Lecture 3: Nonconnective animated modules

3.1 Suspensions and loop spaces

Def: $\mathcal{C}$ ∞-category with terminal object $pt \in \mathcal{C}$
($\mathrm{Maps}_{\mathcal{C}}(X, pt)$ is a contractible anima $\forall X \in \mathcal{C}$)

$f : X \to Y$ morphism in $\mathcal{C}$

$\mathrm{Cofib}(f)$ = cofibre of f is the ~~colimit of~~ pushout in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ pt \arrow[r] & \mathrm{Cofib}(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

(cocartesian square)

$\mathrm{Fib}_y(f)$ = fibre of f at any "point" $y : pt \to Y$

is the pullback in the square:

$$\begin{tikzcd} \mathrm{Fib}_y(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ pt \arrow[r, "y"] & Y \end{tikzcd}$$

cartesian square

Example: C ordinary category (viewed as an ∞-category via N)
Then:
$$\mathrm{Cofib}(f) \simeq \mathrm{Coker}(f) \qquad \mathrm{Fib}_y(f) = \mathrm{Ker}(f)$$

Example: $\mathcal{C}$ ∞-category, $X \in \mathcal{C}$ object.

suspension of X: $\Sigma(X) \in \mathcal{C}$ $\quad \Sigma(X) = \mathrm{Cofib}(X \to pt)$

the loop space $\Omega_x(X) := \mathrm{Fib}_x(pt \xrightarrow{x} X)$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & \Sigma(X) \end{tikzcd} \qquad \begin{tikzcd} \Omega_x(X) \arrow[r] \arrow[d] & pt \arrow[d, "x"] \\ pt \arrow[r, "x"'] & X \end{tikzcd}$$

cocartesian $\qquad\qquad$ cartesian

Rmk: $\mathcal{C}$ ordinary category $\rightsquigarrow$ $\Sigma(X) = pt \quad \Omega_x(X) = pt$

$\forall X, \quad \forall x : pt \to X$

Ex: In the ∞-category Kan $\simeq$ Anim

$\Sigma(\emptyset) = S^0 \qquad S^{n+1} = \Sigma(S^n) \quad \forall n \geq 0$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & pt \sqcup pt \end{tikzcd}$$

Rmk: points of a loop space $\Omega_x(X)$:

$$\begin{tikzcd} pt \arrow[dr, dashed] & & \\ & \Omega_x(X) \arrow[r] \arrow[d] & pt \arrow[d, "x"] \\ & pt \arrow[r, "x"'] & X \end{tikzcd} \iff \begin{tikzcd} pt \arrow[r, "\exists!"] \arrow[d, "\exists!"'] \arrow[dr, "y" description] \arrow[dr, phantom, shift left=3, "\sigma_1"] \arrow[dr, phantom, shift right=3, "\sigma_2"] & pt \arrow[d, "x"] \\ pt \arrow[r, "x"'] & X \end{tikzcd}$$

commutative square

Warning: have to specify 2-simplices σ_1, σ_2 up to which the triangles commute.

$\Delta^1 \times \Delta^1 \to \mathcal{C}$

$\iff (pt \xrightarrow{y} X) \simeq (pt \xrightarrow{x} X) \quad (\sigma_1)$

$(pt \xrightarrow{y} X) \simeq (pt \xrightarrow{x} X) \quad (\sigma_2)$

$\iff x \simeq y \simeq x$ (paths in X)

$\Leftrightarrow$ path ~~loop~~ in X between the two points x and x

$\Leftrightarrow$ loop in X based at $x \in X$

3.2 Infinite loop spaces

<u>Def</u>: <u>pointed anima</u>: pair (X, x_0) $\quad X \in \mathrm{Anim}$ $\quad x_0 : pt \to X$
(~ pointed Kan complex)

$\mathrm{Anim}_* = \{(X, x_0)\}$ = ∞-category of pointed anima

$$\begin{tikzcd} & pt \arrow[dl, "x_0"'] \arrow[d, "y_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(X, x_0) is <u>n-connective</u> if:
$\pi_i(X, x_0) = 0 \quad \forall i < n$

<u>Ex</u>: every $X \in \mathrm{Anim}_*$ is 0-connective.
$X \in \mathrm{Anim}_*$ is 1-connective $\Leftrightarrow$ X connected
$\Leftrightarrow$ $\pi_0(X) = 0$

<u>Theorem</u> (1-fold loop spaces)

- <u>Def</u>: a <u>1-fold loop space</u> is a pair (X_0, X_1) of pointed anima together with an isomorphism

  $$\underbrace{X_0}_{\text{underlying anima}} \simeq \Omega(\underbrace{X_1}_{\text{delooping of } X_0})$$

  (X_0, X_1) is <u>connective</u> if X_i is i-connective $\forall i$
  ($\Leftrightarrow$ X_1 is 1-connective)

- <u>Claim</u>: 1) For every $X \in \mathrm{Anim}_*$, $\Omega(X)$ admits an $\mathbb{E}_1$-group structure.

