ACRing}, x \in X(A))$.

We have the obvious derived versions of Definitions 5.1.7 and 5.2.2:

Definition 8.5.6 (Derived algebraic spaces).

- (i) A derived stack X is *0-Artin*, or a *derived algebraic space*, if its diagonal $X \rightarrow X \times X$ is schematic and a monomorphism, and there exists an étale surjection $U \rightarrow X$ where U is a derived scheme.
- (ii) A morphism $f : X \rightarrow Y$ is *0-Artin*, or *representable*, if for every affine V and every morphism $V \rightarrow Y$, the fibre $X \times_Y^{\mathbf{R}} V$ is a derived algebraic space.

- (iii) A 0-Artin morphism $f : X \rightarrow Y$ is *flat*, *smooth*, or *surjective* if for every affine V and every morphism $V \rightarrow Y$, there exists a derived scheme U and an étale surjection $U \twoheadrightarrow X \times_Y V$ such that the composite $U \rightarrow X \times_Y V \rightarrow V$ has the respective property.

Definition 8.5.7 (Derived algebraic stacks).

- (i) A derived stack X is *1-Artin*, or a *derived algebraic stack*, if its diagonal is representable and there exists a smooth surjection $U \twoheadrightarrow X$ where U is a derived algebraic space.
- (ii) A morphism of derived stacks $f : X \rightarrow Y$ is *1-representable*, or *representable by derived algebraic stacks*, if for every affine derived scheme V and every morphism $v : V \rightarrow Y$, the derived fibre $X \times_Y^{\mathbf{R}} V$ is a derived algebraic stack.
- (iii) A derived algebraic stack is *Deligne–Mumford* if it admits an étale surjection from a derived scheme. Equivalently, its classical truncation is Deligne–Mumford.

Continuing the same procedure, we can define inductively:

Definition 8.5.8 (Derived Artin stacks).

- (i) For $n > 0$, a morphism of derived stacks $f : X \rightarrow Y$ is $(n-1)$ -*Artin* if for every affine V and every morphism $V \rightarrow Y$, $X \times_Y^{\mathbf{R}} V$ is $(n-1)$ -Artin. A derived stack X is n -*Artin* if its diagonal is $(n-1)$ -Artin and there exists a smooth surjection $U \twoheadrightarrow X$ where U is a derived scheme. An $(n-1)$ -Artin morphism $f : X \rightarrow Y$ is *flat*, *smooth*, or *surjective*, if for any affine V and any morphism $V \rightarrow Y$, there exists a derived scheme U and a smooth surjection $U \twoheadrightarrow X \times_Y V$ such that the composite $U \rightarrow X \times_Y V \rightarrow V$ has the respective property.
- (ii) A derived stack is *Artin* if it is n -Artin for some n . A morphism of derived stacks is *Artin* if it is n -Artin for some n . A morphism of derived stacks is *flat*, *smooth*, or *surjective*, if it is n -Artin with the respective property for some n .

Remark 8.5.9. An n -Artin stack takes values in n -groupoids, i.e., in ∞ -groupoids that are n -truncated.

Remark 8.5.10. Classically, the terms *algebraic stack* and *Artin stack* are used interchangeably. We follow Gaitsgory in redefining Artin stacks as “higher” Artin stacks, and algebraic stacks to be 1-Artin stacks.

8.6. Cotangent complexes of algebraic stacks.

Definition 8.6.1. Let $f : X \rightarrow Y$ be a morphism in DStk . Given an animated ring $A \in \text{ACRing}$, an A -point $x \in X(A)$, and a connective derived A -module

$M \in \mathbf{D}(A)_{\geq 0}$, consider the commutative square

$$\begin{array}{ccc} X(A \oplus M) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ Y(A \oplus M) & \longrightarrow & Y(A) \end{array}$$

induced by the morphism of animated rings $A \oplus M \rightarrow A$, $(a, m) \mapsto a$. This gives rise to a map of anima (or ∞ -groupoids)

$$X(A \oplus M) \rightarrow Y(A \oplus M) \times_{Y(A)} X(A).$$

The animus $\mathrm{Der}_x(X/Y, M)$ of *derivations of f at x with values in M* is the fibre at the point (y', x) , where $y' \in Y(A \oplus M)$ is the image of $y = f(x) \in Y(A)$ along the trivial derivation $A \rightarrow A \oplus M$, $a \mapsto (a, 0)$.

Definition 8.6.2. Let $f : X \rightarrow Y$ be a morphism of (derived) stacks and $\mathcal{L} \in \mathbf{D}_{\mathrm{qc}}(X)$ an *eventually connective* quasi-coherent complex on X (i.e., $\mathcal{L}[n]$ is connective for some integer n). We say that \mathcal{L} is a *(relative) cotangent complex* for $f : X \rightarrow Y$ if for every $A \in \mathrm{ACRing}$ and $x : \mathrm{Spec}(A) \rightarrow X$, the inverse image $\mathbf{L}x^*\mathcal{L}$ corepresents the functor $\mathrm{Der}_x(X/Y, -)$. That is, there are isomorphisms

$$\mathrm{Maps}_{\mathbf{D}(A)}(\mathbf{L}x^*\mathcal{L}, M) \simeq \mathrm{Der}_x(X/Y, M)$$

functorial in $M \in \mathbf{D}(A)_{\geq 0}$, where by abuse of notation we identify $\mathbf{L}x^*\mathcal{L} \in \mathbf{D}_{\mathrm{qc}}(\mathrm{Spec}(A))$ with the derived A -module $\mathbf{R}\Gamma(\mathrm{Spec}(A), \mathbf{L}x^*\mathcal{L}) \in \mathbf{D}(A)$.

Theorem 8.6.3. *Let $f : X \rightarrow Y$ be a 1-representable morphism of derived stacks.*

- (i) *There exists a cotangent complex $\mathbf{L}_{X/Y}$ for f .*
- (ii) *The cotangent complex $\mathbf{L}_{X/Y}$ is (-1) -connective. That is, for every derived scheme U and every smooth morphism $p : U \rightarrow X$, the inverse image $p^*\mathbf{L}_{X/Y}$ is (-1) -connective, i.e.*

$$\mathrm{H}^{-i}(U, \mathbf{L}_{X/Y}) = \pi_i \mathbf{R}\Gamma(U, \mathbf{L}_{X/Y}) = \pi_i \mathrm{Maps}_{\mathbf{D}(U)}(\mathcal{O}_U, p^*\mathbf{L}_{X/Y}) = 0$$

for all $i < -1$. If f is representable by derived algebraic spaces (or derived Deligne–Mumford stacks), then $\mathbf{L}_{X/Y}$ is in fact connective.

Proof. To construct a cotangent complex \mathcal{L} for f , it will suffice by definition of $\mathbf{D}_{\mathrm{qc}}(X)$ to construct for every $A \in \mathrm{ACRing}$ and every morphism $x : \mathrm{Spec}(A) \rightarrow X$, a quasi-coherent complex $\mathcal{L}(x) \in \mathbf{D}_{\mathrm{qc}}(\mathrm{Spec}(A)) \simeq \mathbf{D}(A)$ which corepresents the functor $\mathrm{Der}_x(X/Y, -)$, in such a way that $\mathcal{L}(x)$ is stable under derived base change as (A, x) varies.

Let $x : \mathrm{Spec}(A) \rightarrow X$ be a morphism where $A \in \mathrm{ACRing}$. Since f is 1-representable, the derived fibre $F = X \times_Y^{\mathbf{R}} \mathrm{Spec}(A)$ is algebraic. Supposing that $\mathbf{L}_{F/\mathrm{Spec}(A)}$ exists, it is easy to see that the collection

$$(a^*\mathbf{L}_{F/\mathrm{Spec}(A)})_{A,x},$$

where $a = (f, \text{id}) : \text{Spec}(A) \rightarrow F$, defines a cotangent complex for $\mathbf{L}_{X/Y}$. Replacing f by $F \rightarrow \text{Spec}(A)$, we may therefore assume that Y is affine and X is algebraic.

Thus write $Y = \text{Spec}(R)$. Suppose first that X is also affine, say $X = \text{Spec}(A)$. Then we let $\mathbf{L}_{X/Y} \in \mathbf{D}_{\text{qc}}(X)_{\geq 0}$ be the connective quasi-coherent complex corresponding to the connective derived A -module

$$\mathbf{L}_{R/A} := \text{Cofib}(\mathbf{L}_R \otimes_R^{\mathbf{L}} A \rightarrow \mathbf{L}_A) \in \mathbf{D}(A)_{\geq 0}.$$

Using Theorem 8.3.3 it is easy to check that is indeed a cotangent complex for $X \rightarrow Y$.

Suppose next that X is a derived Deligne–Mumford stack. By Proposition 8.4.7, the collection

$$(\mathbf{L}_{\text{Spec}(A)/\text{Spec}(R)})_{(A,x)} \in \varprojlim_{(A,x)} \mathbf{D}(\text{Spec}(A))_{\geq 0} \simeq \mathbf{D}_{\text{qc}}(X)_{\geq 0},$$

where the limit is taken over pairs $(A \in \text{ACRing}, x \in X(A))$ such that $x : \text{Spec}(A) \rightarrow X$ is étale, determines a connective quasi-coherent complex $\mathbf{L}_{X/Y}$ over X with a canonical identification $\mathbf{L}x^* \mathbf{L}_{X/Y} \simeq \mathbf{L}_{R/A}$ for every $A \in \text{ACAlg}_R$ and étale morphism $x : \text{Spec}(A) \rightarrow X$. It is easy to check that $\mathbf{L}_{X/Y}$ is a cotangent complex for f .

Finally consider the case of a derived algebraic stack X . Let $A \in \text{ACRing}$ and $x : \text{Spec}(A) \rightarrow X$ a morphism. Choose a smooth surjection $p : U \twoheadrightarrow X$ where U is a derived algebraic space. Since p is an effective epimorphism, there exists an étale cover $\text{Spec}(A') \twoheadrightarrow \text{Spec}(A)$ up to which x lifts to $u : \text{Spec}(A') \rightarrow U$. By étale descent, it will suffice to construct $\mathcal{L}(\text{Spec}(A') \rightarrow X) \in \mathbf{D}_{\text{qc}}(\text{Spec}(A'))$ instead of $\mathcal{L}(x)$. We thus replace A by A' and assume that x lifts to $u : \text{Spec}(A) \rightarrow U$.

