ÉTALE MICROLOCALIZATION

Thursdays, 13:00–15:00
Seminar Room 638, Astro-Math Building, NTU Campus
Organizers: P.-H. Chuang, A. A. Khan

(1) **Sheaves and local systems. (2026-10-01)**

In the context of topological spaces, recall sheaves (with values in an ∞-category), locally constant sheaves (a.k.a. local systems), monodromy representations, and the equivalence between discrete local systems and covering spaces; follow [Kha, §§2.3, 3.5]. Then explain how the analogous constructions work in the étale setting, following [Hoy, §§2–4] and [BGH, §13.2]. Discuss the examples of Kummer and Artin–Schreier covers in characteristic $p > 0$, following e.g. [SP, Tags 03PK and 0A3J].

(2) **The six operations and constructible sheaves. (2026-10-08)**

Specialize now to sheaves with values in the ∞-category $D(\Lambda)$, where Λ is a commutative ring; in the étale setting, take Λ to be finite and ℓ^k-torsion for some $\ell \neq p$. Define constructible sheaves (of finite type) in the Betti and étale settings, with respect to algebraic stratifications; see [KS, §§8.4–8.5] and [BGH, §13.2]. Summarize the six functor formalism in both settings, following [Kha, §3] and [SGA4$\frac{1}{2}$, Th. finitude, §§1, 4], including the Poincaré duality isomorphism $f^! \simeq f^*(d)[2d]$ for smooth maps of pure relative dimension d (explain the Tate twist; see [SGA4, Exp. XVIII, Th. 3.2.5]) and the Verdier duality functor, and briefly discuss the construction of the functors if time permits. State that constructible sheaves of finite type[1] are preserved by the six functors and by Verdier duality in both settings; see [KS, §8.4] and [SGA4$\frac{1}{2}$, Th. finitude, §§1.5–1.8, 4].

(3) **Nearby and vanishing cycles; universal local acyclicity. (2026-10-15)**

Briefly review the construction of the nearby and vanishing cycles functors in the complex-analytic case, along with the monodromy action, following [KS, §8.6]. Then define the analogous constructions in the étale setting, including the inertia action, following [SGA7, Exp. XIII, §1.3, §2.1]. State the Milnor formula, with the Swan conductor correction, of [SGA7, Exp. XVI, §§1.7–1.8 and Th. 2.4]. Define universal local acyclicity, with both the vanishing cycles characterization in [Bei, §3.12] and the “six operations” criterion in [LZ, Lemma 2.14, Theorem 2.16]. Explain the Artin–Schreier example in [Sai, Prop. 1.1.3(1)].

[1] a.k.a. bounded constructible complexes of finite Tor-dimension, or equivalently with perfect stalks

(4) **Singular support (2026-10-22[2])**

Give the definition of singular support in the complex-analytic case following [KS, §5.1]. Then give the analogous definition in the étale setting, following [Bei, §§1.2–1.3], and state the existence and equidimensionality theorem [Bei, Th. 1.3]. Discuss the examples in [Bei, §1.3, Example] showing that in characteristic $p > 0$, SS need not be Lagrangian, and that even Lagrangian components need not be conormal bundles.

(5) **Radon and Legendre transforms. (2026-10-29)**

Following [XY, §§3.1–3.7] and [Bei, §§1.6.1–1.6.2], explain the Radon and Legendre transforms on $\mathbf{P}(V)$. State and prove [XY, Prop. 3.2]. For the cohomological-correspondence argument in [XY, §3.5], introduce the needed formalism from [LZ, §2.2, Lemma 2.8].

(6) **Specialization to the normal bundle (2026-11-05)**

Briefly review the specialization functor from [KS, §§4.1–4.2]. Then explain Verdier's specialization functor in the étale setting, and state its main properties following [Ver, §8], [Sai, §1.2], [XY, §2].

(7) **The Fourier–Deligne transform (2026-11-12)**

Briefly review the Fourier–Sato transform from [KS, §3.7]. Explain the Fourier–Deligne transform in the étale setting, and state its main properties following [Sai, §1.1], [XY, §3.8]. State and prove [XY, Prop. 3.12].

(8) $\mu\underline{\mathrm{Hom}}$ **and** SS_μ **(2026-11-26)**

Given specialization and a Fourier transform, one can define the microlocalization functor and $\mu\underline{\mathrm{Hom}}(F, G)$ as in [KS, §§4.3–4.4], see [Sai, §2.1]. Define also $\mathrm{SS}_\mu(F) \coloneqq \mathrm{Supp}\, \mu\underline{\mathrm{Hom}}(F, F)$ following [Sai, Def. 2.1.1]; the main goal of the seminar is to compare this with the usual singular support $\mathrm{SS}(F)$. State the first properties of $\mu\underline{\mathrm{Hom}}$ and SS_μ from [Sai, §2] and [XY, Lem. 6.3, Lem. 6.9, Thm. 8.3]. State also the support estimate of [XY, Thm. 5.2], to be proven in Talk 10. Explain in detail the example of $F = j_! L_\chi$ on $\mathbf{A}^1$ in [Sai, §2.2].

(9) $\mathrm{SS}_\mu(F) \subseteq \mathrm{SS}(F)$**. (2026-12-03)**

Explain the proof of [XY, Thm. 4.2].

(10) **Support estimate for the microlocalization. (2026-12-17)**

Develop the Lu–Zheng formalism following [LZ, §§2.2–2.3, Construction 3.3] and [XY, Appendix A], and use it to explain the proof of [XY, Thm. 5.2].

[2]We need to meet on Monday or Tuesday this week due to an organizer's travel plans.

(11) $\mathrm{SS}(F) \subseteq \mathrm{SS}_\mu(F)$. **(2026-12-24)**

Explain the proof of [XY, Thm. 6.2].

References

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