In particular, the functor

$$\{\text{1-fold loop spaces}\} \longrightarrow \mathrm{Anim}_*$$
$$(X_0, X_1) \longmapsto X_0$$

factors through $\{\mathcal{E}_1\text{-groups}\} \xrightarrow{\text{forget}} \mathrm{Anim}_*$.

2) Restricted to the full subcategory of connective loop spaces, this gives an equivalence of ∞-categories

$$\{\text{connective 1-fold loop spaces}\} \xrightarrow{\sim} \{\mathcal{E}_1\text{-groups}\}.$$

$\Rightarrow$ Any $\mathcal{E}_1$-group structure on $X \in \mathrm{Anim}_*$ gives rise to a unique delooping BX

BX - 1-connective pointed anima, $\Omega B(X) \simeq X$.

Rmk:
$$\begin{array}{ccc} \{\text{1-fold loop spaces}\} & \xrightarrow{\sim} & \{\text{pointed anima}\} \\ (X_0, X_1) & \longmapsto & X_1 \\ \cup & & \cup \\ \{\text{connective 1-fold loop spaces}\} & \xrightarrow{\sim} & \{\text{pointed connected anima}\} \\ (\Omega(X), X) & \longleftarrow & X \end{array}$$

Definition: A spectrum is a sequence of pointed anima $X = (X_0, X_1, X_2, \dots)$ together with ~~equivalences~~ isomorphisms $X_n \simeq \Omega(X_{n+1})$ $\forall n \geq 0$.

- X is an infinite delooping of X_0

infinite loop space

A spectrum X is <u>connective</u> if X_n is an <u>n-connective</u> pointed anima. $(\forall n \geq 0)$

Spt = ∞-category of spectra
$\mathrm{Spt}_{\geq 0}$ = full subcategory of connective spectra

$$\mathrm{Spt} \to \cdots \xrightarrow{\Omega} \mathrm{Anim}_* \xrightarrow{\Omega} \mathrm{Anim}_* \xrightarrow{\Omega} \mathrm{Anim}_*$$

limit diagram of ∞-categories

$$\mathrm{Spt}_{\geq 0} \to \cdots \xrightarrow{\Omega} (\mathrm{Anim}_*)_{\geq 2} \xrightarrow{\Omega} (\mathrm{Anim}_*)_{\geq 1} \xrightarrow{\Omega} (\mathrm{Anim}_*)_{\geq 0}$$

full subcategory of 2-connective pointed anima

<u>Rmk</u>: Projections

$$\mathrm{Spt} \xrightarrow{\Omega^{\infty-n}} \mathrm{Anim}_*$$
$$(X_0, X_1, \ldots) \longmapsto X_n$$

<u>Theorem</u> (infinite loop space machine)

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1) $X \in \mathrm{Spt}$ $\quad \Omega^\infty(X) \in \mathrm{Anim}_*$ admits an $\mathcal{E}_\infty$-group str.

$\Omega^\infty : \mathrm{Spt} \to \{\mathcal{E}_\infty\text{-groups}\}$

2) When restricted to $\mathrm{Spt}_{\geq 0}$:

$$\mathrm{Spt}_{\geq 0} \hookrightarrow \mathrm{Spt} \xrightarrow{\Omega^\infty} \{\mathcal{E}_\infty\text{-groups}\}$$ is an equivalence.

3.3 Stable ∞-categories

Def: $\mathcal{C}$ ∞-category is <u>stable</u> if
- admits finite limits and a <u>zero object</u>, (object which is $\Leftrightarrow$ terminal + initial)
- $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence
  ($\Sigma\Omega \simeq id$, $\Omega\Sigma \simeq id$)
  (behave like shift functors in a triangulated category)

Theorem: Spt is a stable ∞-category.

Sketch:

$$\begin{tikzcd}
\mathrm{Spt} \arrow[r, dashed] \arrow[d, "\Omega"'] & \cdots \arrow[r, "\Omega"] & \mathrm{Anim}_* \arrow[r, "\Omega"] \arrow[d, "\Omega"] & \mathrm{Anim}_* \arrow[r, "\Omega"] \arrow[d, "\Omega"] & \mathrm{Anim}_* \arrow[d, "\Omega"] \\
\mathrm{Spt} \arrow[r, dashed] & \cdots \arrow[r, "\Omega"] & \mathrm{Anim}_* \arrow[r, "\Omega"] \arrow[ur, dashed, "id"'] & \mathrm{Anim}_* \arrow[r] \arrow[ur, dashed, "id"'] & \mathrm{Anim}_*
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Spt} \arrow[r, "\Theta"] \arrow[dr, dashed, "\Theta_n"'] & \mathrm{Spt} \arrow[d, "\Omega^{\infty-n}"] \\
& \mathrm{Anim}_*
\end{tikzcd}$$

$\Omega \circ \Theta_n \simeq \Theta_{n+1}$

$\Theta_n := \Omega^{\infty-n-1}$ $\qquad$ $\Omega \circ \Omega^{\infty-n-1} \simeq \Omega^{\infty-n}$

