

SHEAVES AND LOCAL SYSTEMS

1. SHEAVES AND LOCALLY CONSTANT SHEAVES

Let $\mathcal{V}$ be a presentable ∞ -category. Our main examples will be the category Set of sets, the ∞ -category $\mathbf{H}$ of homotopy types, and the ∞ -category $\mathbf{D}(\Lambda)$ of complexes of modules over a commutative ring Λ . We first recall sheaves on topological spaces, following [Kha, §§2.2–2.3].

1.1. Sheaves on topological spaces.

Definition 1.1.1. Let X be a topological space. Let $\mathrm{Op}(X)$ denote the set of open subsets $U \subseteq X$, ordered by inclusion. A *presheaf* on X with values in $\mathcal{V}$ is a presheaf on $\mathrm{Op}(X)$ with values in $\mathcal{V}$. We denote by

$$\mathrm{PShv}(X; \mathcal{V}) := \mathrm{Fun}(\mathrm{Op}(X)^{\mathrm{op}}, \mathcal{V})$$

the ∞ -category of $\mathcal{V}$ -valued presheaves on X . For $\mathcal{V} = \mathbf{D}(\Lambda)$, we also write $\mathrm{PShv}(X; \Lambda) := \mathrm{PShv}(X; \mathbf{D}(\Lambda))$.

Definition 1.1.2. Let $\mathcal{F}$ be a $\mathcal{V}$ -valued presheaf on X . We say that $\mathcal{F}$ is a *sheaf* if for every open $U \subseteq X$ and every collection of opens $\{V_\alpha\}_\alpha$ with $U = \bigcup_\alpha V_\alpha$, the canonical map

$$\Gamma(U, \mathcal{F}) \rightarrow \mathrm{Tot}(\Gamma(V_\bullet, \mathcal{F}))$$

is invertible. Here $V_\bullet$ is the *Čech nerve* of the cover, with

$$\Gamma(V_n, \mathcal{F}) := \prod_{\alpha_0, \dots, \alpha_n} \Gamma(V_{\alpha_0} \cap \dots \cap V_{\alpha_n}, \mathcal{F}).$$

The notation $\mathrm{Tot}(C^\bullet) := \varprojlim_{[n] \in \Delta} C^n$ indicates the *totalization* of a cosimplicial object $C^\bullet$.

Notation 1.1.3. We denote by $\mathrm{Shv}(X; \mathcal{V}) \subseteq \mathrm{PShv}(X; \mathcal{V})$ the full subcategory spanned by $\mathcal{V}$ -valued sheaves. For $\mathcal{V} = \mathbf{D}(\Lambda)$ we also write $\mathrm{Shv}(X; \Lambda) := \mathrm{Shv}(X; \mathbf{D}(\Lambda))$.

Definition 1.1.4. Let $X \in \mathrm{Sch}_{\mathbf{C}}$. Recall that its analytification X^{an} is $X(\mathbf{C})$ with the analytic topology, a locally compact, second countable Hausdorff space with a basis of contractible opens. We define

$$\mathrm{Shv}(X; \mathcal{V}) := \mathrm{Shv}(X^{\mathrm{an}}; \mathcal{V}).$$

In particular, $\mathrm{Shv}(X; \Lambda) := \mathrm{Shv}(X^{\mathrm{an}}; \Lambda)$. These are also called *Betti sheaves*.

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Construction 1.1.5. If $f : Y \rightarrow X$ is a continuous map, denote by

$$f^* : \mathrm{Shv}(X; \mathcal{V}) \rightarrow \mathrm{Shv}(Y; \mathcal{V})$$

the $*$ -pullback functor, obtained by sheafifying the presheaf

$$U \mapsto \varinjlim_{W \supseteq f(U)} \Gamma(W, \mathcal{F}),$$

where W runs through the open neighbourhoods of $f(U)$, ordered by reverse inclusion. For the inclusion $x : \mathrm{pt} \rightarrow X$ of a point, the *stalk* of $\mathcal{F}$ at x is

$$\mathcal{F}_x := x^*(\mathcal{F}) \simeq \varinjlim_{U \ni x} \Gamma(U, \mathcal{F}).$$

1.2. Locally constant sheaves.

Definition/Theorem 1.2.1. The inclusion $\iota : \mathrm{Shv}(X; \mathcal{V}) \hookrightarrow \mathrm{PShv}(X; \mathcal{V})$ admits a left adjoint

$$L : \mathrm{PShv}(X; \mathcal{V}) \rightarrow \mathrm{Shv}(X; \mathcal{V}),$$

called the *sheafification functor*. For $K \in \mathcal{V}$, the constant sheaf on X with value K is the sheafification of the constant presheaf,

$$\underline{K}_X := L(K_{\mathrm{cst}}).$$

Definition 1.2.2.

- (i) A sheaf $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{V})$ is called *constant* if there exists an object $K \in \mathcal{V}$ and an isomorphism $\mathcal{F} \simeq \underline{K}_X$ in $\mathrm{Shv}(X; \mathcal{V})$.
- (ii) A sheaf $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{V})$ is *locally constant* if there exists an open cover $X = \bigcup_{\alpha} U_{\alpha}$ such that each $\mathcal{F}|_{U_{\alpha}} \in \mathrm{Shv}(U_{\alpha}; \mathcal{V})$ is constant.

Remark 1.2.3. Locally constant sheaves are also called *local systems*.

Definition 1.2.4. We say $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$ is a *constant sheaf of finite type* if it is isomorphic to $\underline{K}_X$ for some *perfect* complex $K \in \mathbf{D}(\Lambda)$. Similarly, $\mathcal{F}$ is a *locally constant sheaf of finite type* if there is an open cover $X = \bigcup_{\alpha} U_{\alpha}$ such that each $\mathcal{F}|_{U_{\alpha}}$ is constant of finite type.

Notation 1.2.5. We denote by $\mathrm{Loc}(X; \mathcal{V}) \subseteq \mathrm{Shv}(X; \mathcal{V})$ the full subcategory of locally constant sheaves. We write $\mathrm{Loc}(X; \Lambda) := \mathrm{Loc}(X; \mathbf{D}(\Lambda))$. We denote by $\mathrm{Loc}_{\mathrm{ft}}(X; \Lambda) \subseteq \mathrm{Loc}(X; \Lambda)$ the full subcategory of locally constant sheaves of finite type.

2. MONODROMY REPRESENTATIONS

From now on we adopt the standing conventions that topological spaces are locally compact Hausdorff, second countable, of finite covering dimension¹, and have a basis of contractible opens. For example, this includes manifolds and analytifications of separated finite type $\mathbf{C}$ -schemes.

¹Finite covering dimension guarantees that the ∞ -topos $\mathrm{Shv}(X; \mathbf{H})$ is hypercomplete (see [HTT, Thm. 7.2.3.6, Cor. 7.2.1.12]).

2.1. The fundamental groupoid. Tautologically, any topological space X has an underlying homotopy type in $\mathbf{H}$. One can regard this as a higher version of the fundamental groupoid $\Pi_1(X)$, whence the following notation.

Construction 2.1.1. Let X be a topological space. The *fundamental ∞ -groupoid* $\Pi_\infty(X) \in \mathbf{H}$ is defined as follows.

- Objects are points $x \in X$.
- 1-morphisms $x \rightarrow y$ are paths $\gamma : [0, 1] \rightarrow X$ from $\gamma(0) = x$ to $\gamma(1) = y$.
- 2-morphisms $\gamma_1 \rightarrow \gamma_2$ are endpoint-preserving homotopies, i.e., morphisms $h : [0, 1] \times [0, 1] \rightarrow X$ satisfying the appropriate boundary conditions.
- ...
- n -morphisms are morphisms $[0, 1]^{\times n} \rightarrow X$ satisfying appropriate boundary conditions.
- Composition of morphisms is defined by concatenation of paths.
- Inverses are defined by reversing paths and (higher) homotopies.