Let $M \in \mathbf{D}(A)_{\geq 0}$. Note that there is a canonical base point of $\text{Der}_x(X/Y, M)$ given by the image of $x \in X(A)$ in $X(A \oplus M)$ by the trivial derivation. It is clear from the definitions that we have a fibre sequence

$$\text{Der}_u(U/X, M) \rightarrow \text{Der}_u(U/Y, M) \rightarrow \text{Der}_x(X/Y, M).$$

Since $U \rightarrow X$ and $U \rightarrow X \rightarrow Y$ are representable (by derived algebraic spaces) we have

$$\begin{aligned} \text{Der}_u(U/X, M) &\simeq \text{Maps}_{\mathbf{D}(A)}(\mathbf{L}u^* \mathbf{L}_{U/X}, M) \\ \text{Der}_u(U/Y, M) &\simeq \text{Maps}_{\mathbf{D}(A)}(\mathbf{L}u^* \mathbf{L}_{U/Y}, M) \end{aligned}$$

by the cases already considered. Since p is formally smooth, the map $\text{Der}_u(U/Y, M) \rightarrow \text{Der}_x(X/Y, M)$ is surjective (on π_0). It follows that there is also a fibre sequence

$$\text{Der}_x(X/Y, M) \rightarrow \text{Maps}_{\mathbf{D}(A)}(\mathbf{L}u^* \mathbf{L}_{U/X}, M[1]) \rightarrow \text{Maps}_{\mathbf{D}(A)}(\mathbf{L}u^* \mathbf{L}_{U/Y}, M[1]),$$

since the second map here is a delooping of $\text{Der}_u(U/X, M) \rightarrow \text{Der}_u(U/Y, M)$. Hence

$$\text{Der}_x(X/Y, M) \simeq \text{Maps}_{\mathbf{D}(A)}(\mathcal{L}(x), M)$$

where we set

$$\mathcal{L}(x) = \mathbf{L}u^* \text{Cofib}(\mathbf{L}_{U/X}[-1] \rightarrow \mathbf{L}_{U/Y}[-1]) \simeq \mathbf{L}u^* \text{Fib}(\mathbf{L}_{U/X} \rightarrow \mathbf{L}_{U/Y}).$$

Since $u^*\mathbf{L}_{U/X}$ and $u^*\mathbf{L}_{U/Y}$ are stable under derived base change in u (they obviously assemble to the quasi-coherent complexes $\mathbf{L}_{U/X}$ and $\mathbf{L}_{U/Y}$ respectively), so is $\mathcal{L}(x)$. Since connectivity is stable under colimits and $\mathbf{L}_{U/X}[-1]$ and $\mathbf{L}_{U/Y}[-1]$ are (-1) -connective (by the cases above), it is clear that $\mathcal{L}(x)$ is (-1) -connective. Thus the collection $(\mathcal{L}(x))_{A,x}$ assembles to a (-1) -connective cotangent complex $\mathbf{L}_{X/Y} \in \text{D}_{\text{qc}}(X)$. \square

Theorem 8.2.3 immediately globalizes as follows:

Proposition 8.6.4.

- (i) *Let $f : X \rightarrow Y$ be a morphism of derived stacks over a derived stack S . Then there is an exact triangle*

$$\mathbf{L}f^*\mathbf{L}_{Y/S} \rightarrow \mathbf{L}_{X/S} \rightarrow \mathbf{L}_{X/Y}$$

in $\text{D}_{\text{qc}}(X)$.

- (ii) *For every cartesian square*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow p & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in DStk (exhibiting X' as the derived fibred product $X \times_Y^{\mathbf{R}} Y'$), there is a canonical isomorphism

$$\mathbf{L}p^*\mathbf{L}_{X/Y} \simeq \mathbf{L}_{X'/Y'}.$$

Example 8.6.5. Let G be a smooth group algebraic space over a base scheme S , and let U be a (derived) algebraic stack over S with a G -action. We can describe the cotangent complex of the quotient stack $X = [U/G]$ as follows. Consider the cartesian square (where as usual, all products are fibred over S):

$$\begin{array}{ccc} G \times U & \xrightarrow{\text{pr}} & U \\ \downarrow \text{act} & & \downarrow p \\ U & \xrightarrow{p} & [U/G]. \end{array}$$

We have

$$\mathbf{L}_{U/[U/G]} \simeq \mathbf{L}d^*\mathbf{L}\text{act}^*\mathbf{L}_{U/[U/G]} \simeq \mathbf{L}d^*\mathbf{L}_{G \times U/U} \simeq \mathbf{L}d^*\mathbf{L}\text{pr}_1^*\mathbf{L}_{G/S}$$

where $d = (e, \text{id}) : U \rightarrow G \times U$ and $\text{pr}_1 : G \times U \rightarrow G$, using Proposition 8.6.4(ii) twice. Since $\text{pr}_1 \circ d$ factors as the projection $f : U \rightarrow S$ followed by the identity section $e : S \rightarrow G$, we get

$$\mathbf{L}_{U/[U/G]} \simeq \mathbf{L}f^*\mathbf{L}e^*\mathbf{L}_{G/S} \simeq \mathbf{L}f^*\mathcal{g}$$

where $\mathcal{g} = \mathbf{L}e^*\mathbf{L}_{G/S} \simeq e^*\Omega_{G/S}$ is the dual Lie algebra of G (recall that G is smooth).

Finally, we have by Proposition 8.6.4(i) an exact triangle

$$\mathbf{L}p^*\mathbf{L}_{[U/G]} \rightarrow \mathbf{L}U \rightarrow \mathbf{L}_{U/[U/G]}$$

where $p: U \rightarrow [U/G]$ is the quotient morphism and $\mathbf{L}U$ and $\mathbf{L}_{[U/G]}$ denote the cotangent complexes relative to the base S . Under the equivalence $\mathrm{D}_{\mathrm{qc}}([U/G]) \simeq \mathrm{D}_{\mathrm{qc}}^G(U)$, $\mathbf{L}_{[U/G]}$ may be regarded as the quasi-coherent complex

$$\mathrm{Fib}(\mathbf{L}U \rightarrow \mathbf{L}f^*\mathcal{G}) \in \mathrm{D}_{\mathrm{qc}}(U)$$

with a natural G -action (induced naturally by the action on U). For example, if U is a smooth scheme, then this is a 2-term complex with Ω_U in degree 0 and $f^*\mathcal{G}$ in (homological) degree -1 . Note that if G is finite (and hence étale), we have $\mathcal{G} \simeq 0$; this is compatible with the fact that $\mathbf{L}_{[U/G]}$ should be connective (since $[U/G]$ is Deligne–Mumford).

Specializing to the absolute case, the following tautological reformulation will be useful in practice:

Proposition 8.6.6. *Let X be a derived stack over a commutative ring R . Then X admits a cotangent complex $\mathbf{L}_X := \mathbf{L}_{X/R}$ over $\mathrm{Spec}(R)$ if and only if the following conditions hold:*

- (i) *For every animated R -algebra A and every $x \in X(A)$, denote by $F_x(N)$ the fibre at x of the map*

$$X(A \oplus N) \rightarrow X(A)$$

for every $N \in \mathrm{D}(A)_{\geq 0}$. Then the functor $F_x(-)$ is corepresented by an eventually connective derived A -module M_x .

- (ii) *For every morphism of animated R -algebras $A \rightarrow B$ and every connective derived B -module N , the commutative square*

$$\begin{array}{ccc} X(A \oplus N) & \longrightarrow & X(B \oplus N) \\ \downarrow & & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array}$$

is cartesian.

Moreover, under these conditions we have $x^\mathbf{L}_X \simeq M_x$ for every $A \in \mathrm{ACAlg}_R$ and $x \in X(A)$ (modulo the equivalence $\mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(A)) \simeq \mathrm{D}(A)$).*

8.7. Smoothness.

Proposition 8.7.1. *Let $f: X \rightarrow Y$ be a morphism of derived algebraic stacks. Then f is smooth if and only if $f_{\mathrm{cl}}: X_{\mathrm{cl}} \rightarrow Y_{\mathrm{cl}}$ is locally of finite presentation and $\mathbf{L}_{X/Y}$ is perfect of Tor-amplitude ≤ 0 (or equivalently, $[-1, 0]$), i.e., if and only if $\mathbf{L}_{X/Y}$ is a perfect complex such that*

$$\pi_i(\mathbf{L}_{X/Y} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}) = 0$$

for all $\mathcal{F} \in \mathrm{QCoh}(X)$ and all $i > 0$.

Example 8.7.2. If X is a smooth algebraic stack over a commutative ring R , then by definition there exists a smooth scheme U and a smooth representable surjection $p : U \twoheadrightarrow X$. We have the exact triangle (Proposition 8.6.4)

$$p^* \mathbf{L}_X \rightarrow \mathbf{L}_U \rightarrow \mathbf{L}_{U/X}.$$

Since U is smooth over $\mathrm{Spec}(R)$, $\mathbf{L}_U \simeq \Omega_U^1$ (= cotangent sheaf of U over $\mathrm{Spec}(R)$) is locally free and has Tor-amplitude in $[0, 0]$. Since $p : U \twoheadrightarrow X$ is smooth and representable, $\mathbf{L}_{U/X}$ is also has Tor-amplitude in $[0, 0]$. It follows that the fibre $p^* \mathbf{L}_X$ has Tor-amplitude in $[-1, 0]$. Since p is smooth surjective, it follows that \mathbf{L}_X has Tor-amplitude in $[-1, 0]$.

This suggests the following generalization of smoothness:

Definition 8.7.3. Let $f : X \rightarrow Y$ be a morphism of derived algebraic stacks. We say that f is *homotopically smooth* if f_{cl} is locally of finite presentation and $\mathbf{L}_{X/Y}$ is a perfect complex. We say that f is *homotopically n -smooth*, $n \geq 0$, if moreover $\mathbf{L}_{X/Y}$ is of Tor-amplitude $\leq n$.

Example 8.7.4. We say that $f : X \rightarrow Y$ is *quasi-smooth* if it is homotopically 1-smooth. This admits the following equivalent characterization: there exists a smooth surjection $U \twoheadrightarrow X$ such that $f|_U$ factors via a smooth morphism $Y' \rightarrow Y$ and a closed immersion $U \rightarrow Y'$ which exhibits U as the derived zero locus of a section s of a vector bundle E over Y' .

$$\begin{array}{ccccc} U & \longrightarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow s & & \\ Y' & \xrightarrow{0} & E & & \end{array}$$

See [KRy, Prop. 2.3.14]. In fact, it is possible to generalize this to characterize homotopical n -smoothness by taking into account “shifted” vector bundles $E[-i]$, $0 \leq i < n$.

