# Lecture 4: Sheaves

Announcement: Next semester: course by Denis Nardin on stable homotopy theory (Spectra).

## Addendum to § 2.4 (Derived functors)

Recall $\phi: A \to B$ ring homo

$\phi^*: \mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_B : \phi_*$ ← restriction of scalars

extension of scalars → $\phi^*$

$(- \otimes_A B)$

Note: $\phi_*$ admits a right adjoint, co-extension of scalars
$( M \in \mathrm{Mod}_A \mapsto \mathrm{Hom}_A(B, M) \in \mathrm{Mod}_B)$

$\Rightarrow \phi_*$ also preserves colimits.

$\Rightarrow L\phi_* : D(B)_{\geq 0} \to D(A)_{\geq 0}$ left-derived functor
commutes with colimits
extends $F_B \hookrightarrow \mathrm{Mod}_B \to \mathrm{Mod}_A \to D(A)$

Claim: 1) $\mathbb{L}\phi_*$ preserves limits and colimits
2) $\mathbb{L}\phi_*$ preserves underlying anima
3) $\mathbb{L}\phi_*$ conservative (detects isos.)

Rmk. 2) $\Rightarrow$ preserves + detects discreteness

Proof of 2. $N \in D(B)_{\geq 0}$

$$\mathbb{L}\phi_*(N)^\circ = \mathrm{Maps}_{D(A)_{\geq 0}}(A, \mathbb{L}\phi_*(N)) \simeq \mathrm{Maps}_{D(B)_{\geq 0}}(\mathbb{L}\phi^*(A), N) \simeq \mathrm{Maps}_{D(B)_{\geq 0}}(B, N) \simeq N^\circ$$

$\mathbb{L}\phi^*(A) \simeq \phi^*(A) \simeq B$
$A \in F_A$

**Proof of 3:** Follows from 2) and the fact that

$$\begin{aligned} D(A)_{\geq 0} &\longrightarrow \mathrm{Anim} \\ M &\longmapsto M^0 \end{aligned} \quad \text{is conservative}$$

because $F_A$ is generated by $A$ under finite coproducts.

**Notation:**

$$\begin{tikzcd} \phi_* = \mathbb{L}\phi_* : D(B)_{\geq 0} \arrow[r] & D(A)_{\geq 0} \\ \mathrm{Mod}_B \arrow[u, phantom, "\cup"] \arrow[r, "\phi_*"] & \mathrm{Mod}_A \arrow[u, phantom, "\cup"] \end{tikzcd}$$

[ Conservativity of $M \mapsto M^0$. $\alpha : M \longrightarrow N$ in $D(A)_{\geq 0}$

Assume that $M^0 \xrightarrow{\sim} N^0$ iso in Anim

$M^0 = M(A)$ $\qquad M : F_A^{op} \to \mathrm{Anim}$

$\simeq \mathrm{Maps}(A, M)$

want $M(A^{\oplus n}) \xrightarrow{\alpha} N(A^{\oplus n})$ iso $\forall n$

$\phantom{want} \cong M(A)^{\times n} \xrightarrow{\sim} \cong N(A)^{\times n}$

**Constr.** $- \overset{\mathbb{L}}{\otimes} -$ $\quad$ $\mathbb{R}\mathrm{Hom}$ right-adjoint bifunctor to

$\overset{\mathbb{L}}{\otimes} : D(A)_{\geq 0} \times D(A)_{\geq 0} \to D(A)_{\geq 0}$

{ left-derived functor of

$- \otimes - : \mathrm{Mod}_A \times \mathrm{Mod}_A \to \mathrm{Mod}_A$

**Rmk.** $\phi_* \phi^* \simeq (-) \otimes_A B$ as endofunctors of $\mathrm{Mod}_A$

$$\Rightarrow \mathbb{L}\phi_* \mathbb{L}\phi^* \simeq \mathbb{L}(\phi_* \phi^*) \simeq (-) \overset{\mathbb{L}}{\otimes}_A B$$