$\Theta \circ \Omega \simeq id$: $\quad \underbrace{\Omega^{\infty-n}\,\Theta\circ\Omega} \simeq \Omega^{\infty-n-1}\circ\Omega \simeq \Omega^{\infty-n}$

$\Omega\circ\Theta \simeq id$ $\quad \Omega^{\infty-n}\,\Omega\circ\Theta \simeq \Omega^{\infty-n+1}\circ\Theta \simeq \Omega^{\infty-n}$

($\Rightarrow \Theta \simeq \Sigma$)

Prop: $\mathcal{C}$ ∞-category. TFAE:
- $\mathcal{C}$ is stable
- $\mathcal{C}$ admits finite colimits + zero obj. and $\Sigma : \mathcal{C} \to \mathcal{C}$ equivalence.
- $\mathcal{C}$ admits finite colimits + finite limits + zero obj. and any commutative square is cartesian <u>iff</u> it is cocartesian.

Def: $\mathcal{C}$ stable. an exact triangle in $\mathcal{C}$ is a comm. square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

which is co/cartesian.

zero obj. $\longrightarrow 0$

Notation: $X \xrightarrow{f} Y \xrightarrow{g} Z$

Warning: Have to specify $g \circ f \sim 0$ (null-homotopy) as part of the data.

Rmk: The homotopy category $h(\mathcal{C})$ is triangulated:

- $h(\mathcal{C})$ additive

$\pi_0 \, Maps_{\mathcal{C}}(X, Y)$ are ab. groups

$\pi_0 \, \underbrace{Maps_{\mathcal{C}}(\Sigma(X), Y)}_{\simeq \, \Omega \, Maps_{\mathcal{C}}(X,Y)} = \pi_1 (Maps_{\mathcal{C}}(X, Y))$ is a group

$\pi_0 \, (Maps_{\mathcal{C}}(\Sigma^2(X), Y)) \simeq \pi_2 (Maps_{\mathcal{C}}(X,Y))$ is an abelian group

$\forall \; X \simeq \Sigma^2(\Omega^2(X))$ ($\mathcal{C}$ stable)

- $[n] := \Sigma^n$ if $n > 0$ shift/translation
  $:= \Omega^n$ if $n < 0$

- exact triangles comma from image of $\mathcal{C} \to h(\mathcal{C})$

3.4 Animations of additive categories

Rmk: $\mathcal{C}$ algebraic category

$\mathcal{C}$ additive $\Rightarrow$ $\mathcal{C} \simeq \mathrm{Fun}_{\Pi}(F_{\mathcal{C}}^{op}, \mathrm{Ab})$

Idea: $X \in \mathcal{C} \Leftrightarrow X : F_{\mathcal{C}}^{op} \to \mathrm{Set}$ automatically takes values in Ab

If $X \in \mathcal{C}$ representable ($\mathrm{Hom}(-, X)$)
In general, X is built out of filt. colims and reflexive coeq's of representable objs.
$\mathrm{Ab} \to \mathrm{Set}$ preserves such colimits

Prop: $\mathcal{C}$ additive algebraic category

Then: $\mathrm{Anim}(\mathcal{C}) \simeq \mathrm{Fun}_{\Pi}(F_{\mathcal{C}}^{op}, \mathrm{Spt}_{\geq 0})$

In particular: $\mathrm{Anim}(\mathcal{C}) \hookrightarrow \mathrm{Fun}_{\Pi}(F_{\mathcal{C}}^{op}, \mathrm{Spt}) =: \mathrm{Anim}^{nc}(\mathcal{C})$
fully faithful embedding with stable target and w/ ess. image closed under (finite) colimits and extensions.

Sketch: • Every $X \in \mathrm{Anim}(\mathcal{C}) \Leftrightarrow X : F_{\mathcal{C}}^{op} \to \mathrm{Anim}$ takes values in $\mathbb{E}_\infty$-groups

• $\{\mathbb{E}_\infty\text{-groups}\} \simeq \mathrm{Spt}_{\geq 0} \hookrightarrow \mathrm{Spt}$. ∎

Rmk: $\mathrm{Anim}^{nc}(\mathcal{C}) \to \cdots \to \mathrm{Anim}(\mathcal{C}) \xrightarrow{\Omega} \mathrm{Anim}(\mathcal{C})$ limit

Def: A nonconnective animated A-module is an object of ~~Anim that~~ $\mathrm{Anim}^{nc}(\mathrm{Mod}_A)$.

$D(A) := \mathrm{Anim}^{nc}(\mathrm{Mod}_A)$ stable ∞-category

$\cup$

$D(A)_{\geq 0} = \mathrm{Anim}(\mathrm{Mod}_A)$ prestable ∞-category

$\cup$

$D(A)^{\heartsuit}_{\geq 0} = \mathrm{Mod}_A$ abelian category

<u>References:</u>

- [HA] Lurie, Higher Algebra, Chap. 1
- [SAG] Lurie, Spectral Alg. Geom, App C.1.5.7.