Notation 2.1.2. We write

$$\Pi_1(X) := \mathrm{h} \Pi_\infty(X), \quad \pi_1(X, x) := \mathrm{Aut}_{\Pi_1(X)}(x).$$

Proposition 2.1.3. *The homotopy type $\Pi_\infty(X)$ is uniquely determined by canonical isomorphisms*

$$\mathrm{Maps}_{\mathbf{H}}(\Pi_\infty(X), H) \simeq \Gamma(X, \underline{H}),$$

natural in $H \in \mathbf{H}$.

Proof. See [HPT, Prop. 2.5] (with $S = \mathrm{pt}$). Uniqueness follows from the Yoneda lemma. (Cf. [HA, Ex. A.4.18].) $\square$

2.2. Monodromy representations.

Theorem 2.2.1. *Suppose that X is contractible, and denote by $a : X \rightarrow \mathrm{pt}$ the projection. Then the functor*

$$a^* : \mathcal{V} \simeq \mathrm{Shv}(\mathrm{pt}; \mathcal{V}) \rightarrow \mathrm{Shv}(X; \mathcal{V})$$

is fully faithful, with essential image $\mathrm{Loc}(X; \mathcal{V})$. In particular, every locally constant sheaf on X is constant.

Proof. See [HPT, Cor. 2.16] (cf. [HA, Prop. A.2.1, Lem. A.2.2]). $\square$

Construction 2.2.2. Let $\mathcal{F}$ be a locally constant sheaf on X . Let $\gamma : [0, 1] \rightarrow X$ be a path from a point $x = \gamma(0)$ to $y = \gamma(1)$. Consider the $*$ -pullback

$$\gamma^*(\mathcal{F}) \in \mathrm{Shv}([0, 1]; \mathcal{V}).$$

Since the interval $[0, 1]$ is contractible, Theorem 2.2.1 implies that $\gamma^*(\mathcal{F}) \simeq \underline{K}_{[0, 1]}$, where $K = \Gamma([0, 1], \gamma^*(\mathcal{F})) \in \mathcal{V}$. Denote by $i_0, i_1 : \mathrm{pt} \rightarrow [0, 1]$ the

inclusions of the endpoints 0 and 1, respectively, and consider the diagram:

$$\begin{array}{ccccccc} x^*(\mathcal{F}) & \xlongequal{\quad} & i_0^* \gamma^*(\mathcal{F}) & \xlongequal{\quad} & i_0^*(\underline{K}) & \xlongequal{\quad} & K \\ \downarrow & & & & & & \parallel \\ y^*(\mathcal{F}) & \xlongequal{\quad} & i_1^* \gamma^*(\mathcal{F}) & \xlongequal{\quad} & i_1^*(\underline{K}) & \xlongequal{\quad} & K, \end{array}$$

where the right-hand vertical identification is the identity map, and we write $x, y : \text{pt} \rightarrow X$ for the inclusions of the respective points. We find that there exists an essentially unique isomorphism in $\mathcal{V}$

$$\rho_{\mathcal{F}}(\gamma) : x^*(\mathcal{F}) \rightarrow y^*(\mathcal{F})$$

making the diagram above commute. We call $\rho_{\mathcal{F}}(\gamma)$ the *parallel transport* associated with the path γ .

Remark 2.2.3. Suppose given an endpoint-preserving homotopy $h : [0, 1] \times [0, 1] \rightarrow X$ between two paths γ_1 and γ_2 from x to y . Then, using the contractibility of $[0, 1] \times [0, 1]$ one can imitate Construction 2.2.2 to construct an invertible 2-morphism $\rho_{\mathcal{F}}(h)$ between $\rho_{\mathcal{F}}(\gamma_1)$ and $\rho_{\mathcal{F}}(\gamma_2)$. We may visualize h and $\rho_{\mathcal{F}}(h)$ as homotopies witnessing the commutativity of the squares

$$\begin{array}{ccc} x & \xrightarrow{\gamma_1} & y \\ \parallel & & \parallel \\ x & \xrightarrow{\gamma_2} & y, \end{array} \quad \begin{array}{ccc} x^*(\mathcal{F}) & \xrightarrow{\rho(\gamma_1)} & y^*(\mathcal{F}) \\ \parallel & & \parallel \\ x^*(\mathcal{F}) & \xrightarrow{\rho(\gamma_2)} & y^*(\mathcal{F}), \end{array}$$

respectively.

Construction 2.2.4. More generally, one can use the contractibility of $[0, 1]^{\times n}$ to construct the parallel transport $\rho_{\mathcal{F}}(\theta)$ for all n -morphisms θ of $\Pi_{\infty}(X)$. That is, one may construct a functor

$$\rho_{\mathcal{F}} : \Pi_{\infty}(X) \rightarrow \mathcal{V}, \quad (2.2.5)$$

naturally in $\mathcal{F}$, which we call the *monodromy representation* of the locally constant sheaf $\mathcal{F} \in \text{Loc}(X; \mathcal{V})$.

The following theorem asserts that locally constant sheaves are completely determined by their monodromy representations.

Theorem 2.2.6. *The functor*

$$\text{Loc}(X; \mathcal{V}) \rightarrow \text{Fun}(\Pi_{\infty}(X), \mathcal{V}), \quad \mathcal{F} \mapsto \rho_{\mathcal{F}},$$

is an equivalence of ∞ -categories. For $\mathcal{V} = \mathbf{D}(\Lambda)$, it restricts to an equivalence

$$\text{Loc}_{\text{ft}}(X; \Lambda) \simeq \text{Fun}(\Pi_{\infty}(X), \text{Perf}(\Lambda)).$$

Moreover, these equivalences are natural in X .

Proof. See [HPT, Cor. 3.7, Obs. 3.9 and 3.11] (cf. [HA, Thm. A.4.19]). We briefly sketch the inverse construction. Given a functor $\rho : \Pi_{\infty}(X) \rightarrow \mathcal{V}$, we define a presheaf $\mathcal{F}_{\rho}$ whose sections over an open $U \subseteq X$ are given by

$$\Gamma(U, \mathcal{F}_{\rho}) := \varprojlim (\Pi_{\infty}(U) \rightarrow \Pi_{\infty}(X) \xrightarrow{\rho} \mathcal{V}).$$

One checks that it is a sheaf, that it is locally constant, and that $\rho \mapsto \mathcal{F}_\rho$ is inverse to $\mathcal{F} \mapsto \rho_{\mathcal{F}}$. $\square$

Example 2.2.7. Take the space $X = \mathbf{C}^\times$. Since $\mathbf{C}^\times$ is connected, with $\pi_1(\mathbf{C}^\times, 1) \simeq \mathbf{Z}$ and $\pi_i(\mathbf{C}^\times, 1) \simeq 0$ for $i \geq 2$, we may identify $\Pi_\infty(\mathbf{C}^\times)$ with the groupoid $B\mathbf{Z}$. Thus any functor $\rho : \Pi_\infty(\mathbf{C}^\times) \rightarrow \mathcal{V}$ is determined by its image on the positive generator $\gamma(t) := \exp(2\pi it)$ of $\pi_1(\mathbf{C}^\times, 1)$; that is, ρ amounts to the object $V := \rho(1)$ of $\mathcal{V}$ together with the automorphism $T := \rho(\gamma) : V \xrightarrow{\sim} V$. By Theorem 2.2.6, a locally constant sheaf $\mathcal{F}$ on $\mathbf{C}^\times$ thus amounts to the pair

$$(V_{\mathcal{F}}, T_{\mathcal{F}}), \quad V_{\mathcal{F}} := \mathcal{F}_1, \quad T_{\mathcal{F}} := \rho_{\mathcal{F}}(\gamma) : V_{\mathcal{F}} \xrightarrow{\sim} V_{\mathcal{F}},$$

where $\mathcal{F}_1$ is the stalk at $1 \in \mathbf{C}^\times$ and $T_{\mathcal{F}}$ is parallel transport along γ . We call $T_{\mathcal{F}}$ the *monodromy operator* of $\mathcal{F}$ around 0.

