WEAVES

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Abstract. We introduce the theory of weaves, an axiomatization of the six operations formalism on various derived categories of sheaves. *Very preliminary version.*

1. Introduction

In this paper we introduce a notion called *weaves*, which is our axiomatization of a sheaf theory equipped with Grothendieck's formalism of six operations.

Definition 1.1. A *preweave* $\mathbf{D}$ on the ∞-category of derived schemes $\mathcal{S}$ is a lax symmetric monoidal functor

$$\mathbf{D}^*_! : \mathrm{Corr}(\mathcal{S}) \to \mathrm{Cat}_\infty$$

valued in the ∞-category of ∞-categories, which factors through the subcategory containing only left adjoint functors.

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Here $\mathrm{Corr}(\mathcal{S})$ is the ∞-category of *correspondences*, whose objects are derived schemes and whose morphisms $X \to Y$ are diagrams

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \\ Y & & \end{array}$$

where g is locally of finite type. The functor $\mathbf{D}^*_!$ therefore encodes both the (contravariant) operation f^* and the (covariant) operation $g_!$. Functoriality with respect to correspondences (which are composed by forming fibre products) encodes the base change formula between $g_!$ and f^*. The lax symmetric monoidal structure encodes the tensor product $\otimes$ and the projection formula for $g_!$. Moreover, it encodes the right adjoint operations f_*, $g^!$, and $\underline{\mathrm{Hom}}$ (internal Hom).

Definition 1.2. A *weave* is a preweave which admits $*$-direct image for proper morphisms and $\sharp$-direct image for smooth morphisms.

See Definition 2.52. Informally speaking, $\mathbf{D}$ admits $*$-direct image for proper morphisms f if f^* admits a right adjoint f_* satisfying the base change and projection formulas, and moreover commuting with the !-operation. Dually, it admits $\sharp$-direct image for smooth morphisms f if f^* admits a left adjoint $f_\sharp$ satisfying the base change and projection formulas, and moreover commuting with the !-operation. Due to an asymmetry (unlike for proper morphisms, the diagonal of a smooth morphism is typically no longer smooth) the latter should be interpreted as also requiring that for every section s of a smooth morphism f, the operation $f_\sharp s_!$ is invertible. See Definitions 2.18 and 2.30 for the precise definitions.

Remark 1.3. We do not make it explicit here, but $\mathcal{S}$ can be replaced by any reasonable category, including say the category of (locally compact Hausdorff) topological spaces or the ∞-category of derived complex-analytic spaces.

The idea to encode six functor formalisms via ∞-categories of correspondences seems to first appear in [Ga] where it was attributed to J. Lurie. In [GR], Gaitsgory and Rozenblyum showed that the $(\infty, 2)$-category of correspondences admits a universal property that allows one to easily define functors out of it. For that reason they considered an $(\infty, 2)$-categorical version of Definition 1.1; lax symmetrical monoidal structures were later incorporated into the definition by Richarz and Scholbach in order to encode projection formulas coherently (see [RS, App. A]).

The work of Liu and Zheng [LZ1, LZ2] appeared shortly after [Ga] and provides an alternative approach to the construction of functors out of ∞-categories of correspondences. Their approach has the advantage of not relying on the theory of $(\infty, 2)$-categories (which at the time of writing is not yet fully established, see Warning 2.56), though at the cost of being technically much more involved and not yielding any *uniqueness* of the resulting functor. Recently, L. Mann [Man, App. A.5] combined aspects of all these ideas, looking at functors out of correspondences to encode base change formulas, and incorporating lax symmetric monoidality to encode projection formulas, but disregarding the $(\infty, 2)$-categorical structures as in Liu–Zheng. His notion of "6-functor formalism" defined in [Man, Def. A.5.7] is what we have called *preweaves* here.

For us, a preweave does not yet incorporate all the features a six functor formalism should have. Our notion of *weave* (Definition 1.2) is designed to encode the further properties we believe are missing, such as smooth and proper base change formulas and Poincaré duality.

We lay out the basic aspects of the theory in Sect. 2. Once the definitions are made, the arguments are very standard for those accustomed to working with abstract six functor formalisms.

In Sect. 3 we axiomatize sheaf theories that are modelled on the Betti sheaf theory in some sense (in contrast with quasi/ind-coherent sheaves and related theories). We call these *topological* weaves: they include Betti sheaves, étale sheaves (with torsion or ℓ-adic coefficients), and various categories of motivic sheaves. In fact, we show that topological weaves are equivalent to Voevodsky's axiomatization of "motivic" six functor formalisms (Corollary 3.33). In particular, topological weaves are equivalent to "motivic ∞-categories" [Kha1, CD, Ayo] or $(*, \sharp, \otimes)$-formalisms satisfying Voevodsky's conditions [Kha4]. Again, the arguments here are very well-known to experts, who will hardly find anything original here.

In Sect. 4 we sketch a proof of a result announced a year ago, which explains how to extend weaves from schemes or algebraic spaces to Artin stacks, even without étale descent. For example, we can form the lisse-extension of the motivic stable homotopy category to Artin stacks. In particular, all the results of [Kha3] on virtual fundamental classes can be extended to arbitrary generalized cohomology theories. (This is different from the "genuine" extensions considered in [KhRa1].)

Conventions and notation. We fix an implicit base scheme B; the term "stack" will mean "derived stack over B" throughout the paper, and similarly for "scheme" and "algebraic space".

We say that an algebraic space is *decent* if it is Zariski-locally quasi-separated in the sense of [SP, Tag 02X5]. This is equivalent to being Nisnevich-locally a scheme by [Knu, Chap. II, Thm. 6.4]. (It is stronger than the notion of decent algebraic space introduced in [SP, Tag 03I8], see [SP, Tag 03JX].)

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f^* and $g_!$ are "woven" together via commutative diagrams

$$\begin{array}{ccccccc}
\mathbf{D}(X_{0,n}) & \xrightarrow{f^*_{0,n-1}} & \mathbf{D}(X_{0,n-1}) & \xrightarrow{f^*_{0,n-2}} & \cdots & \xrightarrow{f^*_{0,0}} & \mathbf{D}(X_{0,0}) \\
\downarrow{\scriptstyle g_{0,n,!}} & & \downarrow{\scriptstyle g_{0,n-1,!}} & & \downarrow{\scriptstyle g_{0,1,!}} & & \downarrow{\scriptstyle g_{0,0,!}} \\
\mathbf{D}(X_{1,n}) & \xrightarrow{f^*_{1,n-1}} & \mathbf{D}(X_{1,n-1}) & \xrightarrow{f^*_{1,n-2}} & \cdots & \xrightarrow{f^*_{1,0}} & \mathbf{D}(X_{1,0}) \\
\downarrow{\scriptstyle g_{1,n,!}} & & \downarrow{\scriptstyle g_{1,n-1,!}} & & \downarrow & & \downarrow{\scriptstyle g_{1,0,!}} \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
\downarrow{\scriptstyle g_{n-1,n,!}} & & \downarrow{\scriptstyle g_{n-1,n-1,!}} & & \downarrow & & \downarrow{\scriptstyle g_{n-1,0,!}} \\
\mathbf{D}(X_{n,n}) & \xrightarrow{f^*_{n,n-1}} & \mathbf{D}(X_{n,n-1}) & \xrightarrow{f^*_{n,n-2}} & \cdots & \xrightarrow{f^*_{n,0}} & \mathbf{D}(X_{n,0})
\end{array}$$

I would like to thank Marco Volpe for discussions about the six operations in topology, and Bogdan Zavyalov for some questions about the proof of Poincaré duality in [Kha] (see also his related paper [BZ][1] about axiomatizing Poincaré duality).

2. Weaves

2.1. **Preweaves.**

Notation 2.1. We let $\mathcal{S}$ be a full subcategory of the ∞-category of stacks which is closed under finite coproducts and finite limits. Let $\mathcal{S}' \subseteq \mathcal{S}$ be a subcategory such that:

(i) Every isomorphism in $\mathcal{S}$ lies in $\mathcal{S}'$ (so that in particular, $\mathcal{S}'$ contains all objects of $\mathcal{S}$).

(ii) For every morphism $f : X \to Y$ in $\mathcal{S}'$ and any morphism $q : Y' \to Y$ in $\mathcal{S}$, the fibred product $X \times_Y Y'$ exists in $\mathcal{S}$ and the morphism $X \times_Y Y' \to Y'$ belongs to $\mathcal{S}'$.

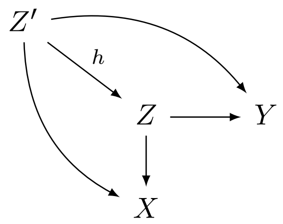
When not otherwise specified, $\mathcal{S}'$ will default to the subcategory of locally of finite type morphisms.

Definition 2.2. Given X and Y in $\mathcal{S}$, a *correspondence* from X to Y is a diagram

$$\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
\downarrow{\scriptstyle g} & & \\
Y & &
\end{array}$$

[1]The papers are not entirely independent, since they began as a joint project before B.Z. chose to write [BZ] on his own.

where g belongs to $\mathcal{S}'$. An (iso)morphism of correspondences $(X \leftarrow Z' \rightarrow Y) \rightarrow (X \leftarrow Z \rightarrow Y)$ is a commutative diagram in $\mathcal{S}$



where h is an (iso)morphism. For X and Y fixed, correspondences from X to Y (and isomorphisms between them) form an ∞-groupoid $\mathrm{Corr}(X,Y)$. Moreover, there exists an ∞-category $\mathrm{Corr}(\mathcal{S})$ whose objects are those of $\mathcal{S}$ and whose mapping anima are given by

$$\mathrm{Maps}_{\mathrm{Corr}(\mathcal{S})}(X,Y) = \mathrm{Corr}(X,Y)^{\simeq},$$

with composition law defined by forming fibred products. Moreover, $\mathrm{Corr}(\mathcal{S})$ admits a canonical symmetric monoidal structure. See e.g. [GR, Chap. 7].

Definition 2.3. Given a symmetric monoidal ∞-category $\mathcal{V}$, a *left preweave with values in* $\mathcal{V}$ is a lax symmetric monoidal functor

$$\mathrm{Corr}(\mathcal{S}) \rightarrow \mathcal{V}.$$

A *left preweave* is a left preweave with values in Cat_∞. A *right preweave* is a left preweave with values in $\mathrm{Cat}_\infty^{\mathrm{op}}$. Given a left (resp. right) preweave $\mathbf{D}$, we will denote the corresponding functor by

$$\mathbf{D}^*_! : \mathrm{Corr}(\mathcal{S}) \rightarrow \mathrm{Cat}_\infty, \quad \text{resp. } \mathbf{D}^!_* : \mathrm{Corr}(\mathcal{S})^{\mathrm{op}} \rightarrow \mathrm{Cat}_\infty. \tag{2.4}$$

Definition 2.5. A *preweave* is a left preweave such that the functor $\mathrm{Corr}(\mathcal{S}) \rightarrow \mathrm{Cat}_\infty$ factors through the subcategory of Cat_∞ containing only left adjoint functors. Equivalently, it is a right preweave such that $\mathrm{Corr}(\mathcal{S})^{\mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ factors through the subcategory of Cat_∞ containing only right adjoint functors. Given a preweave $\mathbf{D}$, we will write

$$\mathbf{D}^*_! : \mathrm{Corr}(\mathcal{S}) \rightarrow \mathrm{Cat}_\infty \quad \text{and} \quad \mathbf{D}^!_* : \mathrm{Corr}(\mathcal{S})^{\mathrm{op}} \rightarrow \mathrm{Cat}_\infty \tag{2.6}$$

for the corresponding left and right preweaves, respectively.

Remark 2.7. More precisely, we will refer to preweaves on $\mathcal{S}$ or even $(\mathcal{S},\mathcal{S}')$ when there is possible ambiguity.

Remark 2.8. Implicit in Definition 2.3 is the choice of Grothendieck universe with respect to which the objects of Cat_∞ are small. When we wish to make this explicit, we will speak of *small* (left/right) weaves, *large* (left/right) weaves, and so on.