**Lemma:** $\phi : A \to B$ flat ring homo
Then $L\phi^* : D(A)_{\geq 0} \longrightarrow D(B)_{\geq 0}$ preserves discrete objects

(Write $\phi = L\phi^*$ in this case.
$$\begin{array}{ccc} D(A)_{\geq 0} & \xrightarrow{\phi^*} & D(B)_{\geq 0} \\ \cup\! \mid & & \cup\! \mid \\ \mathrm{Mod}_A & \xrightarrow{\phi^*} & \mathrm{Mod}_B \end{array}$$
.)

**Proof:** Recall $\phi_*$ detects discreteness.
So suff to show $\phi_* L\phi^* \; D(A)_{\geq 0} \longrightarrow D(A)_{\geq 0}$ preserves discreteness.
$$\phi_* L\phi^* \cong (-) \overset{L}{\underset{A}{\otimes}} B$$

B flat A-module $\Longrightarrow$ B is a filtered colimit of f.g. free A-modules (Lazard)

Rmk: filtered colimits are preserved by the embedding $\mathrm{Mod}_A \hookrightarrow D(A)_{\geq 0}$.

$$\Rightarrow \underbrace{M}_{\text{discrete}} \overset{L}{\underset{A}{\otimes}} B \simeq M \overset{L}{\underset{A}{\otimes}} (\varinjlim_\alpha \underbrace{N_\alpha}_{\text{fg free}}) \simeq \varinjlim_\alpha (M \otimes N_\alpha) \text{ discrete.} \quad \blacksquare$$

## Addendum to §3.4 (Animations of additive categories)

Def: $\mathcal{C}$ $\infty$-category with finite colimits and zero object
$\mathcal{C}$ is prestable if the following equivalent conditions hold

- $\mathcal{C}$ admits a fully faithful embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ where $\mathcal{D}$ stable, such that the essential image is closed under finite colimits and extensions.
- $\Sigma : \mathcal{C} \to \mathcal{C}$ fully faithful $(\Leftrightarrow \Omega\Sigma \simeq \mathrm{id})$

Ex. $\mathrm{Anim}(\mathcal{C})$ is prestable if $\mathcal{C}$ additive.

Ex. $D(A)_{\geq 0}$ is prestable $\forall A \in \mathrm{CRing}$.

Rmk. Can use universal property of the construction

$$\mathrm{Anim}(\mathcal{C}) \rightsquigarrow \mathrm{Anim}^{nc}(\mathcal{C}) = \varprojlim \left( \cdots \to \mathrm{Anim}(\mathcal{C}) \xrightarrow{\Omega} \mathrm{Anim}(\mathcal{C}) \right)$$

to extend derived functors to nonconnective objects.

## 4.1 Reminder on sheaves of sets

Reminder: $X$ top space, $\mathcal{B}$ basis
Assume $\mathcal{B}$ is intersection-closed: $\forall U \subseteq X$ open, $\exists\, U = \bigcup U_i$, $U_i \in \mathcal{B}$ such that $U_{i_0} \cap \cdots \cap U_{i_n} \in \mathcal{B}$ for all finite subsets $\{i_0, \ldots, i_n\} \in I$.

For any presheaf $\mathcal{F}$ on $X$ the following are equiv

1) $\mathcal{F}$ is a sheaf
2) $\forall\, U = \bigcup U_i$ covering with $U_i \in \mathcal{B}$ we have
$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$ limit diagram

**Example:** If $X$ coherent top. space then $\mathcal{U}_c(X)$ is an intersection-closed basis, so we get.

(the collection $\mathcal{U}_c(X)$ of compact opens of $X$ is ~~closed~~ under intersections and forms a basis of $X$)

$\mathcal{F}$ is a sheaf on $X$

$\iff$ $U = \bigcup_i U_i$ finite covering $U_i \in \mathcal{B}$

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

$\iff$ $U = U_1 \cup U_2$ $U_i \in \mathcal{B}$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}(U_1) \arrow[d] \\ \mathcal{F}(U_2) \arrow[r] & \mathcal{F}(U_1 \cap U_2) \end{tikzcd}$$ Cartesian square