We now specialize to $\mathcal{V} = \mathbf{D}(\Lambda)$ and fix a perfect complex $K \in \text{Perf}(\Lambda)$.

Notation 2.2.8. Denote by $\text{Perf}(\Lambda)^\simeq$ the maximal ∞ -groupoid in $\text{Perf}(\Lambda)$, obtained by discarding all non-invertible morphisms. The connected component containing K can be identified with

$$B \text{Aut}(K) \subseteq \text{Perf}(\Lambda)^\simeq,$$

where $\text{Aut}(K) := \text{Aut}_\Lambda(K) \subseteq \text{Maps}_{\mathbf{D}(\Lambda)}(K, K)$ is the type of Λ -linear automorphisms, and $B \text{Aut}(K)$ is its classifying type, consisting of a single object K whose automorphism type is $\text{Aut}(K)$.

Proposition 2.2.9. *Let $\text{Loc}_K(X; \Lambda) \subseteq \text{Loc}_{\text{ft}}(X; \Lambda)$ denote the full subcategory spanned by sheaves locally isomorphic to $\underline{K}_X$. The monodromy equivalence induces an equivalence of ∞ -groupoids*

$$\text{Loc}_K(X; \Lambda)^\simeq \simeq \text{Fun}(\Pi_\infty(X), B \text{Aut}(K)).$$

Proof. Every functor $\Pi_\infty(X) \rightarrow \text{Perf}(\Lambda)$ sends all morphisms to isomorphisms, so factors through $\text{Perf}(\Lambda)^\simeq$. It factors through the component $B \text{Aut}(K)$ if and only if it takes every object to an object isomorphic to K . Under monodromy, this says exactly that every stalk of the corresponding locally constant sheaf is isomorphic to K . This is equivalent to being locally isomorphic to $\underline{K}_X$. $\square$

3. RANK ONE LOCAL SYSTEMS

Using the monodromy equivalence, we have the following recipe to cook up rank one local systems on a space.

Construction 3.0.1. Let X be a space.

- (i) Let G be a (discrete) group. Recall that the type BG is the classifying groupoid for G -torsors: any G -torsor $f : Y \rightarrow X$ determines a map $c_f : \Pi_\infty(X) \rightarrow BG$ in $\mathbf{H}$ (see e.g. [Kha2, Ex. 4.5.6]). By Theorem 2.2.6,

we obtain a discrete local system $\mathcal{S}_f$ on X , with stalks isomorphic to the set G , and monodromy representation

$$\rho_{\mathcal{S}_f} : \Pi_\infty(X) \xrightarrow{c_f} BG \hookrightarrow \text{Set}^G,$$

where $\text{Set}^G := \text{Fun}(BG^{\text{op}}, \text{Set})$ is the category of G -sets (i.e., sets with right G -action)². Since the G -torsor $f : Y \rightarrow X$ is tautologically trivialized by base change along f , the composite

$$\Pi_\infty(Y) \xrightarrow{f} \Pi_\infty(X) \xrightarrow{c_f} BG$$

is homotopic to the constant functor factoring through $* \rightarrow BG$ (the quotient map, classifying the trivial G -torsor). Thus $f^*\mathcal{S}_f$ is the constant sheaf on Y with value G .

- (ii) Given any character³ $\chi : G \rightarrow \Lambda^\times \simeq \text{Aut}(\Lambda)$, we consider the composite

$$\rho_\chi : \Pi_\infty(X) \xrightarrow{c_f} BG \xrightarrow{B\chi} B\text{Aut}(\Lambda) \subseteq \text{Perf}(\Lambda),$$

which sends a point $x \in X$ to Λ , and a loop γ based at x to multiplication by $\chi(c_f(\gamma)) \in \Lambda^\times$. By Theorem 2.2.6, this corresponds to a locally constant sheaf of finite type

$$\mathcal{L}_\chi \in \text{Loc}_{\text{ft}}(X; \Lambda),$$

with stalks isomorphic to Λ and monodromy representation $\rho_{\mathcal{L}_\chi} = \rho_\chi$. Moreover, $f^*\mathcal{L}_\chi \simeq \underline{\Lambda}_Y \in \text{Loc}_{\text{ft}}(Y; \Lambda)$ is *constant*.

Remark 3.0.2. Concretely, $\mathcal{S}_f \in \text{Shv}(X; \text{Set}^G)$ is the sheaf of sections of the G -torsor f : its sections over an open $U \subseteq X$ are continuous maps $s : U \rightarrow Y$ with $f \circ s = [U \hookrightarrow X]$.

Similarly, $\mathcal{L}_\chi \in \text{Shv}(X; \Lambda)$ is the discrete sheaf of Λ -modules whose sections over an open $U \subseteq X$ are locally constant functions $\phi : f^{-1}(U) \rightarrow \Lambda$ with $\phi(y \cdot g) = \chi(g)^{-1} \cdot \phi(y)$ for all $y \in f^{-1}(U)$ and $g \in G$.

Example 3.0.3. Let $n \geq 1$ be an integer, and $\mu_n \subseteq \mathbf{C}^\times$ the group of n th roots of unity. The map

$$f : \mathbf{C}^\times \rightarrow \mathbf{C}^\times, \quad z \mapsto z^n,$$

is a μ_n -torsor, with μ_n acting on the source by $z \cdot \zeta := \zeta z$. Indeed, $(\zeta z)^n = z^n$, so μ_n acts over the target; it acts freely and transitively on each fibre, since two n th roots of w differ by a unique element of μ_n . For local triviality, choose a branch of the logarithm on a small disk $U \subseteq \mathbf{C}^\times$ and put $s(w) := \exp(\log(w)/n)$. Then

$$U \times \mu_n \xrightarrow{\sim} f^{-1}(U), \quad (w, \zeta) \mapsto \zeta s(w),$$

is a μ_n -equivariant homeomorphism over U .

Proposition 3.0.4.

²The embedding is Yoneda. Explicitly, it sends $* \mapsto G$ on objects, and $g \in G$ to the multiplication map $g : G \rightarrow G$.

³By a character of G , we will always mean a group homomorphism $G \rightarrow \Lambda^\times$, where $\Lambda^\times$ is the multiplicative group of units of Λ .