Remark 8.7.5. The notion of homotopical smoothness is not as interesting when X and Y are required to be classical (underived). For example, if X and Y are classical noetherian schemes and $i : X \hookrightarrow Y$ is a closed immersion of finite Tor-amplitude (this is automatic say if Y is regular), then the following conditions are equivalent (see [Avr, Thm. 1.3]):

- (a) i is homotopically smooth, i.e., the relative cotangent complex $\mathbf{L}_{X/Y}$ is a perfect complex;
- (b) i is homotopically 1-smooth, i.e., the relative cotangent complex $\mathbf{L}_{X/Y}$ is a perfect complex of Tor-amplitude $[0, 1]$;
- (b') i is a regular (or *lci*) closed immersion.

Whereas homotopical smoothness is a rather restrictive condition in classical algebraic geometry, we will see that in the derived setting it often comes for free: natural derived enhancements of singular moduli functors that arise in practice typically tend to be homotopically smooth. The following fact may be regarded as a conceptual explanation for this phenomenon: it

turns out that homotopical smoothness is equivalent to being locally of finite presentation in a derived sense:

Theorem 8.7.6 (Lurie). *Let $f : X \rightarrow Y$ be a morphism of derived algebraic stacks. Then f is homotopically smooth if and only if f is locally homotopically of finite presentation, i.e., if for every affine derived scheme $V = \mathrm{Spec}(A)$ and every morphism $V \rightarrow Y$, the derived fibre $X \times_Y^{\mathbf{R}} V$ preserves filtered colimits when regarded as a functor $\mathrm{ACAlg}_A \rightarrow \mathrm{Grpd}_\infty$.*

Here we used the following observation to make sense of the definition of “locally homotopically of finite presentation”:

Remark 8.7.7. Let $X \rightarrow Y$ be a morphism of derived stacks where $Y = \mathrm{Spec}(A)$ is an affine derived scheme. Then X may be regarded equivalently as a functor

$$X : \mathrm{ACAlg}_A \rightarrow \mathrm{Grpd}_\infty,$$

where $\mathrm{ACAlg}_A = \mathrm{ACRing}_{A \setminus -}$ is the ∞ -category of A -algebras, via the canonical equivalence

$$\mathrm{Shv}(\mathrm{ACRing}^{\mathrm{op}}; \mathrm{Grpd}_\infty)_{/\mathrm{Spec}(A)} \simeq \mathrm{Shv}(\mathrm{ACAlg}_A^{\mathrm{op}}; \mathrm{Grpd}_\infty).$$

Remark 8.7.8. By way of comparison, let us recall that a morphism of classical stacks $f : X \rightarrow Y$ is locally of finite presentation if and only if for every affine scheme $V = \mathrm{Spec}(A)$ and every morphism $V \rightarrow Y$, the fibre $X \times_Y V$ preserves filtered colimits when regarded as a functor $\mathrm{CAlg}_A \rightarrow \mathrm{Grpd}$.

8.8. Tangent and cotangent bundles.

Definition 8.8.1. Let X be a derived stack. Given a perfect complex $\mathcal{E} \in \mathrm{Perf}(X)$, we denote by $\mathbf{V}(\mathcal{E})$ the stack of cosections of \mathcal{E} , or equivalently sections of \mathcal{E}^\vee . That is, given a derived scheme T over X , the T -points of $\mathbf{V}(\mathcal{E})$ over X are morphisms $\mathcal{E}|_T \rightarrow \mathcal{O}_T$ in $\mathrm{Perf}(T)$: its functor of points $\mathrm{DSch}_X^{\mathrm{op}} \rightarrow \mathrm{Grpd}_\infty$ is

$$(T \xrightarrow{t} X) \mapsto \mathrm{Maps}_{\mathrm{D}_{\mathrm{qc}}(T)}(\mathbf{L}t^*\mathcal{E}, \mathcal{O}_T).$$

We call $\mathbf{V}(\mathcal{E})$ the *derived vector bundle* associated with \mathcal{E} .⁹

Theorem 8.8.2. *If \mathcal{E} is $(-n)$ -connective, then $\mathbf{V}(\mathcal{E})$ is n -Artin over X .*

Definition 8.8.3. Let X be a homotopically smooth derived Artin stack over a commutative ring R . The *tangent* and *cotangent bundles* of X (over R) are the derived vector bundles

$$T_X := \mathbf{V}(\mathbf{L}_X), \quad T_X^* := \mathbf{V}(\mathbf{L}_X^\vee),$$

respectively. Similarly, given a homotopically smooth morphism $f : X \rightarrow Y$, the relative tangent and cotangent bundles are the derived vector bundles

$$T_{X/Y} := \mathbf{V}(\mathbf{L}_{X/Y}), \quad T_{X/Y}^* := \mathbf{V}(\mathbf{L}_{X/Y}^\vee),$$

over X .

⁹This agrees with Grothendieck’s convention for vector bundles, in which $\mathcal{E} \mapsto \mathbf{V}(\mathcal{E})$ is contravariant, rather than the dual convention taking the assignment $\mathcal{E} \mapsto \mathrm{Spec}(\mathrm{Sym}(\mathcal{E}^\vee))$.

Example 8.8.4. If X is a smooth scheme over R , then $\mathbf{L}_X \simeq \Omega_X^1[0]$ is the cotangent sheaf in degree zero, so T_X and T_X^* are the usual tangent and cotangent bundles.

Example 8.8.5. Suppose $X = [U/G]$ where G is a smooth group scheme over R and U is a smooth R -scheme with G -action. We saw in Example 8.6.5 that the cotangent complex $\mathbf{L}_{[U/G]}$ is the 2-term G -equivariant complex

$$[\Omega_U \rightarrow \mathfrak{g} \boxtimes \mathcal{O}_U]$$

with Ω_U in degree 0, $\mathfrak{g} \boxtimes \mathcal{O}_U$ in (homological) degree -1 , and with differential the “infinitesimal coaction” map. Let $\mathfrak{g} := \mathbf{V}(\mathfrak{g})$ and $\mathfrak{g}^* := \mathbf{V}(\mathfrak{g}^\vee)$. We see that $T_{[U/G]} \times_{[U/G]} U = \mathbf{V}(\mathbf{L}_{[U/G]}|_U)$ is the stacky quotient $[T_U/\mathfrak{g} \times U]$ of the infinitesimal action map $\mathfrak{g} \times U \rightarrow T_U$, and the tangent bundle $T_{[U/G]}$ is the further quotient by G :

$$T_{[U/G]} \simeq [[T_U/G]/[\mathfrak{g} \times U/G]] \quad (8.8.6)$$

where $[\mathfrak{g} \times U/G]$ is the adjoint quotient stack. Its dual, the cotangent bundle $T_{[U/G]}^*$, is the derived zero locus of the infinitesimal coaction map, further quotiented by G . That is, form the homotopy cartesian square

$$\begin{array}{ccc} \mu^{-1}(0) & \longrightarrow & T_U^* \\ \downarrow & & \downarrow \\ U & \xrightarrow{0} & \mathfrak{g}^* \times U. \end{array}$$

Here $\mu^{-1}(0)$ is equivalently the derived zero locus of the moment map $\mu: T_U^* \rightarrow \mathfrak{g}^* \times U \rightarrow \mathfrak{g}^*$. Thus $T_{[U/G]}^*$ is the derived Hamiltonian reduction

$$T_{[U/G]}^* \simeq [\mu^{-1}(0)/G]. \quad (8.8.7)$$

Remark 8.8.8. In Example 8.8.5, the derived zero locus $\mu^{-1}(0)$ is rarely classical, and $T_{[U/G]}^*$ is quasi-smooth but rarely smooth. Thus even if one only cares about smooth stacks X , it becomes necessary to use the language of derived algebraic geometry as soon as we want to look at the cotangent bundle T_X^* (without truncating important information).

Definition 8.8.9. Let $f: X \rightarrow Y$ be a homotopically smooth morphism of derived Artin stacks. The *normal* and *conormal bundles* are

$$N_{X/Y} := T_{X/Y}[1] := \mathbf{V}(\mathbf{L}_{X/Y}[-1]), \quad N_{X/Y}^* := T_{X/Y}^*[-1] := \mathbf{V}(\mathbf{L}_{X/Y}^\vee[1]),$$

respectively.

Example 8.8.10. If $f: X \rightarrow Y$ is a regular closed immersion between schemes, then the cotangent complex $\mathbf{L}_{X/Y} \simeq \mathcal{N}_{X/Y}[1]$ is the shifted conormal sheaf, so $N_{X/Y}$ is nothing else than the usual normal bundle.

9. DEFORMATION THEORY OF PERFECT COMPLEXES

9.1. The moduli stack of perfect complexes.

Construction 9.1.1. Let $\mathcal{M}_{\text{Perf}}$ denote the functor

$$\text{ACRing} \rightarrow \text{Grpd}_{\infty}, \quad A \mapsto \text{D}_{\text{perf}}(A)^{\simeq}$$

sending an animated ring to the ∞ -groupoid of perfect derived A -modules. It follows from Theorem 8.4.2 that this satisfies étale descent, hence defines a derived stack.

Remark 9.1.2. The derived stack $\mathcal{M}_{\text{Perf}}$, or at least its classical truncation, was first studied by A. Hirschowitz and C. Simpson [HS], and analyzed in more detail by B. Toën and M. Vaquié [TV].

Construction 9.1.3. The *universal perfect complex* is the perfect complex

$$\mathcal{G}^{\text{univ}} \in \text{D}_{\text{perf}}(\mathcal{M}_{\text{Perf}})$$

defined such that for every animated ring A and every morphism $x : \text{Spec}(A) \rightarrow \mathcal{M}_{\text{Perf}}$ classifying a perfect complex $\mathcal{G} \in \text{D}_{\text{perf}}(A)$, we have

$$x^*(\mathcal{G}^{\text{univ}}) \simeq \mathcal{G}.$$

It follows formally that for *every* derived stack X and every perfect complex $\mathcal{G} \in \text{D}_{\text{perf}}(X)$, there is a unique morphism $f : X \rightarrow \mathcal{M}_{\text{Perf}}$ and an isomorphism $f^*(\mathcal{G}^{\text{univ}}) \simeq \mathcal{G}$.