## 4.2. Sheaves with values in $\infty$-categories

**Def:** $X$ top space $\bigcup_i U_i = X$

**Čech nerve** of the family $(U_i)$, the simplicial diagram

$$\cdots \underbrace{\coprod_{i,j,k} U_{i,j,k}}_{= U_i \cap U_j \cap U_k} \substack{\rightarrow\\\rightarrow\\\rightarrow} \underbrace{\coprod_{i,j} U_{i,j}}_{= U_i \cap U_j} \rightrightarrows \coprod_i U_i$$

denoted $\check{C}(U_i/X)$.

**Rmk:** View as a diagram in presheaves $\mathcal{U}(X)^{op} \to Set$.

$\mathcal{F}$ presheaf $\mathcal{U}(X)^{op} \to \mathcal{V}$ where $\mathcal{V}$ = $\infty$-category with limits

$\mathcal{F}$ satisfies <u>descent</u> ($\mathcal{F}$ is a <u>sheaf</u>) if $\forall\, U \subseteq X$ open, $U = \bigcup_i U_i$:

$$\mathcal{F}(U) \xrightarrow{\sim} \mathrm{Tot}(\underbrace{\mathcal{F}(\check{C}(U_i/U)_\bullet)}_{\text{cosimplicial diagram}}) \quad = \quad \text{totalization of the cosimp diagram}$$

is an iso. in $\mathcal{V}$

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \substack{\to\\\to\\\to} \prod_{i,j,k} \mathcal{F}(U_{i,j,k}) \substack{\to\\\to\\\to\\\to}$$

is a limit diagram

<u>Ex</u>: If $\mathcal{V}$ is an ordinary category this limit is the same as the equalizer

$$\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij}).$$

<u>Theorem</u>: $X$ coherent top space
$\mathcal{V}$ $\infty$-cat with limits
$\mathcal{F}: \mathcal{U}(X)^{op} \to \mathcal{V}$ presheaf

TFAE:

1) $\mathcal{F}$ is a sheaf
2) $\forall\, U, V \subseteq X$ compact open subsets

$\mathcal{F}(\emptyset) = pt$

$$\begin{array}{ccc} \mathcal{F}(U \cup V) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cap V) \end{array}$$

is cartesian in $\mathcal{V}$

**Theorem.** $X$ top. space, $\mathcal{B} \subseteq \mathcal{U}(X)$ basis $\cap$-closed s.th. every $U \in \mathcal{B}$ is compact

$\mathcal{F}: \mathcal{U}(X)^{op} \to \mathcal{V}$

TFAE:

1) $\mathcal{F}$ satisfies descent (is a sheaf).

2) $\forall\ U_1, \dots, U_n \subset \mathcal{B}$ s.th. $U = \bigcup U_i \in \mathcal{B}$

$$\mathcal{F}(U) \xrightarrow{\sim} \varprojlim_{S \neq \emptyset} \mathcal{F}(U_S)$$

limit over nonempty subsets $S \subseteq \{1, \dots, n\}$

where $U_S = \bigcap_{i \in S} U_i$

**Example:** (animated modules)

$X$ affine scheme $= \mathrm{Spec}(A)$

$\exists !$ Zariski sheaf of $\infty$-categories on $X_{Zar}$ (small Zar site)

$$D: (X_{Zar})^{op} \to \infty\text{-Cat}$$

whose values on elementary opens $U(f)$ are

$$D(U(f)) \simeq D(A[f^{-1}]).$$

complement of $V(f) \subseteq \mathrm{Spec}(A)$

Moreover: $D(U)^{\heartsuit} \simeq$ quasi-coherent sheaves on $U$

Proof: next lecture

## 4.3 Sheaves on sites

Recall site = category $\mathcal{C}$ with Grothendieck topology $\tau$

Then can talk about sheaves on $\mathcal{C}$.