- (i) Consider the local system of G -sets $\mathcal{S}_f \in \text{Loc}(\mathbf{C}^\times; \text{Set}^G)$ associated to the μ_n -torsor $f : \mathbf{C}^\times \rightarrow \mathbf{C}^\times$, $z \mapsto z^n$. Then $\mathcal{S}_f$ has stalks isomorphic to μ_n , and monodromy operator $T : \mu_n \xrightarrow{\sim} \mu_n$ given by multiplication by $\zeta_n := \exp(2\pi i/n)$. Moreover, $f^*\mathcal{S}_f$ is the constant sheaf $\underline{\mu_n}_{\mathbf{C}^\times}$.
- (ii) Let $\chi : \mu_n \rightarrow \Lambda^\times$ be a character, and consider the local system of Λ -modules $\mathcal{L}_\chi \in \text{Loc}_{\text{ft}}(\mathbf{C}^\times; \Lambda)$ associated with f . Then $\mathcal{L}_\chi$ has stalks isomorphic to Λ , and monodromy operator $T : \Lambda \rightarrow \Lambda$ given by multiplication by $\chi(\zeta_n) \in \Lambda^\times$. Moreover, $f^*\mathcal{L}_\chi$ is the constant sheaf $\underline{\Lambda}_{\mathbf{C}^\times}$.
- (iii) The assignment $\chi \mapsto \mathcal{L}_\chi$ determines a bijection between characters of μ_n and isomorphism classes of local systems $\mathcal{L} \in \text{Loc}_\Lambda(\mathbf{C}^\times; \Lambda)$ on $\mathbf{C}^\times$ with monodromy operator satisfying $T^{\text{on}} = \text{id}$.

Proof.

- (i) Let $\rho_{\mathcal{S}_f} : \Pi_\infty(\mathbf{C}^\times) \rightarrow \text{Fun}(B\mu_n^{\text{op}}, \text{Set})$ be the monodromy representation. Then $T_{\mathcal{S}_f} := \rho_{\mathcal{S}_f}(\gamma)$, where $\gamma(t) := \exp(2\pi it)$ is the positive generator of $\pi_1(\mathbf{C}^\times, 1)$. By definition this is the composite of the bijections

$$\rho_{\mathcal{S}_f}(\gamma) : f^{-1}(1) \xrightarrow{\text{ev}_0^{-1}} \Gamma([0, 1], \gamma^*\mathcal{S}_f) \xrightarrow{\text{ev}_1} f^{-1}(1).$$

Concretely, $\gamma^*\mathcal{S}_f$ is constant on $[0, 1]$, and its global sections are the lifts $\tilde{\gamma} : [0, 1] \rightarrow \mathbf{C}^\times$ of γ along f ; $\rho_{\mathcal{S}_f}(\gamma)$ sends $\zeta \in f^{-1}(1)$ to the endpoint of the unique lift of γ starting at ζ . Since ev_0 is bijective we can label the lifts $\tilde{\gamma}$ by their starting point $\tilde{\gamma}(0)$: for $\zeta \in f^{-1}(1) = \mu_n$, the lift is $\tilde{\gamma}_\zeta(t) := \zeta \exp(2\pi it/n)$, and its endpoint is $\zeta \cdot \zeta_n$, where $\zeta_n := \exp(2\pi i/n)$. Thus $T := \rho_{\mathcal{S}_f}(\gamma)$ is multiplication by ζ_n on μ_n . In other words, the functor $c_f : B\mathbf{Z} \simeq \Pi_\infty(\mathbf{C}^\times) \rightarrow B\mu_n$ is induced by the group homomorphism $\mathbf{Z} \simeq \pi_1(\mathbf{C}^\times, 1) \rightarrow \mu_n$ sending the positive generator γ to $\zeta_n \in \mu_n$.

- (ii) This follows from the previous claim by unwinding definitions.
- (iii) By Proposition 2.2.9 (and Example 2.2.7), there is an equivalence $\text{Loc}_\Lambda(\mathbf{C}^\times; \Lambda)^\simeq \simeq \text{Fun}(B\mathbf{Z}, B\Lambda^\times)$. In particular, an object $\mathcal{L}$ is determined by the monodromy operator $T_{\mathcal{L}} \in \text{Aut}(\Lambda) \simeq \Lambda^\times$, and isomorphisms $\mathcal{L} \simeq \mathcal{L}'$ correspond to units $u \in \Lambda^\times$ with $uT_{\mathcal{L}}u^{-1} = T_{\mathcal{L}'}$. It follows, since $\Lambda^\times$ is abelian, that isomorphism classes of $\mathcal{L}$ are in bijection with elements of $\Lambda^\times$.

Since ζ_n generates μ_n , every $\lambda \in \Lambda^\times$ with $\lambda^n = 1$ is $\chi(\zeta_n)$ for a unique χ . Since $T_{\mathcal{L}_\chi} = \chi(\zeta_n)$ by (ii), $\chi \mapsto \mathcal{L}_\chi$ is a bijection onto the isomorphism classes with $T_{\mathcal{L}}^{\text{on}} = 1$. Since f induces multiplication by n on π_1 , so that $T_{f^*\mathcal{L}} = T_{\mathcal{L}}^{\text{on}}$, this is equivalently the condition that $f^*\mathcal{L}$ is constant.

□

Example 3.0.5. Since $\mathbf{C}$ is contractible, Theorem 2.2.1 shows that every locally constant sheaf on $\mathbf{C}$ is constant. In particular, for a nontrivial character χ , the local system $\mathcal{L}_\chi$ on $\mathbf{C}^\times$ does not extend to $\mathbf{C}$: the restriction to $\mathbf{C}^\times$ of a constant sheaf has monodromy operator $T = \text{id}$, whereas $\mathcal{L}_\chi$ has $T = \chi(\zeta_n)$.

Geometrically, this reflects the fact that the map

$$f : \mathbf{C} \rightarrow \mathbf{C}, \quad z \mapsto z^n,$$

is not a μ_n -torsor: μ_n fixes 0, so it does not act freely on the fibre over 0; one says that f is *ramified* at 0. As we will see below, this is in contrast with the affine line in characteristic $p > 0$ (Example 6.0.6).

4. ÉTALE SHEAVES

We now explain the analogues of these constructions for the étale topology. We follow [Hoy, §§2–4] and [BGH, §13.2].

4.1. Sheaves. Let k be a field. We write Sch_k for the category of separated finite type k -schemes.

Definition 4.1.1. For $X \in \mathrm{Sch}_k$ we write $X_{\mathrm{\acute{e}t}}$ for the *étale site* of X : its underlying category is the full subcategory of $(\mathrm{Sch}_k)_{/X}$ spanned by étale k -morphisms $U \rightarrow X$,⁴ and its Grothendieck topology is the étale topology, generated by finite covering families $(U_\alpha \rightarrow U)_\alpha$ in $X_{\mathrm{\acute{e}t}}$ such that $\coprod_\alpha U_\alpha \rightarrow X$ is surjective.

Definition 4.1.2. Let $\mathcal{F} : X_{\mathrm{\acute{e}t}}^{\mathrm{op}} \rightarrow \mathcal{V}$ be a presheaf. We say that $\mathcal{F}$ is an *étale sheaf* if, for every finite covering family $\{U_\alpha \rightarrow U\}_\alpha$ in $X_{\mathrm{\acute{e}t}}$, the canonical map

$$\Gamma(U, \mathcal{F}) \longrightarrow \mathrm{Tot} \left(\prod_{\alpha_0, \dots, \alpha_\bullet} \Gamma(U_{\alpha_0} \times_U \cdots \times_U U_{\alpha_\bullet}, \mathcal{F}) \right)$$

is invertible.