Remark 9.1.4. We can similarly consider the larger stacks $\mathcal{M}_{\text{D}_{\text{coh}}}$ and $\mathcal{M}_{\text{D}_{\text{pscoh}}}$ sending $A \in \text{ACRing}_R$ to $\text{D}_{\text{coh}}(A)^{\simeq}$ or $\text{D}_{\text{pscoh}}(A)^{\simeq}$, respectively. Here $\text{D}_{\text{pscoh}}(A) \subseteq \text{D}_{\text{qc}}(A)$ is the full subcategory of *pseudocoherent* derived A -modules (sometimes called *almost perfect* A -modules) and $\text{D}_{\text{coh}}(A) \subseteq \text{D}_{\text{pscoh}}(A)$ is the full subcategory of *coherent* derived A -modules.¹⁰ There are open immersions of derived stacks

$$\mathcal{M}_{\text{D}_{\text{perf}}} \hookrightarrow \mathcal{M}_{\text{D}_{\text{coh}}} \hookrightarrow \mathcal{M}_{\text{D}_{\text{pscoh}}},$$

but these larger stacks do not admit a cotangent complex (because the perfect complexes are precisely the dualizable objects in $\text{D}_{\text{pscoh}}(X)$).

9.2. The cotangent complex of the moduli of perfect complexes.

Theorem 9.2.1. *The perfect complex*

$$\mathbf{L}_{\mathcal{M}_{\text{Perf}}} := \mathcal{G}^{\text{univ}} \otimes^{\mathbf{L}} \mathcal{G}^{\text{univ}, \vee}[-1]$$

is a cotangent complex for the derived stack $\mathcal{M}_{\text{Perf}}$.

Lemma 9.2.2. *Let A be an animated ring and $M \in \text{D}_{\text{perf}}(A)$ a perfect derived A -module. For every $N \in \text{D}_{\text{perf}}(A)_{\geq 0}$ denote by $F_M(N)$ the fibre at M of the map of anima*

$$\text{D}_{\text{perf}}(A \oplus N)^{\simeq} \rightarrow \text{D}_{\text{perf}}(A)^{\simeq}$$

given by extending scalars along the canonical homomorphism $A \oplus N \rightarrow A$. Then we have canonical isomorphisms

$$F_M(N) \simeq \text{Maps}_{\text{D}(A)}(M \otimes_A^{\mathbf{L}} M^{\vee}[-1], N),$$

¹⁰When A is noetherian these are defined as follows: $M \in \text{D}(A)$ is pseudocoherent if it is homologically bounded above ($\pi_i(M) = 0$ for $i \gg 0$) and its homotopy groups $\pi_i(M)$ are finitely generated $\pi_0(A)$ -modules (see [Lur2, Def. 7.2.4.10]); it is *coherent* if it is additionally bounded below ($\pi_i(M) = 0$ for $i \ll 0$).

natural in N .

Proof. By definition, $F_M(N)$ is the ∞ -groupoid of deformations of M along $A \oplus N \rightarrow A$; that is, it is the ∞ -groupoid of pairs (\widetilde{M}, θ) where \widetilde{M} is an $A \oplus N$ -module and θ is an isomorphism $\widetilde{M} \otimes_{A \oplus N}^{\mathbf{L}} A \simeq M$. Since the square

$$\begin{array}{ccc} A \oplus N & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus N[1] \end{array}$$

is cartesian, this is equivalent to the ∞ -groupoid of deformations of $M \otimes_A^{\mathbf{L}} (A \oplus N[1]) \simeq M \oplus (M \otimes_A^{\mathbf{L}} N[1])$ along the trivial derivation $A \rightarrow A \oplus N[1]$. Equivalently, this is the ∞ -groupoid of automorphisms of $M \oplus (M \otimes_A^{\mathbf{L}} N[1])$ over $A \oplus N[1]$ which extend to the identity $\text{id}_M : M = M$ along $A \oplus N[1] \rightarrow A$. That is,

$$F_M(N) \simeq \text{End}_{A \oplus N[1]}(M \oplus (M \otimes_A^{\mathbf{L}} N[1])) \times_{\text{End}_{D(A)}(M)} \{\text{id}_M\}$$

where we can write End instead of Aut since every such endomorphism is necessarily invertible. Thus we have

$$\begin{aligned} F_M(N) &\simeq \text{Maps}_{D(A)}(M, (M \oplus (M \otimes_A^{\mathbf{L}} N[1])) \times_M 0) \\ &\simeq \text{Maps}_{D(A)}(M, M \otimes_A^{\mathbf{L}} N[1]) \simeq \text{Maps}_{D(A)}(M \otimes_A^{\mathbf{L}} M^\vee[-1], N) \end{aligned}$$

where the last isomorphism follows from the fact that M is perfect, hence dualizable. \square

Proof of Theorem 9.2.1. Let $A \in \text{ACRing}$ be an animated ring and $x : \text{Spec}(A) \rightarrow \mathcal{M}_{\text{Perf}}$ an A -point classifying a perfect derived A -module $M \in D_{\text{perf}}(A)$. By Lemma 9.2.2, the animum of derivations $\text{Der}_x(X, M)$ (relative to $\text{Spec}(\mathbf{Z})$) is corepresented by $M \otimes_A^{\mathbf{L}} M^\vee[-1]$. Moreover, if $A \rightarrow B$ is a morphism of animated rings, then we have an isomorphism

$$(M \otimes_A^{\mathbf{L}} M^\vee[-1]) \otimes_A^{\mathbf{L}} B \simeq N \otimes_B^{\mathbf{L}} N^\vee[-1]$$

where $N = M \otimes_A^{\mathbf{L}} B$ is the perfect derived B -module classified by

$$\text{Spec}(B) \rightarrow \text{Spec}(A) \xrightarrow{x} \mathcal{M}_{\text{Perf}}.$$

It follows that the collection $(M \otimes_A^{\mathbf{L}} M^\vee[-1])$ assembles, as (A, x) varies, into a perfect complex on $\mathcal{M}_{\text{Perf}}$ which is a cotangent complex for $\mathcal{M}_{\text{Perf}}$. By construction, this perfect complex is nothing else than $\mathcal{G}^{\text{univ}} \otimes^{\mathbf{L}} \mathcal{G}^{\text{univ}, \vee}[-1] \in D_{\text{perf}}(\mathcal{M}_{\text{Perf}})$. \square

10. MODULI STACKS OF COMPLEXES, SHEAVES, AND BUNDLES

10.1. Mapping stacks.

Construction 10.1.1. Let S be a derived stack and let X and Y be derived stacks over S . The *derived mapping stack* $\underline{\mathrm{Maps}}_S(X, Y)$ is the functor

$$\underline{\mathrm{Maps}}_S(X, Y) : \mathrm{ACRing} \rightarrow \mathrm{Grpd}_\infty, \quad R \mapsto \mathrm{Maps}_S(X \times_S^{\mathbf{R}} \mathrm{Spec}(R), Y)$$

sending R to the ∞ -groupoid of S -morphisms $X_R := X \times_S^{\mathbf{R}} \mathrm{Spec}(R) \rightarrow Y$.

Remark 10.1.2. If X is *flat* over S , then the classical truncation of $\underline{\mathrm{Maps}}_S(X, Y)$ coincides with the classical mapping or Hom stack:

$$\underline{\mathrm{Maps}}_S(X, Y)_{\mathrm{cl}} \simeq \underline{\mathrm{Hom}}_{S_{\mathrm{cl}}}(X_{\mathrm{cl}}, Y_{\mathrm{cl}}).$$

Definition 10.1.3. The *evaluation morphism*

$$\mathrm{ev} : \mathrm{Maps}_S(X, Y) \times_S^{\mathbf{R}} X \rightarrow Y$$

is the morphism classified by the identity $\mathrm{id} : \mathrm{Maps}_S(X, Y) \rightarrow \mathrm{Maps}_S(X, Y)$.

Proposition 10.1.4. *Let $f : X \rightarrow Y$ be a morphism of derived algebraic stacks. If f is proper, representable, and of finite Tor-amplitude, then $\mathbf{L}f^* : \mathrm{D}_{\mathrm{qc}}(Y) \rightarrow \mathrm{D}_{\mathrm{qc}}(X)$ admits a left adjoint f_{\sharp} . For perfect complexes $\mathcal{F} \in \mathrm{D}_{\mathrm{perf}}(X)$, it is given by the formula $f_{\sharp}(\mathcal{F}) := \mathbf{R}f_*(\mathcal{F}^\vee)^\vee$.*

Proof. If $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X)$ is perfect, hence dualizable, we may set

$$f_{\sharp}(\mathcal{F}) := \mathbf{R}f_*(\mathcal{F}^\vee)^\vee.$$

It is clear that this defines a left adjoint to $\mathbf{L}f^* : \mathrm{D}_{\mathrm{perf}}(Y) \rightarrow \mathrm{D}_{\mathrm{perf}}(X)$:

$$\begin{aligned} \mathrm{Maps}(\mathbf{R}f_*(\mathcal{F}^\vee)^\vee, \mathcal{G}) &\simeq \mathrm{Maps}(\mathcal{O}_Y, \mathcal{G} \otimes^{\mathbf{L}} \mathbf{R}f_*(\mathcal{F}^\vee)) \\ &\simeq \mathrm{Maps}(\mathcal{G}^\vee, \mathbf{R}f_*(\mathcal{F}^\vee)) \\ &\simeq \mathrm{Maps}(\mathbf{L}f^*(\mathcal{G})^\vee, \mathcal{F}^\vee) \\ &\simeq \mathrm{Maps}(\mathcal{F}, \mathbf{L}f^*(\mathcal{G})), \end{aligned}$$

for every perfect complex $\mathcal{G} \in \mathrm{D}_{\mathrm{perf}}(Y)$, where we recall that for a dualizable object A the functors $A \otimes^{\mathbf{L}} (-)$ and $A^\vee \otimes^{\mathbf{L}} (-)$ are adjoint to one another.

If \mathcal{F} is not perfect, to define $f_{\sharp}(\mathcal{F}) \in \mathrm{D}_{\mathrm{qc}}(Y)$ is equivalent to define $\mathbf{L}v^* f_{\sharp}(\mathcal{F})$ for every morphism $v : \mathrm{Spec}(R) \rightarrow Y$ where $R \in \mathrm{ACRing}$. By left transposition from the derived base change formula $\mathbf{L}f^* \mathbf{R}v_* \simeq \mathbf{R}u_* \mathbf{L}f_R^*$, we have $\mathbf{L}v^* f_{\sharp}(\mathcal{F}) \simeq f_{R, \sharp} \mathbf{L}u^*(\mathcal{F})$ whenever f_{\sharp} and $f_{R, \sharp}$ exist. Therefore we may replace Y by $\mathrm{Spec}(R)$ and assume Y is affine.