Definition 2.9. A left preweave is *(finitely) cocomplete*, *(finitely) complete*, *(finitely) bicomplete*, *presentable*, or *compactly generated* if every ∞-category $\mathbf{D}(X)$ has the respective property for every $X \in \mathcal{S}$. A weave has one of these properties if its underlying left preweave does.

Notation 2.10. Given a left preweave $\mathbf{D}$, the functor $\mathbf{D}^*_! : \mathrm{Corr}(\mathcal{S}) \to \mathrm{Cat}_\infty$ gives rise to two functors

$$\mathbf{D}^* : \mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty \quad \text{and} \quad \mathbf{D}_! : \mathcal{S}' \to \mathrm{Cat}_\infty. \tag{2.11}$$

In particular, for every morphism $f : X \to Y$ in $\mathcal{S}$ we have a functor $f^* : \mathbf{D}(Y) \to \mathbf{D}(X)$ and, if f belongs to $\mathcal{S}'$, a functor $f_! : \mathbf{D}(X) \to \mathbf{D}(Y)$. Dually, if $\mathbf{D}$ is a *right preweave* then the functor $\mathbf{D}^!_* : \mathrm{Corr}(\mathcal{S})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ gives rise to functors

$$\mathbf{D}_* : \mathcal{S} \to \mathrm{Cat}_\infty \quad \text{and} \quad \mathbf{D}^! : \mathcal{S}'^{\mathrm{op}} \to \mathrm{Cat}_\infty \tag{2.12}$$

encoding operations f_* and (if f belongs to $\mathcal{S}'$) $f^!$.

Notation 2.13. Let $\mathbf{D}$ be a left preweave. For every cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y, \end{array} \tag{2.14}$$

the functor $\mathbf{D}^*_!$ encodes a canonical isomorphism

$$\mathrm{Ex}^*_! : q^* f_! \simeq g_! p^* \tag{2.15}$$

called the *base change formula*.

Notation 2.16. Let $\mathbf{D}$ be a left preweave. The lax symmetric monoidal structure on $\mathbf{D}^*_!$ encodes some further data. For example, there is a symmetric monoidal structure on $\mathbf{D}(X)$ for every $X \in \mathcal{S}$. Given a morphism $f : X \to Y$ in $\mathcal{S}$, the functor f^* is symmetric monoidal. Given a morphism $f : X \to Y$ in $\mathcal{S}'$, the functor $f_!$ is $\mathbf{D}(X)$-linear (where $\mathbf{D}(Y)$ is a module over $\mathbf{D}(X)$ via the symmetric monoidal functor f^*); that is, there is a canonical isomorphism

$$\mathrm{Pr}^*_! : f_!(-) \otimes (-) \simeq f_!(- \otimes f^*(-)) \tag{2.17}$$

called the *projection formula*.

2.2. **Proper axioms.**

Definition 2.18. Let $\mathbf{D}$ be a left preweave. We say that $\mathbf{D}$ *admits $*$-direct image* for a morphism $f : X \to Y$ in $\mathcal{S}$ if the following conditions hold:

(Pr1) For every base change g of f (along a morphism in $\mathcal{S}$), the functor g^* admits a right adjoint g_*.

(Pr2) For every base change g of f (along a morphism in $\mathcal{S}$), the functor g_* satisfies the projection formula. That is, the canonical morphism

$$\mathrm{Pr}^*_* : g_*(-) \otimes (-) \to g_*(- \otimes g^*(-))$$

is invertible.

(Pr3) The functor f_* commutes with $*$-inverse image. That is, for every cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y, \end{array}$$

the canonical morphism

$$\mathrm{Ex}^*_* : q^* f_* \xrightarrow{\text{unit}} g_* g^* q^* f_* \simeq g_* p^* f^* f_* \xrightarrow{\text{counit}} g_* p^* \tag{2.19}$$

is invertible.

(Pr4) The functor f_* commutes with !-direct image. That is, for every cartesian square as above, the exchange transformation

$$\mathrm{Ex}_{!,*} : q_! g_* \xrightarrow{\text{unit}} f_* f^* q_! g_* \simeq f_* p_! g^* g_* \xrightarrow{\text{counit}} f_* p_!. \tag{2.20}$$

is invertible.

Lemma 2.21. *Let* $\mathbf{D}$ *be a left preweave admitting $*$-direct image for proper morphisms. Let $f : X \to Y$ be an n-truncated*[2] *morphism in $\mathcal{S}$ with proper diagonal, where $n \geqslant -2$. Then there is a canonical morphism $\epsilon_f : f^* f_! \to \mathrm{id}$. Moreover, when f is proper, the right transpose $f_! \to f_*$ is invertible.*

Proof. If $n = -2$, then f is an isomorphism and the claim is obvious. If $n \geqslant -1$, assume the claim holds for $(n-1)$-truncated morphisms. Since the diagonal Δ_f is $(n-1)$-truncated and proper, we have the isomorphism $\Delta_! \simeq \Delta_*$ by assumption. The cartesian square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\mathrm{pr}_2} & X \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

gives rise to a natural transformation

$$\epsilon_f : f^* f_! \simeq \mathrm{pr}_{2,!} \mathrm{pr}_1^* \xrightarrow{\text{unit}} \mathrm{pr}_{2,!} \Delta_! \Delta^* \mathrm{pr}_1^* \simeq \mathrm{id}. \tag{2.22}$$

If f is proper, then the right transpose of ϵ_f is the natural transformation

$$f_! \simeq f_! \mathrm{pr}_{2,*} \Delta_* \xrightarrow{\mathrm{Ex}_{!,*}} f_* \mathrm{pr}_{1,!} \Delta_* \simeq f_* \mathrm{pr}_{1,!} \Delta_! \simeq f_* \tag{2.23}$$

where $\mathrm{Ex}_{!,*} : f_! \mathrm{pr}_{2,*} \to f_* \mathrm{pr}_{1,!}$ is invertible by (Pr4). □

Lemma 2.24. *Let* $\mathbf{D}$ *be a left preweave. Let $f : X \to Y$ be a proper morphism in $\mathcal{S}$ with diagonal $\Delta : X \to X \times_Y X$. Suppose that there exist morphisms*

$$c_f : \mathbf{1}_Y \to f_!(\mathbf{1}_X),$$
$$c_\Delta : \mathbf{1}_{X \times_Y X} \to \Delta_!(\mathbf{1}_X)$$

[2] For example, any n-representable morphism of stacks is n-truncated. Any morphism of n-Artin stacks is n-representable.

in $\mathbf{D}(Y)$ *and* $\mathbf{D}(X\times_Y X)$*, respectively, and a commutative square*

$$\begin{array}{ccccc} f^*(\mathbf{1}_Y) & = & = & = & \mathbf{1}_X \\ \downarrow{\scriptstyle c_f} & & & & \| \\ f^*f_!(\mathbf{1}_X) & = & \mathrm{pr}_{2,!}(\mathbf{1}_{X\times_Y X}) & \xrightarrow{c_\Delta} & \mathrm{pr}_{2,!}\Delta_!(\mathbf{1}_X). \end{array} \tag{2.25}$$

Then the natural transformations

$$\eta_f : \mathrm{id} \simeq (-)\otimes \mathbf{1}_Y \xrightarrow{(-)\otimes c_f} (-)\otimes f_!(\mathbf{1}_X) \simeq f_! f^*$$

and

$$\epsilon_f : f^* f_! \simeq \mathrm{pr}_{2,!}\mathrm{pr}_1^* \xrightarrow{\eta_\Delta} \mathrm{pr}_{2,!}\Delta_!\Delta^*\mathrm{pr}_1^* \simeq \mathrm{id},$$

where η_Δ *is defined like* η_f *using* c_Δ*, exhibit* $f_!$ *as a right adjoint of* f^**.*

Proof. We need to show that the composites

$$f^* \xrightarrow{f^* * \eta_f} f^* f_! f^* \xrightarrow{\epsilon_f * f^*} f^* \tag{2.26}$$

$$f_! \xrightarrow{\eta_f * f_!} f_! f^* f_! \xrightarrow{f_! * \epsilon_f} f_! \tag{2.27}$$

are both identity. Note that the evaluation of (2.26) on the unit object, call it θ, is the counterclockwise loop around $f^*(\mathbf{1}_Y)$ in (2.25). In particular, it is homotopic to the identity by assumption. By the projection formula for $f_!$, (2.26) is itself identified with $f^*(-)\otimes\theta$, hence also homotopic to the identity; similarly, (2.27) is identified with $f_!(-\otimes\theta)$. □

Corollary 2.28. *Let* $\mathbf{D}$ *be a left preweave. The following conditions are equivalent:*

(i) *Every* n*-truncated proper morphism admits* $*$*-direct image in* $\mathbf{D}$*.*

(ii) *There exists a collection of morphisms* $c_f : \mathbf{1}_Y \to f_!(\mathbf{1}_X)$ *associated with every proper morphism* $f : X \to Y$ *in* $\mathcal{S}$*, which is stable under base change and composition.*

Proof. It is easy to see that the condition is necessary. Conversely, suppose we have the collection $(c_f)_f$. Then for every f, the conditions of Lemma 2.24 are satisfied. Indeed, the commutative square (2.25) can be subdivided as follows:

$$\begin{array}{ccccc} f^*(\mathbf{1}_Y) & = & \mathbf{1}_X & = & \mathbf{1}_X \\ \downarrow{\scriptstyle c_f} & & \downarrow{\scriptstyle c_{\mathrm{pr}_2}} & & \| \\ f^*f_!(\mathbf{1}_X) & = & \mathrm{pr}_{2,!}(\mathbf{1}_{X\times_Y X}) & \xrightarrow{c_\Delta} & \mathrm{pr}_{2,!}\Delta_!(\mathbf{1}_X) \end{array}$$

where the left-hand square commutes by base change and the right-hand square commutes by functoriality. □

Remark 2.29. If $\mathcal{S}$ has morphisms that are not n-truncated for any n, it would be appropriate to require an additional axiom (Pr5) requiring invertibility of the endofunctor $g_* s_!$, for any section s of any base change g of f; compare (Sm5). Then, one could extend Corollary 2.28 to a necessary and sufficient condition for $*$-direct image for arbitrary proper morphisms, where in (ii) we have to incorporate some $\otimes$-invertible object σ just as in the dual statement Lemma 2.45.

2.3. **Smooth axioms.**

Definition 2.30. Let $\mathbf{D}$ be a left preweave. We say that $\mathbf{D}$ *admits* $\sharp$*-direct image* for a morphism $f : X \to Y$ in $\mathcal{S}$, or that f *admits a* $\sharp$*-direct image in* $\mathbf{D}$, if the following conditions hold:

(Sm1) For every base change g of f (along a morphism in $\mathcal{S}$), the functor g^* admits a left adjoint $g_\sharp$.

(Sm2) For every base change g of f (along a morphism in $\mathcal{S}$), the functor $g_\sharp$ satisfies the projection formula. That is, the canonical morphism

$$\mathrm{Pr}^*_\sharp : g_\sharp(- \otimes g^*(-)) \to g_\sharp(-) \otimes (-)$$

is invertible.

(Sm3) The functor $f_\sharp$ commutes with $*$-inverse image. That is, for every cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y, \end{array} \tag{2.31}$$

the canonical morphism

$$\mathrm{Ex}^*_\sharp : g_\sharp p^* \xrightarrow{\text{unit}} g_\sharp p^* f^* f_\sharp \simeq g_\sharp g^* q^* f_\sharp \xrightarrow{\text{counit}} q^* f_\sharp \tag{2.32}$$

is invertible.

(Sm4) The functor $f_\sharp$ commutes with !-direct image. That is, for every cartesian square as above, the exchange transformation

$$\mathrm{Ex}_{\sharp,!} : f_\sharp p_! \xrightarrow{\text{unit}} f_\sharp p_! g^* g_\sharp \simeq f_\sharp f^* q_! g_\sharp \xrightarrow{\text{counit}} q_! g_\sharp. \tag{2.33}$$

is invertible.

(Sm5) For every section s of every base change g of f (along a morphism in $\mathcal{S}$), the functor $g_\sharp s_!$ is an equivalence.