Notation 4.1.3. For a presentable ∞ -category $\mathcal{V}$, we write $\mathrm{Shv}(X_{\mathrm{\acute{e}t}}; \mathcal{V})$ for the ∞ -category of $\mathcal{V}$ -valued sheaves on $X_{\mathrm{\acute{e}t}}$. For a commutative ring Λ , we write

$$\mathrm{Shv}(X_{\mathrm{\acute{e}t}}; \Lambda) := \mathrm{Shv}(X_{\mathrm{\acute{e}t}}; \mathbf{D}(\Lambda)).$$

As in the Betti case, the inclusion admits a left adjoint

$$L_{\mathrm{\acute{e}t}} : \mathrm{PShv}(X_{\mathrm{\acute{e}t}}; \mathcal{V}) \rightarrow \mathrm{Shv}(X_{\mathrm{\acute{e}t}}; \mathcal{V}),$$

called *étale sheafification*. For $K \in \mathcal{V}$, the *constant étale sheaf* with value K is $\underline{K}_X := L_{\mathrm{\acute{e}t}}(K_{\mathrm{cst}})$. The constant-sheaf functor is left adjoint to global sections:

$$\mathrm{Maps}_{\mathrm{Shv}(X_{\mathrm{\acute{e}t}}; \mathcal{V})}(\underline{K}_X, \mathcal{F}) \simeq \mathrm{Maps}_{\mathcal{V}}(K, \Gamma(X, \mathcal{F})).$$

Construction 4.1.4. For an étale morphism $f : X \rightarrow Y$, restriction along the forgetful functor $X_{\mathrm{\acute{e}t}} \rightarrow Y_{\mathrm{\acute{e}t}}$ defines the $*$ -pullback functor

$$f^* : \mathrm{Shv}(Y_{\mathrm{\acute{e}t}}; \mathcal{V}) \rightarrow \mathrm{Shv}(X_{\mathrm{\acute{e}t}}; \mathcal{V}), \quad \Gamma(U, f^* \mathcal{G}) := \Gamma(U, \mathcal{G}).$$

One can show that f^* preserves étale-local equivalences, and hence $f^* L_Y \simeq L_X f^*$. In particular, $f^* \underline{K}_Y \simeq \underline{K}_X$.

⁴Note that any morphism $f : U \rightarrow V$ in this category is automatically étale.

Definition 4.1.5. Let x be a geometric point of X , i.e. a morphism $x : \mathrm{Spec}(\Omega) \rightarrow X$ where Ω is a separably closed field. An *étale neighbourhood* of x is an étale map $U \rightarrow X$ together with a lift $u : \mathrm{Spec}(\Omega) \rightarrow U$ of x . The *stalk* of $\mathcal{F}$ at x is

$$\mathcal{F}_x := \varinjlim_{(U,u)} \Gamma(U, \mathcal{F}) \in \mathcal{V},$$

where (U, u) runs through the étale neighbourhoods of x .

4.2. Locally constant étale sheaves.

Definition 4.2.1. A sheaf $\mathcal{F} \in \mathrm{Shv}(X_{\mathrm{\acute{e}t}}; \mathcal{V})$ is *constant* if it is isomorphic to $\underline{K}_X$ for some $K \in \mathcal{V}$. It is *locally constant*, or a *local system*, if there is an étale covering family $\{u_\alpha : U_\alpha \rightarrow X\}_\alpha$ such that each $u_\alpha^*(\mathcal{F})$ is constant. We denote by

$$\mathrm{Loc}(X_{\mathrm{\acute{e}t}}; \mathcal{V}) \subseteq \mathrm{Shv}(X_{\mathrm{\acute{e}t}}; \mathcal{V})$$

the full subcategory of locally constant sheaves.

Definition 4.2.2. A sheaf $\mathcal{F} \in \mathrm{Shv}(X_{\mathrm{\acute{e}t}}; \Lambda)$ is *locally constant of finite type* if it is étale-locally isomorphic to $\underline{K}$ for a perfect complex $K \in \mathrm{Perf}(\Lambda)$. We write

$$\mathrm{Loc}_{\mathrm{ft}}(X_{\mathrm{\acute{e}t}}; \Lambda) \subseteq \mathrm{Loc}(X_{\mathrm{\acute{e}t}}; \Lambda).$$

5. MONODROMY REPRESENTATIONS OF ÉTALE SHEAVES

5.1. Étale homotopy types. We first want an analogue of the homotopy type $\Pi_\infty(X)$ of a topological space X . By Proposition 2.1.3, $\Pi_\infty(X)$ corepresents the functor $H \mapsto \Gamma(X, \underline{H})$. We may thus ask whether the analogous functor is also corepresentable in the étale setting.

Theorem 5.1.1. [HTT, Prop. 5.5.2.7] *Let $\mathcal{C}$ be a presentable ∞ -category and $F : \mathcal{C} \rightarrow \mathbf{H}$ a functor. Then F is corepresentable if and only if F is accessible and preserves limits.*

Lemma 5.1.2. *For a topological space X , the functor $H \mapsto \Gamma(X, \underline{H})$ is accessible and preserves limits.*

Proof. Both $H \mapsto \underline{H}$ and $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ are accessible by [HTT, Prop. 5.4.7.7]. For limit preservation, choose a basis $\mathcal{B}$ for the topology of X by nonempty contractible opens. By [Aok, Thm. A.6] (which applies by Footnote 1), every $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{H})$ is right Kan-extended from $\mathcal{B}^{\mathrm{op}}$; in particular,

$$\Gamma(X, \mathcal{F}) \simeq \varprojlim_{U \in \mathcal{B}^{\mathrm{op}}} \Gamma(U, \mathcal{F}).$$

For each $U \in \mathcal{B}$, the canonical map

$$H \rightarrow \Gamma(U, \underline{H}_X) \simeq \Gamma(U, \underline{H}_U)$$

is an isomorphism by Theorem 2.2.1. By naturality it follows that $\Gamma(X, \underline{H}_X)$ is the limit of the constant diagram $\underline{H}_{\mathcal{B}^{\mathrm{op}}}$ over $\mathcal{B}^{\mathrm{op}}$. Both functors $H \mapsto \underline{H}_{\mathcal{B}^{\mathrm{op}}}$ and $\varprojlim_{\mathcal{B}^{\mathrm{op}}} (-)$ are limit-preserving, whence the claim. $\square$

Lemma 5.1.3. *Let $X \in \text{Sch}_k$ and consider the functor $H \mapsto \Gamma_{\text{ét}}(X, \underline{H})$ ⁵. This is accessible and preserves finite limits, but it need not preserve infinite limits.*

Proof. Accessibility is as in the Betti case. It preserves *finite* limits since $H \mapsto \underline{H}_X$ does, while $\mathcal{F} \mapsto \Gamma_{\text{ét}}(X, \mathcal{F})$ preserves arbitrary ones. Therefore, it preserves arbitrary limits if and only if it preserves infinite products.

Consider a collection of homotopy types of the form $H_n = BG_n$ where G_n are groups. Recall that $B(-)$ commutes with infinite products, i.e. $\prod_n BG_n \simeq B(\prod_n G_n)$, and that $\pi_0 \Gamma_{\text{ét}}(X, \underline{BG}) \simeq H_{\text{ét}}^1(X; G)$ is the set of isomorphism classes of étale G -torsors. Therefore, it will suffice to show that the canonical map

$$H_{\text{ét}}^1(X; G) \rightarrow \prod_n H_{\text{ét}}^1(X; G_n), \quad [P] \mapsto ([P \times^G G_n])_n$$

need not be bijective, where $G := \prod_n G_n$.