In this case, X is a derived algebraic space which is in particular separated over $\mathrm{Spec}(R)$ and hence quasi-compact quasi-separated. For such X , a (generalization of) an important theorem of Thomason asserts that every quasi-coherent complex can be written as a filtered colimit of perfect complexes (see e.g. [Lur3, Prop. 9.6.1.1] or [Kha2, Thm. 1.40]). It follows that there is a unique colimit-preserving functor $f_{\sharp} : \mathrm{D}_{\mathrm{qc}}(X) \rightarrow \mathrm{D}_{\mathrm{qc}}(Y)$ which restricts to $\mathbf{R}f_*(-^\vee)^\vee$ on perfect complexes. This is clearly left adjoint to $\mathbf{L}f^*$: if $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X)$ is a colimit of a filtered system $(\mathcal{F}_\alpha)_\alpha$ of perfect complexes,

then we have:

$$\begin{aligned}
\mathrm{Maps}(f_{\sharp}(\mathcal{F}), \mathcal{G}) &\simeq \mathrm{Maps}(\varinjlim_{\alpha} f_{\sharp}(\mathcal{F}_{\alpha}), \mathcal{G}) \\
&\simeq \varprojlim_{\alpha} \mathrm{Maps}(f_{\sharp}(\mathcal{F}_{\alpha}), \mathcal{G}) \\
&\simeq \varprojlim_{\alpha} \mathrm{Maps}(\mathcal{F}_{\alpha}, \mathbf{L}f^{*}(\mathcal{G})) \\
&\simeq \mathrm{Maps}(\varinjlim_{\alpha} \mathcal{F}_{\alpha}, \mathbf{L}f^{*}(\mathcal{G})) \\
&\simeq \mathrm{Maps}(\mathcal{F}, \mathbf{L}f^{*}(\mathcal{G})),
\end{aligned}$$

for every $\mathcal{G} \in \mathrm{D}_{\mathrm{qc}}(Y)$. \square

Remark 10.1.5. In the situation of Proposition 10.1.4, it is possible to show that the functor $\mathbf{R}f_{*}$ preserves colimits and admits a right adjoint $f^{!}$. The quasi-coherent complex $\omega_{X/Y} := f^{!}(\mathcal{O}_Y) \in \mathrm{D}_{\mathrm{qc}}(X)$ is called the relative *dualizing complex*. One then has the following alternative description of the functor f_{\sharp} :

$$f_{\sharp}(-) \simeq \mathbf{R}f_{*}(- \otimes \omega_{X/Y}).$$

Theorem 10.1.6. *Suppose S is algebraic and X and Y are derived stacks over S . Set $H := \underline{\mathrm{Maps}}_S(X, Y)$ and consider the diagram*

$$H \xleftarrow{\pi} X \times_S^{\mathbf{R}} H \xrightarrow{\mathrm{ev}} Y$$

where π is the projection. If X is proper of finite Tor-amplitude and representable over S , and Y admits a cotangent complex $\mathbf{L}_{Y/S}$ over S , then the perfect complex

$$\mathbf{L}_{H/S} \simeq \pi_{\sharp} \mathbf{L}\mathrm{ev}^{*}(\mathbf{L}_{Y/S})$$

is a relative cotangent complex for H over S .

Proof. For simplicity we take $S = \mathrm{pt} = \mathrm{Spec}(k)$ (and omit it from the notation). Given $R \in \mathrm{ACAlg}_k$, an R -point $h \in H(R)$ classifying a morphism $f : X_R \rightarrow Y$, and a connective derived R -module $M \in \mathrm{D}(R)_{\geq 0}$, derivations of H at h are extensions of the morphism $f : X_R \rightarrow Y$ along $X_R \hookrightarrow X_{R \oplus M}$. Since the latter can be regarded as the trivial square-zero extension of X_R by $p_R^{*}(M)$, where $p_R : X_R \rightarrow \mathrm{Spec}(R)$ is the projection, these are classified by the cotangent complex of Y , i.e.,

$$\mathrm{Der}_h(H, M) \simeq \mathrm{Maps}_{\mathrm{D}(X_R)}(\mathbf{L}f^{*} \mathbf{L}_Y, \mathbf{L}p_R^{*}(M)).$$

The assumptions on X imply that $\mathbf{L}p_R^{*}$ admits a left adjoint $p_{R, \sharp}$, hence $\mathrm{Der}_h(H, -)$ is corepresented by $p_{R, \sharp} \mathbf{L}f^{*} \mathbf{L}_Y$. Now $\mathbf{L}_H := \pi_{\sharp} \mathbf{L}\mathrm{ev}^{*}(\mathbf{L}_Y)$ is the unique perfect complex on H such that $\mathbf{L}h^{*}(\mathbf{L}_H) \simeq p_{R, \sharp} \mathbf{L}f^{*}(\mathbf{L}_Y)$, since we have the commutative diagram

$$\begin{array}{ccccc}
H & \xleftarrow{\pi} & X \times_S^{\mathbf{R}} H & \xrightarrow{\mathrm{ev}} & Y \\
h \uparrow & & \uparrow & & f \uparrow \\
\mathrm{Spec}(R) & \xleftarrow{p_R} & X_R & \xlongequal{\quad} & X_R
\end{array}$$

where the left-hand square is cartesian. It follows that \mathbf{L}_H is a cotangent complex for H . \square

10.2. Moduli of complexes, sheaves, and bundles.

Definition 10.2.1. Let k be a commutative ring and X a derived algebraic stack over k .

- (i) The *moduli stack of perfect complexes on X* is the derived mapping stack

$$\mathcal{M}_{\text{Perf}(X)} = \underline{\text{Maps}}(X, \mathcal{M}_{\text{Perf}})$$

where we omit the subscript $\text{Spec}(k)$ from the notation in the mapping stack. By definition, its R -points for an animated k -algebra R are k -morphisms $X_R := X \times \text{Spec}(R) \rightarrow \mathcal{M}_{\text{Perf}}$, i.e., perfect complexes on X_R .

- (ii) Let G be a smooth group scheme over k . The *moduli stack of G -torsors over X* (a.k.a. *principal G -bundles over X*) is the derived mapping stack

$$\mathcal{M}_{\text{Bun}_G(X)} = \underline{\text{Maps}}(X, BG).$$

By definition, its R -points for an animated k -algebra R are k -morphisms $X_R \rightarrow BG$, i.e., G -torsors on X_R .

- (iii) The *moduli stack of vector bundles over X* is the substack $\mathcal{M}_{\text{Vect}(X)} \subseteq \mathcal{M}_{\text{Perf}(X)}$ defined as follows: for an animated k -algebra R , an R -point of $\mathcal{M}_{\text{Perf}(X)}$ belongs to $\mathcal{M}_{\text{Vect}(X)}$ if and only if the corresponding perfect complex $\mathcal{F} \in \text{D}_{\text{perf}}(X_R)$ is connective and flat (over X_R).

- (iv) The *moduli stack of coherent sheaves on X* is the substack $\mathcal{M}_{\text{Coh}(X)}$ of the derived mapping stack

$$\mathcal{M}_{\text{D}_{\text{coh}}(X)} = \underline{\text{Maps}}(X, \text{D}_{\text{coh}})$$

defined as follows: for an animated k -algebra R , an R -point of $\mathcal{M}_{\text{D}_{\text{coh}}(X)}$ belongs to $\mathcal{M}_{\text{Coh}(X)}$ if and only if the corresponding coherent complex $\mathcal{F} \in \text{D}_{\text{coh}}(X_R)$ is connective and flat over $\text{Spec}(R)$.

Remark 10.2.2. For R an ordinary k -algebra, the ∞ -groupoid of R -points of $\mathcal{M}_{\text{Vect}(X)}$ is equivalent to the 1-groupoid of locally free sheaves of finite rank on X_R . Note that there is a canonical isomorphism of derived stacks

$$\mathcal{M}_{\text{Vect}(X)} \simeq \coprod_{n \geq 0} \mathcal{M}_{\text{Bun}_{\text{GL}_n}(X)},$$

see Example 4.5.1.

Lemma 10.2.3. *Let X be a derived algebraic stack over an animated commutative ring R . Let $\mathcal{F} \in \text{D}_{\text{coh}}(X)$ be a connective pseudocoherent complex. If R is discrete and \mathcal{F} is flat over R , then \mathcal{F} is discrete, i.e., $\mathcal{F} \in \text{Coh}(X)$.*

Proof. Recall that \mathcal{F} is discrete if for any smooth morphism $u : U \rightarrow X$ where U is affine, we have $\pi_i(\mathbf{L}u^*\mathcal{F}) = 0$ for all $i \neq 0$. Since $\mathbf{L}u^*$ preserves pseudocoherence we may assume that $X = \text{Spec}(A)$ is affine. Recall that a

connective complex M over A is flat over R if and only if $\pi_0(M)$ is flat over $\pi_0(R)$, and

$$\pi_i(M) \simeq \pi_0(M) \otimes_{\pi_0(R)}^{\mathbf{L}} \pi_i(R)$$

for all $i \geq 0$. Thus if M is flat over R and R is discrete (i.e. $\pi_i(R) = 0$ for $i \neq 0$), then we have $\pi_i(M) = 0$ for $i \neq 0$. \square

Remark 10.2.4. For R an ordinary k -algebra, Lemma 10.2.3 shows that the ∞ -groupoid of R -points of $\mathcal{M}_{\mathrm{Coh}(X)}$ is equivalent to the 1-groupoid of coherent sheaves on X_R which are flat over $\mathrm{Spec}(R)$.