Remark 2.34. The asymmetry between Definitions 2.18 and 2.30 stems from the fact that the class of smooth morphisms is not stable under formation of diagonals. (For étale morphisms, this is not a problem; cf. Lemma 2.50.)

Notation 2.35. If $\mathbf{D}$ satisfies (Sm1) for a morphism $f : X \to Y$, we set

$$\Sigma_f := \mathrm{pr}_{2,\sharp}\Delta_! : \mathbf{D}(X) \to \mathbf{D}(X) \tag{2.36}$$

where $\Delta : X \to X \times_Y X$ is the diagonal and $\mathrm{pr}_2 : X \times_Y X \to X$ the second projection.

Lemma 2.37. *Let* $\mathbf{D}$ *be a left preweave. Let* $f : X \to Y$ *be a morphism in* $\mathcal{S}$ *satisfying (Sm1). Then we have:*

(i) *If* $\mathbf{D}$ *satisfies (Sm5), then* Σ_f *is an equivalence. In particular, the object* $\Sigma_f(\mathbf{1}_X)$ *is* $\otimes$*-invertible. If* $\mathbf{D}$ *is a preweave, then moreover* $\Omega_f := \Delta_f^! \mathrm{pr}_2^*$ *is inverse to* Σ_f *and* $\Omega_f(\mathbf{1}_X)$ *is a* $\otimes$*-inverse to* $\Sigma_f(\mathbf{1}_X)$*.*

(ii) *If* $\mathbf{D}$ *satisfies (Sm2), then* Σ_f *is* $\mathbf{D}(X)$*-linear. In particular, there is a canonical isomorphism of functors*

$$\Sigma_f(-) \otimes (-) \simeq \Sigma_f(- \otimes -). \tag{2.38}$$

If $\mathbf{D}$ *is a preweave, then there is moreover a canonical isomorphism*

$$\Sigma_f(\underline{\mathrm{Hom}}(-,-)) \simeq \underline{\mathrm{Hom}}(-, \Sigma_f(-)). \tag{2.39}$$

Proof. Follows from the definitions. For (2.39), observe that (2.38) implies the canonical isomorphism $\Omega_f(\underline{\mathrm{Hom}}(-,-)) \simeq \underline{\mathrm{Hom}}(-, \Omega_f(-))$ by adjunction, whence the canonical isomorphism

$$\Sigma_f(\underline{\mathrm{Hom}}(-, \Omega_f(-))) \simeq \Sigma_f \Omega_f(\underline{\mathrm{Hom}}(-,-)) \simeq \underline{\mathrm{Hom}}(-,-).$$

We conclude by inserting $\Sigma_f(-)$ in the second argument. □

Lemma 2.40. *Let* $\mathbf{D}$ *be a left preweave. Let* $f : X \to Y$ *be a morphism in* $\mathcal{S}$ *satisfying (Sm1) and (Sm4). Then there is a canonical isomorphism* $f_\sharp \simeq f_! \Sigma_f$*.*

Proof. The cartesian square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\mathrm{pr}_2} & X \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

gives rise by (Sm4) to an invertible natural transformation

$$\mathrm{Ex}_{\sharp,!} : f_\sharp \mathrm{pr}_{1,!} \to f_! \mathrm{pr}_{2,\sharp},$$

whence an invertible natural transformation

$$f_\sharp \simeq f_\sharp \mathrm{pr}_{1,!} \Delta_! \xrightarrow{\mathrm{Ex}_{\sharp,!}} f_! \mathrm{pr}_{2,\sharp} \Delta_!. \tag{2.41}$$

□

Corollary 2.42. *Let* $\mathbf{D}$ *be a left preweave. Let* $f : X \to Y$ *be a morphism in* $\mathcal{S}$ *satisfying (Sm1), (Sm2) and (Sm4). Then* f *satisfies (Sm5) if and only if* Σ_g *is invertible for every base change* g *of* f*.*

Proof. The condition is clearly necessary. For the other direction, let $g : X' \to Y'$ be a base change of f as in (2.31) and let s be a section. To show that $g_\sharp s_!$ is invertible, it will suffice to show that the object $g_\sharp s_!(\mathbf{1}_{Y'})$ is $\otimes$-invertible because $g_\sharp$ satisfies the projection formula (Sm2). By Lemma 2.40 we have

$$g_\sharp s_!(\mathbf{1}_{Y'}) \simeq g_!(s_!(\mathbf{1}_{Y'}) \otimes \Sigma_g(\mathbf{1}_{X'})) \simeq g_! s_! s^*(\Sigma_g(\mathbf{1}_{X'})) \simeq s^*(\Sigma_g(\mathbf{1}_{X'}))$$

by the projection formula for $s_!$. Since Σ_g is invertible, $\Sigma_g(\mathbf{1}_X)$ is $\otimes$-invertible. □

Corollary 2.43 (Poincaré duality)**.** *Let* $\mathbf{D}$ *be a left preweave. If* $f : X \to Y$ *is a morphism in* $\mathcal{S}$ *admitting* $\sharp$*-direct image in* $\mathbf{D}$*, then* $f_!$ *admits a right adjoint* $f^! := \Sigma_f f^*$*.*

Proof. Let Σ_f^{-1} be an inverse of the equivalence Σ_f. We have $f_\sharp \simeq f_! \Sigma_f$ by Lemma 2.40, hence $f_\sharp \Sigma_f^{-1} \simeq f_!$. Since $\Sigma_f f^*$ is a right adjoint to $f_\sharp \Sigma_f^{-1}$, the claim follows. □

Remark 2.44. By definition, we have the formula $\Sigma_f(\mathbf{1}_X) \simeq f^!(\mathbf{1}_Y)$ whenever $\mathbf{D}$ admits $\sharp$-direct image for f.

Lemma 2.45. *Let* $\mathbf{D}$ *be a left preweave. Let* $f : X \to Y$ *be a morphism in* $\mathcal{S}$ *with diagonal* $\Delta : X \to X \times_Y X$*. Suppose that there exists a* $\otimes$*-invertible object* $\sigma \in \mathbf{D}(X)$ *along with morphisms*

$$c_f : f_!(\sigma) \to \mathbf{1}_Y$$
$$c_\Delta : \Delta_!(\sigma^{\otimes -1}) \to \mathbf{1}_{X \times_Y X}$$

in $\mathbf{D}(Y)$ *and* $\mathbf{D}(X \times_Y X)$*, respectively, such that*

$$c_\Delta : \Delta_!(\mathbf{1}_X) \simeq \Delta_!(\sigma^{\otimes -1} \otimes \sigma) \simeq \Delta_!(\sigma^{\otimes -1} \otimes \Delta^* \mathrm{pr}_1^*(\sigma))$$
$$\simeq \Delta_!(\sigma^{\otimes -1}) \otimes \mathrm{pr}_1^*(\sigma) \xrightarrow{c_\Delta \otimes \mathrm{pr}_1^*(\sigma)} \mathrm{pr}_1^*(\sigma)$$

fits in a commutative square

$$\begin{array}{ccccc} \mathrm{pr}_{2,!}\Delta_!(\mathbf{1}_X) & \xrightarrow{c_\Delta} & \mathrm{pr}_{2,!}\mathrm{pr}_1^*(\sigma) & = & f^* f_!(\sigma) \\ \| & & & & \downarrow{\scriptstyle f^*(c_f)} \\ \mathbf{1}_X & & = & & f^*(\mathbf{1}_Y) \end{array} \tag{2.46}$$

where $\mathrm{pr}_i : X \times_Y X \to X$ *are the projections. Set* $\Xi := (-) \otimes \sigma$*,* $\Xi^{-1} := (-) \otimes \sigma^{\otimes -1}$*, and define a natural transformation*

$$\epsilon_f : f_! \Xi f^* \simeq f_!(\Xi(\mathbf{1}_X)) \otimes (-) \xrightarrow{c_f \otimes (-)} \mathbf{1}_Y \otimes (-) \simeq \mathrm{id},$$

using the linearity of $f_!$ *and* Ξ*, and similarly* $\epsilon_\Delta : \Delta_! \Xi^{-1} \Delta^*(-) \to \mathrm{id}$ *using* c_Δ*. Define also*

$$\eta_f : \mathrm{id} \simeq \mathrm{pr}_{2,!}\Delta_! \Xi^{-1} \Delta^* \mathrm{pr}_1^* \Xi \xrightarrow{\epsilon_\Delta} \mathrm{pr}_{2,!}\mathrm{pr}_1^* \Xi \simeq f^* f_! \Xi.$$

Then η_f *and* ϵ_f *exhibit* $f_! \Xi$ *as a left adjoint to* f^**.*

Proof. This is dual to Lemma 2.24. The claim is that both composites

$$f_! \Xi \xrightarrow{f_! \Xi(\eta_f)} f_! \Xi f^* f_! \Xi \xrightarrow{\epsilon_f * f_! \Xi} f_! \Xi \tag{2.47}$$

$$f^* \xrightarrow{\eta_f * f^*} f^* f_! \Xi f^* \xrightarrow{f^* * \epsilon_f} f^* \tag{2.48}$$

are homotopic to identity. Let θ denote the result of evaluating (2.48) on the unit object. Note that this is the clockwise loop around $\mathbf{1}_X$ in (2.46), hence is homotopic to the identity by assumption. By the projection formula for $f_!$, (2.47) is identified with $f_!(- \otimes \theta)$, hence also homotopic to the identity; similarly, (2.48) is identified with $f^*(-) \otimes \theta$. □

Corollary 2.49. *Let* $\mathbf{D}$ *be a left preweave. The following conditions are equivalent:*

(i) *Every smooth morphism admits* $\sharp$*-direct image in* $\mathbf{D}$.

(ii) (a) *There exists a collection of* $\otimes$*-invertible objects* $\sigma_f \in \mathbf{D}(X)$ *associated with every smoothable*[3] *quasi-smooth morphism* $f : X \to Y$ *in* $\mathcal{S}$*, which is stable under base change and composition. That is,* $p^*\sigma_f \simeq \sigma_g$ *for any cartesian square as in (2.31); if* f *and* g *are composable, we have* $\sigma_{f\circ g} \simeq g^*(\sigma_f) \otimes \sigma_g$.

(b) *There exists a collection of morphisms* $c_f : f_!(\sigma_f) \to \mathbf{1}_Y$ *associated with every smoothable quasi-smooth morphism* $f : X \to Y$ *in* $\mathcal{S}$*, which is stable under base change and functorial (up to incoherent homotopy) as in* [DJK, Def. 2.3.6].

Proof. The condition is clearly sufficient (easy to see). Conversely, any collection as in the statement satisfies the conditions of Lemma 2.45. Note that for every f with diagonal Δ, we have $\sigma_\Delta \simeq \sigma_f^{\otimes -1}$, since $\mathbf{1} \simeq \Delta^*(\sigma_{\mathrm{pr}_2}) \otimes \sigma_\Delta$ and $\sigma_{\mathrm{pr}_2} \simeq \mathrm{pr}_1^*\sigma_f$. Hence we have the left adjoint $f_\sharp := f_!(- \otimes \sigma_f)$ to f^*. It is easy to check that it satisfies the projection formula, commutes with $*$-inverse image, and with !-direct image. For the condition that $g_\sharp s_!$ is invertible for every section s of a base change g of f, note that we have

$$g_\sharp s_!(\mathbf{1}) \simeq g_!(s_!(\mathbf{1}) \otimes \sigma_g) \simeq g_! s_! s^*(\sigma_g) \simeq s^*(\sigma_g)$$

which is $\otimes$-invertible, so we conclude by the projection formula. $\square$

The following is an analogue of Lemma 2.21.