Concretely, take $X = \text{Spec}(k)$ over $k = \mathbf{F}_q$, and $G_n := \mathbf{Z}/n\mathbf{Z}$. For each n , let $Y_n \rightarrow X$ be the $\mathbf{Z}/n\mathbf{Z}$ -torsor $\text{Spec}(\mathbf{F}_{q^n}) \rightarrow \text{Spec}(\mathbf{F}_q)$, where the action $\mathbf{Z}/n\mathbf{Z} \rightarrow \text{Gal}(\mathbf{F}_{q^n}/\mathbf{F}_q)$ sends 1 to the Frobenius $x \mapsto x^q$. Then the collection of isomorphism classes $(Y_n)_n$ does not lie in the image of the above map: every étale torsor over $\text{Spec}(\mathbf{F}_q)$ must split over some $\mathbf{F}_{q^m}$ (any trivializing affine étale cover over $\mathbf{F}_q$ is a finite coproduct $\text{Spec}(K_i)$ where $K_i/\mathbf{F}_q$ are finite separable extensions, necessarily each of the form $\mathbf{F}_{q^m}$ for some m), but a $\prod_n \mathbf{Z}/n\mathbf{Z}$ -torsor P in the preimage has no sections over any $\text{Spec}(\mathbf{F}_{q^m})$ (such a section would yield an $\mathbf{F}_{q^m}$ -point of $Y_n = \text{Spec}(\mathbf{F}_{q^n})$ for every n , forcing $n \mid m$ for all n). $\square$

Remark 5.1.4. The argument from the Betti case does not apply because $X_{\text{ét}}$ does not have a basis of objects $U \in X_{\text{ét}}$ with $\Gamma_{\text{ét}}(U, \underline{H}) \simeq H$. For example, for $X = \text{Spec}(k)$ and $U = \text{Spec}(L) \in X_{\text{ét}}$, where L/k is a finite separable field extension, and $H = BG$ for a finite group G , we have $\pi_0 \Gamma_{\text{ét}}(U, \underline{BG}) \simeq H_{\text{ét}}^1(U; G)$, which is typically nontrivial unless L is separably closed.

Notation 5.1.5. Write $\text{Pro}(\mathbf{H})$ for the ∞ -category of pro-objects in $\mathbf{H}$ (see [SAG, Def. A.8.1.1]). A pro-object P can be represented by a cofiltered diagram $(P_i)_{i \in I}$ of homotopy types. For $H \in \mathbf{H}$, regarded as a constant pro-object, the mapping type is

$$\text{Maps}_{\text{Pro}(\mathbf{H})}(P, H) \simeq \varinjlim_{i \in I^{\text{op}}} \text{Maps}_{\mathbf{H}}(P_i, H).$$

Passing to pro-objects buys us the following tautological version of Theorem 5.1.1 (cf. [Hoy, p. 3]):

Theorem 5.1.6. *Let $\mathcal{C}$ be a presentable ∞ -category and $F : \mathcal{C} \rightarrow \mathbf{H}$ a functor. Then F is pro-corepresentable if and only if F is accessible and preserves finite limits.*

⁵this could also be written $\Gamma(X_{\text{ét}}, \underline{H})$

Proof. The ∞ -category $\mathrm{Pro}(\mathbf{H})$ is by definition the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathbf{H})^{\mathrm{op}}$ spanned by the accessible and finite-limit-preserving functors. $\square$

Thus by Lemma 5.1.3 we have:

Definition/Theorem 5.1.7. [Hoy, Def. 2.3] *The étale shape of $X \in \mathrm{Sch}_k$ is the pro-homotopy type $\Pi_{\infty}^{\mathrm{ét}}(X) \in \mathrm{Pro}(\mathbf{H})$ uniquely determined by canonical isomorphisms*

$$\mathrm{Maps}_{\mathrm{Pro}(\mathbf{H})}(\Pi_{\infty}^{\mathrm{ét}}(X), H) \simeq \Gamma_{\mathrm{ét}}(X, \underline{H}),$$

natural in $H \in \mathbf{H}$.

5.2. The profinite étale shape.

Definition 5.2.1. A homotopy type H is π -finite if it has finitely many connected components, all its homotopy groups are finite, and they vanish in sufficiently high degrees. We denote by $\mathbf{H}_{\pi} \subseteq \mathbf{H}$ the full subcategory of π -finite homotopy types. A *profinite homotopy type* is an object of $\mathrm{Pro}(\mathbf{H}_{\pi})$.

Example 5.2.2. A finite set, regarded as a discrete homotopy type, is π -finite. If G is a finite group, its classifying type BG is π -finite. For a finite abelian group A and $n \geq 1$, the Eilenberg–Mac Lane type $K(A, n)$ is π -finite.

Definition 5.2.3 ([BGH, Def. 4.4.2]). The *profinite étale shape* of X is the profinite completion

$$\widehat{\Pi}_{\infty}^{\mathrm{ét}}(X) := \Pi_{\infty}^{\mathrm{ét}}(X)^{\wedge} \in \mathrm{Pro}(\mathbf{H}_{\pi}).$$

Equivalently, it is the pro-object characterized by the natural isomorphisms

$$\mathrm{Maps}_{\mathrm{Pro}(\mathbf{H}_{\pi})}(\widehat{\Pi}_{\infty}^{\mathrm{ét}}(X), H) \simeq \Gamma_{\mathrm{ét}}(X, \underline{H}), \quad H \in \mathbf{H}_{\pi}.$$

Remark 5.2.4. By [BGH, Rec. 2.8.2], the profinite completion $\widehat{\Pi}_{\infty}^{\mathrm{ét}}(X)$ is the pro-object corepresenting the restriction of $\mathrm{Maps}_{\mathrm{Pro}(\mathbf{H})}(\Pi_{\infty}^{\mathrm{ét}}(X), -) : \mathbf{H} \rightarrow \mathbf{H}$ to $\mathbf{H}_{\pi}$. Thus for every $H \in \mathbf{H}_{\pi}$ there is a canonical isomorphism

$$\mathrm{Maps}_{\mathrm{Pro}(\mathbf{H}_{\pi})}(\widehat{\Pi}_{\infty}^{\mathrm{ét}}(X), H) \simeq \mathrm{Maps}_{\mathrm{Pro}(\mathbf{H})}(\Pi_{\infty}^{\mathrm{ét}}(X), H).$$

For example, for a finite group G , $\pi_0 \mathrm{Maps}_{\mathrm{Pro}(\mathbf{H}_{\pi})}(\widehat{\Pi}_{\infty}^{\mathrm{ét}}(X), BG) \simeq \pi_0 \Gamma_{\mathrm{ét}}(X, \underline{BG}) \simeq H_{\mathrm{ét}}^1(X, G)$ is the set of isomorphism classes of G -torsors on $X_{\mathrm{ét}}$.

5.3. Monodromy representations.

Definition 5.3.1 ([BGH, Def. 13.2.1]). An ∞ -category $\mathcal{V}$ is *locally π -finite* if $\mathrm{Maps}_{\mathcal{V}}(K, L)$ is π -finite for every pair of objects $K, L \in \mathcal{V}$.

Example 5.3.2. The category FinSet is locally π -finite, since its Hom sets are finite. The ∞ -category $\mathbf{H}_{\pi}$ is locally π -finite, and if Λ is a finite ring, then so is $\mathrm{Perf}(\Lambda)$ [BGH, Exs. 13.2.2]. Indeed, the derived Hom between two perfect complexes is represented by a bounded complex of finite modules, so its underlying mapping type has finite homotopy groups and is truncated.

Definition 5.3.3. Let $P = (P_i)_{i \in I} \in \text{Pro}(\mathbf{H}_\pi)$ and let $\mathcal{V}$ be an essentially small locally π -finite ∞ -category. We put

$$\text{Fun}_{\text{cont}}(P, \mathcal{V}) := \varinjlim_{i \in I^{\text{op}}} \text{Fun}(P_i, \mathcal{V}),$$

where the colimit is taken in Cat_∞ , with transition functors given by pullback along refinements. We call its objects *continuous representations* of P in $\mathcal{V}$.