Lemma 10.2.5. *If X is smooth over k , then we have an inclusion $\mathcal{M}_{\mathrm{Coh}(X)} \subseteq \mathcal{M}_{\mathrm{Perf}(X)}$.*

Proof. Let $R \in \mathrm{ACAlg}_k$. An R -point of $\mathcal{M}_{\mathrm{Coh}(X)}$ corresponds to a coherent complex $\mathcal{F} \in \mathrm{D}_{\mathrm{coh}}(X_R)$ which is connective and flat over R . To show that \mathcal{F} is perfect it is enough to check that for every point $s : \mathrm{Spec}(\kappa) \rightarrow \mathrm{Spec}(R)$, \mathcal{F} restricts to a perfect complex $\mathbf{L}i_s^*(\mathcal{F})$ over the derived fibre $X_s := X \otimes_k^{\mathbf{L}} \kappa$:

$$\begin{array}{ccc} X_s & \xrightarrow{i_s} & X_R \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\kappa) & \xrightarrow{s} & \mathrm{Spec}(R). \end{array}$$

Note that $\mathbf{L}i_s^*(\mathcal{F})$ is pseudocoherent (because $*$ -inverse image always preserves pseudocoherence). By the projection and base change formulas we have

$$\mathbf{R}i_{s,*} \mathbf{L}i_s^*(\mathcal{F}) \simeq \mathcal{F} \otimes^{\mathbf{L}} \mathbf{R}i_{s,*}(\mathcal{O}_{X_s}) \simeq \mathcal{F} \otimes^{\mathbf{L}} \mathbf{L}f^* \mathbf{R}s_*(\mathcal{O}_{\mathrm{Spec}(\kappa)}),$$

which is discrete because $\mathbf{R}s_*(\mathcal{O}_{\mathrm{Spec}(\kappa)})$ is discrete and \mathcal{F} is flat over R . It follows that $\mathbf{L}i_s^*(\mathcal{F})$ is itself discrete, and hence is a coherent sheaf. Since X_s is smooth over κ and hence regular, we conclude that $\mathbf{L}i_s^*(\mathcal{F})$ is perfect as claimed. \square

Remark 10.2.6. In the proof of Lemma 10.2.5 we implicitly used the notion of regularity of algebraic stacks and the fact that a smooth algebraic stack over a field is regular. Let us say by definition that a derived algebraic stack X is *regular* if every coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X)$ is a perfect complex, i.e., its image in $\mathrm{D}_{\mathrm{qc}}(X)$ belongs to the subcategory $\mathrm{D}_{\mathrm{perf}}(X)$. Then we have:

- (i) For a noetherian ring R , $\mathrm{Spec}(R)$ is regular in our sense if and only if its local rings are regular in the sense of Serre [Se].
- (ii) If X admits a smooth surjection $p : \mathrm{Spec}(R) \twoheadrightarrow X$ where R is a regular noetherian ring, then X is regular (recall that $\mathcal{F} \in \mathrm{Coh}(X)$ is perfect if and only if $p^*(\mathcal{F})$ is perfect and use (i)).
- (iii) If $X \rightarrow \mathrm{Spec}(R)$ is a smooth morphism of finite presentation where R is regular noetherian, then X is regular. Indeed, for any smooth atlas $\mathrm{Spec}(A) \twoheadrightarrow X$, A is a smooth R -algebra and hence is regular.

Proposition 10.2.7. *The inclusion*

$$\mathcal{M}_{\mathrm{Vect}(X)} \rightarrow \mathcal{M}_{\mathrm{Perf}(X)}$$

is an open immersion; that is, for any animated k -algebra R and any morphism $\mathrm{Spec}(R) \rightarrow \mathcal{M}_{\mathrm{Perf}(X)}$, the base change

$$\mathcal{M}_{\mathrm{Vect}(X)} \times_{\mathcal{M}_{\mathrm{Perf}(X)}}^{\mathbf{R}} \mathrm{Spec}(R) \rightarrow \mathrm{Spec}(R)$$

is an open immersion of derived schemes. If X is smooth, then the same holds for the inclusion

$$\mathcal{M}_{\mathrm{Coh}(X)} \rightarrow \mathcal{M}_{\mathrm{Perf}(X)}.$$

Proof. Note first that both morphisms are monomorphisms, since for every animated k -algebra R the induced functors $\mathcal{M}_{\mathrm{Vect}(X)}(R) \rightarrow \mathcal{M}_{\mathrm{Coh}(X)}(R) \rightarrow \mathcal{M}_{\mathrm{Perf}(X)}(R)$ are monomorphisms of ∞ -groupoids (see Example 5.1.2). Now if $\mathrm{Spec}(R) \rightarrow \mathcal{M}_{\mathrm{Perf}(X)}$ classifies a perfect complex $\mathcal{F} \in \mathrm{D}_{\mathrm{perf}}(X_R)$, then the base changes in the statement are exactly the open loci where \mathcal{F} is connective and flat (over X_R or $\mathrm{Spec}(R)$, respectively). \square

Notation 10.2.8. By construction, we have an “evaluation” morphism $\mathrm{ev} : X \times \mathcal{M}_{\mathrm{Perf}(X)} \rightarrow \mathcal{M}_{\mathrm{Perf}}$. We denote by $\mathcal{E}_X := \mathbf{L}\mathrm{ev}^*(\mathcal{E}^{\mathrm{univ}})$ the inverse image of the universal perfect complex along this map. By abuse of notation, we also denote its restriction to $\mathcal{M}_{\mathrm{Coh}(X)}$ and $\mathcal{M}_{\mathrm{Vect}(X)}$ in the same way.

Corollary 10.2.9. *Let X be as in Definition 10.2.1. The derived stack $\mathcal{M}_{\mathrm{Perf}(X)}$ admits a relative cotangent complex*

$$\mathbf{L}\mathcal{M}_{\mathrm{Perf}(X)} = \mathrm{pr}_{2,\sharp}(\mathcal{E}_X \otimes^{\mathbf{L}} \mathcal{E}_X^\vee[1]) \simeq \mathrm{pr}_{2,*}(\mathcal{E}_X \otimes^{\mathbf{L}} \mathcal{E}_X^\vee[1] \otimes^{\mathbf{L}} \mathrm{pr}_1^*(\omega_X))$$

where pr_i are the two projections from $X \times \mathcal{M}_{\mathrm{Perf}(X)}$. Moreover, we have the same formula for the cotangent complexes of $\mathcal{M}_{\mathrm{Coh}(X)}$ and $\mathcal{M}_{\mathrm{Vect}(X)}$ (modulo the abuse of notation mentioned in 10.2.8).

Proof. Combine Theorems 10.1.6 and 9.2.1. The second isomorphism follows from Remark 10.1.5. \square

Remark 10.2.10. Once we know that $\mathcal{M}_{\mathrm{Coh}(X)}$ and $\mathcal{M}_{\mathrm{Vect}(X)}$ are algebraic and locally finitely presented (on classical truncations), it will follow from Corollary 10.2.9 that they are homotopically smooth. Moreover, it will be homotopically $(n-1)$ -smooth if X is of dimension $\leq n$ (using Proposition 8.7.1). For example, it will be smooth (and classical) when X is a curve and quasi-smooth when X is a surface. Note in contrast that when X is a surface, the *classical truncations* of $\mathcal{M}_{\mathrm{Coh}(X)}$ and $\mathcal{M}_{\mathrm{Vect}(X)}$ are neither smooth nor homotopically smooth.

Remark 10.2.11. One can similarly write down the cotangent complex of $\mathcal{M}_{\mathrm{Bun}_G(X)}$ using Theorem 10.1.6 and our computation $\mathbf{L}BG \simeq \mathfrak{g}^\vee[-1]$ (Example 8.6.5). In particular, it will also be homotopically smooth, and smooth when X is of dimension ≤ 1 .

10.3. The Artin–Lurie representability theorem.

Theorem 10.3.1 (Artin–Lurie). *Let k be a commutative ring, which we assume is excellent or more generally a G -ring. Let X be a derived stack over k . Then X is algebraic if and only if the following conditions hold:*

- (i) X admits a cotangent complex \mathbf{L}_X (relative to k).
- (ii) The restriction of X to ordinary k -algebras takes values in 1-groupoids.
- (iii) Almost of finite presentation. The functor $X : \mathrm{ACAlg}_k \rightarrow \mathrm{Grpd}_\infty$ preserves filtered colimits when restricted to n -truncated algebras for all $n \geq 0$. (For example, X is locally homotopically of finite presentation, see Theorem 8.7.6.)
- (iv) Integrability. For every complete local noetherian k -algebra R , the canonical map $X(R) \rightarrow \varprojlim_n X(R/\mathfrak{m}^n)$ is invertible, where $\mathfrak{m} \subseteq R$ is the maximal ideal.
- (v) Nil-completeness. For every $R \in \mathrm{ACAlg}_k$, the canonical map $X(R) \rightarrow \varprojlim_n X(\tau_{\leq n}(R))$ is invertible.
- (vi) Infinitesimal cohesion. For every cartesian square in ACAlg_k

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

such that $A \rightarrow B$ and $B' \rightarrow B$ are surjective on π_0 with nilpotent kernel, X sends the square to a cartesian square.

Remark 10.3.2. Theorem 10.3.1 gives criteria for a derived stack to be algebraic. This also implies that its classical truncation is algebraic.

Once we know that X is algebraic, we can further detect whether it is Deligne–Mumford using its diagonal (Theorem 5.4.1), or alternatively by checking that its cotangent complex is connective. Similarly, we can check whether it is an algebraic space by checking that its restriction to ordinary algebras takes values in sets instead of groupoids (Theorem 5.4.1).

It is possible to check the conditions of Theorem 10.3.1 for the moduli stacks we have been considering (except for $\mathcal{M}_{\mathrm{Perf}(X)}$, which doesn't satisfy condition (ii)). The main point is the computation of the cotangent complex (Corollary 10.2.9); we leave the other verifications to the reader.

Theorem 10.3.3. Let k be a G -ring and X an algebraic space which is proper and of finite Tor-amplitude over k . Then the following derived stacks are algebraic:

- (i) The moduli stack $\mathcal{M}_{\mathrm{Vect}(X)}$ of vector bundles over X .
- (ii) The moduli stack $\mathcal{M}_{\mathrm{Bun}_G(X)}$ of G -bundles over X , for every smooth group scheme G over k .
- (iii) The moduli stack $\mathcal{M}_{\mathrm{Coh}(X)}$ of coherent sheaves on X , if X is smooth over k .

Moreover, they are smooth (hence classical) if X is a relative curve, and quasi-smooth if X is a relative surface.

The last claim also follows from our computation of the cotangent complex, see Remark 10.2.10.

11. COHOMOLOGY OF STACKS

11.1. Abelian sheaves. Let k denote a base field and Sch_k the category of locally of finite type k -schemes. Given $X \in \text{Sch}_k$ and a commutative ring Λ we denote by $D(X; \Lambda)$ either:

- (a) If $k = \mathbf{C}$, the stable ∞ -category $\text{Shv}(X(\mathbf{C}), D(\Lambda))$ of sheaves on the topological space $X(\mathbf{C})$ with values in the derived ∞ -category of Λ -modules.
- (b) If $\Lambda = \mathbf{Z}/n\mathbf{Z}$ where the $n \neq \text{char}(k)$, the stable ∞ -category $\text{Shv}(X_{\text{ét}}, D(\Lambda))$ of sheaves on the small étale site $X_{\text{ét}}$ with values in the derived ∞ -category of Λ -modules.