Lemma 2.50. *Let* $\mathbf{D}$ *be a left preweave admitting* $\sharp$*-direct image for étale morphisms. For every* n*-truncated étale morphism* $f : X \to Y$ *in* $\mathcal{S}$*, there is a canonical isomorphism* $f_\sharp \simeq f_!$ *(in particular,* $f^! \simeq f^*$*).*

Proof. If $n = -2$, then f is an isomorphism and the claim is obvious. Assume $n \geqslant -1$ and that the claim holds for $(n-1)$-truncated morphisms. Since the diagonal $\Delta : X \to X \times_Y X$ is $(n-1)$-truncated and étale, we have the isomorphism $\Delta_\sharp \simeq \Delta_!$ by assumption. The cartesian square

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\mathrm{pr}_2} & X \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

gives rise to an invertible natural transformation

$$\mathrm{Ex}_{\sharp,!} : f_\sharp \mathrm{pr}_{2,!} \to f_! \mathrm{pr}_{1,\sharp},$$

whence an invertible natural transformation

$$f_\sharp \simeq f_\sharp \mathrm{pr}_{2,!}\Delta_! \xrightarrow{\mathrm{Ex}_{\sharp,!}} f_!\mathrm{pr}_{1,\sharp}\Delta_! \simeq f_!\mathrm{pr}_{1,\sharp}\Delta_\sharp \simeq f_!. \tag{2.51}$$

[3] Recall that a morphism f is *smoothable* if it admits a global factorization through a closed immersion i followed by a smooth morphism p; given such a factorization, if f is quasi-smooth, then i is necessarily quasi-smooth.

□

2.4. **Weaves.**

Definition 2.52. A *left weave* is a left preweave which admits $*$-direct image for proper morphisms belonging to $\mathcal{S}'$ and $\sharp$-direct image for smooth morphisms belonging to $\mathcal{S}'$. A *right weave* is a left preweave with values in $\mathrm{Cat}_\infty^{\mathrm{op}}$ which admits $*$-direct image for proper morphisms and $\sharp$-direct image for smooth morphisms.[4] A *weave* is a preweave $\mathbf{D}$ whose underlying left preweave $\mathbf{D}^*_!$ is a left weave (or equivalently, its underlying right preweave $\mathbf{D}^!_*$ is a right weave).

2.5. **Construction of weaves.**

We recall a (slight variant of) [Kha4, Def. 2.2] and [Kha1, Chap. 2, Def. 3.5.2]:

Definition 2.53. A $(*, \sharp, \otimes)$-*formalism* on $\mathcal{S}$ is a presheaf of symmetric monoidal ∞-categories $\mathbf{D}^* : \mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ such that for every smooth morphism f, the functor $f^* = \mathbf{D}(f)$ admits a left adjoint $f_\sharp$ satisfying the projection formula and the base change formula. We say that $\mathbf{D}$ is *adjointable* if it factors through the subcategory of ∞-categories and left adjoint functors (i.e., the functor f^* admits a right adjoint f_* for every morphism f in $\mathcal{S}$). We say that $\mathbf{D}$ satisfies *Thom stability* if it is cocomplete and for every smooth separated morphism f with a section s, the endofunctor $f_\sharp s_*$ is invertible.

Theorem 2.54 (Gaitsgory–Rozenblyum)**.** *The functor*

$$\mathrm{Fun}(\mathrm{Corr}(\mathcal{S}), \mathrm{Cat}_\infty) \to \mathrm{Fun}(\mathcal{S}^{\mathrm{op}}, \mathrm{Cat}_\infty) \tag{2.55}$$

restricts to an equivalence from the ∞-category of left weaves (resp. of weaves) to the ∞-category of (resp. adjointable) $(*, \sharp, \otimes)$-formalisms satisfying the proper base change and projection formulas and Thom stability.

Proof. This follows from the results of [GR, Pt. III]; see [RS, Lem. A.7] (which corrects a mistake in the way the author attempted to encode the projection formula in [Kha1, Thm. 4.2.2]). □

Warning 2.56. The proof of Theorem 2.54 relies on the universal property of the $(\infty, 2)$-category of correspondences proven in [GR, Pt. III, Ch. 7, 3.2.2]. The proof in *op. cit.* is based on certain facts in $(\infty, 2)$-category theory, the proofs of which have not at the time of writing appeared yet in the literature. However, the machinery developed by Y. Liu and W. Zheng [LZ1, LZ2] (see also [Man, Prop. A.5.10]) produces a canonical section of the functor (2.55), which is enough for our purposes.

2.6. **Twists.**

Definition 2.57 (Thom twist)**.** Let $\mathbf{D}$ be a left preweave which admits $\sharp$-direct image for vector bundles. Given a vector bundle $\pi : E \to X$ in $\mathcal{S}$, let

[4] To make sense of this definition, observe that Definitions 2.18 and 2.30 make sense using the 2-categorical structure of $\mathrm{Cat}_\infty^{\mathrm{op}}$ (hence more generally for left preweaves with values in a symmetric monoidal $(\infty, 2)$-category).

$0 : X \to E$ denote the zero section and consider the invertible endofunctor $\Sigma^E : \mathbf{D}(X) \to \mathbf{D}(X)$ defined by

$$\Sigma^E := \pi_\sharp 0_!. \tag{2.58}$$

Remark 2.59. Under the canonical equivalence

$$\mathrm{End}_{\mathrm{Mod}_{\mathbf{D}(X)}(\mathrm{Cat}_\infty)}(\mathbf{D}(X))^{\simeq} \to \underline{\mathrm{Pic}}(\mathbf{D}(X))$$

between the ∞-groupoid of $\mathbf{D}(X)$-linear invertible endofunctors of $\mathbf{D}(X)$ and the Picard ∞-groupoid of $\otimes$-invertible objects in $\mathbf{D}(X)$, this is the same data as that of the $\otimes$-invertible object $\Sigma^E(\mathbf{1}_X)$

Proposition 2.60. *Given an exact sequence of vector bundles*

$$0 \to E' \xrightarrow{i} E \xrightarrow{p} E'' \to 0$$

over X, there is a canonical isomorphism

$$\Sigma^E \to \Sigma^{E''}\Sigma^{E'}. \tag{2.61}$$

Proof. The cartesian square

$$\begin{array}{ccc} E' & \xrightarrow{i} & E \\ \downarrow{\scriptstyle \pi_{E'}} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{0_{E''}} & E'' \end{array}$$

gives rise to the invertible exchange transformation $\mathrm{Ex}_{\sharp,!} : p_\sharp i_! \to 0_{E'',!}\pi_{E',!}$, whence the canonical isomorphism

$$\Sigma^E = \pi_{E,\sharp}0_{E,!} \simeq \pi_{E'',\sharp}p_\sharp i_! 0_{E',!} \xrightarrow{\mathrm{Ex}_{\sharp,!}} \pi_{E'',\sharp}0_{E'',!}\pi_{E',!}0_{E',!} = \Sigma^{E''}\Sigma^{E'}.$$

□

In fact, the isomorphisms of Proposition 2.60 are homotopy coherent. More precisely, suppose given an n-gapped object of the ∞-category $\mathrm{Vect}(X)$ of vector bundles in the sense of [Lur2, Lect. 16], i.e., a commutative diagram of cocartesian squares

$$\begin{array}{ccccccc} E_{0,0} & \xrightarrow{i_{0,0}} & E_{0,1} & \xrightarrow{i_{0,1}} & \cdots & \xrightarrow{i_{0,n-1}} & E_{0,3} \\ & & \downarrow{\scriptstyle p_{0,1}} & & \downarrow{\scriptstyle p_{0,n-1}} & & \downarrow{\scriptstyle p_{0,n}} \\ & & E_{1,1} & \xrightarrow{i_{1,1}} & \cdots & \xrightarrow{i_{1,n-1}} & E_{1,n} \\ & & & & \downarrow & & \downarrow{\scriptstyle p_{1,n}} \\ & & & & \cdots & \longrightarrow & \cdots \\ & & & & & & \downarrow{\scriptstyle p_{n-1,n}} \\ & & & & & & E_{n,n} \end{array}$$

where $E_{k,k}$ is the zero bundle for every $0 \leqslant k \leqslant n$. We get the commutative diagram

$$\begin{array}{ccccccc}
\mathbf{D}(E_{0,0}) & \xrightarrow{i_{0,0,!}} & \mathbf{D}(E_{0,1}) & \xrightarrow{i_{0,1,!}} & \cdots & \xrightarrow{i_{0,2,!}} & \mathbf{D}(E_{0,n}) \\
 & & \downarrow{\scriptstyle p_{0,1,\sharp}} & & \downarrow{\scriptstyle p_{0,n-1,\sharp}} & & \downarrow{\scriptstyle p_{0,n,\sharp}} \\
 & & \mathbf{D}(E_{1,1}) & \xrightarrow{i_{1,1,!}} & \cdots & \xrightarrow{i_{1,n-1,!}} & \mathbf{D}(E_{1,n}) \\
 & & & & \downarrow & & \downarrow{\scriptstyle p_{1,n,\sharp}} \\
 & & & & \cdots & \longrightarrow & \cdots \\
 & & & & & & \downarrow{\scriptstyle p_{n-1,n,\sharp}} \\
 & & & & & & \mathbf{D}(E_{n,n}),
\end{array}$$

by passing to left adjoints vertically from the diagram expressing homotopy coherence of the base change isomorphisms between p^* and $i_!$ (which is encoded by the functor $\mathbf{D}^*_!$ by definition).

In particular, by definition of Waldhausen's $S_\bullet$-construction this gives rise to a canonical map

$$\mathrm{K}(\mathrm{Vect}(X)) \to \underline{\mathrm{Pic}}(\mathbf{D}(X)) \tag{2.62}$$

sending the class of $E \in \mathrm{Vect}(X)$ to $\Sigma^E(\mathbf{1}_X)$, for every $X \in \mathcal{S}$. Similarly, the homotopy coherence of the base change isomorphisms between !-direct image (resp. $\sharp$-direct image) with $*$-inverse image give rise to homotopy coherence of the isomorphisms $f^*\Sigma^E \simeq \Sigma^{f^*E} f^*$ for any morphism $f : X' \to X$ in $\mathcal{S}$. In summary, we have:

Proposition 2.63. *There exists a canonical map of presheaves of $\mathcal{E}_\infty$-groups on $\mathcal{S}$*

$$\mathrm{K}(\mathrm{Vect}(-)) \to \underline{\mathrm{Pic}}(\mathbf{D}(-)) \tag{2.64}$$

which restricts to the functor $\mathrm{Vect}(-)^\simeq \to \underline{\mathrm{Pic}}(\mathbf{D}(-))$, $(E \to X) \mapsto \Sigma^E(\mathbf{1}_X)$.

Remark 2.65. Let $\mathrm{K}(-) = \mathrm{K}(\mathrm{Perf}(-))$ denote the presheaf sending a stack X to the ∞-category of algebraic K-theory of perfect complexes on X. Over affines, or more generally over stacks which admit the derived resolution property [Kha5, Def. 1.32], the map $\mathrm{K}(\mathrm{Vect}(-)) \to \mathrm{K}(-)$ restricts to an isomorphism. If $\mathbf{D}$ (and hence $\underline{\mathrm{Pic}}(\mathbf{D}(-))$) is right Kan extended from a full subcategory whose objects have the derived resolution property, it follows that (2.64) factors via a canonical map of presheaves of $\mathcal{E}_\infty$-groups on $\mathcal{S}$

$$\mathrm{K}(-) \to \underline{\mathrm{Pic}}(\mathbf{D}(-)). \tag{2.66}$$

For example, this is the case if $\mathbf{D}$ satisfies Nisnevich descent and $\mathcal{S}$ consists of qcqs algebraic spaces, tame Deligne–Mumford stacks, or nicely/linearly scalloped Artin stacks [KhRa1, Def. 2.9].

2.7. **Orientations.**

Definition 2.67. Let $\mathbf{D}$ be a left preweave which admits $\sharp$-direct image for vector bundles. An *orientation* of $\mathbf{D}$ is a commutative diagram of $\mathcal{E}_\infty$-groups

$$\begin{array}{ccc} \mathrm{K}(\mathrm{Vect}(-)) & \xrightarrow{(2.64)} & \underline{\mathrm{Pic}}(\mathbf{D}(-)) \\ & {\scriptstyle \mathrm{rk}}\searrow \quad \nearrow & \\ & \mathbf{Z} & \end{array}$$

where $\mathrm{rk} : \mathrm{K}(\mathrm{Vect}(X)) \to \mathbf{Z}$ is the rank map (valued in the discrete ∞-groupoid whose points are integers). Informally speaking, this amounts to the choice of Thom isomorphisms $\Sigma^E(\mathbf{1}) \simeq \Sigma^{\mathbf{A}^{\mathrm{rk}(E)}}(\mathbf{1})$ which are functorial and compatible with exact sequences up to coherent homotopy. We say that $\mathbf{D}$ is *oriented* if it admits an orientation.