Notation 5.3.4 ([BGH, Def. 13.2.6]). For an essentially small ∞ -category $\mathcal{V}$, we use the Yoneda embedding $\mathcal{V} \hookrightarrow \widehat{\mathcal{V}} := \text{PShv}(\mathcal{V}; \mathbf{H})$ to define $\text{Loc}(X_{\text{ét}}; \mathcal{V})$ as the full subcategory of $\text{Loc}(X_{\text{ét}}; \widehat{\mathcal{V}})$ spanned by sheaves locally of the form $\underline{K}$ where $K \in \mathcal{V}$.

Theorem 5.3.5 (Monodromy). *For an essentially small locally π -finite ∞ -category $\mathcal{V}$, there is a canonical equivalence*

$$\text{Loc}(X_{\text{ét}}; \mathcal{V}) \simeq \text{Fun}_{\text{cont}}(\widehat{\Pi}_\infty^{\text{ét}}(X), \mathcal{V}),$$

natural in X . In particular, for a finite commutative ring Λ , there is an equivalence

$$\text{Loc}_{\text{ft}}(X_{\text{ét}}; \Lambda) \simeq \text{Fun}_{\text{cont}}(\widehat{\Pi}_\infty^{\text{ét}}(X), \text{Perf}(\Lambda)).$$

Proof. This is [BGH, Th. 13.2.11], specialized to the one-point stratification, combined with the identification of [BGH, Ex. 13.2.9]. See also [Hoy, Th. 4.3] (combined with [BGH, Ex. 13.2.8]) for the case $\mathcal{V} = \mathbf{H}_\pi$. $\square$

Definition 5.3.6. The inverse functor of Theorem 5.3.5 associates to $\mathcal{F} \in \text{Loc}_{\text{ft}}(X_{\text{ét}}; \Lambda)$ its *monodromy representation*,

$$\rho_{\mathcal{F}} \in \text{Fun}_{\text{cont}}(\widehat{\Pi}_\infty^{\text{ét}}(X), \text{Perf}(\Lambda)).$$

Fix a finite commutative ring Λ and a perfect complex $K \in \text{Perf}(\Lambda)$.

Lemma 5.3.7. *The homotopy type $B \text{Aut}(K)$ is π -finite.*

Proof. By Example 5.3.2, the type $\text{Maps}_{\mathbf{D}(\Lambda)}(K, K)$ is π -finite. Its union of invertible components $\text{Aut}(K)$ is therefore π -finite as well. Delooping shifts the positive homotopy groups by one, and $B \text{Aut}(K)$ is connected. $\square$

Proposition 5.3.8. *Let $\text{Loc}_K(X_{\text{ét}}; \Lambda)$ be the full subcategory of locally constant sheaves étale-locally isomorphic to $\underline{K}_X$. There are canonical equivalences of ∞ -groupoids*

$$\begin{aligned} \text{Loc}_K(X_{\text{ét}}; \Lambda) &\simeq \text{Maps}_{\text{Pro}(\mathbf{H}_\pi)}(\widehat{\Pi}_\infty^{\text{ét}}(X), B \text{Aut}(K)) \\ &\simeq \Gamma_{\text{ét}}(X, \underline{B \text{Aut}(K)}). \end{aligned}$$

Proof. Under Theorem 5.3.5, the sheaves in question correspond to representations with all fibres isomorphic to K . After passing to maximal ∞ -groupoids, these are the continuous functors landing in $B \text{Aut}(K) \subseteq \text{Perf}(\Lambda)^\simeq$. Since $B \text{Aut}(K)$ is π -finite, their type is the mapping type from $\widehat{\Pi}_\infty^{\text{ét}}(X)$ to $B \text{Aut}(K)$ in $\text{Pro}(\mathbf{H}_\pi)$. The second equivalence is the defining property of the profinite shape. $\square$

6. RANK ONE ÉTALE LOCAL SYSTEMS

Using the étale analogue of the monodromy equivalence (Theorem 5.3.5), we adapt the recipe of Sect. 3.

Construction 6.0.1. Let $f : Y \rightarrow X$ be an étale G -torsor in Sch_k , for G a finite group.

- (i) Recall that the type BG is the classifying groupoid for G -torsors, and it is π -finite. By Remark 5.2.4, the G -torsor f determines a map $c_f : \widehat{\Pi}_\infty^{\text{ét}}(X) \rightarrow BG$ in $\text{Pro}(\mathbf{H}_\pi)$. The category $\text{FinSet}^G := \text{Fun}(BG^{\text{op}}, \text{FinSet})$ of finite G -sets has finite Hom sets, so it is locally π -finite. By Theorem 5.3.5, we obtain a discrete local system $\mathcal{S}_f \in \text{Loc}(X_{\text{ét}}; \text{FinSet}^G)$, with stalks isomorphic to G , and monodromy representation

$$\rho_{\mathcal{S}_f} : \widehat{\Pi}_\infty^{\text{ét}}(X) \xrightarrow{c_f} BG \hookrightarrow \text{FinSet}^G.$$

Since the G -torsor f is tautologically trivialized by base change along f , the composite $\widehat{\Pi}_\infty^{\text{ét}}(Y) \rightarrow \widehat{\Pi}_\infty^{\text{ét}}(X) \xrightarrow{c_f} BG$ is homotopic to the constant map factoring through $*$ $\rightarrow BG$. Thus $f^* \mathcal{S}_f$ is the constant sheaf on Y with value G .

- (ii) Let Λ be a finite commutative ring. Given a character $\chi : G \rightarrow \Lambda^\times$, the composite

$$\rho_\chi : \widehat{\Pi}_\infty^{\text{ét}}(X) \xrightarrow{c_f} BG \xrightarrow{B\chi} B \text{Aut}(\Lambda) \subseteq \text{Perf}(\Lambda)$$

corresponds by Theorem 5.3.5 to a locally constant sheaf of finite type

$$\mathcal{L}_\chi \in \text{Loc}_{\text{ft}}(X_{\text{ét}}; \Lambda),$$

with stalks isomorphic to Λ and monodromy representation $\rho_{\mathcal{L}_\chi} = \rho_\chi$. Moreover, $f^* \mathcal{L}_\chi \simeq \underline{\Lambda}_Y$ is constant.

Remark 6.0.2. Concretely, $\mathcal{S}_f$ is the sheaf $U \mapsto \text{Hom}_X(U, Y)$ on $X_{\text{ét}}$ represented by Y , with the right G -action induced by the action on Y . Similarly, $\mathcal{L}_\chi$ is the sheaf of sections of $Y \times^G \Lambda$, the quotient of $Y \times \Lambda$ by $(y \cdot g, \lambda) \sim (y, \chi(g)\lambda)$. Its sections over $U \in X_{\text{ét}}$ are the locally constant functions $\phi : Y \times_X U \rightarrow \Lambda$ with $\phi(y \cdot g) = \chi(g)^{-1} \cdot \phi(y)$ for all $g \in G$.

Notation 6.0.3 ([SP, Tag 0BNC]). Let X be connected and x a geometric point. The *étale fundamental group* $\pi_1^{\text{ét}}(X, x)$ is the automorphism group of the fibre functor $Y \mapsto Y_x$ on finite étale X -schemes. It is a profinite group [SP, Tag 0BMR]. By Theorem 5.3.5 with $\mathcal{V} = \text{FinSet}$ and [SP, Tag 03RV], $Y \mapsto Y_x$ identifies finite étale X -schemes with finite continuous $\pi_1(\widehat{\Pi}_\infty^{\text{ét}}(X), x)$ -sets, compatibly with the fibre functors. Hence the monodromy action on stalks gives an isomorphism $\pi_1(\widehat{\Pi}_\infty^{\text{ét}}(X), x) \simeq \pi_1^{\text{ét}}(X, x)$ (cf. [BGH, 11.1.7]), which we use to identify the two.