Theorem 11.1.1. *The presheaf $D^* : \text{Sch}_k^{\text{op}} \rightarrow \text{Cat}_{\infty}$ determined by the assignment*

$$X \mapsto D(X; \Lambda), \quad f \mapsto f^* \tag{11.1.2}$$

satisfies descent for the étale topology.

Remark 11.1.3. By Example 4.1.11, it follows moreover that D^* satisfies descent for smooth surjections.

Construction 11.1.4. Let AlgStk denote the ∞ -category of algebraic stacks locally of finite type over k . By Theorem 11.1.1, there exists a unique étale sheaf $D^* : \text{AlgStk}_k^{\text{op}} \rightarrow \text{Cat}_{\infty}$ extending (11.1.2). More precisely, it is the right Kan extension, given on $X \in \text{AlgStk}_k$ by the formula

$$D(X) \simeq \varprojlim_{(T, t)} D(T)$$

where the limit is taken over the category of pairs (T, t) where T is a scheme and $t : T \rightarrow X$ is a smooth morphism.

Theorem 11.1.5 (Six operations). *We have the following operations on the ∞ -categories $D(X)$ for $X \in \text{AlgStk}_k$:*

- (i) *An adjoint pair of bifunctors*

$$\begin{aligned} \otimes & : D(X) \times D(X) \rightarrow D(X), \\ \underline{\text{Hom}} & : D(X) \times D(X) \rightarrow D(X) \end{aligned}$$

for all $X \in \text{AlgStk}_k$.

- (ii) *For every morphism $f : X \rightarrow Y$ in AlgStk_k , an adjoint pair*

$$f^* : D(Y) \rightarrow D(X), \quad f_* : D(X) \rightarrow D(Y).$$

- (iii) *For every morphism $f : X \rightarrow Y$ in AlgStk_k , an adjoint pair*

$$f_! : D(X) \rightarrow D(Y), \quad f^! : D(Y) \rightarrow D(X).$$

Moreover, they satisfy the following properties:

(i) Base change formula: *For every cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow p & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

there is a canonical isomorphism

$$q^* f_! \simeq g_! p^*.$$

(ii) Projection formula: *For every morphism $f : X \rightarrow Y$ in AlgStk_k , there is a canonical isomorphism*

$$f_!(-) \otimes (-) \simeq f_!(- \otimes f^*(-)).$$

(iii) Forgetting supports: *If f has proper diagonal, there is a canonical morphism*

$$f_! \rightarrow f_*$$

which is invertible when f is proper.

(iv) Étale pull-back: *If f is étale, there is a canonical isomorphism $f^! \simeq f^*$.*

(v) Localization: *If $X \in \text{Stk}_k$ and $i : Z \hookrightarrow X$ is a closed immersion with complementary open immersion $j : U \hookrightarrow X$, then there are canonical exact triangles*

$$\begin{aligned} j_! j^* &\rightarrow \text{id} \rightarrow i_! i^* \\ i_* i^! &\rightarrow \text{id} \rightarrow j_* j^! \end{aligned}$$

This theorem was proven by Y. Liu and W. Zheng, see [LZ].

Remark 11.1.6. There is a unique way to extend all the above constructions to *derived* algebraic stacks in such a way that we still have localization triangles: since the inclusion of the classical truncation $i : X_{\text{cl}} \hookrightarrow X$ is a surjective closed immersion, we must have $D(X) \simeq D(X_{\text{cl}})$. By base change formulas, all four operations associated with a morphism $f : X \rightarrow Y$ must also be identified with the corresponding operations for $f_{\text{cl}} : X_{\text{cl}} \rightarrow Y_{\text{cl}}$.

Remark 11.1.7. Moreover, if we extend $D(-)$ to *higher* Artin stacks (and thus to all derived Artin stacks) with the same definition, then we still have the six operations in this generality.

11.2. Co/homology. Given a (derived) algebraic stack X locally of finite type over k , let $a_X : X \rightarrow \text{Spec}(k)$ denote the projection. We define

$$\begin{aligned} C^\bullet(X; \Lambda) &:= R\Gamma(f_* f^* \Lambda) \simeq R\Gamma(X; \Lambda_X), \\ C_\bullet^{\text{BM}}(X; \Lambda) &:= R\Gamma(f_* f^! \Lambda) \simeq R\Gamma(X; \omega_X), \end{aligned}$$

where $\Lambda_X = f^* \Lambda$ and $\omega_X = f^! \Lambda$ denote the constant and dualizing sheaves, respectively. These are the complexes of cochains and Borel–Moore chains

on X , respectively. We also write

$$\begin{aligned} H^*(X; \Lambda) &:= H^*(C^\bullet(X; \Lambda)) \simeq H^*(X; \Lambda_X), \\ H_*^{\text{BM}}(X; \Lambda) &:= H^{-*}(C_\bullet^{\text{BM}}(X; \Lambda)) \simeq H^{-*}(X; \omega_X). \end{aligned}$$

Theorem 11.1.5 yields the following consequences:

Proposition 11.2.1.

- (i) Proper push-forward: *Let $f : X \rightarrow Y$ be a proper morphism in AlgStk_k . Then there is a canonical morphism*

$$f_* : C_\bullet^{\text{BM}}(X; \Lambda) \rightarrow C_\bullet^{\text{BM}}(Y; \Lambda).$$

- (ii) Étale pull-back: *Let $f : X \rightarrow Y$ be an étale morphism in AlgStk_k . Then there is a canonical morphism*

$$f^! : C_\bullet^{\text{BM}}(Y; \Lambda) \rightarrow C_\bullet^{\text{BM}}(X; \Lambda).$$

- (iii) Localization triangle: *Let $X \in \text{Stk}_k$ and $i : Z \hookrightarrow X$ a closed immersion with complementary open immersion $j : U \hookrightarrow X$. Then there is a canonical exact triangle*

$$C_\bullet^{\text{BM}}(Z; \Lambda) \xrightarrow{i_*} C_\bullet^{\text{BM}}(X; \Lambda) \xrightarrow{j^!} C_\bullet^{\text{BM}}(U; \Lambda).$$

We also have the following consequence of Theorem 11.1.1:

Corollary 11.2.2. *On the ∞ -category AlgStk_k , the presheaves*

$$\begin{aligned} X &\mapsto C^\bullet(X; \Lambda), & f &\mapsto f^* \\ X &\mapsto C_\bullet^{\text{BM}}(X; \Lambda), & f &\mapsto f^! \end{aligned}$$

satisfy descent for the étale topology.

11.3. Intersection theory. We are finally in position to see how working with complexes of chains (as objects in the derived ∞ -category) rather than their homology groups leads to a streamlined approach to (virtual, stacky) intersection theory. Details of the following constructions can be found in [Kha1].

Recall the normal bundle from Definition 8.8.9. The following is a generalization of Verdier's deformation to the normal bundle [Ver]:

Definition/Theorem 11.3.1. *Let $f : X \rightarrow Y$ be a homotopically smooth morphism of derived Artin stacks. The normal deformation $D_{X/Y}$ is the derived mapping stack*

$$D_{X/Y} = \underline{\text{Maps}}_{Y \times \mathbf{A}^1}(Y \times \{0\}, X \times \mathbf{A}^1).$$

- (i) *If X and Y are n -Artin, then $D_{X/Y}$ is $(n+1)$ -Artin.*

(ii) *There is a commutative diagram of cartesian squares*

$$\begin{array}{ccccc}
 X & \xrightarrow{0} & X \times \mathbf{A}^1 & \longleftarrow & X \times \mathbf{G}_m \\
 \downarrow 0 & & \downarrow \widehat{f} & & \downarrow f \times \text{id} \\
 N_{X/Y} & \longrightarrow & D_{X/Y} & \longleftarrow & Y \times \mathbf{G}_m \\
 \downarrow & & \downarrow & & \downarrow \\
 \{0\} & \longrightarrow & \mathbf{A}^1 & \longleftarrow & \mathbf{G}_m.
 \end{array}$$

See [Kha1, §1.4] and [HKR].

Construction 11.3.2. Let $f : X \rightarrow Y$ be a homotopically smooth morphism of derived algebraic stacks locally of finite type over k . There is a canonical map

$$\text{sp}_{X/Y} : C_{\bullet}^{\text{BM}}(Y; \Lambda) \rightarrow C_{\bullet}^{\text{BM}}(N_{X/Y}; \Lambda) \quad (11.3.3)$$

defined as the composite

$$\begin{aligned}
 C_{\bullet}^{\text{BM}}(Y; \Lambda) &\xrightarrow{\text{incl}} C_{\bullet}^{\text{BM}}(Y; \Lambda) \oplus C_{\bullet}^{\text{BM}}(Y; \Lambda)(1)[1] \\
 &\simeq C_{\bullet}^{\text{BM}}(Y \times \mathbf{G}_m; \Lambda)[-1] \xrightarrow{\partial} C_{\bullet}^{\text{BM}}(N_{X/Y}; \Lambda)
 \end{aligned}$$

where the splitting comes from the unit section of \mathbf{G}_m and ∂ is the boundary map in the localization triangle

$$C_{\bullet}^{\text{BM}}(N_{X/Y}; \Lambda) \rightarrow C_{\bullet}^{\text{BM}}(D_{X/Y}; \Lambda) \rightarrow C_{\bullet}^{\text{BM}}(Y \times \mathbf{G}_m; \Lambda) \xrightarrow{\partial}$$

Notation 11.3.4. For an integer $d \in \mathbf{Z}$, we set $\langle d \rangle := (d)[2d]$, where (d) denotes the Tate twist.

Construction 11.3.5. Let $f : X \rightarrow Y$ be a morphism in AlgStk_k . Suppose $f : X \rightarrow Y$ is quasi-smooth, i.e., homotopically 1-smooth (Example 8.7.4), of relative virtual dimension d . Then $\mathbf{L}_{X/Y}$ is in Tor-amplitude $[-1, 1]$ and $N_{X/Y}$ is a “vector bundle stack”. We have the generalized homotopy invariance isomorphism

$$C_{\bullet}^{\text{BM}}(X; \Lambda) \simeq C_{\bullet}^{\text{BM}}(N_{X/Y}; \Lambda)\langle d \rangle.$$

since the projection $N_{X/Y} \rightarrow X$ is of relative dimension $-d$. The *quasi-smooth pull-back*, or *virtual pull-back*, is the canonical map

$$f^! : C_{\bullet}^{\text{BM}}(Y; \Lambda) \xrightarrow{\text{sp}_{X/Y}} C_{\bullet}^{\text{BM}}(N_{X/Y}; \Lambda) \simeq C_{\bullet}^{\text{BM}}(X; \Lambda)\langle -d \rangle. \quad (11.3.6)$$

Remark 11.3.7. Note that, even if X and Y are schemes, the above construction passes through the algebraic stacks $N_{X/Y}$ and $D_{X/Y}$ (which are not schemes unless $f : X \rightarrow Y$ is a closed immersion). Similarly, if X and Y are 1-Artin, we need to make use of the extension of $D(-)$ and the six operations to higher Artin stacks.