Remark 2.68. An *incoherent orientation* of $\mathbf{D}$ is a commutative diagram of 1-groupoids

$$\begin{array}{ccc} \tau_{\leqslant 1}\mathrm{K}(\mathrm{Vect}(-)) & \longrightarrow & \tau_{\leqslant 1}\underline{\mathrm{Pic}}(\mathbf{D}(-)) \\ & {\scriptstyle \mathrm{rk}}\searrow \quad \nearrow & \\ & \mathbf{Z} & \end{array}$$

commute. This is compatible with the notion of orientation considered in [CD, Def. 2.4.38].

Remark 2.69. The above notion of orientation could also be called GL-*orientation*. We can similarly define SL-*orientations* by replacing $\mathrm{K}(\mathrm{Vect}(-))$ by the K-theory of principal SL-bundles, and similarly for SL^c or Sp (cf. [DFJK, §7]).

2.8. **Descent.**

Definition 2.70. We say that a left preweave $\mathbf{D}$ satisfies *additivity* if for every finite family $(X_i)_i$ in $\mathcal{S}$, the canonical morphism

$$\mathbf{D}(\sqcup_i X) \to \prod_i \mathbf{D}(X_i)$$

is invertible. In particular, $\mathbf{D}(\varnothing)$ is a terminal object of $\mathcal{V}$.

Definition 2.71. Let $(p_\alpha : U_\alpha \to X)_\alpha$ be a collection of morphisms in $\mathcal{S}$. We say that a presheaf $F : \mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is *separated* with respect to $(p_\alpha)_\alpha$ if the family of functors $p_\alpha^* = F(p_\alpha)$ is jointly conservative. If $\mathbf{D}$ is additive, we say that it satisfies *Čech descent* along $(p_\alpha)_\alpha$ if the following is a limit diagram in Cat_∞:

$$F(X) \to \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_{\alpha,\beta}) \substack{\to\\\to\\\to} \prod_{\alpha,\beta,\gamma} F(U_{\alpha,\beta,\gamma}) \substack{\to\\\to\\\to\\\to} \cdots.$$

Note that Čech descent implies separation. Note also that this definition makes sense as long as F is defined on a subcategory of $\mathcal{S}$ that contains the morphisms p_α.

Lemma 2.72. *Let* $\mathbf{D}$ *be a left preweave satisfying additivity. Let* $(p_\alpha : U_\alpha \to X)_\alpha$ *be a collection of morphisms such that* $\mathbf{D}$ *admits* $\sharp$*-direct image for every morphism obtained by base change and composition from morphisms in* $(p_\alpha)_\alpha$*. Given a subset of indices* $\alpha_1, \ldots, \alpha_n$*, we write*

$$p_{\alpha_1,\ldots,\alpha_n} : U_{\alpha_1,\ldots,\alpha_n} := U_\alpha \underset{U}{\times} \cdots \underset{U}{\times} U_{\alpha_n} \to X.$$

Consider the following conditions:

(i) *The presheaf* $\mathbf{D}^*$ *satisfies Čech descent along* $(U_\alpha \to X)_\alpha$*.*

(ii) *For every* $\mathcal{F} \in \mathbf{D}(X)$ *the following is a colimit diagram in* $\mathbf{D}(X)$*:*

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \bigoplus_{\alpha,\beta} p_{\alpha,\beta,\sharp} p^*_{\alpha,\beta}(\mathcal{F}) \to \bigoplus_{\alpha} p_{\alpha,\sharp} p^*_\alpha(\mathcal{F}) \rightrightarrows \mathcal{F}.$$

(iii) *The presheaf* $\mathbf{D}^*$ *is separated with respect to* $(U_\alpha \to X)_\alpha$*.*

If $\mathbf{D}$ *is a weave, then we also consider the following condition:*

(ii') For every $\mathcal{F} \in \mathbf{D}(X)$ *the following is a limit diagram in* $\mathbf{D}(X)$*:*

$$\mathcal{F} \to \prod_{\alpha} p_{\alpha,*} p^*_\alpha(\mathcal{F}) \rightrightarrows \prod_{\alpha,\beta} p_{\alpha,\beta,*} p^*_{\alpha,\beta}(\mathcal{F}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

Then (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii). If* $\mathbf{D}$ *is a weave, then all four listed conditions are equivalent.*

Proof. Note that (i) holds if and only if the canonical functor

$$F^* : \mathbf{D}(X) \to \operatorname{Tot}\Big(\prod_{\alpha_1,\ldots,\alpha_\bullet} \mathbf{D}(U_{\alpha_1,\ldots,\alpha_\bullet}) \Big)$$

is an equivalence. This admits a left adjoint $F_\sharp$ given informally by

$$\big((\mathcal{F}_{\alpha_1,\ldots,\alpha_n})_{\alpha_1,\ldots,\alpha_n} \big)_{[n]\in\mathbf{\Delta}} \mapsto \varinjlim_{[n]\in\mathbf{\Delta}} \bigoplus_{\alpha_1,\ldots,\alpha_n} p_{\alpha_1,\ldots,\alpha_n,\sharp}(\mathcal{F}_{\alpha_1,\ldots,\alpha_n}).$$

If $\mathbf{D}$ is a weave, then F^* also admits a right adjoint F_* given informally by

$$\big((\mathcal{F}_{\alpha_1,\ldots,\alpha_n})_{\alpha_1,\ldots,\alpha_n} \big)_{[n]\in\mathbf{\Delta}} \mapsto \varprojlim_{[n]\in\mathbf{\Delta}} \prod_{\alpha_1,\ldots,\alpha_n} p_{\alpha_1,\ldots,\alpha_n,*}(\mathcal{F}_{\alpha_1,\ldots,\alpha_n}).$$

Note also that (ii) holds if and only if the counit $F_\sharp F^* \to \mathrm{id}$ is invertible, and similarly (when $\mathbf{D}$ is a weave), (ii') holds if and only if the unit $\mathrm{id} \to F_* F^*$ is invertible. It is thus clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and (i) $\Rightarrow$ (ii') $\Rightarrow$ (iii) when $\mathbf{D}$ is a weave.

Suppose $\mathbf{D}$ is a weave. For p a morphism admitting $\sharp$-direct image in $\mathbf{D}$, the functor p^* is both a left and a right adjoint, hence in particular preserves limits and colimits. Moreover, its right adjoint p_* commutes with $*$-inverse image by (passing to right adjoints from) (Sm3). Hence (the dual of) [Lur, Cor. 4.7.5.3] shows that F_* is fully faithful, and that F^* is an equivalence if and only if the functors p^*_α are jointly conservative. In particular, (iii) $\Rightarrow$ (i). □

Definition 2.73. Let Q be a commutative square in $\mathcal{S}$. We say that a presheaf $F : \mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ satisfies *excision* with respect to Q if the commutative square $F(Q)$ is cartesian. Note also that this definition makes sense as long as F is defined on a subcategory of $\mathcal{S}$ that contains every morphism appearing in Q.

Lemma 2.74. *Let* $\mathbf{D}$ *be a left preweave. Let* Q *be a commutative square of the form*

$$\begin{array}{ccc} W & \xrightarrow{p'} & V \\ \downarrow{\scriptstyle q'} & \searrow^{r} & \downarrow{\scriptstyle q} \\ U & \xrightarrow{p} & X \end{array}$$

in $\mathcal{S}$ *and assume* $\mathbf{D}$ *admits* $\sharp$*-direct image for every morphism appearing in* Q*. Consider the following conditions:*

(i) *The presheaf* $\mathbf{D}^*$ *satisfies excision with respect to* Q*.*

(ii) *For every* $\mathcal{F} \in \mathbf{D}(X)$ *the following square is cocartesian:*

$$\begin{array}{ccc} r_\sharp r^*(\mathcal{F}) & \longrightarrow & q_\sharp q^*(\mathcal{F}) \\ \downarrow & & \downarrow \\ p_\sharp p^*(\mathcal{F}) & \longrightarrow & \mathcal{F}. \end{array}$$

(iii) *The presheaf* $\mathbf{D}^*$ *is separated with respect to* (p, q)*.*

If $\mathbf{D}$ *is a weave, then we also consider the following condition:*

(ii') *For every* $\mathcal{F} \in \mathbf{D}(X)$ *the following square is cartesian:*

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & p_* p^*(\mathcal{F}) \\ \downarrow & & \downarrow \\ q_* q^*(\mathcal{F}) & \longrightarrow & r_* r^*(\mathcal{F}). \end{array}$$

Then (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii). If* $\mathbf{D}$ *is a weave, then (i)* $\Rightarrow$ *(ii')* $\Rightarrow$ *(iii).*

Proof. Let $F^* : \mathbf{D}(X) \to \mathbf{D}(U) \times_{\mathbf{D}(W)} \mathbf{D}(V)$ denote the canonical functor. It admits a left adjoint $F_\sharp$ given by the formula

$$(\mathcal{F}_U, \mathcal{F}_V, \mathcal{F}_W) \mapsto p_\sharp(\mathcal{F}_U) \underset{r_\sharp(\mathcal{F}_W)}{\sqcup} q_\sharp(\mathcal{F}_V),$$

and, if $\mathbf{D}$ is a weave, a right adjoint F_* given by

$$(\mathcal{F}_U, \mathcal{F}_V, \mathcal{F}_W) \mapsto p_*(\mathcal{F}_U) \underset{r_*(\mathcal{F}_W)}{\times} q_*(\mathcal{F}_V).$$

Then (i) is the condition that the adjunction $(F^*, F_\sharp)$ is an equivalence (equivalently, if $\mathbf{D}$ is a weave, that $(F^*, F_\sharp)$ is an equivalence). The condition (ii) (resp. (ii')) is the assertion that the counit $F_\sharp F^* \to \mathrm{id}$ (resp. the unit $\mathrm{id} \to F_* F^*$) is invertible. $\square$

3. Topological weaves

3.1. **Localization.** Let $\mathbf{D}$ be a left preweave satisfying additivity and admitting $*$-direct image (resp. $\sharp$-direct image) for closed immersions (resp. open immersions). We assume that $\mathbf{D}$ is finitely bicomplete (Definition 2.9).

Lemma 3.1. *The ∞-category $\mathbf{D}(X)$ is pointed, i.e., admits an (essentially unique) zero object, for every $X \in \mathcal{S}$.*

Proof. Since $i : \varnothing \to X$ is a closed immersion, $i_! \simeq i_*$ is a right adjoint. Since it is an open immersion, $i_! \simeq i_\sharp$ is also a left adjoint. In particular, it sends the zero object $0 \in \mathbf{D}(\varnothing)$ to a zero object of $\mathbf{D}(X)$. □

Remark 3.2. Let $i : Z \to X$ be a closed immersion in $\mathcal{S}$ such that the complementary open immersion $j : X \smallsetminus Z \to X$ also belongs to $\mathcal{S}$. By additivity, the base change isomorphism for the cartesian square

$$\begin{array}{ccc} \varnothing & \longrightarrow & X \smallsetminus Z \\ \downarrow & & \downarrow j \\ Z & \xrightarrow{i} & X \end{array}$$

takes the form $j^* i_! \simeq 0$, where 0 is the constant functor on the zero object (Lemma 3.1); since $i_! \simeq i_*$ and $j^* \simeq j^*$ are right adjoints, they preserve zero objects. Thus we get a commutative square

$$\begin{array}{ccc} j_! j^* & \longrightarrow & \mathrm{id} \\ \downarrow & & \downarrow \\ 0 \simeq j_! j^* i_! i^* & \longrightarrow & i_! i^*. \end{array} \tag{3.3}$$

Definition 3.4 (Localization)**.** Let $\mathbf{D}$ be a left preweave taking values in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ which satisfies additivity and admits $*$-direct image (resp. $\sharp$-direct image) for closed immersions (resp. open immersions). We say that $\mathbf{D}$ satisfies the *localization property* if for every complementary closed-open pair $i : Z \to X$, $j : X \smallsetminus Z \to X$ in $\mathcal{S}$, the square (3.3) is cocartesian.