Let $f : Y \rightarrow X$ be an étale G -torsor and $y \in Y_x$. Since Y_x is a G -torsor, each $\sigma \in \pi_1^{\text{ét}}(X, x)$ determines $c_f(\sigma) \in G$ by

$$\sigma(y) = y \cdot c_f(\sigma).$$

Since σ commutes with the action of G by X -automorphisms, $\sigma\tau(y) = \sigma(y \cdot c_f(\tau)) = y \cdot c_f(\sigma)\tau$, so $c_f : \pi_1^{\text{ét}}(X, x) \rightarrow G$ is a homomorphism. It is continuous, since its kernel is the stabilizer of y , which is open. Replacing y by $y \cdot g$ replaces c_f by its conjugate $g^{-1}c_f g$. Since $\mathcal{S}_f$ is represented by Y (Remark 6.0.2), under the identification $G \simeq Y_x$, $h \mapsto y \cdot h$, σ acts on the stalk of $\mathcal{S}_f$ by left multiplication by $c_f(\sigma)$. Thus this is the homomorphism induced by $c_f : \widehat{\Pi}_\infty^{\text{ét}}(X) \rightarrow BG$ on fundamental groups.

Example 6.0.4 ([SP, Tag 03PK]). Let $n \geq 1$ be invertible in k , and suppose that the group $\mu_n := \mu_n(k)$ of n th roots of unity in k has n elements. The map

$$f : \mathbf{G}_m \rightarrow \mathbf{G}_m, \quad y \mapsto y^n,$$

is finite étale of degree n . On coordinate rings it is $k[t, t^{-1}] \rightarrow k[y, y^{-1}]$, $t \mapsto y^n$. The ring $k[y, y^{-1}]$ is free of rank n over $k[t, t^{-1}]$, with basis $1, y, \dots, y^{n-1}$, and the derivative ny^{n-1} is invertible. The group μ_n acts by $y \cdot \zeta := \zeta y$ over the target, and f is a μ_n -torsor: two n th roots of t differ by a unique element of μ_n . This is the étale analogue of Example 3.0.3. If instead $\text{char}(k) = p$ divides n , then f is not étale: its derivative vanishes.

Proposition 6.0.5. *Let k be separably closed, n invertible in k , $f : \mathbf{G}_m \rightarrow \mathbf{G}_m$ the μ_n -torsor of Example 6.0.4, and $\bar{1}$ the geometric point $1 \in \mathbf{G}_m(k)$.*

- (i) *The local system $\mathcal{S}_f \in \text{Loc}(\mathbf{G}_{m, \text{ét}}; \text{FinSet}^{\mu_n})$ has stalks isomorphic to μ_n , and $c_f : \pi_1^{\text{ét}}(\mathbf{G}_m, \bar{1}) \rightarrow \mu_n$ is surjective. Moreover, $f^* \mathcal{S}_f$ is the constant sheaf $\underline{\mu_n}_{\mathbf{G}_m}$.*
- (ii) *Let Λ be a finite commutative ring and $\chi : \mu_n \rightarrow \Lambda^\times$ a character. Then $\mathcal{L}_\chi \in \text{Loc}_{\text{ft}}(\mathbf{G}_{m, \text{ét}}; \Lambda)$ has monodromy representation $\chi \circ c_f$, and $f^* \mathcal{L}_\chi$ is the constant sheaf $\underline{\Lambda}_{\mathbf{G}_m}$.*
- (iii) *The assignment $\chi \mapsto \mathcal{L}_\chi$ determines a bijection between characters of μ_n and isomorphism classes of local systems $\mathcal{L} \in \text{Loc}_\Lambda(\mathbf{G}_{m, \text{ét}}; \Lambda)$ such that $f^* \mathcal{L}$ is constant.*

Proof.

- (i) The fibre of f over $\bar{1}$ is μ_n . Since the source $\mathbf{G}_m$ of f is connected, $\pi_1^{\text{ét}}(\mathbf{G}_m, \bar{1})$ acts transitively on μ_n ([SGA1, Exp. V, §7], [SP, Tag 0BN3]). Since $\sigma(1) = 1 \cdot c_f(\sigma) = c_f(\sigma)$, this is the surjectivity of c_f .
- (ii) This follows from the previous claim by unwinding definitions.
- (iii) By Proposition 5.3.8, isomorphism classes of $\mathcal{L}$ are in bijection with $\pi_0 \text{Maps}_{\text{Pro}(\mathbf{H}_\pi)}(\widehat{\Pi}_\infty^{\text{ét}}(\mathbf{G}_m), B\Lambda^\times)$. Since $\mathbf{G}_m$ is connected and $\Lambda^\times$ is abelian, this is the set of continuous homomorphisms $\rho_{\mathcal{L}} : \pi_1^{\text{ét}}(\mathbf{G}_m, \bar{1}) \rightarrow \Lambda^\times$. The homomorphism $\pi_1^{\text{ét}}(f)$ identifies $\pi_1^{\text{ét}}(\mathbf{G}_m, \bar{1})$ of the source with the stabilizer of $1 \in f^{-1}(\bar{1})$ [SGA1, Exp. V, Prop. 6.13 and §7], which is $\ker(c_f)$ since μ_n acts freely on $f^{-1}(\bar{1})$. By naturality of Theorem 5.3.5, $\rho_{f^* \mathcal{L}} = \rho_{\mathcal{L}} \circ \pi_1^{\text{ét}}(f)$. Thus $f^* \mathcal{L}$ is constant if and only if $\rho_{\mathcal{L}}$ kills $\ker(c_f)$. Since c_f is surjective by (i), this holds if and only if $\rho_{\mathcal{L}} = \chi \circ c_f$ for a unique character χ , i.e., $\mathcal{L} \simeq \mathcal{L}_\chi$ by (ii).

□

Example 6.0.6 ([SP, Tag 0A3J]). Suppose $\text{char}(k) = p > 0$. The *Artin–Schreier map*

$$q : \mathbf{A}_k^1 \rightarrow \mathbf{A}_k^1, \quad y \mapsto y^p - y,$$

is finite étale of degree p . Indeed, its coordinate algebra over $k[t]$ is $k[t, y]/(y^p - y - t)$, which is free with basis $1, y, \dots, y^{p-1}$. The derivative of the defining polynomial with respect to y is -1 , so the map is étale everywhere, including over 0. The constant group $\mathbf{F}_p$ acts by $y \cdot a := y + a$, and q is an $\mathbf{F}_p$ -torsor, since two roots of $y^p - y - t$ differ by an element of $\mathbf{F}_p$.

For a geometric point x , the source of q is connected, so $c_q : \pi_1^{\text{ét}}(\mathbf{A}_k^1, x) \rightarrow \mathbf{F}_p$ is surjective, as in Proposition 6.0.5(i). Hence, for a finite commutative ring Λ , $\mathcal{L}_\chi$ is nonconstant for every nontrivial character $\chi : \mathbf{F}_p \rightarrow \Lambda^\times$. This is in contrast with Example 3.0.5, even though $y \mapsto y^n$ on $\mathbf{A}_k^1$ is still ramified at 0 for $n > 1$ invertible in k .

Construction 6.0.7. Let Λ be a finite commutative ring and $\psi : \mathbf{F}_p \rightarrow \Lambda^\times$ a character. The *Artin–Schreier sheaf* is the local system $\mathcal{L}_\psi \in \text{Loc}_{\text{ft}}(\mathbf{A}_{k, \text{ét}}^1; \Lambda)$ of Construction 6.0.1 for the torsor q and the character $\chi = \psi$. Its monodromy representation is $\psi \circ c_q$. A nontrivial character exists if and only if $\Lambda^\times$ has an element of order p .

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