Definition 11.3.8. Let X be a quasi-smooth derived algebraic stack of relative virtual dimension d over $\text{Spec}(k)$. The projection $a_X : X \rightarrow \text{Spec}(k)$ gives rise to the pull-back

$$a_X^! : C_{\bullet}^{\text{BM}}(\text{Spec}(k)) \rightarrow C_{\bullet}^{\text{BM}}(X)\langle -d \rangle$$

and hence to the canonical element

$$[X] \in C_{\bullet}^{\text{BM}}(X)\langle -d \rangle \rightsquigarrow [X] \in H_{2d}^{\text{BM}}(X)\langle -d \rangle$$

called the *virtual fundamental class* of X .

Remark 11.3.9. The element $[X] \in C_{\bullet}^{\text{BM}}(X)\langle -d \rangle$ corresponds to a canonical morphism

$$\Lambda_X \langle d \rangle \rightarrow a_X^!(\Lambda)$$

in $D(X; \Lambda)$. This gives rise to a natural transformation

$$a_X^*(-)\langle d \rangle \rightarrow a_X^*(-) \otimes a_X^!(\Lambda) \xrightarrow{\text{can}} a_X^!(-) \quad (11.3.10)$$

or by adjunction a trace map $a_{X,!}a_X^*\langle d \rangle \rightarrow \text{id}$. In the relative case, where $f : X \rightarrow Y$ is a quasi-smooth morphism of relative virtual dimension d , we similarly get a natural transformation

$$\text{tr}_f : f_!f^*\langle d \rangle \rightarrow \text{id}. \quad (11.3.11)$$

Theorem 11.3.12 (Poincaré duality).

- (i) If $f : X \rightarrow Y$ is smooth, then the natural transformation $f^*(-)\langle d \rangle \rightarrow f^!(-)$ is invertible. Equivalently, tr_f is the counit of an adjunction $(f_!, f^*\langle d \rangle)$.
- (ii) For any smooth algebraic stack X in AlgStk_k , cap product with $[X]$ determines a canonical isomorphism

$$(-) \cap [X] : C_{\bullet}(X) \rightarrow C_{\bullet}^{\text{BM}}(X)\langle -d \rangle.$$

Proof. If $f : X \rightarrow Y$ is smooth, then the diagonal $\Delta : X \rightarrow X \times_Y X$ is still quasi-smooth. Thus we have a natural transformation tr_{Δ} , which gives rise to a unit for the adjunction $(f_!, f^*\langle d \rangle)$. The second statement follows from the first. See [Kha3]. \square

Example 11.3.13. Let \mathcal{M}_S denote the moduli stack $\mathcal{M}_{\text{Coh}(S)}$ (or $\mathcal{M}_{\text{Vect}(S)}$, $\mathcal{M}_{\text{Bun}_G(S)}$) for S an algebraic surface. Then since \mathcal{M}_S is quasi-smooth (Theorem 10.3.3), we have constructed a (virtual) fundamental class $[\mathcal{M}_S] \in H_*^{\text{BM}}(\mathcal{M}_S)$. We remark that the traditional method [BF] does not apply here since \mathcal{M}_S is far from being Deligne–Mumford.

In this framework it is easy to prove the following formula for intersection products. If X is a smooth k -scheme, the cap product in cohomology gives rise by Poincaré duality to an intersection product

$$C_{\bullet}^{\text{BM}}(X)\langle -p \rangle \otimes C_{\bullet}^{\text{BM}}(X)\langle -q \rangle \rightarrow C_{\bullet}^{\text{BM}}(X)\langle -p - q + d \rangle.$$

If Y is quasi-smooth of virtual dimension d and proper over X , the virtual fundamental class gives rise to a class in $C_{\bullet}^{\text{BM}}(X)\langle -d \rangle$ by proper push-forward.

Theorem 11.3.14 (Non-transverse Bézout formula). *Let Y and Z be smooth or lci closed subvarieties of X , of dimension p and q respectively. Then there is a canonical homotopy*

$$[Y] \cdot [Z] \simeq [Y \times_X^{\mathbf{R}} Z]$$

in $C_{\bullet}^{\text{BM}}(X)\langle -p - q + d \rangle$.

Note that while the left-hand side consists of usual cycle classes, the right-hand side is genuinely virtual unless the intersection is transverse (that is to say, unless the derived intersection $Y \times_X^{\mathbf{R}} Z$ reduces to the classical scheme-theoretic intersection).

11.4. Quotient stacks.

Definition 11.4.1. Let G be a linear algebraic group over the base field k . Let $X \in \text{AlgStk}_k$ be an algebraic stack with G -action. The complex of equivariant Borel–Moore chains is defined by

$$C_{\bullet}^{\text{BM},G}(X) := R\Gamma([X/G], f^!(\Lambda_{BG})) \simeq C_{\bullet}^{\text{BM}}([X/G]; \Lambda)\langle g \rangle$$

where $f : [X/G] \rightarrow BG$ is the projection of the quotient stack to the classifying stack, $g = \dim(G)$, and the isomorphism is Poincaré duality for BG .

The following two statements, proven in [KRa], show that this construction can be described by (algebraic approximations to) the Borel construction.

Choose a filtered system $(V_{\alpha})_{\alpha}$ of G -representations where the transition maps $V_{\alpha} \hookrightarrow V_{\beta}$ are monomorphisms. Let $W_{\alpha} \subseteq V_{\alpha}$ be G -invariant closed subschemes such that:

- (a) G acts freely on $U_{\alpha} := V_{\alpha} \setminus W_{\alpha}$,
- (b) $U_{\alpha} \subseteq U_{\alpha+1}$ for all α ,
- (c) We have $\text{codim}_{V_{\alpha}}(W_{\alpha}) \rightarrow \infty$ as $n \rightarrow \infty$.

Let U_{∞} denote the ind-algebraic space $\{U_{\alpha}\}_{\alpha}$. For example, for $G = \mathbf{G}_m$ the obvious choices give $[U_{\alpha}/G] = \mathbf{P}_k^{\infty}$.

Theorem 11.4.2. *There is a canonical isomorphism*

$$C_{\bullet}^{\text{BM},G}(X) \simeq C_{\bullet}^{\text{BM}}(X \times^G U_{\infty})\langle -\dim(U_{\infty}/G) \rangle := \varprojlim_{\alpha} C_{\bullet}^{\text{BM}}(X \times^G U_{\alpha})\langle -d_{\alpha} \rangle$$

where $X \times^G U_{\alpha} := [(X \times U_{\alpha})/G]$ is the quotient by the (free) diagonal action and $d_{\alpha} = \dim(U_{\alpha}/G)$.

Theorem 11.4.3. *There is a cartesian square of ∞ -categories*

$$\begin{array}{ccc} \mathrm{D}([X/G]) & \hookrightarrow & \mathrm{D}(X \times^G U_{\infty}) \\ \downarrow & & \downarrow \\ \mathrm{D}(X) & \hookrightarrow & \mathrm{D}(X \times U_{\infty}) \end{array}$$

where every arrow is \star -pullback, and the horizontal arrows are fully faithful.

Informally speaking, this means that a sheaf on $[X/G]$ amounts to the data of a sheaf \mathcal{F} on X , a sheaf \mathcal{G} on $X \times^G U_{\infty}$, and an isomorphism $\mathcal{F}|_{X \times U_{\infty}} \simeq \mathcal{G}|_{X \times U_{\infty}}$.

11.5. Concentration and localization. Let $X \in \text{AlgStk}_k$ and let $i : Z \hookrightarrow X$ be a closed immersion. Let Σ be a set of line bundles on X .

Question 11.5.1 (Concentration). When is the induced map

$$i_* : C_{\bullet}^{\text{BM}}(Z; \Lambda)[c_1(\Sigma)^{-1}] \rightarrow C_{\bullet}^{\text{BM}}(X; \Lambda)[c_1(\Sigma)^{-1}] \quad (11.5.2)$$

invertible?

The following was proven in [AKLPR]:

Theorem 11.5.3. *Assume that X has affine stabilizers. Suppose that for every point $x \in X \setminus Z$ there exists a line bundle $L \in \Sigma$ whose restriction along $B \text{Aut}(x) \hookrightarrow X$ is trivial. Then concentration holds, i.e., (11.5.2) is invertible.*

Corollary 11.5.4. *Let T be a split algebraic torus acting on a Deligne–Mumford stack $X \in \text{AlgStk}_k$. Let Z be the closed substack of fixed points¹¹. Then concentration holds with Σ the set of all nontrivial characters of BT (pulled back to $[X/T]$): in particular, we have a canonical isomorphism*

$$i_* : C_{\bullet}^{\text{BM},T}(Z; \Lambda)[c_1(\Sigma^{-1})] \rightarrow C_{\bullet}^{\text{BM},T}(X; \Lambda)[c_1(\Sigma)^{-1}].$$

The localization triangle (Proposition 11.2.1) gives a very useful way to prove results of this form, since it reduces the problem to Σ -acyclicity of Borel–Moore chains on the complement $X \setminus Z$.

From this one can derive:

Corollary 11.5.5 (Virtual localization). *Let T be a split algebraic torus acting on a Deligne–Mumford stack $X \in \text{AlgStk}_k$. Assume X is quasi-smooth and let Z be the fixed locus as in Corollary 11.5.4. Then we have a canonical homotopy*

$$[X] \simeq i_*([Z] \cap e(N_{Z/X})^{-1})$$

in $C_{\bullet}^{\text{BM},T}(X)[c_1(\Sigma^{-1})]$.

When X is smooth this is the Atiyah–Bott localization formula. In the quasi-smooth case it is the virtual localization formula of Graber–Pandharipande [GP]. Unlike *op. cit.* we do not need to assume X admits a global embedding into an ambient smooth stack, or that the cotangent complex $\mathbf{L}_{Z/X}$ admits a global resolution by vector bundles. Again, these improvements are possible because we work at the level of Borel–Moore chains as objects of the derived ∞ -category $\mathbf{D}(\Lambda)$.

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¹¹Since X is a stack, the appropriate definition of Z here is subtle. Briefly, Z is the homotopy fixed point stack with respect to an appropriate reparametrization of the torus action. See [AKLPR, Cor. 3.7].

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