Remark 3.5. If $\mathbf{D}$ satisfies the localization property and $\mathbf{D}$ is stable (i.e., it takes values in $\mathrm{Cat}_\infty^{\mathrm{stab}}$), then the functors i^* and j^* are jointly conservative.

Lemma 3.6. *Let $\mathbf{D}$ be a left preweave satisfying the localization property. Then for every surjective closed immersion $i : Z \to X$ in $\mathcal{S}$, the unit $\mathrm{id} \to i_! i^*$ is invertible.*

Proof. Since i is surjective, the complementary open immersion is $j : \varnothing \to X$. By additivity, we have $j_! j^* \simeq 0$. By localization, the square (3.3) is cocartesian so the claim follows. □

Lemma 3.7. *Let $\mathbf{D}$ be a left preweave admitting $*$-direct image for a proper morphisms. Let $f : X \to Y$ be an n-truncated proper morphism in $\mathcal{S}$ with diagonal $\Delta : X \to X \times_Y X$. Consider the following conditions:*

(i) *The natural transformation* $\mathrm{counit} : f^* f_! \to \mathrm{id}$ *(Lemma 2.21) is invertible; that is,* $f_!$ *is fully faithful.*

(ii) *The natural transformation* $\mathrm{unit} : \mathrm{id} \to \Delta_! \Delta^*$ *(Lemma 2.21) is invertible; that is,* Δ^* *is fully faithful.*

Then we have (ii) $\implies$ *(i).*

Proof. The counit $f^* f_! \to \mathrm{id}$ factors as follows:

$$\epsilon_f : f^* f_! \simeq \mathrm{pr}_{2,!} \mathrm{pr}_1^* \xrightarrow{\eta_\Delta} \mathrm{pr}_{2,!} \Delta_! \Delta^* \mathrm{pr}_1^* \simeq \mathrm{id}.$$

See e.g. [Hoy, Lem. B.1]. □

Corollary 3.8. *Let* $\mathbf{D}$ *be a left preweave satisfying the localization property. Then for every closed immersion* $i : Z \to X$ *in* $\mathcal{S}$*, the natural transformation*

$$\mathrm{counit} : i^* i_! \to \mathrm{id}$$

is invertible.

Proof. Let $\Delta : Z \to Z \times_X Z$ denote the diagonal. By Lemma 3.6, the localization property implies that the unit $\mathrm{id} \to \Delta_! \Delta^*$ is invertible. Hence the counit $i^* i_! \to \mathrm{id}$ is invertible by Lemma 3.7. □

Corollary 3.9 (Derived invariance)**.** *Let* $\mathbf{D}$ *be a left preweave satisfying the localization property. Let* $X \in \mathcal{S}$ *and denote by* $i : X_{\mathrm{cl}} \to X$ *the classical truncation (resp. by* $X_{\mathrm{red}} \to X$ *the reduced classical truncation). If* X_{cl} *(resp.* X_{red}*) belongs to* $\mathcal{S}$*, then the functors* i^* *and* $i_!$ *are both equivalences.*

Proof. Combine Lemma 3.6 and Corollary 3.8. □

Proposition 3.10. *If* $\mathbf{D}$ *is stable and satisfies the localization property, then it satisfies étale excision.*

Proof. Suppose given a cartesian square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & \searrow^{r} & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

where j is an open immersion and p is an étale morphism inducing an isomorphism over $|X \smallsetminus U|$. It will suffice (see the proof of Lemma 2.74) to show that the canonical morphisms

$$\mathcal{F}_U \to j^*(j_\sharp(\mathcal{F}_U) \underset{r_\sharp(\mathcal{F}_W)}{\sqcup} p_\sharp(\mathcal{F}_V))$$

$$\mathcal{F}_V \to p^*(j_\sharp(\mathcal{F}_U) \underset{r_\sharp(\mathcal{F}_W)}{\sqcup} p_\sharp(\mathcal{F}_V))$$

are invertible for all $(\mathcal{F}_U, \mathcal{F}_V, \mathcal{F}_W) \in \mathbf{D}(U) \times_{\mathbf{D}(W)} \mathbf{D}(V)$, and that the canonical morphism

$$j_\sharp j^*(\mathcal{F}) \underset{r_\sharp r^*(\mathcal{F})}{\sqcup} p_\sharp p^*(\mathcal{F}) \to \mathcal{F}$$

is invertible for all $\mathcal{F} \in \mathbf{D}(X)$. The first two follow easily from the fact that i_* and $j_\sharp$ commute with $*$-inverse image. For the last, it is enough by the localization property to check after $*$-inverse image to U or $X \smallsetminus U$ (Remark 3.5), after which we again use the commutativity of i_* and $j_\sharp$ with $*$-inverse image. □

Corollary 3.11. *If* $\mathbf{D}$ *is stable and satisfies the localization property, then it satisfies Nisnevich descent on the full subcategory of* $\mathcal{S}$ *spanned by stacks that are qcqs* 1*-Artin.*

Proof. By [HK, Prop. 2.9], the condition of Nisnevich descent on qcqs 1-Artin stacks is equivalent to étale excision. Hence this follows from Proposition 3.10. □

Corollary 3.12. *Suppose* $\mathbf{D}$ *is stable and satisfies Zariski descent and the localization property. If* $\mathcal{S}$ *consists of* 1*-Artin stacks, then* $\mathbf{D}$ *satisfies Nisnevich descent on* $\mathcal{S}$*.*

Proof. Every 1-Artin stack admits a Zariski cover by one that is quasi-compact and quasi-separated, so the claim follows from Corollary 3.11. □

Proposition 3.13. *Suppose* $\mathbf{D}$ *satisfies localization and continuity and is constructibly generated (in the sense of* [DFJK, App. A]*). Then for any locally noetherian*[5] $S \in \mathcal{S}$*, the family of functors* $s^* : \mathbf{D}(S) \to \mathbf{D}(\mathrm{Spec}(\kappa))$ *is jointly conservative as* $s : \mathrm{Spec}(\kappa) \to S$ *varies.*

Proof. We employ the extension of the !-inverse image functor to essentially of finite presentation morphisms in [DFJK, App. B.3]. We first demonstrate conservativity of the family $(s^!)_s$, arguing as in [DFJK, Prop. B.20] but using noetherian induction instead of induction on the dimension. By [DFJK, Prop. B.19] we have $i_s^! \simeq h_s^! j_s^*$ where $h_s : \{s\} \hookrightarrow \mathrm{Spec}(\mathcal{O}_{S,s})$, so we deduce conservativity of $(j_s^*)_s$. We are thus reduced to the case where the classical truncation of S is a noetherian local scheme, hence in particular of some finite Krull dimension d. Then the claim follows by induction on d, applying Remark 3.5 to the inclusion of the closed point and its open complement. □

3.2. **Homotopy invariance.** In this subsection we assume that $\mathcal{S}$ contains the affine line $\mathbf{A}^1$ (over our implicit base scheme B).

Definition 3.14. Let $\mathbf{D}$ be a left preweave admitting $\sharp$-direct image for vector bundles. We say that $\mathbf{D}$ satisfies *homotopy invariance* if for every vector bundle $\pi : E \to X$ in $\mathcal{S}$, the counit $\pi_\sharp \pi^* \to \mathrm{id}$ is invertible.

3.3. **Topological weaves.** We assume that $\mathcal{S}$ satisfies the following: if $X \in \mathcal{S}$ and $X' \to X$ is a schematic morphism of finite type, then $X' \in \mathcal{S}$. We also assume $\mathcal{S}'$ contains all schematic morphisms of finite type.

[5] It is enough for the classical truncation to be locally noetherian. When it is both locally noetherian and of finite Krull dimension, Proposition 3.13 is well-known (see e.g. [CD, Prop. 4.3.17]).

Definition 3.15. Let $\mathbf{D}$ be a left preweave admitting $\sharp$-direct image for open immersions and vector bundles and $*$-direct image for closed immersions. We say that $\mathbf{D}$ is *topological* if satisfies localization, homotopy invariance, and Nisnevich descent.

Remark 3.16. If all stacks in $\mathcal{S}$ are 1-Artin, then the condition of Nisnevich descent in Definition 3.15 is equivalent to Zariski descent. If moreover all stacks in $\mathcal{S}$ are qcqs, then we may omit the condition entirely. See Corollary 3.11.

Proposition 3.17. *Every topological left preweave* $\mathbf{D}$ *is stable.*

Proof. Let $\pi : X \times \mathbf{A}^1 \to X$ be the projection for some $X \in \mathcal{S}$. Since $\mathbf{D}$ admits $\sharp$-direct image for π, the object $\mathcal{T} = \pi_\sharp 0_!(\mathbf{1}_X)$ is $\otimes$-invertible, where 0 is the zero section. By localization (and since $\pi_\sharp$ preserves colimits, as a left adjoint), there is a cofibre sequence

$$\pi^\circ_\sharp(\mathbf{1}) \to \pi_\sharp(\mathbf{1}) \to \pi_\sharp 0_!(\mathbf{1}) = \mathcal{T}$$

where $\pi^\circ : X \times (\mathbf{A}^1 \smallsetminus \{0\}) \to X$. By homotopy invariance, $\pi_\sharp(\mathbf{1}) \simeq \pi_\sharp\pi^*(\mathbf{1}) \simeq \mathbf{1}$. We deduce a canonical isomorphism $\mathcal{T} \simeq \Sigma(\pi^\circ_\sharp(\mathbf{1})) \simeq \pi^\circ_\sharp(\mathbf{1}) \otimes \Sigma(\mathbf{1})$, where $\Sigma \simeq (-) \otimes \Sigma(\mathbf{1})$ denotes suspension in the pointed ∞-category $\mathbf{D}(X)$ (since $\mathbf{D}$ takes values in $\mathrm{Cat}^{\mathrm{ex}}_\infty$, the tensor product is right-exact in each argument). Since $\mathcal{T}$ is $\otimes$-invertible, so is $\Sigma(\mathbf{1})$. In particular, $\Sigma : \mathbf{D}(X) \to \mathbf{D}(X)$ is an equivalence. □

Definition 3.18 (Tate twist)**.** Let $\mathbf{D}$ be a topological left preweave. For every $X \in \mathcal{S}$ and every integer $n \in \mathbf{Z}$, define the invertible $\mathbf{D}(X)$-linear endofunctor $\mathcal{F} \mapsto \mathcal{F}(n)$ on $\mathbf{D}(X)$ by

$$\mathcal{F}(n) = \mathcal{F} \otimes \pi^\circ_\sharp(\mathbf{1})^{\otimes n}[-n]$$

where $\pi^\circ : X \times (\mathbf{A}^1 \smallsetminus \{0\}) \to X$ is the projection and $[-1]$ denotes the inverse to the shift in the stable ∞-category $\mathbf{D}(X)$. Then we have

$$\Sigma^{X \times \mathbf{A}^n}(\mathcal{F}) \simeq \mathcal{F}(n)[2n] \tag{3.19}$$

by the proof of Proposition 3.17, where $\Sigma^{X \times \mathbf{A}^n} : \mathbf{D}(X) \to \mathbf{D}(X)$ is the Thom twist with respect to the trivial bundle of rank n (Definition 2.57).

Theorem 3.20 (Robalo)**.** *On the category of decent algebraic spaces, the weave* $\mathbf{SH}$*, sending* S *to the stable* ∞*-category* $\mathbf{SH}(S)$ *of motivic spectra over* S*, is the initial topological weave.*

See Subsect. 3.5 below for the proof.

3.4. **Twists.**

Theorem 3.21. *Let* $\mathbf{D}$ *be a topological left weave (resp. satisfying étale descent). Let* $f : X \to Y$ *be a smooth schematic morphism of* Nis*-Artin stacks (resp. of Artin stacks) in* $\mathcal{S}$ *and* $s : Y \to X$ *a section. Then there is a canonical isomorphism*

$$f_\sharp s_! \simeq \Sigma^{s^* T_f} \tag{3.22}$$

where $T_f \coloneqq \mathbf{V}_X(\mathcal{L}_f)$ is the relative tangent bundle.

