Lecture 0: Overview

Reminder: A comm. ring

abelian groups $\quad K_0(A) := K_0(\mathrm{Proj}_A) \simeq K_0(\mathrm{Perf}_A)$

$K_0(X) := K_0(\mathrm{Vect}(X))$ (X scheme)

spaces (E_∞-spaces): $\quad K(A) =$ group completion of $\mathrm{Proj}_A^{\simeq}$

$K^{\mathrm{naive}}(X) := K(\mathrm{Vect}(X))$ Quillen K-theory of vector bundles

$K(X) := K(\mathrm{Perf}(X))$ Waldhausen K-theory of perfect complexes

X has resolution property $\quad K(X) \simeq K^{\mathrm{naive}}(X)$

$K(\mathrm{Spec}(A)) \simeq K(A)$

Theorem (Thomason):

Algebraic K-theory satisfies descent.
That is: $X \mapsto K(X)$ is a "sheaf" in some sense.

Warning: $X \mapsto K_0(X)$ presheaf of sets/ab. groups is not a sheaf.

<u>Reminder</u> (sheaf of sets)

X top. space $\qquad \mathcal{U}(X) :=$ poset of open subsets $U \subseteq X$ (ordered by inclusion)

a <u>presheaf</u> on X (view as a category)

is a presheaf on $\mathcal{U}(X)$, i.e. a functor $\mathcal{U}(X)^{op} \to Set$

$F : \mathcal{U}(X)^{op} \to Set$ is a <u>sheaf</u> if:
for all $X = \bigcup_{i \in I} U_i$ open covers the diagram

$$F(X) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j)$$

is a limit diagram.

<u>Ex</u>: X scheme, $E \xrightarrow{a} X$ vector bundle

$\Gamma(-, E)$ <u>sheaf</u> of sections of E

$$\underset{U \subseteq X}{\Gamma(U, E)} = \left\{ s : U \to E \;\middle|\; \begin{tikzcd} & E \arrow[d] \\ U \arrow[ur, "s"] \arrow[r, hook] & X \end{tikzcd} \right\}$$

<u>Ex</u>: $X \mapsto Vect(X)/\simeq \; = \{$iso. classes of vector bundles$\}$
is <u>not</u> a sheaf.

<u>Problem</u>: vector bundles have nontrivial automorphisms

<u>Solution</u>:

<u>Solution</u>: $X \mapsto \mathrm{Vect}(X)^{\simeq}$ = <u>groupoid</u> of v.b.'s

is a <u>sheaf</u> in a 2-categorical sense:

$$\mathrm{Vect}(X)^{\simeq} \longrightarrow \prod_i \mathrm{Vect}(U_i)^{\simeq} \rightrightarrows \prod_{i,j\in I} \mathrm{Vect}(U_i \cap U_j)^{\simeq} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cancel{\prod_{i,j,k} \mathrm{Vect}(U_i\cap U_j\cap U_k)}$$

$$\substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} \mathrm{Vect}(U_i \cap U_j \cap U_k)^{\simeq}$$

is a <u>2-limit</u> diagram in the 2-category of groupoids.

<u>Ex</u>: $X \mapsto \mathrm{Perf}(X)^{\simeq}$

<u>Problem</u>: now have derived/higher automorphisms.

<u>Solution</u>: consider the <u>∞-groupoid</u> of perfect complexes

groupoids $\rightarrow$ 2-groupoids $\rightarrow \cdots \rightarrow$ ∞-groupoid $\simeq$ Spaces (Kan complexes, CW complexes)

<u>Theorem</u>: $X \mapsto \mathrm{Perf}(X)^{\simeq}$ is a sheaf of ∞-groupoids.

$$\mathrm{Perf}(X)^{\simeq} \longrightarrow \underbrace{\prod_i \mathrm{Perf}(U_i)^{\simeq} \rightrightarrows \prod_{i,j} \mathrm{Perf}(U_i\cap U_j)^{\simeq} \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k} \mathrm{Perf}(U_i\cap U_j\cap U_k)^{\simeq} \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots}_{\text{cosimplicial diagram indexed on } \Delta}$$

Theorem (Thomason): $X \mapsto K(X)$ is a sheaf of ∞-groupoids.

(over quasi-compact quasi-separated schemes)

Corollary (Mayer-Vietoris): $X = U \cup V$ Zar. cover

$$\cdots \xrightarrow{\partial} K_n(X) \to K_n(U) \oplus K_n(V) \to K_n(U \cap V) \xrightarrow{\partial} K_{n-1}(X) \to \cdots$$