Ex: $X$ top space $\rightsquigarrow$ $\mathcal{U}(X)$ topology generated by families $(U_i \to U)_i$ where $U_i \subseteq U$ opens $U = \bigcup U_i$.

Ex $X_{Zar}$ = small Zariski site $X$ scheme
as a category: $\mathcal{U}(X) = \{U \subseteq X \text{ opens}\}$
w/ induced topology

Ex (Big affine Zariski site)

$Sch^{aff} = \{$cat of affine schemes$\}$
$\tau$ = topology generated by finite Zar covering families $(U_i \hookrightarrow X)_{i \in I}$ open immersions which is jointly surjective. ($\coprod_i U_i \twoheadrightarrow X$ faithfully flat)

Def $\mathcal{C}$ site, $\mathcal{B} \subseteq \mathcal{C}$ full subcat is a basis for $\mathcal{C}$ if $\forall X \in \mathcal{C}$ $\exists$ family $(Y_\alpha \to X)_\alpha$ $Y_\alpha \in \mathcal{B}$ which generates a covering sieve.

- $\mathcal{B} \subseteq \mathcal{C}$ is $\cap$-closed if $\mathcal{C}$ admits fibred products, $\mathcal{B} \subseteq \mathcal{C}$ is closed under fibred products, and.

$\forall X \in \mathcal{C}$ $\exists$ family $(Y_\alpha \to X)_{\alpha \in A}$ $Y_\alpha \in \mathcal{B}$ and
$Y_{\alpha_0} \times_X \cdots \times_X Y_{\alpha_n}$

$$Y_{\alpha_0} \times_X \cdots \times_X Y_{\alpha_n} \in \mathcal{B} \quad \forall \{\alpha_0, \dots, \alpha_n\} \subseteq \Lambda.$$

(which generates a covering sieve).

<u>Rmk</u> $\mathcal{B} \subseteq \mathcal{C}$ basis $\Longrightarrow$ induced topology on $\mathcal{B}$ from $\mathcal{C}$
(a sieve is covering $\Longleftrightarrow$ image in $\mathcal{C}$ is covering)

<u>Theorem</u>: $\mathcal{C}$ site, $\mathcal{B} \subseteq \mathcal{C}$ $\cap$-closed basis
$\mathcal{F} \; \mathcal{C}^{op} \to \mathcal{V}$ presheaf
TFAE:
1) $\mathcal{F}$ is a sheaf on $\mathcal{C}$
2) $\mathcal{F}|_{\mathcal{B}}$ is a sheaf on $\mathcal{B}$

[Hoyois, Quadratic refinem App. C]

<u>Rmk</u>. Under assumptions on the topology of $\mathcal{C}$, can prove that descent ~~#~~ can be checked using only squares (Voevodsky)

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V}) & \xrightarrow{i^*} & \mathrm{Fun}(\mathcal{B}^{op}, \mathcal{V}) \\ \cup\!\!| & & \cup\!\!| \\ \mathrm{Shv}_{\mathcal{V}}(\mathcal{C}) & \xrightarrow{i^*} & \mathrm{Shv}_{\mathcal{V}}(\mathcal{B}) \end{array} \qquad i : \mathcal{B} \hookrightarrow \mathcal{C}$$

<u>Fact</u>. $i^* : \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{B}^{op}, \mathcal{V})$ admits a right adjoint: RKE = right Kan extension.

$$\mathcal{F} = \mathrm{RKE}(\mathcal{F}_0) : \mathcal{C}^{op} \to \mathcal{V} \qquad \mathcal{F}(X) \cong \varprojlim \mathcal{F}(X_0).$$

(given $\mathcal{F}_0 \; \mathcal{B}^{op} \to \mathcal{V}$)

taken over pairs $(X_0, u)$
$X_0 \in \mathcal{B}$, $u : X_0 \to X$ morphism