The following is essentially a reformulation of a result of Morel–Voevodsky [MV].

Corollary 3.23 (Relative purity)**.** *Let $\mathbf{D}$ be a topological left weave (resp. satisfying étale descent). Then for every smooth schematic morphism $f : X \to Y$ of* Nis-*Artin stacks (resp. of Artin stacks) in $\mathcal{S}$, there is a canonical isomorphism $\Sigma_f \simeq \Sigma^{T_f}$. In particular, there are canonical isomorphisms*

$$f_\sharp \simeq f_! \Sigma^{T_f}, \qquad f^! \simeq \Sigma^{T_f} f^*. \tag{3.24}$$

Proof. Recall that $\Sigma_f \simeq \mathrm{pr}_{2,\sharp}\Delta_!$ by definition. Hence by Theorem 3.21, $\Sigma_f \simeq \Sigma^{\Delta^* T_{\mathrm{pr}_2}}$. Moreover, $\Delta^* T_{\mathrm{pr}_2} \simeq \Delta^* \mathrm{pr}_1^* T_f \simeq T_f$. □

Remark 3.25. In Theorem 3.21 and Corollary 3.23, the assumption that f is schematic is not necessary.

The following notation will be useful in the proof of Theorem 3.21.

Notation 3.26. Fix a stack $Y \in \mathcal{S}$. Given a pair $(f' : X' \to Y', s')$ where Y' is smooth over Y, $f' : X' \to Y'$ is a smooth morphism, and $s' : Y' \to X'$ is a section, we set

$$P(X', Y') \coloneqq P(f' : X' \to Y', s') \coloneqq f'_\sharp s'_!.$$

Given a pair $(f' : X' \to Y', s')$ and a morphism $v : Y'' \to Y'$ we get a diagram of cartesian squares

$$\begin{array}{ccccc} Y'' & \xrightarrow{s''} & X'' & \xrightarrow{f''} & Y'' \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ Y' & \xrightarrow{s'} & X' & \xrightarrow{f'} & Y' \end{array}$$

which we regard as a morphism of pairs $v : (f'', s'') \to (f', s')$. This gives rise to a canonical isomorphism

$$P(X'', Y'') \circ v^* = f''_\sharp s''_! v^* \xrightarrow{\mathrm{Ex}^*_!} f''_\sharp u^* s'_! \xrightarrow{\mathrm{Ex}^*_\sharp} v^* f'_\sharp s'_! = v^* \circ P(X', Y'). \tag{3.27}$$

Furthermore, the commutative square

$$\begin{array}{ccc} Y'' & \xrightarrow{a''} & Y \\ \downarrow & & \parallel \\ Y' & \xrightarrow{a'} & Y \end{array}$$

gives rise to a canonical natural transformation

$$a''_\sharp \circ P(X'', Y'') \circ v^* \simeq a''_\sharp v^* P(X', Y') \xrightarrow{\mathrm{Ex}^*_\sharp} a'_\sharp \circ P(X', Y'). \tag{3.28}$$

If v is smooth, we also get the isomorphism

$$v_\sharp P(X'', Y'') = v_\sharp f''_\sharp s''_! \simeq f'_\sharp u_\sharp s''_! \xrightarrow{\mathrm{Ex}_{\sharp,!}} f'_\sharp s'_! v_\sharp = P(X', Y') \circ v_\sharp. \tag{3.29}$$

Proof of Theorem 3.21. By [Ryd, Thm. 1.2], s factors canonically through a closed immersion $t : Y \hookrightarrow X'$ and an étale morphism $e : X' \to X$ (where X' is an algebraic space). Since s is a local immersion, e is a local isomorphism (in particular, it is schematic so $X' \in \mathcal{S}$ by our assumptions). Since e is étale we have a canonical isomorphism

$$f_\sharp s_! \simeq f_\sharp e_! t_! \simeq f_\sharp e_\sharp t_! \simeq g_\sharp t_!$$

where $g = f \circ e : X' \to X \to Y$, and $e^* T_f \simeq T_g$ so that $\Sigma^{s^* T_f} \simeq \Sigma^{t^* T_g}$. Replacing f and s by g and t, respectively, we may therefore assume that s is a closed immersion.

Note that $P(X, Y) = f_\sharp s_!$ and

$$P(N_s, Y) = \pi_\sharp 0_! =: \Sigma^{N_s} \simeq \Sigma^{s^* T_f}$$

where $\pi : N_s = \mathbf{V}_Y(\mathcal{L}_s[-1]) \to Y$ is the normal bundle and $0 : Y \to N_s$ is the zero section. Let D denote the deformation to the normal bundle associated with s [KhRy, Thm. 4.1.13]. This is equipped with a closed immersion $Y \times \mathbf{A}^1 \to D$ and a smooth affine morphism $D \to X \times \mathbf{A}^1$, whose composite is $s \times \mathrm{id}$. In particular, $D \in \mathcal{S}$ (by our assumptions) and $Y \times \mathbf{A}^1 \to D$ is a section of the smooth morphism $D \to X \times \mathbf{A}^1 \to Y \times \mathbf{A}^1$. The morphisms of pairs

$$(N_s, Y) \to (D, Y \times \mathbf{A}^1) \leftarrow (X, Y),$$

induced by the inclusions $i_\epsilon : Y \times \{\epsilon\} \to Y \times \mathbf{A}^1$, where $\epsilon \in \{0, 1\}$, give rise to canonical morphisms (3.28)

$$P(N_s, Y) \circ i_0^* \to p_\sharp \circ P(D, Y \times \mathbf{A}^1) \leftarrow P(X, Y) \circ i_1^* \tag{3.30}$$

where $p : X \times \mathbf{A}^1 \to X$ is the projection. It will suffice to show that both maps are invertible.

Let τ stand for the Nisnevich (resp. étale) topology. Since $\mathbf{D}$ satisfies τ-descent, the smooth base change formula implies that the claim is τ-local on Y. Since Y is τ-Artin, it admits a smooth atlas $V \twoheadrightarrow Y$ with τ-local sections. Replacing Y by V, we may assume that Y (and hence X) is a scheme.

Localizing further on Y and replacing X by some open neighbourhood of $s(Y)$, we may assume that there exists an étale morphism $q : X \to Y \times \mathbf{A}^n$ over Y, for some $n \geqslant 0$, such that $s : Y \to X$ is identified with the inclusion of the zero locus $q^{-1}(Y \times \{0\})$ (see e.g. [Kha2, Lem. 4.2.3]). Under the isomorphisms (3.27), we may therefore assume that f is the projection $Y \times \mathbf{A}^n \to Y$ and s is the zero section. In this case, there are cartesian squares

$$\begin{array}{ccccc} Y \times \mathbf{A}^1 & \longrightarrow & D & \longrightarrow & Y \times \mathbf{A}^1 \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{0} & Y \times \mathbf{A}^n & \longrightarrow & Y \end{array}$$

and (3.30) is identified with

$$P(\mathbf{A}^n_Y, Y) \circ i_0^* \to P(\mathbf{A}^n_Y, Y) \circ p_\sharp \to P(\mathbf{A}^n_Y, Y) \circ i_1^*$$

under the isomorphism $p_\sharp \circ P(\mathbf{A}^n_Y, Y) \simeq P(\mathbf{A}^n_Y, Y) \circ p_\sharp$ (3.29). By homotopy invariance, both arrows are identified with the identity of $P(\mathbf{A}^n_Y, Y)$ after applying p^* on the right. In particular, they induce isomorphisms on the unit object, so we conclude by the projection formula. □

3.5. **Topological weaves vs. Voevodsky formalisms.** Assume that for every $X \in \mathcal{S}$, we have: every quasi-compact open $U \subseteq X$ also belongs to $\mathcal{S}$; every closed substack $Z \subseteq X$ also belongs to $\mathcal{S}$; and every projective bundle $\mathbf{P}_X(\mathcal{E})$, where $\mathcal{E}$ is a finite locally free sheaf on X, also belongs to $\mathcal{S}$.

Definition 3.31. Let $\mathbf{D}^*$ be an adjointable $(*, \sharp, \otimes)$-formalism on $\mathcal{S}$. Consider the following conditions:

(i) *Homotopy invariance.* For every vector bundle $\pi : E \to X$ in $\mathcal{S}$, the unit $\mathrm{id} \to \pi_*\pi^*$ is invertible. Equivalently, the counit $\pi_\sharp\pi^* \to \mathrm{id}$ is invertible.

(ii) *Localization.* For every closed immersion $i : Z \to X$ in $\mathcal{S}$ with complementary open immersion $j : U \to X$, the functor i_* is fully faithful with essential image spanned by objects in the kernel of j^*. Equivalently, i_* is fully faithful and the following square is cartesian:

$$\begin{array}{ccc} j_\sharp j^* & \longrightarrow & \mathrm{id} \\ \downarrow & & \downarrow \\ 0 \simeq j_\sharp j^* i_* i^* & \longrightarrow & i_* i^*, \end{array}$$

where the isomorphism $j^* i_* \simeq 0$ is the left transpose of the smooth base change formula $i^* j_\sharp \simeq 0$.

If $\mathbf{D}^*$ satisfies Thom stability, homotopy invariance, and localization, then we say that it *satisfies the Voevodsky conditions.* We also say simply that $\mathbf{D}^*$ is a *Voevodsky formalism* on $\mathcal{S}$.

Theorem 3.32 (Voevodsky, Ayoub, Cisinski–Déglise)**.** *Assume that $\mathcal{S}$ consists of decent algebraic spaces. Every Voevodsky formalism satisfies the proper base change and projection formulas.*

Proof. See [CD, Thms. 2.4.26, 2.4.28], [Kha1, Chap. 2, Thm. 3.5.4], [Kha4, Thm. 2.24]. □

Corollary 3.33. *Assume that $\mathcal{S}$ consists of decent algebraic spaces. The functor*

$$\mathrm{Fun}(\mathrm{Corr}(\mathcal{S}), \mathrm{Cat}_\infty) \to \mathrm{Fun}(\mathcal{S}^{\mathrm{op}}, \mathrm{Cat}_\infty) \tag{3.34}$$

restricts to an equivalence from the ∞-category of topological weaves to the ∞-category of Voevodsky formalisms.

Proof. By Theorem 2.54 and the definitions, the functor restricts to an equivalence from the ∞-category of topological weaves to the ∞-category of adjointable $(*, \sharp, \otimes)$-formalisms satisfying the proper base change and projection formulas and the Voevodsky conditions. By Theorem 3.32, the latter is the same as the ∞-category of Voevodsky formalisms. □

Remark 3.35. Warning 2.56 applies also to Corollary 3.33. However, the section of (3.34) produced by the machinery of [LZ1] sends Voevodsky formalisms to topological weaves; in particular, every Voevodsky formalism admits a canonical extension to a topological weave, which is enough for our purposes.

Proof of Theorem 3.20. It is proven in [Rob, Cor. 2.39] (see also [DG]) that **SH** is the initial Voevodsky formalism. Hence the result follows from Corollary 3.33. □

3.6. **Orientations.**

Example 3.36. Let $\mathcal{S}$ be the ∞-category of decent algebraic spaces. By [BH, Prop. 16.28, Ex. 16.30], the weave of MGL-modules admits a canonical orientation. In particular, any weave $\mathbf{D}$ equipped with a morphism $\mathbf{D}_{\mathrm{MGL}} \to \mathbf{D}$ inherits an orientation. For example, this is the case for the weave **DM** of integral Voevodsky motives, the weave of KGL-modules, the weave of étale sheaves with $\mathbf{Z}/n\mathbf{Z}$ or ℓ-adic coefficients (on the ∞-category of algebraic spaces over $\mathbf{Z}[1/n]$, resp. $\mathbf{Z}[1/\ell]$).

Conjecture 3.37. *The weave* $\mathbf{D}_{\mathrm{MGL}}$ *is the initial oriented topological weave.*

Remark 3.38. By [EHKSY], $\mathbf{D}_{\mathrm{MGL}}$ is the initial topological weave equipped with traces $\mathrm{tr}_f : f_!(\mathbf{1}_X) \to \mathbf{1}_Y$ for finite flat quasi-smooth morphisms $f : X \to Y$, which are compatible with base change and composition up to coherent homotopy. Thus the claim is equivalent to the assertion that an orientation of a topological weave $\mathbf{D}$ is the same datum as a homotopy coherent system of traces for finite flat quasi-smooth morphisms. By [DJK] there exists for any topological weave a canonical system of traces $\mathrm{tr}_f : f_!(\Sigma^{T_f}(\mathbf{1}_X)) \to \mathbf{1}_Y$, compatible with composition only up to incoherent homotopy (where T_f is the virtual tangent bundle), which takes the form $\mathrm{tr}_f : f_!(\mathbf{1}_X) \to \mathbf{1}_Y$ in the presence of an orientation. Note that the homotopy coherence (of both orientations and traces) is automatic if, for instance, $\mathbf{D}(X)$ admits a t-structure for which the objects $\Sigma^E(\mathbf{1}_X)$ belong to the heart, for every $X \in \mathcal{S}$ and every vector bundle $E \to X$ (e.g., for étale or Betti sheaves).

4. Lisse extension

Let $\tau \in \{\mathrm{Nis}, \mathrm{\acute{e}t}\}$ stand for the Nisnevich or étale topology. We define (τ, n)-Artin and τ-Artin stacks as in [KhRa2, 0.2.2]. A stack is $(\mathrm{\acute{e}t}, 0)$-Artin, resp. $(\mathrm{Nis}, 0)$-Artin, if it is a (resp. decent) algebraic space. For $n > 0$, X is (τ, n)-*Artin* if it has $(\tau, n-1)$-representable diagonal and admits a smooth morphism $U \twoheadrightarrow X$ with τ-local sections from some scheme U. A stack is τ-*Artin* if it is (τ, n)-Artin for some n. For $\tau = \mathrm{\acute{e}t}$, these are the usual notions of n-Artin stacks and Artin stacks, while e.g. $(\mathrm{Nis}, 1)$-Artin stacks are the same as quasi-separated 1-Artin stacks with separated diagonal.

Definition 4.1. Assume that every stack in $\mathcal{S}$ is τ-Artin. Denote by $\mathcal{S}_0$ the full subcategory of $\mathcal{S}$ spanned by $(\tau, 0)$-Artin stacks. We say that a preweave

$\mathbf{D}$ is *lisse-extended* if $\mathbf{D}^*$ is right Kan-extended from its restriction to $\mathcal{S}_0$ and $\mathbf{D}^!$ (2.12) is right Kan-extended from its restriction to $\mathcal{S}_0$.

The following is a generalization of [Kha3, Thm. A.5]. In particular, it gives a new construction of the six functor formalism for étale sheaves on Artin stacks [LZ2].

Theorem 4.2. *Denote by $\mathcal{S}_0$ the full subcategory of $\mathcal{S}$ spanned by $(\tau,0)$-Artin stacks. Then for every weave $\mathbf{D}$ on $\mathcal{S}_0$ satisfying τ-descent, there exists a unique extension of $\mathbf{D}$ to a lisse-extended weave $\mathbf{D}^{\triangleleft}$ on $\mathcal{S}$.*

The main ingredient is the following:

Theorem 4.3. *With notation as above, denote by $\mathbf{D}^{*,\triangleleft}$ the right Kan extension to $\mathcal{S}$ of $\mathbf{D}^*$ and by $\mathbf{D}^{!,\triangleleft}$ the right Kan extension of $\mathbf{D}^!$ to $\mathcal{S}$. Then for every τ-Artin stack $X \in \mathcal{S}$, there is a canonical equivalence*

$$\mathbf{D}^{*,\triangleleft}(X) \simeq \mathbf{D}^{!,\triangleleft}(X).$$

We will thus define $\mathbf{D}^{\triangleleft}(X) := \mathbf{D}^{*,\triangleleft}(X)$ for $X \in \mathcal{S}$. Theorem 4.3 then gives the canonical functors $\mathbf{D}^{*,\triangleleft} : \mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ and $\mathbf{D}^{\triangleleft}_! : \mathcal{S} \to \mathrm{Cat}_\infty$ (since $\mathbf{D}^{!,\triangleleft}$ factors through the subcategory of right adjoint functors). It is not difficult to show that the resulting operations f^* and $f_!$ still satisfy the base change formula. A full proof of Theorem 4.2 requires the homotopy coherence of this base change formula, which we do not undertake here.

Let $\mathrm{Pt}(X)$ denote the ∞-category of pairs (T,t) where T is $(\tau,0)$-Artin and $t : T \to X$ is a morphism. Morphisms $f : (T',t') \to (T,t)$ are commutative triangles

$$\begin{array}{ccc} T' & \xrightarrow{\quad f \quad} & T \\ & {\scriptstyle t'}\searrow \quad \swarrow{\scriptstyle t} & \\ & X. & \end{array}$$

By construction, $\mathbf{D}^{*,\triangleleft}(X)$ is the limit

$$\mathbf{D}^{*,\triangleleft}(X) = \varprojlim_{(T,t)} \mathbf{D}(T), \tag{4.4}$$

taken over $\mathrm{Pt}(X)^{\mathrm{op}}$, where the transition functors are $*$-inverse image. Similarly, $\mathbf{D}^{!,\triangleleft}(X)$ is given by the colimit

$$\mathbf{D}^{!,\triangleleft}(X) = \varprojlim_{(T,t)} \mathbf{D}(T) \tag{4.5}$$

where the transition functors are !-inverse image.

Denote by $\mathrm{Lis}(X)$ the full subcategory of $\mathrm{Pt}(X)$ spanned by pairs (T,t) where $t : T \to X$ is smooth, and by $\mathrm{Lis}'(X)$ the wide subcategory of $\mathrm{Lis}(X)$ spanned by only smooth morphisms (i.e., morphisms $(T',t') \to (T,t)$ are smooth morphisms $f : T' \to T$ with $t \circ f = t'$). We will require the following lemma (a slight generalization of [KhRa2, Prop. 1.4]):

Lemma 4.6. *Let F be a presheaf on $\mathcal{S}_0$ with values in a complete ∞-category $\mathcal{V}$. Denote by $F^{\triangleleft}$ its right Kan extension to $\mathcal{S}$. Let $X \in \mathcal{S}$ be a τ-Artin stack. If F satisfies τ-descent, then for any $(\tau,0)$-representable morphism $f : X' \to X$, the canonical maps*

$$F^{\triangleleft}(X') \to \varprojlim_{(T,t)\in \mathrm{Lis}_X} F^{\triangleleft}(T \underset{X}{\times} X') \to \varprojlim_{(T,t)\in \mathrm{Lis}'_X} F^{\triangleleft}(T \underset{X}{\times} X') \tag{4.7}$$

are invertible.

Proof. Let $p : U \twoheadrightarrow X$ be a smooth morphism admitting τ-level sections where U is a scheme. Denote by $U_\bullet$ the Čech nerve of p. Then $U_\bullet \times_X X'$ is identified with the Čech nerve of $U \times_X X' \twoheadrightarrow X'$, so by [KhRa2, Lem. 1.5] there is a canonical equivalence

$$\mathrm{Tot}(F(U_\bullet \underset{X}{\times} X')) \simeq F^{\triangleleft}(X').$$

This defines a simplicial diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Lis}'_X$, so by projection there is a canonical map

$$\varprojlim_{(T,t)\in \mathrm{Lis}'(X)} F(T \underset{X}{\times} X') \to \mathrm{Tot}(F(U^s_\bullet \underset{X}{\times} X')) \simeq F(X),$$

which is inverse to (4.7). (The same argument works for limit over Lis_X instead of Lis'_X.) ☐

Following [Ga, 11.4.4], we consider a presheaf of ∞-categories $\mathbf{D}^{*!}$ on the product $\mathrm{Lis}'_X \times \mathrm{Pt}_X$. We first give an informal description when X is $(\tau,1)$-Artin. On objects, it is given by the assignment

$$(T,t,T',t') \mapsto \mathbf{D}(T \underset{X}{\times} T'),$$

where $T \times_X T'$ is $(\tau,0)$-Artin since $t : T \to X$ is 0-representable. Given a morphism of pairs $(f,g) : (T_1,t_1,T'_1,t'_1) \to (T_2,t_2,T'_2,t'_2)$, with $f : (T_1,t_1) \to (T_2,t_2)$ in Lis'_X and $g : (T'_1,t'_1) \to (T'_2,t'_2)$ in Pt_X, $\mathbf{D}^{*!}$ sends (f,g) to the composite

$$\mathbf{D}(T_2 \underset{X}{\times} T'_2) \xrightarrow{(f\times \mathrm{id})^!} \mathbf{D}(T_1 \underset{X}{\times} T'_2) \xrightarrow{(\mathrm{id}\times g)^*} \mathbf{D}(T_1 \underset{X}{\times} T_2).$$

That this is compatible with composition requires the observation that the exchange transformation $\mathrm{Ex}^{*!} : (\mathrm{id}\times g)^*(f\times\mathrm{id})^! \to (f\times\mathrm{id})^!(\mathrm{id}\times g)^*$ of functors $\mathbf{D}(T_2 \times_X T'_2) \to \mathbf{D}(T_1 \times_X T'_1)$, associated to the cartesian square

$$\begin{array}{ccc} T_1 \times_X T'_1 & \xrightarrow{f\times\mathrm{id}} & T_2 \times_X T'_1 \\ {\scriptstyle \mathrm{id}\times g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times g} \\ T_1 \times_X T'_2 & \xrightarrow{f\times\mathrm{id}} & T_2 \times_X T'_2, \end{array}$$

is invertible because f is smooth. That it is *homotopy coherently* compatible with composition, and hence defines a functor of ∞-categories as claimed, requires the homotopy coherence of the exchange transformations $\mathrm{Ex}^{*!}$ for compositions of cartesian squares. This is equivalent to the same property for the base change isomorphisms $\mathrm{Ex}^*_!$, since the former are obtained formally from the latter (cf. [Ga, Prop. 7.6.7]; more precisely, follow [Ga, 11.4.4] using

$\mathbf{D}^*_! : \mathrm{Corr}(\mathcal{S}) \to \mathrm{Cat}_\infty$ as input in the proof of Prop. 7.6.7). For higher Artin stacks, proceed by induction as in [Ga, 11.5.4].

Now form the limit of ∞-categories

$$\varprojlim_{(T,t,T',t') \in \mathrm{Lis}'_X \times \mathrm{Pt}_X} \mathbf{D}^{*!}(T \underset{X}{\times} T'). \tag{4.8}$$

We can calculate this in two different ways. First, we have

$$\begin{aligned}
\varprojlim_{(T,t,T',t')} \mathbf{D}^{*!}(T \underset{X}{\times} T') &\simeq \varprojlim_{(T',t') \in \mathrm{Pt}_X} \varprojlim_{(T,t) \in \mathrm{Lis}'_X} \mathbf{D}(T \underset{X}{\times} T') \\
&\simeq \varprojlim_{(T',t') \in \mathrm{Pt}_X} \mathbf{D}(T') \\
&\simeq \mathbf{D}^{*,\triangleleft}(X),
\end{aligned}$$

where the transition functors in the systems $\{(T,t)\}$ and $\{(T',t')\}$ are !- and $*$-inverse image, respectively, and the second isomorphism is Lemma 4.6. Symmetrically, we also have

$$\begin{aligned}
\varprojlim_{(T,t,T',t')} \mathbf{D}^{*!}(T \underset{X}{\times} T') &\simeq \varprojlim_{(T,t) \in \mathrm{Lis}'_X} \varprojlim_{(T',t') \in \mathrm{Pt}_X} \mathbf{D}(T \underset{X}{\times} T') \\
&\simeq \varprojlim_{(T,t) \in \mathrm{Lis}'_X} \mathbf{D}(T) \\
&\simeq \mathbf{D}^{!,\triangleleft}(X).
\end{aligned}$$

This concludes the proof of Theorem 4.3.

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